Brief project report

Project: P 28154-N13

Spectral analysis of the $\bar{\partial}$-Neumann operator

The $\overline{\partial}$-Neumann operator associated to a bounded pseudoconvex domain is a central object in complex analysis. The corresponding operator on weighted spaces $L^2(\mathbb{C}^n, e^{-\varphi})$ opens interesting connections to Schrödinger operators with magnetic field and to the Witten Laplacian. In addition, we investigate complete Hermitian manifolds and the spectral analysis of the $\bar{\partial}$-complex and, with the Segal Bargmann space as underlying Hilbert space, the spectral analysis of the ∂-complex together with its ramifications to differential geometry.

Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a plurisubharmonic $\mathcal{C}^2$-function and let

$$L^2(\mathbb{C}^n, e^{-\varphi}) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \text{ measurable} : \int_{\mathbb{C}^n} |f|^2 e^{-\varphi}\, d\lambda < \infty\}.$$

We consider the weighted $\overline{\partial}$-complex

$$L^2_{(0,q-1)}(\mathbb{C}^n, e^{-\varphi}) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q+1)}(\mathbb{C}^n, e^{-\varphi})$$

and set

$$\Box_{\varphi,q} = \overline{\partial}\,\overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi \overline{\partial}.$$

Suppose that the sum $s_q(z)$ of the smallest q eigenvalues of the Levi matrix

$$M_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}\right)_{j,k=1}^n$$

has the property $\lim_{|z|\to\infty} s_q(z) = +\infty$.
Then the inverse operator

$$N_{\varphi,q} : L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi})$$

of $\Box_{\varphi,q}$ exists and is compact.

In the paper "On some spectral properties of the weighted $\overline{\partial}$-Neumann problem", by Franz Berger and the project leader (J. Math. Kyoto Univ. 59 (2019), 441–453), necessary conditions for compactness of the weighted $\overline{\partial}$-Neumann operator on the space $L^2(\mathbb{C}^n, e^{-\varphi})$ for a plurisubharmonic function φ are studied.
Let $A^2(\mathbb{C}^n, e^{-\varphi})$ be the Bergman space of entire functions f belonging to $L^2(\mathbb{C}^n, e^{-\varphi})$. If one applies $\Box_{\varphi,q}$ to $(0,q)$-forms with coefficients belonging to $A^2(\mathbb{C}^n, e^{-\varphi})$, one finds out that $\Box_{\varphi,q}$ restricts to a multiplication operator on this space. In this way, one gets the following result: let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic C^2 function and suppose

that the corresponding weighted space $A^2(\mathbb{C}^n, e^{-\varphi})$ of entire function is infinite dimensional. If there is $1 \le q \le n$ such that the $\overline{\partial}$-Neumann operator $N_{\varphi,q}$ is compact, then $\limsup_{|z|\to\infty} \operatorname{tr}(M_\varphi(z)) = +\infty$.
But one can also use the fact that $\square_{\varphi,n}$ has compact resolvent on $L^2_{0,n}(\mathbb{C}^n, e^{-\varphi})$ if and only if a certain magnetic Schrödinger operator $-\Delta_A + V$ has compact resolvent on $L^2(\mathbb{R}^{2n})$. From the theory of Schrödinger operators with magnetic field one obtains now a stronger necessary condition for compactness: if there is $1 \le q \le n$ such that $\square_{\varphi,q}$ has compact resolvent, then $\lim_{|z|\to\infty} \int_{B(z,1)} \operatorname{tr}(M_\varphi)^2 \, d\lambda = +\infty$.
For so-called decoupled weight functions

$$\varphi(z) = \varphi_1(z_1) + \cdots + \varphi_n(z_n),$$

where all φ_j are subharmonic and such that $\Delta\varphi_j$ defines a nontrivial doubling measure, one can use the tensor product structure and a formula for the essential spectrum of $\square_{\varphi,q}$ to show that $N_{\varphi,q}$ fails to be compact for $1 \le q \le n-1$, and $N_{\varphi,n}$ is compact if and only if $\lim_{|z|\to\infty} \int_{B_1(z)} \operatorname{tr}(M_\varphi) \, d\lambda = \infty$.

