Integration and microlocal analysis in Colombeau algebras of generalized functions

Günther Hörmann
Institut für Mathematik
Universität Wien
Guenther.Hoermann@univie.ac.at

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Abstract

We study integration and Fourier transform in the Colombeau algebra $\mathcal{G}_\tau$ of tempered generalized functions using a general damping factor. This unifies different settings described earlier by Colombeau, Nedeljkov-Pilipović, and Damsma (for a simplified version). Further we prove characterizations of regularity for generalized functions in two situations: compactly supported or in the image of $\mathcal{S}'$ inside $\mathcal{G}_\tau$. Finally we investigate the notion of wave front set in the Colombeau algebra $\mathcal{G}(\Omega)$, Ω an open subset of $\mathbb{R}^n$, and show that it is in fact independent of the damping measure used for Fourier transform.

1 Introduction and basic notation

Generalized functions in the sense of Colombeau (cf.[1]) extend classical distribution theory to a consistent and efficient method for solving problems in nonlinear analysis. One main field of application and further development is the systematic investigation of partial differential operators involving nonlinear operations with singular objects (e.g. linear PDO with singular coefficients acting on distributions, nonlinear PDE with singular initial data etc.; cf. [9], and [4] for a survey of recent developments and results).

In distribution theory a refined tool to study propagation of singularities by PDO — and even the character of singularities — was introduced by Hörmander's definition of wave front set ([5], Ch.8). Furthermore it can be used to extend the operations of composition and multiplication in certain situations without leaving distribution theory. The notion of wave front set rests on spectral analysis of singularities, i.e. investigation of smoothness of a distribution near a point in $\mathbb{R}^n$ by decay properties of the Fourier transform of "localizations" around this point. This concept was carried over to Colombeau algebras by Pilipović in a consistent way, i.e. on the subspace of distributions the notion of wave front set remains unchanged (cf. [8] for a detailed representation).

As in the classical theory one therefore uses an extension of Fourier transform to tempered generalized functions. Such extensions were defined in different ways: directly on the Colombeau algebra of tempered generalized functions $\mathcal{G}_\tau$ by Colombeau ([1], Ch.4; considerably extended in [12]), Pilipović et.al. (cf. [8], Ch.1; here, slightly more general weight functions for spatial decay are considered), and Damsma ([2]: simplified Colombeau algebras with null ideal changed), or on variants of such algebras as in [3] (Fourier ultrafunctions, having as representatives sequences in the Schwartz space $\mathcal{S}$), and in [10, 11] (Fourier transform on $\mathcal{S}$ lifted to algebras of quotients of $\mathcal{S}$-sequence spaces).

All approaches to Fourier transform in tempered Colombeau algebras cited above use an extension of the classical integral formula on $S(\mathbb{R}^n)$. This seems natural since the classical duality method for extension to $\mathcal{S}'$ is not applicable in this more general context but elements are represented by sequences of smooth functions. It motivated various definitions of integration as a linear map from the algebra to the ring of generalized complex numbers (cf. [1], Ch.4, and [7] or [8], Ch.1).

In this paper we first unify integration theory and Fourier transform on the Colombeau algebra of tempered generalized functions and also point out some differences of the special settings (Sect. 2 and 3). Then we investigate and develop basic notions of regularity theory and microlocal analysis in this context (Sect. 4 and 5). It turns out that these are independent of the integration method used in computing Fourier transforms.

Throughout this paper we use notions and notation from Colombeau's theory of generalized functions as developed and described in [1] Ch.1,2,4, [9] Ch.III, [6] Ch.1, and [8] Ch.1.

Let us just recall the definition of tempered generalized functions in detail (cf. [1], Ch.4). $\mathcal{A}_m$ ($m \in \mathbb{N}_0$) denotes the set of tensor products of test functions with vanishing moments up to order m and integral equal to 1. For a test function ϕ we use the notation $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$.

Definition 1.1. $\mathcal{E}_{\mathrm{M},\tau}$ is the set of all maps $R\colon \mathcal{A}_0 \times \mathbb{R}^n \to \mathbb{C}$ with the following property: $\forall \alpha \in \mathbb{N}_0^n\ \exists N \in \mathbb{N}\ \forall \phi \in \mathcal{A}_N\ \exists c > 0, \eta > 0$ such that

$$|\partial^\alpha R(\phi_\varepsilon, x)| \le c(1+|x|)^N \varepsilon^{-N} \qquad \forall x \in \mathbb{R}^n, 0 < \varepsilon < \eta\ . \tag{1}$$

$\mathcal{N}_\tau$ is the subset of all $R \in \mathcal{E}_{\mathrm{M},\tau}$ with the property: $\forall \alpha \in \mathbb{N}_0^n\ \exists N \in \mathbb{N}\ \forall q \ge N$ and $\phi \in \mathcal{A}_q\ \exists c > 0, \eta > 0$:

$$|\partial^\alpha R(\phi_\varepsilon, x)| \le c(1+|x|)^N \varepsilon^{q-N} \qquad \forall x \in \mathbb{R}^n, 0 < \varepsilon < \eta\ . \tag{2}$$

$\mathcal{N}_\tau$ is an ideal in $\mathcal{E}_{\mathrm{M},\tau}$ and we define the algebra of *tempered generalized functions by*

$$\mathcal{G}_\tau = \mathcal{E}_{\mathrm{M},\tau}/\mathcal{N}_\tau\ . \tag{3}$$

Recall that the space of tempered distributions $\mathcal{S}'$ is embedded into $\mathcal{G}_\tau$ by convolution, i.e. $u \mapsto (u * \phi)_\phi$ factors to an injective map. On the subspace $\mathcal{O}_{\mathrm{C}}$ of smooth functions with uniform polynomial growth order for all derivatives – therefore in particular on $\mathcal{S}$ – this embedding is equal to the map induced by $u \mapsto (u)_\phi$. We denote by $\overline{\mathbb{C}}$ the ring of generalized complex numbers (i.e.

constants in $\mathcal{G}$). Frequently, if $R: \mathcal{A}_0 \times \mathbb{R}^n \to \mathbb{C}$ is moderate we denote its class in the Colombeau algebra by $[(R(\phi, .)_\phi]$ or $R + \mathcal{N}_\tau$ (as used in [6]). Fourier transform in $\mathcal{S}'$ will always be denoted by $\widehat{}$ and follow the conventions of [5], Ch.7.

