Chapter 1

Unbounded operators on Hilbert spaces

Definition 1.1. Let H_1, H_2 be Hilbert spaces and $T : \mathrm{dom}(T) \longrightarrow H_2$ be a densely defined linear operator, i.e. $\mathrm{dom}(T)$ is a dense linear subspace of H_1. Let $\mathrm{dom}(T^*)$ be the space of all $y \in H_2$ such that $x \mapsto (Tx, y)_2$ defines a continuous linear functional on $\mathrm{dom}(T)$. Since $\mathrm{dom}(T)$ is dense in H_1 there exists a uniquely determined element $T^*y \in H_1$ such that $(Tx, y)_2 = (x, T^*y)_1$ (Riesz representation theorem). The map $y \mapsto T^*y$ is linear and $T^* : \mathrm{dom}(T^*) \longrightarrow H_1$ is the adjoint operator to T.

T is called a closed operator, if the graph

$$\mathcal{G}(T) = \{(f, Tf) \in H_1 \times H_2 \, : \, f \in \mathrm{dom}(T)\}$$

is a closed subspace of $H_1 \times H_2$.

The inner product in $H_1 \times H_2$ is

$$((x, y), (u, v)) = (x, u)_1 + (y, v)_2.$$

If $\tilde{V}$ is a linear subspace of H_1 which contains $\mathrm{dom}(T)$ and $\tilde{T}x = Tx$ for all $x \in \mathrm{dom}(T)$ then we say that $\tilde{T}$ is an extension of T.

An operator T with domain $\mathrm{dom}(T)$ is said to be closable if it has a closed extension $\tilde{T}$.

Lemma 1.2. *Let T be a densely defined closable operator. Then there is a closed extension $\overline{T}$, called its closure, whose domain is smallest among all closed extensions.*

Proof. Let $\mathcal{V}$ be the set of $x \in H_1$ for which there exist $x_k \in \mathrm{dom}(T)$ and $y \in H_2$ such that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} Tx_k = y$. Since $\tilde{T}$ is a closed extension of T it follows that $x \in \mathrm{dom}(\tilde{T})$ and $\tilde{T}x = y$. Therefore y is uniquely determined by x. We define $\overline{T}x = y$ with $\mathrm{dom}(\overline{T}) = \mathcal{V}$. Then $\overline{T}$ is an extension of T and every closed extension of T is also an extension of $\overline{T}$. The graph of $\overline{T}$ is the closure of the graph of T in $H_1 \times H_2$. Hence $\overline{T}$ is a closed operator. □

Lemma 1.3. *Let $T_1 : dom(T_1) \longrightarrow H_2$ be a densely defined operator and $T_2 : H_2 \longrightarrow H_3$ be a bounded operator. Then $(T_2 T_1)^* = T_1^* T_2^*$, which includes that $dom((T_2 T_1)^*) = dom(T_1^* T_2^*)$.*

Proof. Note that

$$\mathrm{dom}(T_1^* T_2^*) = \{f \in \mathrm{dom}(T_2^*) \,:\, T_2^*(f) \in \mathrm{dom}(T_1^*)\}.$$

Let $f \in \mathrm{dom}(T_1^* T_2^*)$ and $g \in \mathrm{dom}(T_2 T_1)$. Then

$$(T_1^* T_2^* f, g) = (T_2^* f, T_1 g) = (f, T_2 T_1 g),$$

hence $\mathrm{dom}(T_1^* T_2^*) \subseteq \mathrm{dom}((T_2 T_1)^*)$.

Now let $f \in \mathrm{dom}((T_2 T_1)^*)$. As T_2^* is bounded and everywhere defined on H_3, and for all $g \in \mathrm{dom}(T_2 T_1) = \mathrm{dom}(T_1)$ we have

$$((T_2 T_1)^* f, g) = (f, T_2 T_1 g) = (T_2^* f, T_1 g).$$

Hence $T_2^* f \in \mathrm{dom}(T_1^*)$ and $f \in \mathrm{dom}(T_1^* T_2^*)$. □

Lemma 1.4. *Let T be a densely defined operator on H and let S be a bounded operator on H. Then $(T + S)^* = T^* + S^*$.*

Proof. Let $f \in \mathrm{dom}(T^* + S^*) = \mathrm{dom}(T^*)$. Then for all $g \in \mathrm{dom}(T + S) = \mathrm{dom}(T)$ we have

$$((T^* + S^*)f, g) = (T^* f, g) + (S^* f, g) = (f, Tg) + (f, Sg) = (f, (T + S)g),$$

hence $f \in \mathrm{dom}((T + S)^*)$ and $(T + S)^* f = T^* f + S^* f$.

If $f \in \mathrm{dom}((T + S)^*)$, then for all $g \in \mathrm{dom}(T + S) = \mathrm{dom}(T)$ we have

$$([(T + S)^* - S^*]f, g) = (f, (T + S)g) - (f, Sg) = (f, Tg),$$

therefore $f \in \mathrm{dom}(T^*)$ and $\mathrm{dom}((T + S)^*) = \mathrm{dom}(T^* + S^*) = \mathrm{dom}(T^*)$. □

Lemma 1.5. *Let $T : dom(T) \longrightarrow H_2$ be a densely defined linear operator and define $V : H_1 \times H_2 \longrightarrow H_2 \times H_1$ by $V((x, y)) = (y, -x)$. Then*

$$\mathcal{G}(T^*) = [V(\mathcal{G}(T))]^\perp = V(\mathcal{G}(T)^\perp);$$

in particular T^* is always closed.