In the paper "Sobolev spaces for the weighted $\overline{\partial}$-Neumann operator" (International Journal of Math., 28, No.9 (2017), 1740007) the project leader describes an approach to obtain the compactness estimates for the weighted $\overline{\partial}$-Neumann operator by defining appropriate weighted Sobolev spaces and proving an appropriate Rellich - Kondrachov lemma. As an application, the spectral analysis of related Pauli operators is investigated: "Pauli operators and the $\overline{\partial}$-Neumann problem", Ufa Mathematical Journal, 9, No.3 (2017), 1-7.

The project leader finished his monography **"Complex Analysis - a functional analytic approach"**, De Gruyter Graduate 2018. In this book, a modern approach to complex analysis of one and several variables is given. The second part of the book contains Fréchet and Hilbert spaces of holomorphic functions, the Bergman kernel and the basics of unbounded operators on Hilbert spaces in order to tackle the inhomogeneous Cauchy-Riemann equations and the $\overline{\partial}$-Neumann operator.

Franz Berger, PhD student supported by the funds of the project, finished his doctoral studies in April 2018. The paper "Discreteness of spectrum for the $\overline{\partial}$-Neumann Laplacian on manifolds of bounded geometry"(arXiv:1808.02730) originates from his thesis. In this paper he shows that some known results on the spectral properties of this operator on pseudoconvex domains in $\mathbb{C}^n$ continue to hold on Kähler manifolds satisfying certain bounded geometry assumptions. In particular, he considers the Dolbeault complex for forms with values in a line bundle, where known results from magnetic Schrödinger operator theory can be applied.

Together with the post-docs Gian Maria Dall'Ara and Duong Ngoc Son, both supported by the project, he uses an appropriate version of the Bochner–Kodaira–Nakano basic identity in order to prove the interesting results of the paper "Exponential decay of

Bergman kernels on complete Hermitian manifolds with Ricci curvature bounded from below" (Complex Variables and Elliptic Equations, 65 (2020), 2086-2111).
In the paper "On noncompactness of the $\overline{\partial}$-Neumann problem on pseudoconvex domains in $\mathbb{C}^3$," (J. of Math. Analysis and Applications 457 (2018), 233-247) Gian Maria Dall'Ara proves that a smooth bounded pseudoconvex domain with a one-dimensional complex manifold M in its boundary has a noncompact $\overline{\partial}$-Neumann operator on $(0,1)$-forms, under the additional assumption that $b\Omega$ has finite regular D'Angelo 2-type at a point of M.

Supported by a travel grant by the Austrian WTZ , a collaboration of the project leader with colleagues from Montenegro (David Kalaj and Djordjije Vujadinovic) resulted in the paper "Sharp pointwise estimates for Fock spaces" (Computational methods and Function Theory 21 (2021), 343–359), where sharp pointwise estimates for arbitrary derivatives of functions in Fock spaces are established.

In the last two years, the project leader concentrated on a study of the ∂-complex, where the underlying Hilbert space is the Segal Bargmann space

$$A^2(\mathbb{C}^n, e^{-|z|^2}) = \{u : \mathbb{C}^n \longrightarrow \mathbb{C} \text{ entire } : \int_{\mathbb{C}^n} |u(z)|^2 \, e^{-|z|^2} \, d\lambda(z) < \infty\}.$$