2 Integration in $\mathcal{G}_\tau$

For generalized functions with compact support the integral can be defined by componentwise integration of a representative. It is easily seen that it gives a linear map $\mathcal{G}_c \to \overline{\mathbb{C}}$. In order to extend integration to a wider class of generalized functions one idea is to introduce a damping factor in the integrals of the representatives. Thus, if a generalized function is given by a representative $(\phi, x) \mapsto R(\phi, x)$ we want to define

$$\int (R + \mathcal{N}) dx := [(\int R(\phi, x) S(\phi, x) dx)_\phi] \,, \tag{4}$$

where $S(\phi, x)$ on the one hand produces convergence and on the other hand must not cut away too much information about R (i.e. should in some sense be "close to 1").

This can be achieved if the growth rate of (representatives of) the generalized functions with respect to x can be controlled, as it is the case for elements of $\mathcal{G}_\tau$. Several approaches of this kind exist (cf. [1], Ch.4, [7], [8], Sect.1.3, [2]) but were developed separately. Since all these yield very similar results concerning basic properties of integration and Fourier transform it is tempting to collect these into one theory.

Definition 2.1. A map $S: \mathcal{A}_0 \times \mathbb{R}^n \to \mathbb{C}$ is called a *damping measure if it has the following properties:*

(i) $\forall \phi \in \mathcal{A}_0$*:* $S(\phi, .) \in \mathcal{S}(\mathbb{R}^n)$

(ii) $\forall \phi \in \mathcal{A}_q$*,* $\forall p \in \mathbb{N}$*:* $\exists c, \eta > 0$*:* $|S(\phi_\varepsilon, x)| \le c\,\varepsilon^{-p}(1 + |x|)^{-p}$ *for all* $x \in \mathbb{R}^n$ *and* $0 < \varepsilon < \eta$

(iii) $\forall \phi \in \mathcal{A}_q$*:* $\exists c, \eta > 0$*:* $|S(\phi_\varepsilon, x) - 1| \le c\,\varepsilon^{q+1}|x|^{q+1}$ *for all* $x \in \mathbb{R}^n$ *and* $0 < \varepsilon < \eta$*.*

We have to check that with this definition equation (4) gives a well defined generalized number if R is a representative of an element in $\mathcal{G}_\tau$. Let $I_S(R, \phi) := \int_{\mathbb{R}^n} R(\phi, x) S(\phi, x) dx$ and set $I_S(R) := (I_S(R, \phi))_\phi$.

Lemma 2.2. Let S be a damping measure. If R is (a representative of) an element in $\mathcal{G}_\tau$ then $I_S(R)$ defines uniquely a generalized complex number, i.e. an element of $\overline{\mathbb{C}}$. $R \mapsto I_S(R)$ factors to a linear map $\mathcal{G}_\tau \to \overline{\mathbb{C}}$.

Proof. by definition of $\mathcal{E}_{\mathrm{M},\tau}$ and property (ii) of the above definition

$$|I_S(R, \phi_\varepsilon)| \le \int |R(\phi_\varepsilon, x)||S(\phi_\varepsilon, x)| dx \le c\,\varepsilon^{-N-p} \int (1 + |x|)^{N-p} dx$$

for fixed N and p arbitrary hence $I_S(R)$ is moderate; if $R \in \mathcal{N}_\tau$ we have $|I_S(R,\phi_\varepsilon)| \leq c\,\varepsilon^{q-N-p} \int (1+|x|)^{N-p} dx$ for fixed N, p arbitrary, and for all $q \geq N$ hence $I_S(R) = 0$ in $\overline{\mathbb{C}}$. The second assertion follows immediately. $\square$

We shortly discuss how former approaches to integration theory in $\mathcal{G}_\tau$ fit into our framework:

Example 2.3.

(i) Colombeau ([1], Ch.4) introduces a damping function of the form $S(\phi, x) = \widehat{\phi(x)}$. Essential properties are reflected in the fact that for the scaled delta nets we have $\widehat{\phi_\varepsilon}(x) = \widehat{\phi}(\varepsilon x)$. Therefore the damping factor is equal to the constant 1 in the limit $\varepsilon \to 0$. More precisely, by definition of $\mathcal{A}_q$ we have $|1 - \widehat{\phi_\varepsilon}(x)| = O\big((\varepsilon|x|)^{q+1}\big)$ if $\phi \in \mathcal{A}_q$. Further, since $\widehat{\phi}$ is in $\mathcal{S}$ for all $k \in \mathbb{N}$ an estimate

$$|\widehat{\phi_\varepsilon}(x)| \leq c_k(1+|\varepsilon x|)^{-k} \leq c_k \varepsilon^{-k}(1+|x|)^{-k}$$

holds. Note that $\widehat{\phi}$ cannot have compact support if $\phi \in \mathcal{A}_0$.

(ii) In [7] Nedeljkov and Pilipović developed a theory of integration, convolution, and Fourier Transform in the more general context of weighted spaces of generalized functions (we refer to the presentation in [8], Sect.1.3). For convenience of comparison and notational simplicity we concentrate on the special case of the weight function $t(r) = 1 + r$ which exactly reproduces $\mathcal{G}_\tau$. Two slightly different approaches are studied (cf. [8], Def.1.42):

(a) choose $\mu \in \mathcal{D}$ with $\mu = 1$ near 0 and define $S(\phi, x) = \mu(d(\phi)x)$ where $d(\phi) = \sup\{|y| \mid \phi(y) \neq 0\}$ is the support number of ϕ; the family $\mu^\varepsilon(.) = \mu(\varepsilon\, .)$ $(0 < \varepsilon \leq 1)$ is called *unit net;*

(b) a net $(\mu^\varepsilon)_{0<\varepsilon\leq 1}$ *is called* special unit net if

- $0 \leq \mu^\varepsilon \leq 1$ for all ε
- there exist constants b, $r > 0$ such that for all ε

$$\mu^\varepsilon(x) = \begin{cases} 1 & \text{if } |x| < b/\varepsilon \\ 0 & \text{if } |x| > b/\varepsilon + r \end{cases}$$

- all derivatives of μ^ε are bounded independently of ε; if $(\mu^\varepsilon)_\varepsilon$ is a special unit net set $S(\phi, x) = \mu^{d(\phi)}(x)$.

In both cases we have the following estimate (note that $d(\phi_\varepsilon) = \varepsilon d(\phi)$ and set $b = d(\mu)/d(\phi)$ in case (a))

$$|\mu^{d(\phi_\varepsilon)}(x)| \leq Ce^{-\varepsilon|x|/b} = C(\sum_l \frac{\varepsilon^l |x|^l}{b^l k!})^{-1} \leq C_k \varepsilon^{-k}(1+|x|)^{-k}$$

for arbitrary k if ε is small. Observe also that trivially we have $|\mu^{d(\phi_\varepsilon)}(x) - 1| = O(\varepsilon^{q+1}|x|^{q+1})$ for all q simply by the scaling-like properties of the supports in both cases.

