THE WEYL–ALGEBRA ON THE TWO–TORUS

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One of the most elaborate examples of a non–commutative manifold is the irrational rotation–C^*–algebra or so–called non–commutative torus. A detailed description from the point of view of non–commutative differential geometry is given in [2] and related papers.

A different approach to this interesting object appeared in a recent paper on quantum–ergodic theory (see [1]), where the non–commutative two–torus is constructed as a quantum analogue of a well known classical dynamical system. The basic idea is to describe the algebraic structure by Weyl type relations depending on a quantization parameter, which justifies the notion Weyl–algebra on the two–torus. We will follow this approach and derive a result on its representation theory.

1 Definitions

We start with the dynamical system (X, τ, T) consisting of the compact space $X = (\mathbb{R}\, mod\, \mathbb{Z})^2$, the two–torus, equipped with

Lebesgue–probability measure τ and a homeomorphism $T : X \to X$, given by a matrix $(T_{ij}) \in GL(2,\mathbb{Z})$. This model can be described equivalently in an algebraic way by the commutative von Neumann–algebra (abbreviated as vNA) $M_0 = L^\infty(X)$, the state $\tau(f) = \int_X f(x)d\tau(x)$ (we use the same symbol for the measure and the corresponding state) and the automorphism $\alpha(f) = f \circ T$, building the W^*–dynamical system (M_0, τ, α).

The algebra M_0 is generated by the functions $W(n)(x) = e^{2\pi i\langle n|x\rangle}$ ($n \in \mathbb{Z}^2$, $\langle .|.\rangle$ denotes the scalar product), which satisfy the relations

$$W(n)W(m) = W(n+m) \quad (1)$$

$$\tau(W(n)) = \delta_{0n}. \quad (2)$$

Let $\theta \in [0,1)$ and define $\sigma : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z},\ (n,m) \mapsto n_1m_2 - n_2m_1$. The idea of the following is to construct a non–commutative version of the above algebra by turning equations (1) and (2) into

$$W(n)W(m) = e^{2\pi i\theta\sigma(n,m)}W(n+m) \quad (3)$$

$$\tau(W(n)) = \delta_{0n}. \quad (4)$$

For a realization of this structure we consider the vector space $\mathcal{F} = \{f : \mathbb{Z}^2 \to \mathbb{C} \mid supp(f)\ is\ finite\}$ and define an algebra multiplication by

$$fg(n) = \sum_{m\in\mathbb{Z}^2} f(m)g(n-m)e^{2\pi i\theta\sigma(m,n)} \quad (5)$$

and an involution

$$f^*(n) = \overline{f(-n)}. \quad (6)$$

In this way we get a *–algebra $\mathcal{W}_\theta$ with unit $\mathbb{1} = \delta_0$, which is generated by the unitary elements $\delta_n(k) = \delta_{nk}$ with the relations (use equation (5))

$$\delta_n\delta_m = e^{2\pi i\theta\sigma(n,m)}\delta_{n+m}\ , \quad (7)$$

the desired Weyl–type relations (3). The definition $\tau(f) = f(0)$ $\forall f \in \mathcal{W}_\theta$ gives a tracial state on $\mathcal{W}_\theta$ with the property

$$\tau(\delta_n) = \delta_{0n}, \tag{8}$$

hence we have constructed a non–commutative (if $\theta \neq 0, 1/2$) analogue of the classical system at an algebraic level.

In order to get a W^*–dynamical system one can consider the GNS–representation of $\mathcal{W}_\theta$ corresponding to the state τ. Since the left–kernel of τ is trivial ($\tau(f^* f) = \sum_n \mid f(n) \mid = 0 \;\Leftrightarrow\; f = 0$) and $\tau(f^* g) = \sum_n \overline{f(n)} g(n)$, the GNS–Hilbert–space is exactly $l^2(\mathbb{Z}^2)$. The action of the generating elements δ_n is given by

$$(\pi_\tau(\delta_n)\xi)(m) = e^{2\pi i\theta\sigma(n,m)}\xi(m-n) \quad \forall \xi \in l^2(\mathbb{Z}^2), \tag{9}$$

which is similar to the action of the Weyl–operators in the usual Schrödinger–representation.

It turns out ([1]) that the structure of the generated vNA $\mathcal{M}_\theta$ over $l^2(\mathbb{Z}^2)$ depends on the parameter θ, i.e. for rational θ $\mathcal{M}_\theta$ is of type I_n ($n < \infty$) with nontrivial center and is the hyperfinite type–II_1–factor for irrational values of θ. The next section will show that in the rational case any representation leads to a type–I vNA.