Differentiation with respect to z_j defines an unbounded densely defined operator on $A^2(\mathbb{C}^n, e^{-|z|^2})$, its adjoint is the multiplication operator by z_j, again unbounded and densely defined. These are the annihilation and creation operators of quantum mechanics. In several complex variables one has the ∂-operator and its adjoint ∂^* acting on (p,0)-forms with coefficients in the Segal Bargmann space. The project leader considers the corresponding ∂-complex and studies spectral properties of the corresponding complex Laplacian $\tilde{\Box} = \partial\partial^* + \partial^*\partial$. In addition he investigates the more general complex Laplacian $\tilde{\Box}_D = DD^* + D^*D$, where D is a differential operator of polynomial type, to find the canonical solutions to the inhomogeneous equations $Du = \alpha$ and $D^*v = \beta$, see "The ∂-complex on the Segal-Bargmann space" (Ann. Polon. Mat. 123 (2019), 295-317) and "The generalized ∂-complex on the Segal Bargmann space" (Operator Theory, Functional Analysis and Applications, Editors M. Amélia Bastos, Luís Castro, Alexei Yu. Karlovich, Volume 282, Birkhäuser Basel (2021), 317–328).
Together with the postdoc Duong Ngoc Son he also considers the ∂-complex on weighted Bergman spaces on Hermitian manifolds satisfying a certain holomorphicity/duality condition. In addition they investigate the existence of real-valued weight functions with real holomorphic gradient fields on Kähler and conformally Kähler manifolds and their relationship to the ∂-complex on weighted Bergman spaces. In this context, the holomorphicity of the torsion tensor T^{rs}_p is of importance, see "The ∂-complex on weighted Bergman spaces on Hermitian manifolds"(J. Math. Anal. Appl. 487(1) (2020) 123994) and "The ∂-Operator and Real Holomorphic Vector Fields" (arXiv:2007.14764).

Duration of the project: 01.09.2016 – 31.03.2021.

Personnel:
Berger Franz, PhD student, 01.09.2016–13.05.2018
Berger Franz, Postdoc, 14.05.2018– 31.01.2019
Dall'Ara Gian Maria, Postdoc, 01.09.2016–30.04.2019
Duong Ngoc Son, Postdoc, 01.09.2020–30.11.2020

Personnel development

Franz Berger is a highly gifted PhD student. He already obtained interesting new results on spectral analysis of the $\overline{\partial}$-Neumann operator, the first part of his thesis has already been accepted for publication in the Journal of Functional Analysis. After his defense (with referee Siqi Fu) he decided to accept a research position at DeepL in Köln for the development of a neural machine translation engine.

Gian Maria Dall'Ara finished his PhD studies at Scuola Normale Superiore di Pisa under the supervision of Fulvio Ricci. His impressing thesis is closely related to the topic of the project, parts of it were published in Advances of Mathematics and in the Journal of Functional Analysis. His approach came from partial differential equations in mathematical physics and surely enhanced the complex analytic aspects of the $\overline{\partial}$-Neumann problem. Especially, in collaboration with Franz Berger and Duong Ngoc Son he established interesting new differential geometric aspects of the $\overline{\partial}$-Neumann problem. In May 2019 he was appointed to a permanent research position at Indam - Scuola Normale Superiore Pisa.

Duong Ngoc Son recently finished his habilitation at the University of Vienna. His expertise in differential geometry was inspiring for the whole complex analysis group in Vienna. He is also very active in Bernhard Lamel's group.

Participation in conferences
Project leader: Lectures on the $\overline{\partial}$-Neumann problem and on the ∂-complex.
2017: Doha (Texas A& M University), American University Beirut, University of Ufa (Russia), Steklov Institute (Moscow), University of Montenegro (Podgorica).
2018: University of Sao Paulo (Brazil), University of Dublin, Columbia University New York, Rutgers University Philadelphia.
2019 : AMS-meeting Honolulu, University of Montenegro Podgorica, IWOTA Lissabon, BIRS Conference Banff (Canada), University of Brno , CIRM Conference Luminy.
2021: Virtual East-West Several Complex Variables seminar

The project leader retired from his university position in October 2017 because he reached the age limit, but was allowed to continue his research project until March 2021.