(iii) Damsma (cf.[2]) describes integration and Fourier transform in a simplified version of the algebra of tempered generalized functions (however, he also changes the definition of the null ideal substantially). The damping function $S(\varepsilon, x)$ itself (by obvious adaption of notation) then defines itself an element of the algebra and has the following properties:

- for all multi-indices α and for all p_0 there is $p \geq p_0$ such that $|\partial_x^\alpha S(\varepsilon, x)| \leq c\varepsilon^{-p}(1+|x|)^{-p}$ holds for small ε
- there exists $R > 0$ such that $S(\varepsilon, x) = 1$ if $|x| < R/\varepsilon$.

In [3] he introduces the algebra of Fourier ultrafunctions consisting only of rapidly decreasing representatives. There is no need for a damping factor in integrals then.

This motivates the following

Definition 2.4. Let S be a damping measure. Then the linear map $\mathcal{G}_\tau \to \overline{\mathbb{C}}$, $R + \mathcal{N}_\tau \mapsto [I_S(R)]$, is called S-*integral. We will use the more suggestive notation* $\int R\, d_S x$ *instead.*

Remark 2.5.

(i) Clearly the integral could be defined over arbitrary Lebesgue-measurable subsets of $\mathbb{R}^n$ (cf. [8], Sect.1.3.1) but here we will concentrate on extension of Fourier transforms and wave front sets where cut-off is achieved by appropriate functions.

(ii) Since in Def.2.1 we did not include conditions on x-Derivatives we can not expect to prove a result on partial integration like in [1], Prop.4.2.9. If necessary it would be easy to vary conditions (ii) and (iii) to hold for derivatives also.

First we want to check consistency with usual integration in special cases: action of $\mathcal{G}_\tau$ on Schwartz functions, integrals of functions in $\mathcal{S}$ or $\mathcal{G}_c$. Recall that if $G \in \mathcal{G}_\tau$ and $f \in \mathcal{S}$ then $\langle G, f\rangle := [(\int G(\phi, x) f(x) dx)_\phi]$ is well-defined in $\overline{\mathbb{C}}$ (cf. [8], p.53). From now on S will always denote a damping measure.

Proposition 2.6.

(i) If $G \in \mathcal{G}_\tau$ and $f \in \mathcal{S}$ then $\langle G, f\rangle = \int Gf\, d_S x$.

(ii) If $f \in \mathcal{S}$ then $\int f d_S x = \int f dx$ in $\mathbb{C} \hookrightarrow \overline{\mathbb{C}}$.

(iii) If $G \in \mathcal{G}$ has compact support then $\int G dx = \int G d_S x$.

Proof. (ii) follows from (i) by setting $G = 1$; to prove (i) we estimate the difference of representatives (neglecting terms of the null ideal)

$$|\int G(\phi_\varepsilon, x) f(x)(1 - S(\phi_\varepsilon, x)) dx| \leq \int |G(\phi_\varepsilon, x)||f(x)||1 - S(\phi_\varepsilon, x)| dx$$

which is dominated by $c\,\varepsilon^{q+1-N} \int |x|^{q+1}(1+|x|)^{N-p} dx$ for fixed N, p arbitrary, and $\phi \in \mathcal{A}_q$ with $q \geq N$ arbitrary. (iii): clearly $\mathcal{G}_c \hookrightarrow \mathcal{G}_\tau$ (cf. [1],4.1.7); again estimating the difference of typical representatives we simply use the fact that on the support of G by property (ii) of S we have $|1 - S(\phi_\varepsilon, x)| \leq C\varepsilon^{q+1}$ if $\phi \in \mathcal{A}_q$. $\square$

Because most proofs of "standard properties" are more or less just restatements of already worked-out proofs of the special settings described above we will quickly proceed to the notions of regularity and wave front set. In the next section we just summarize elementary facts about the Fourier transform in our framework. First, however, we reconsider an example of [1], 4.2.8, in order to illustrate some differences in varying the damping measure S.

Example 2.7. The function $f(x) = x^k$ is in $\mathcal{O}_{\mathrm{C}}(\mathbb{R})$ and can therefore be identified with $[(f)_\phi]$ in $\mathcal{G}_\tau$. We compute $\int x^k d_S x$: by the classical Fourier inversion theorem applied to $x \mapsto S(\phi, x)$

$$\int x^k S(\phi, x) dx = \frac{i^k}{2\pi} \int \Big(\widehat{S(\phi, .)}^{(k)}\Big)\widehat{\ }(-x) dx = i^k \widehat{S(\phi, .)}^{(k)}(0) \ .$$

In Colombeau's setting $\widehat{S(\phi, .)}^{(k)} = 2\pi(-1)^k \phi^{(k)}(-.)$ and therefore

$$\int x^k d_S x = 2\pi(-i)^k [(\phi^{(k)}(0))_\phi] = 2\pi(-i)^k \delta^{(k)}(0) \ ,$$

considered as generalized point value of δ at 0. This also nicely reflects a similar computation in $\mathcal{S}'$ (i.e. Fourier transform of x^k). If S is given by a unit net according to setting (ii)(a) described above we get

$$\int x^k d_S x = [(\frac{i^k}{d(\phi)^{k+1}} \widehat{\mu}^{(k)}(0))_\phi] \ .$$

Now set $k = 0$ and consider the difference of both results on ϕ_ε. This is equal to

$$\frac{1}{\varepsilon}\big(2\pi\phi(0) - \frac{\widehat{\mu}(0)}{d(\phi)}\big)$$

which is in the null ideal if and only if $2\pi d(\phi)\phi(0) = \widehat{\mu}(0)$ for all $\phi \in \mathcal{A}_q$ for some q. This is impossible to hold for fixed μ because the left hand side will take on (purely) imaginary and real values for certain choices of ϕ.

3 Fourier transform in $\mathcal{G}_\tau$

Turning now to the definition of Fourier transform, let $R \in \mathcal{E}_{\mathrm{M},\tau}$ and define $\mathcal{F}_S R \colon \mathcal{A}_0 \times \mathbb{R}^n \to \mathbb{C}$ by

$$(\mathcal{F}_S R)(\phi, x) = \int e^{-i\langle x|y\rangle} R(\phi, y) S(\phi, y) dy \ , \tag{5}$$

or by abuse of notation $\mathcal{F}_S R(x) = \int e^{-ixy} R(y) d_S y$. It is immediately seen that this defines again an element of $\mathcal{G}_\tau$ (i.e. $\mathcal{F}_S R \in \mathcal{E}_{\mathrm{M},\tau}$ and $\mathcal{F}_S R \in \mathcal{N}_\tau$ if $R \in \mathcal{N}_\tau$). We denote the induced linear map again by $\mathcal{F}_S \colon \mathcal{G}_\tau \to \mathcal{G}_\tau$ and call it *generalized S-Fourier transform. The original version according to Colombeau's damping measure is given in his book [1]. Extensive further investigation of its properties and new results are presented by Soraggi (cf. [12]).*

The following consistency results hold and are immediate consequences of the definition and of Prop.2.6.