2 Representations

Let $\pi : \mathcal{W}_\theta \to \mathcal{B}(\mathcal{H})$ be an arbitrary representation of $\mathcal{W}_\theta$ (unital *–homomorphism). Define a map $W : \mathbb{Z}^2 \to \mathcal{U}(\mathcal{H}),\; n \mapsto W(n) = \pi(\delta_n)$, where $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators. The equation (compare with equ.(3) and (7))

$$W(n)W(m) = e^{2\pi i\theta\sigma(n,m)}W(n+m) \tag{10}$$

shows that W defines a projective (unitary) representation of the group $\mathbb{Z}^2$ with cocycle $c(n,m) = e^{2\pi i\theta\sigma(n,m)}$.

Using the central extension $G = \mathbb{Z}^2 \times \mathbb{T}$ ($\mathbb{T}$ denoting the complex numbers of modulus 1) with the multiplication law

$$(n,v) \circ (m,u) = (n+m, e^{2\pi i\theta\sigma(n,m)}uv) \tag{11}$$

one can turn the projective representation W into a representation $T : G \to \mathcal{U}(\mathcal{H})$ of the extended group G by setting

$$T(n,v) = vW(n), \tag{12}$$

since then

$$\begin{aligned} T(n,v)T(m,u) &= vuW(n)W(m) = \\ e^{2\pi i\theta\sigma(n,m)}vuW(n+m) &= T((n,v) \circ (m,u)). \end{aligned}$$

Now consider the vNA $\mathcal{M}$ generated by the representation π. $\mathcal{M}$ can be defined as the strong closure of the set $\pi(\mathcal{W}_\theta)$ or equivalently $T(G)$. Since $T(\{0\} \times \mathbb{T})$ consists only of scalar multiples of the identity, $\mathcal{M}$ can actually be written as strong closure of $T(\mathbb{Z}^2 \times \{1\})$. Therefore it suffices to consider the restriction of T to the closed subgroup $H \subseteq G$ generated by the elements of $\mathbb{Z}^2 \times \{1\}$.

If $\theta \notin \mathbb{Q}$ then $H = G$, since $\{e^{2\pi i\theta k} \mid k \in \mathbb{Z}\}$ is dense in $\mathbb{T}$. Hence we get nothing new in this case.

If $\theta \in \mathbb{Q}$ ($\theta \neq 0$) we can write $\theta = p/q$ with $p, q \in \mathbb{N}$ and get the countable discrete set $H = \mathbb{Z}^2 \times \{e^{2\pi ipk/q} | k = 0, \ldots, q-1\}$. The subgroup H is isomorphic to $\mathbb{Z}^2 \times \mathbb{Z}_q$ ($\mathbb{Z}_q$ denotes the cyclic group of order q) with the operation

$$(n,\bar{l}) \circ (m,\bar{k}) = (n+m, \overline{l+k+\sigma(n,m)}). \tag{13}$$

At this point we can use a result from representation theory of discrete groups concerning groups of type I. A group is said to be of *type I* (or *tame*) if all (unitary) representations generate type–I vNA. A characterization of the type I groups amoung all countable discrete groups is given by the following

Lemma: (see [3]) Let H be a countable discrete group. H is of type I if and only if there exists a commutative normal subgroup $N \subseteq H$ with finite index.

If we set in the above situation $N = (q\mathbb{Z}^2) \times \mathbb{Z}_q$ then N defines a commutative normal subgroup of H, since this is the center of H. Furthermore

$$H/N \cong (\mathbb{Z}_q)^2 \tag{14}$$

and this set has exactly q^2 elements, hence N has finite index in H. Now the above lemma tells us that the vNA generated by $T(H)$ is of type I and therefore the same is true for $\mathcal{M}$, which is actually the same vNA. With a view at the special representation of section 1 we have in conclusion the following

Theorem:
Exactly for $\theta \in \mathbb{Q}$ each representation of $\mathcal{W}_\theta$ generates a vNA of type I.

References

[1] F.Benatti/H.Narnhofer/G.L.Sewell: Lett.Math.Phys. 21 (1991), 157

[2] A.Connes/M.Rieffel: Contemp. Math. 62 (1987), 237

[3] E.Thoma: Math.Ann. 153 (1964), 111