Some properties of differential operators of gradient type associated with spherical harmonics

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Abstract

In this talk we would like to discuss some problems connected with a particular type of first order differential opeartors, closely related to generalized Cauchy–Riemann operators of Stein and Weiss. The operators we consider form an infinite sequence

$$\cdots \xrightarrow{S_{l-2}} \mathcal{E}(\mathbf{R}^d;\mathcal{H}^{l-1}) \xrightarrow{S_{l-1}} \mathcal{E}(\mathbf{R}^d;\mathcal{H}^{l}) \xrightarrow{S_{l}} \mathcal{E}(\mathbf{R}^d;\mathcal{H}^{l+1}) \xrightarrow{S_{l+1}} \cdots,$$

where $\mathcal{H}^l$ denotes the space of homogeneous harmonic polynomials of degree l on $\mathbf{R}^d$ and $\mathcal{E}(\mathbf{R}^d;\mathcal{H}^l)$ denotes the space of smooth functions on $\mathbf{R}^d$ with values in $\mathcal{H}^l$. There is also a similar sequence of adjoints to S_l's in the form

$$\cdots \xleftarrow{S^*_{l-1}} \mathcal{E}(\mathbf{R}^d;\mathcal{H}^{l-1}) \xleftarrow{S^*_{l}} \mathcal{E}(\mathbf{R}^d;\mathcal{H}^{l}) \xleftarrow{S^*_{l+1}} \mathcal{E}(\mathbf{R}^d;\mathcal{H}^{l+1}) \xleftarrow{S^*_{l+2}} \cdots$$

Under the standard identification $\mathcal{H}^1 \cong \mathbf{C}^d$ one sees that S_0 is the gradient and S^*_1 is the divergence. These operators were investigated previously by M. Reimann, T. Branson and others, and are referred to as generalized gradient operators.

We study certain properties of those operators, in particular ther composition properties, and state an analog in this context of the so called Helmholtz decomposition for vector fields. We also derive a closed formula for the projection onto the space of generalized solenoidal fields and give an integrability condition for the equation

$$S_{l-1}U = F$$

as a singular integral equation to be satisfied by F.