Proposition 3.1.

(i) If $f \in \mathcal{S}$ then $\mathcal{F}_S f = \widehat{f}$.

(ii) Let $R \in \mathcal{G}_\tau$ and $f \in \mathcal{S}$ then $\langle \mathcal{F}_S R, f\rangle = \langle R, \widehat{f}\rangle$.

(iii) R, f as in (ii) then $\int \mathcal{F}_S R \cdot f \, d_S x = \int R \cdot \widehat{f} \, d_S x$.

The natural candidate for S-Fourier inversion clearly is given by $(\mathcal{F}_S^* R)(x) = (1/2\pi)^n \int e^{i\langle x|y\rangle} R(y) d_S y$. *An analogous version of the above proposition is then valid for* $\mathcal{F}_S^*$. *However, as we know from [1], Rem.4.3.9, in general an inversion theorem in the strict sense cannot hold. We will show below that actually it can never hold for* $\mathcal{F}_S$. *Nevertheless a weak form of the inversion theorem is true. Recall that two elements* U *and* V *of* $\mathcal{G}_\tau$ *are said to be* equal in the sense of generalized tempered distributions, denoted by $U \stackrel{\text{gtd}}{=} V$, iff $\langle U, \psi\rangle = \langle V, \psi\rangle$ for all $\psi \in \mathcal{S}$. Hence as an immediate consequence of Prop.3.1 and its analogue for $\mathcal{F}_S^*$ we state

Corollary 3.2. For all $R \in \mathcal{G}_\tau$: $\quad \mathcal{F}_S^* \mathcal{F}_S R \stackrel{\text{gtd}}{=} R \stackrel{\text{gtd}}{=} \mathcal{F}_S \mathcal{F}_S^* R$.

Finally we list basic properties concerning exchange of differentiation with multiplication by Fourier transform and also weakened consistency with $\mathcal{S}'$.

Theorem 3.3. Let $U \in \mathcal{G}_\tau$

(i) $\mathcal{F}_S(iy_j U) = -\partial_j \mathcal{F}_S U$ and $\mathcal{F}_S^*(iy_j U) = \partial_j \mathcal{F}_S^* U$

(ii) $\mathcal{F}_S(\partial_j U) \stackrel{\text{gtd}}{=} ix_j \mathcal{F}_S U$ and $\mathcal{F}_S^*(\partial_j U) \stackrel{\text{gtd}}{=} -ix_j \mathcal{F}_S^* U$

(iii) If $U \in \mathcal{G}_\mathrm{c}$ then $\mathcal{F}_S(\partial_j U) = ix_j \mathcal{F}_S U$ and $\mathcal{F}_S^*(\partial_j U) = -ix_j \mathcal{F}_S^* U$

(iv) If $f \in \mathcal{S}'$ then for all $\psi \in \mathcal{S}$: $\langle \iota(\widehat{f}), \psi\rangle \approx \langle \mathcal{F}_S \ \iota(f), \psi\rangle$ (association in $\overline{\mathbb{C}}$). Here, ι denotes the canonical embedding $\mathcal{S}' \to \mathcal{G}_\tau$, $f \mapsto [(f * \phi)_\phi]$.

Proof. (i) as in the proof of [1], Prop.4.3.5, the typical representatives of both sides are equal.

(ii) using (i) and Cor.3.2 we have $\mathcal{F}_S^*(ix_j \mathcal{F}_S U) = \partial_j \mathcal{F}_S^* \mathcal{F}_S U \stackrel{\text{gtd}}{=} \partial_j U$ and therefore $ix_j \mathcal{F}_S U \stackrel{\text{gtd}}{=} \mathcal{F}_S \mathcal{F}_S^*(ix_j \mathcal{F}_S U) \stackrel{\text{gtd}}{=} \mathcal{F}_S(\partial_j U)$; analogously for $\mathcal{F}_S^*$.

(iii) by Prop.2.6(iii) the integrals may be computed without the damping factor $S(\phi, x)$; but then partial integration translates a representative of the left hand side into one of the right hand side; the computations for $\mathcal{F}_S^*$ are completely analogous.

(iv) we just need to observe that $\langle \widehat{f} * \phi_\varepsilon, \psi\rangle - \langle f * \phi_\varepsilon, \widehat{\psi}\rangle = \langle \widehat{f}, \check{\phi}_\varepsilon * \psi\rangle - \langle f, \check{\phi}_\varepsilon * \widehat{\psi}\rangle$ tends to $\langle \widehat{f}, \psi\rangle - \langle f, \widehat{\psi}\rangle = 0$ for $\varepsilon \to 0$.

$\square$

Colombeau's example ([1], Rem.4.3.7.), i.e. $S(\phi, .) = \widehat{\phi}$ and $U = 1 \in \mathcal{G}_\tau(\mathbb{R})$, shows that Thm.3.3(ii) cannot hold with equality in $\mathcal{G}_\tau$ in general. As a consequence Cor.3.2 cannot hold in the strong sense since together with Thm.3.3(i) this would yield a contradiction (as in [1], Rem.4.3.9). But his example can also serve to show that for no damping measure S a strict inversion theorem can hold: with S an arbitrary damping measure we have $\mathcal{F}_S \frac{d}{dx} U = \mathcal{F}_S 0 = 0$; on the other hand a representative of $ix\mathcal{F}_S U$ is given by

$$ix \int e^{-ixy} S(\phi, y) dy = ix\widehat{S(\phi, .)}(x) \ .$$

If this lies in the null ideal $\mathcal{N}_\tau$ then there exist $N \in \mathbb{N}$ and for all $q \geq N$ constants $c, \eta > 0$ such that

$$|(ix\widehat{S(\phi_\varepsilon, .)}(x))'| \leq c(1 + |x|)^N \varepsilon^{q-N} \qquad \forall x, \varepsilon < \eta \ ;$$

the left hand side evaluated at $x = 0$ gives $|\widehat{S(\phi_\varepsilon, .)}(0)|$ which is unbounded by condition (iii) of Def.2.1. But the right hand side tends to 0 as $\varepsilon \to 0$.

Corollary 3.4. For every damping measure S there exists $U \in \mathcal{G}_\tau$ such that $\mathcal{F}_S \partial_j U \neq ix_j \mathcal{F}_S U$. Furthermore, Fourier inversion by $\mathcal{F}_S^*$ does not hold strictly in $\mathcal{G}_\tau$.

Remark 3.5.

(i) There are important results concerning Fourier transform of convolutions which are valid for example in case of special unit nets but not in general (cf. [8], Thm.1.11(c) and Ex.1.19).

(ii) The above impossibility result fits nicely with Damsma's observations in [3], Lemma 1 and 2. This was one reason to introduce the algebra of Fourier ultrafunctions in his paper. Fourier transform on this algebra is a linear isomorphism with many classical properties.

(iii) Radyno et.al. (cf. [10, 11]) also considered Fourier transform lifted to algebras built up by equivalence classes of moderate sequences in S. In this case the exact formula of interchange of partial differentiation with multiplication by variables is still valid.

(iv) In [12], Thm.3.6, Soraggi proved that Fourier transform in the sense of Colombeau is injective when restricted to $\mathcal{G}_c(\mathbb{R}^n)$. Using Prop.2.6, (iii), it follows from his result that this also holds for $\mathcal{F}_S$.

4 Regular generalized tempered functions

In [9], Sect.25, regularity theory is developed intrinsic to the algebra $\mathcal{G}$ of generalized functions. It turns out that the subalgebra $\mathcal{G}^\infty$ of elements (having representatives) of uniform ε-growth order for all derivatives is an appropriate substitute for the subspace C^∞ inside $\mathcal{D}'$. This is emphasized by results on hypoelliptic operators, propagation of singularities, and the remarkable proof of the equality ([9],Thm.25.2)

$$\mathcal{G}^\infty \cap \mathcal{D}' = \mathrm{C}^\infty \tag{6}$$

(valid on arbitrary open subsets of $\mathbb{R}^n$). Hence there is a concept of (generalized) singular support and local regularity in the sense of $\mathcal{G}^\infty$. Since $\mathcal{G}_c \hookrightarrow \mathcal{G}_\tau$ this local concept can be carried over to $\mathcal{G}_\tau$ locally by appropriate cut-off with test functions. This was refined and developed further in [8] towards microlocal analysis of generalized functions. A global variant of $\mathcal{G}^\infty$ was defined as follows (cf. [8], Def.1.46).

Definition 4.1. $\mathcal{G}_\tau^\infty$ is the subalgebra of those elements in $\mathcal{G}_\tau$ having representatives R with the following property: $\exists N \in \mathbb{N}_0 : \forall \alpha \in \mathbb{N}_0^n \; \exists M \in \mathbb{N}_0$: $\forall \phi \in \mathcal{A}_M$ $\exists c, \eta > 0$ such that

$$|\partial^\alpha R(\phi_\varepsilon, x)| \leq c\varepsilon^{-N}(1+|x|)^M \qquad \forall x, \varepsilon < \eta \; . \tag{7}$$

We will show that there is an analogue to equ.(6) for $\mathcal{G}_\tau^\infty$ with $\mathcal{S}'$ instead of $\mathcal{D}'$. Before we will state a lemma of M. Oberguggenberger which is actually included in the proof of [9], Thm.25.2.

Lemma 4.2. Let $v \in \mathcal{E}'$, $\phi \in \mathcal{A}_0$. If there exists $N \in \mathbb{N}_0$ such that for all $m \in \mathbb{N}$ there are positive constants c, η such that

$$\|\Delta^m v * \phi_\varepsilon\|_{\mathrm{L}^1} \leq c\varepsilon^{-N} \qquad (0 < \varepsilon < \eta) \tag{8}$$

(Δ is the Laplace operator) then v is smooth.

Proof. (adapted from [9], pp.275-277) Taking Fourier transform we have for some constant c'

$$\| |\xi|^{2m} \widehat{v}\widehat{\phi_\varepsilon}\|_{\mathrm{L}^\infty} \leq c'\varepsilon^{-N} \; .$$

We want to show that $\widehat{v}$ is rapidly decreasing. Therefore we estimate

$$\varepsilon^N |\xi|^{2m} |\widehat{v}(\xi)| \leq \varepsilon^N |\xi|^{2m} |(\widehat{v}\widehat{\phi_\varepsilon})(\xi)| + \varepsilon^N |\xi|^{2m} |\widehat{v}(\xi)| |1 - \widehat{\phi}(\varepsilon\xi)| \; .$$

The first term on the right hand side is dominated by a constant. For the second one we use $|1 - \widehat{\phi}(\varepsilon\xi)| \leq c''\varepsilon|\xi|$ and that $\widehat{v}$ is polynomially bounded, of order $K-1$ say. Hence we get

$$\varepsilon^N |\xi|^{2m} |\widehat{v}(\xi)| \leq c_1 + c_2 \varepsilon^{N+1}(1+|\xi|)^{2m+K} \tag{9}$$

for some positive constants c_1, c_2 and ε small.

We assert that $a(\xi) := |\xi|^{(2m-KN)/(N+1)} |\widehat{v}(\xi)|$ is bounded. Otherwise there exists a sequence $(\xi_j)_j$ with $|\xi_j| \to \infty$ $(j \to \infty)$ such that $a(\xi_j) \to \infty$. Setting $\varepsilon_j = 1/|\xi_j|^{(2m+K)/(N+1)}$ yields a contradiction in estimate (9) because the right hand side would stay finite whereas the left hand side tends to infinity.

Since m is arbitrary we conclude from the boundedness of a that $\widehat{v}$ is rapidly decreasing and hence that v is smooth. ∎

Theorem 4.3. $\qquad \mathcal{G}_\tau^\infty \cap \mathcal{S}' = \mathcal{O}_\mathrm{M}$

Proof. If $f \in \mathcal{O}_\mathrm{M}$ then it has a representative $(f * \phi)_\phi$ in $\mathcal{E}_{\mathrm{M},\tau}$; if $\alpha \in \mathbb{N}_0^n$ there is $M \in \mathbb{N}$ and $c > 0$ such that $|\partial^\alpha f(y)| \leq c(1+|y|^2)^M$ and therefore

$$|\partial^\alpha f * \phi_\varepsilon(x)| = |\int \partial^\alpha f(x - \varepsilon z)\phi(z) dz| \leq c \int (1 + |x - \varepsilon z|^2)^M |\phi(z)| dz \; .$$

By Peetre's inequality (applied to $x-\varepsilon z$, εz) and since $\varepsilon \le 1$ we have $(1+|x-\varepsilon z|^2)^M \le 2^M(1+|x|^2)^M(1+|z|^2)^M$ hence the above integral can be dominated by some constant times $(1+|x|^2)^M$. Therefore $[(f*\phi)_\phi]$ is in $\mathcal{G}_\tau^\infty$.

Let $U \in \mathcal{S}' \cap \mathcal{G}_\tau^\infty$ with typical representative $(u*\phi)_\phi$ for some $u \in \mathcal{S}'$. By definition of $\mathcal{G}_\tau^\infty$ we have the following: $\exists N$: $\forall \alpha\ \exists M\ \forall \phi \in \mathcal{A}_M$

$$|\partial^\alpha u * \phi_\varepsilon(x)| \le c\,\varepsilon^{-N}(1+|x|)^M \tag{10}$$

for some positive constant c and ε small.

First we will show that $u \in \mathrm{C}^\infty$. It is enough to show that χu is smooth for all $\chi \in \mathcal{D}$. Since $\mathrm{supp}(\chi u) * \phi_\varepsilon$ is compact we have by (10)

$$\|\partial^\alpha(\chi u) * \phi_\varepsilon\|_{\mathrm{L}^1} \le c_{\chi,\alpha}\,\varepsilon^{-N}$$

and setting $\alpha = (2m,\dots,2m)$ it follows from Lemma 4.2 that χu is smooth.

Finally we have to show that u is actually in $\mathcal{O}_\mathrm{M}$. Set $\alpha = \beta + \gamma$ with $|\gamma| = N$ for N chosen in (10) and β arbitrary. Then

$$c\,(1+|x|)^M\varepsilon^{-N} \ge |\partial^\beta u * \partial^\gamma(\phi_\varepsilon)(x)| = \frac{1}{\varepsilon^{|\gamma|}}\Big|\int \partial^\beta u(x-\varepsilon y)\partial^\gamma\phi(y)dy\Big|$$

and therefore

$$c\,(1+|x|)^M \ge \Big|\int \partial^\beta u(x-\varepsilon y)\partial^\gamma\phi(y)dy\Big| \to C_\phi\,|\partial^\beta u(x)| \qquad \varepsilon \to 0$$

for $\phi \in \mathcal{A}_M$ arbitrary. We conclude that u is in $\mathcal{O}_\mathrm{M}$. $\square$

Classically smoothness of a compactly supported distribution can be tested by decrease properties of its Fourier transform. We introduce the appropriate notion in the following definition (in a generality useful for microlocal analysis studied in the next section).

Definition 4.4. Let Γ be a cone in $\mathbb{R}^n$. $R \in \mathcal{G}_\tau$ is said to be *rapidly decreasing in Γ if it has a representative $R(\phi,x)$ with the following property: $\exists N \in \mathbb{N}$: $\forall p \in \mathbb{N}_0\ \exists M \in \mathbb{N}_0$: $\forall \phi \in \mathcal{A}_M$ there are positive constants c, η such that*

$$|R(\phi_\varepsilon,x)| \le c\,\varepsilon^{-N}(1+|x|)^{-p} \qquad \forall x \in \Gamma, 0<\varepsilon<\eta\,.$$

If $\Gamma = \mathbb{R}^n$ we simply say that R is rapidly decreasing.

Theorem 4.5. If $U \in \mathcal{G}_c$ and S is a damping measure then the following are equivalent:

(i) $U \in \mathcal{G}^\infty$

(ii) $\mathcal{F}_S U$ is rapidly decreasing

Proof.
(i)→(ii): follows from Thm.3.3(iii) as in the classical case.

(ii)→(i): let $U(\phi, x)$ be a representative with compact support and $\psi \in \mathcal{S}$ arbitrary then we compute (neglecting terms of the null ideal) using Prop.3.1(ii), Thm.3.3(iii), and the decrease property for $\mathcal{F}_S U$ successively

$$|\langle \partial^\alpha U(\phi_\varepsilon, .), \widehat{\psi}\rangle| = |\langle \mathcal{F}_S \partial^\alpha U(\phi_\varepsilon, .), \psi\rangle = |\langle \xi^\alpha \mathcal{F}_S U(\phi_\varepsilon, .), \psi\rangle| \leq$$

$$c\int |\xi|^{|\alpha|} |\mathcal{F}_S U(\phi_\varepsilon, \xi)||\psi(\xi)| d\xi \leq c'\, \varepsilon^{-N} \int \frac{|\xi|^{|\alpha|}}{(1+|\xi|)^p} |\psi(\xi)| d\xi \leq c''\, \varepsilon^{-N} \|\psi\|_{\mathrm{L}^1}$$

for $p > |\alpha|$, ε small, and $c'' > 0$ depending only on α, ϕ, and p. Since ψ was arbitrary it follows that

$$\|\partial^\alpha U(\phi_\varepsilon, .)\|_{\mathrm{L}^\infty} \leq \tilde{c}\, \varepsilon^{-N}$$

with N independent of α. Hence U is in $\mathcal{G}^\infty$. □

Remark 4.6.

(i) Note that for $U \in \mathcal{G}_c$ the properties to be in $\mathcal{G}^\infty$ or $\mathcal{G}^\infty_\tau$ are equivalent and that condition (i) of Thm.4.5 is independent of the damping measure S. We will further investigate this in the next section.

(ii) The above characterization is just a pointer to Paley-Wiener like results. Much progress in this direction was already achieved in settings with more specified damping measures. We refer to [8] and [12] instead of just copying the results and arguments (nearly literally) to our framework.

(iii) A result related to Thm.4.5 is given in [8], Prop.3.13, using the term $\mathcal{G}^\infty$-rapidly decreasing ([8], Def.3.12) for a tempered generalized function G having a representative $G(\phi, .)$ with the following property: $\exists N \in \mathbb{N}$: $\forall \alpha \in \mathbb{N}_0^n$ $\forall p \in \mathbb{N}$ $\exists M \in \mathbb{N}_0$: $\forall \phi \in \mathcal{A}_M$

$$|\partial^\alpha G(\phi_\varepsilon, x)| \leq C \frac{\varepsilon^{-N}}{(1+|x|^2)^{p/2}} \forall x \in \mathbb{R}^n \tag{11}$$

for some constant $C > 0$ and small ε. Then it is proved that the Fourier transform (using a unit net as damping measure) of a $\mathcal{G}^\infty$-rapidly decreasing function has again this property (clearly, the converse is also true). This is an analogue of the fact that the classical Fourier transform maps S into S.

(iv) It follows directly from Thm.3.3(iii) that in Thm.4.5 the two conditions are then actually equivalent to $\mathcal{F}_S U$ being $\mathcal{G}^\infty$-rapidly decreasing.

5 Equivalence of basic microlocal properties

Microlocal analysis and pseudo-differential calculus in Colombeau algebras are presented in [8], Ch.3, based on Fourier transform and integration using damping measures defined by (special) unit nets. The aim of this section is to show that concerning the basic definitions and facts this fits into the current setting and at the same time maintains the complete range of generality.

We proceed along the lines of classical theory as presented in [5], Sect.8.1. If $U \in \mathcal{G}_c$ we define the cone $\Sigma_S(U) \subseteq \mathbb{R}^n \setminus 0$ to be the complement of those points having open conic neighborhoods Γ such that $\mathcal{F}_S U$ is rapidly decreasing in Γ (and 0 excluded in any case). $\Sigma_S(U)$ is closed in $\mathbb{R}^n \setminus 0$ and by Thm.4.5 is empty if and only if $U \in \mathcal{G}^\infty$. As one would guess from Thm.4.5 this notion is actually independent of S.

Lemma 5.1. If S_1 and S_2 are damping measures and $U \in \mathcal{G}_c$ then $\Sigma_{S_1}(U) = \Sigma_{S_2}(U)$. Hence we may write $\Sigma_{\mathrm{g}}(U)$ instead.

Proof. Assume that $\mathcal{F}_{S_1} U$ is rapidly decreasing in the open cone Γ; let $U(\phi, .)$ be a representative of U then there exist $N \in \mathbb{N}_0$ and for all $p \in \mathbb{N}_0$ a constant c such that (for small ε)

$$\|(1+|\xi|)^p \mathcal{F}_{S_1} U(\phi_\varepsilon, .)\|_{\mathrm{L}^\infty(\Gamma)} \leq c\, \varepsilon^{-N} \ .$$

If $\psi \in \mathcal{S}$ with $\operatorname{supp}\psi \subseteq \Gamma$ we have therefore

$$c\, \varepsilon^{-N} \|\psi\|_{\mathrm{L}^1} \geq |\langle (1+|\xi|)^p \mathcal{F}_{S_1} U(\phi_\varepsilon, .), \psi\rangle| = |\langle \mathcal{F}_{S_1} U(\phi_\varepsilon, .), (1+|\xi|)^p \psi\rangle| \ ,$$

interpreted in the pairing with $\mathrm{L}^\infty(\Gamma)$ as dual of $\mathrm{L}^1(\Gamma)$. But according to Prop.3.1(ii) the right hand side is equal to $|\langle U(\phi_\varepsilon, .), (\psi(1+|\xi|)^p)\widehat{\ }\rangle|$ apart from some null ideal terms in $\overline{\mathbb{C}}$; again by the same arguments this differs from the expression $|\langle (1+|\xi|)^p \mathcal{F}_{S_2} U(\phi_\varepsilon, .), \psi\rangle|$ only by terms of order ε^q for arbitrary q. Since the set of all $\psi \in \mathcal{S}$ with $\operatorname{supp}\psi \subseteq \Gamma$ is dense in $\mathrm{L}^1(\Gamma)$ we conclude that $(1+|\xi|)^p \mathcal{F}_{S_2} U(\phi_\varepsilon, .)$ is bounded in Γ by some constant times ε^{-N} and hence $\mathcal{F}_{S_2} U$ is rapidly decreasing in Γ. Interchanging the roles of S_1 and S_2 completes the proof. □

If $\varphi \in \mathcal{D}$ and $U \in \mathcal{G}_c$ there is a representative of $\mathcal{F}_S(\varphi U)$ of the form $\widehat{\varphi} * \widehat{U(\phi, .)}$ (use Prop.2.6(iii)) and the reasoning in the proof of [5], Lemma 8.1.1, can be applied to show

$$\Sigma_{\mathrm{g}}(\varphi U) \subseteq \Sigma_{\mathrm{g}}(U) \ . \tag{12}$$

Moreover one can also copy the proof of the following property: let $x_0 \in \mathbb{R}^n$ and $(\varphi_\nu)_\nu$ be a net in $\mathcal{D}$ with $\varphi_\nu(x_0) \neq 0$, $\operatorname{supp}\varphi_\nu \to \{x_0\}$ then

$$\lim_\nu \Sigma_{\mathrm{g}}(\varphi_\nu U) = \bigcap_{\varphi \in \mathcal{D}, \varphi(x_0) \neq 0} \Sigma_{\mathrm{g}}(\varphi U) \tag{13}$$

in the sense that $\Sigma_{\mathrm{g}}(\varphi_\nu U)$ will finally be contained in any open cone containing the right hand side. This enables us to transfer Hörmander's definition of wave front set ([5], Def.8.1.2) to the Colombeau algebra. We stick to the notation of [8] because the definition turns out to be equivalent.

Definition 5.2. Let Ω be open in $\mathbb{R}^n$ and $U \in \mathcal{G}(\Omega)$. Then define the *cone of irregular directions at* x_0 *by*

$$\Sigma_{\mathrm{g},x_0}(U) = \bigcap_{\varphi \in \mathcal{D}(\Omega), \varphi(x_0) \neq 0} \Sigma_{\mathrm{g}}(\varphi U) \ . \tag{14}$$

The wave front set of U is the set

$$\mathrm{WF}_{\mathrm{g}}(U) = \{(x,\xi) \in \Omega \times \mathbb{R}^n \setminus 0 \mid \xi \in \Sigma_{\mathrm{g},x}(U)\} \ . \tag{15}$$

Remark 5.3. Completely analogous to the classical case one has $\Sigma_{\mathrm{g},x}(U) \neq \emptyset$ if and only if $x \in \mathrm{singsupp}_{\mathrm{g}} U$. As a consequence the projection of $\mathrm{WF}_{\mathrm{g}}(U)$ to the first component is equal to $\mathrm{singsupp}_{\mathrm{g}} U$. Furthermore the wave front set is a closed subset of $\Omega \times \mathbb{R}^n \setminus 0$ which is conic in the second component. If U has compact support then the projection of $\mathrm{WF}_{\mathrm{g}}(U)$ to the second component is $\Sigma_{\mathrm{g}}(U)$ (proof exactly as in [5], Prop.8.1.3).

By Lemma 5.1 the cone $\Sigma_{g,x_0}(U)$ of irregular directions at a point and hence the wave front set $\mathrm{WF}_g(U)$ is independent of the damping measure S chosen for computation of $\mathcal{F}_S$. We will prove that Def.5.2 is equivalent to the definition of wave front set according to [8], Def.3.14. To this end we use [8], Cor.3.2, which gives formulae completely analogous to (15) and (14) but with $\Sigma_g(\varphi U)$ defined in another way ([8], Def.3.13; note that in the last line of that definition it should read $\psi \mathcal{F} G$ instead of ψG). Therefore we have to show that both definitions for $\Sigma_g(U)$ coincide when U is in $\mathcal{G}_c$. This is the content of the following lemma.

Lemma 5.4. Let $U \in \mathcal{G}_c$ and S be a damping measure. Then $\Sigma_g(U)$ is equal to the complement of all points ξ_0 in $\mathbb{R}^n \setminus 0$ having a conic convex open neighborhood Γ with the following property: $\exists\ \psi \in \mathrm{C}^\infty(\mathbb{R}^n)$ with $\mathrm{supp}\,\psi \subset \Gamma$, ψ positive-homogeneous of degree 0 outside some ball of radius r, $0 < r < \xi_0$, and $\psi = 1$ near ξ_0 such that $\psi \mathcal{F}_S U$ is $\mathcal{G}^\infty$-rapidly decreasing (in the sense of Rem.4.6, (iii)).

Proof. First we note that $\Sigma_g(U)$ is contained in the described set since the stated decay in Γ and the properties of ψ imply rapid decrease of $\mathcal{F}_S U$ in Γ.

To prove the reverse inclusion we assume that $\xi_0 \notin \Sigma_g(U)$ and nonzero. By Rem.4.6, (iv), we conclude that $\mathcal{F}_S U$ is $\mathcal{G}^\infty$-rapidly decreasing in an open convex conic neighborhood Γ of ξ_0. This means exactly that all derivatives $\partial^\alpha \mathcal{F}_S U$ are rapidly decreasing in Γ. Choose a smooth function ψ with $\mathrm{supp}\,\psi \subset \Gamma$, $\psi(\xi) = 1$ for $|\xi| \geq |\xi_0|/2$ and ξ in some closed conic neighborhood Γ_0 of ξ_0 contained in Γ. Then $\psi \partial^\alpha \mathcal{F}_S U$ is rapidly decreasing in $\mathbb{R}^n$ for all $\alpha \in \mathbb{N}_0^n$. For $|\xi|$ large derivatives of order > 0 of ψ vanish in Γ_0 hence $\partial^\alpha\big(\psi \mathcal{F}_S U\big)$ is rapidly decreasing by the Leibniz rule. ∎

To summarize we arrive at the very satisfying conclusion that Def.5.2 gives an equivalent notion of wave front to that introduced in [8], Sect.3.2.2.

Corollary 5.5. The wave front set of a generalized function in $\mathcal{G}(\Omega)$ according to Definition 5.2 can equivalently be determined by the methods of [8], Section 3.2.2.

Hence all results of [8], pp.124-131, are valid also in the presented context. Especially consistency with the distributional wave front set is valid (proved in [8], Sect.3.2.3). More explicitely, if Ω is open in $\mathbb{R}^n$ and ι denotes the canonical embedding $\mathcal{D}'(\Omega) \hookrightarrow \mathcal{G}(\Omega)$, we have the following

Theorem 5.6. If $u \in \mathcal{D}'(\Omega)$ then $\mathrm{WF}(u) = \mathrm{WF}_g(\iota(u))$.

Finally we give an example illustrating the notion in a situation beyond distributions arising from multiplication of distributions.

Example 5.7. In $\mathcal{G}(\mathbb{R})$ let $U = \iota(1/(x+i0))$ and $V = \iota(1/(x-i0))$ and consider $W = U \cdot V$. Note that $\mathrm{WF_g}(U) = \mathrm{WF}(\frac{1}{x+i0}) = \{0\} \times \mathbb{R}^+$ and $\mathrm{WF_g}(V) = \mathrm{WF}(\frac{1}{x-i0}) = \{0\} \times \mathbb{R}^-$. Therefore Hörmander's wave front rule for defining a product in $\mathcal{D}'$ is not applicable. More general with respect to the hierarchy of distributional products in [9], p.69, even the model product does not exist (argument similar to [9], Ex.1.4, Case 3). In $\mathcal{G}$ a representative $W(\phi, x)$ of W is formed as product of the typical representatives (use $1/(x \pm i0) = \mathrm{vp}(1/x) \mp i\pi\delta$)

$$\frac{1}{x \pm i0} * \phi(x) = \int_0^\infty \frac{\phi(x-y) - \phi(x+y)}{y} dy \mp i\pi\phi(x)$$

of U and V. We want to determine the wave front set of W in the sense of Def.5.2.

Assertion 1: $\mathrm{singsupp_g} W = \{0\}$

The k-th derivative of $W(\phi_\varepsilon, x)$ can be written as a polynomial of order $\leq k$ in expressions of the form

$$A_\varepsilon(x) = (\frac{d}{dx})^l \int_0^\infty \frac{\phi(\frac{x-y}{\varepsilon}) - \phi(\frac{x+y}{\varepsilon})}{\varepsilon y} dy \quad \text{and} \quad B_\varepsilon(x) = \frac{1}{\varepsilon^{l+1}} \phi^{(l)}(\frac{x}{\varepsilon}) \; ,$$

where $0 \leq l \leq k$. Let K be compact in $\mathbb{R}$; assume first that $0 \in K$; then the values $A_\varepsilon(0) = \langle \mathrm{vp}(1/y), \phi^{(l)}(y/\varepsilon) \rangle / \varepsilon^{l+1}$, $B_\varepsilon(0) = \phi^{(l)}(0)/\varepsilon^{l+1}$ can never be dominated by $O(\varepsilon^{-N})$ for fixed N for all k and ϕ; otherwise if 0 has distance $\alpha > 0$ from K then B_ε will vanish on K if ε is small enough because $\mathrm{supp}\,\phi$ is compact; for the same reason in expression A_ε the integration variable y will be bounded away from 0 and then splitting into a difference of integrals and substitution $((x \pm y)/\varepsilon)$ will bring each integral into the form

$$(\frac{d}{dx})^l \int_{-\infty}^\infty \frac{\phi(y)}{x - \varepsilon y} dy = \mathrm{const} \cdot \int \frac{\phi(y)}{(x - \varepsilon y)^{l+1}} dy \; ;$$

if ε is so small that $\alpha - \varepsilon d(\phi) \geq \alpha/2$ this expression can be dominated by $C(2/\alpha)^{l+1}$ independent of ε.

Assertion 2: $\Sigma_{\mathrm{g},0}(W) = \mathbb{R} \setminus 0$

$0 \in \mathrm{singsupp_g} W$ therefore $\Sigma_{\mathrm{g},0}(W) \neq \emptyset$; let $\psi \in \mathcal{D}$, $\psi(0) = 1$, ψ real valued, and set $W_1 = \psi W$. $\mathcal{F}_S W_1$ has the representative

$$\int e^{-i\xi x} \Big(\langle \mathrm{vp}(\frac{1}{y}), \phi(x+y) \rangle^2 + \pi^2 \phi(x)^2 \Big) \psi(x) dx \; ;$$

we have $\mathcal{F}_S W_1(\phi, -\xi) = \overline{\mathcal{F}_S W_1(\overline{\phi}, \xi)}$ (this reflects the fact that $\mathrm{vp}(1/y)$ and δ respect complex conjugation) and hence rapid decrease on one side of the real line would imply the same on the other side; but this would imply that $\Sigma_{\mathrm{g},0}(W)$ is empty – a contradiction.

Summarizing we conclude that

$$\mathrm{WF_g}(W) = \{0\} \times \mathbb{R} \setminus 0 \; .$$

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