Proof. $(y, z) \in \mathcal{G}(T^*) \Leftrightarrow (Tx, y)_2 = (x, z)_1$ for each $x \in \mathrm{dom}(T)$ $\Leftrightarrow ((x, Tx), (-z, y)) = 0$ for each $x \in \mathrm{dom}(T)$ $\Leftrightarrow V^{-1}((y, z)) = (-z, y) \in \mathcal{G}(T)^\perp$. Hence $\mathcal{G}(T^*) = V(\mathcal{G}(T)^\perp)$ and since V is unitary we have $V^* = V^{-1}$ and $[V(\mathcal{G}(T))]^\perp = V(\mathcal{G}(T)^\perp)$.

□

Lemma 1.6. *Let $T : dom(T) \longrightarrow H_2$ be a densely defined, closed linear operator. Then*

$$H_2 \times H_1 = V(\mathcal{G}(T)) \oplus \mathcal{G}(T^*).$$

Proof. $\mathcal{G}(T)$ is closed, therefore, by Lemma 1.5: $\mathcal{G}(T^*)^\perp = V(\mathcal{G}(T))$. □

Lemma 1.7. *Let $T : dom(T) \longrightarrow H_2$ be a densely defined, closed linear operator. Then $dom(T^*)$ is dense in H_2 and $T^{**} = T$.*

Proof. Let $z \perp \text{dom}(T^*)$. Hence $(z, y)_2 = 0$ for each $y \in \text{dom}(T^*)$. We have

$$V^{-1} : H_2 \times H_1 \longrightarrow H_1 \times H_2$$

where $V^{-1}((y, x)) = (-x, y)$, and $V^{-1}V = \text{Id}$. Now, by Lemma 1.6, we have

$$H_1 \times H_2 \cong V^{-1}(H_2 \times H_1) = V^{-1}(V(\mathcal{G}(T)) \oplus \mathcal{G}(T^*)) \cong \mathcal{G}(T) \oplus V^{-1}(\mathcal{G}(T^*)).$$

Hence $(z, y)_2 = 0 \Leftrightarrow ((0, z), (-T^*y, y)) = 0$ for each $y \in \text{dom}(T^*)$ implies $(0, z) \in \mathcal{G}(T)$ and therefore $z = T(0) = 0$, which means that $\text{dom}(T^*)$ is dense in H_2.

Since T and T^* are densely defined and closed we have by Lemma 1.5

$$\mathcal{G}(T) = \mathcal{G}(T)^{\perp\perp} = [V^{-1}\mathcal{G}(T^*)]^\perp = \mathcal{G}(T^{**}),$$

where $-V^{-1}$ corresponds to V in considering operators from H_2 to H_1. □

Lemma 1.8. *Let $T : dom(T) \longrightarrow H_2$ be a densely defined linear operator. Then $kerT^* = (imT)^\perp$, which means that $kerT^*$ is closed.*

Proof. Let $v \in \text{ker}T^*$ and $y \in \text{im}T$, which means that there exists $u \in \text{dom}(T)$ such that $Tu = y$. Hence

$$(v, y)_2 = (v, Tu)_2 = (T^*v, u)_1 = 0,$$

and $\text{ker}T^* \subseteq (\text{im}T)^\perp$.

And if $y \in (\text{im}T)^\perp$, then $(y, Tu)_2 = 0$ for each $u \in \text{dom}(T)$, which implies that $y \in \text{dom}(T^*)$ and $(y, Tu)_2 = (T^*y, u)_1$ for each $u \in \text{dom}(T)$. Since each $\text{dom}(T)$ is dense in H_1 we obtain $T^*y = 0$ and $(\text{im}T)^\perp \subseteq \text{ker}T^*$. □

Lemma 1.9. *Let $T : dom(T) \longrightarrow H_2$ be a densely defined, closed linear operator. Then $kerT$ is a closed linear subspace of H_1.*

Proof. We use Lemma 1.8 for T^* and get $\text{ker}T^{**} = (\text{im}T^*)^\perp$. Since, by Lemma 1.7, $T^{**} = T$ we obtain $\text{ker}T = (\text{im}T^*)^\perp$ and that $\text{ker}T$ is a closed linear subspace of H_1.

□

Lemma 1.10. *Let $T : H_1 \longrightarrow H_2$ be a bounded linear operator. $T(H_1)$ is closed if and only if $T|_{(kerT)^\perp}$ is bounded from below, i.e.*

$$\|Tf\| \geq C\|f\| \ , \ \forall f \in (kerT)^\perp.$$

Proof. If $T(H_1)$ is closed, then the mapping

$$T : (\ker T)^{\perp} \longrightarrow T(H_1)$$

is bijective and continuous and, by the open-mapping theorem, also open. This implies the desired inequality.

To prove the other direction, let $(f_n)_n$ be a sequence in H_1 with $Tf_n \to y$ in H_2. We have to show, that there exists $h \in H_1$ with $Th = y$. Decompose $f_n = g_n + h_n$, where $g_n \in \ker T$ and $h_n \in (\ker T)^{\perp}$. By assumption we have

$$\|h_n - h_m\| \leq C\|Th_n - Th_m\| = C\|Tf_n - Tf_m\| < \epsilon,$$

for all sufficiently large n and m. Hence $(h_n)_n$ is a Cauchy sequence. Let $h = \lim_{n\to\infty} h_n$. Then we have

$$\|Tf_n - Th\| = \|Th_n - Th\| \leq \|T\| \, \|h_n - h\|,$$

and therefore

$$y = \lim_{n\to\infty} Tf_n = Th.$$

□

Lemma 1.11. *Let T be as before. $T(H_1)$ is closed if and only if $T^*(H_2)$ is closed.*

Proof. Since $T^{**} = T$, it suffices to show one direction. We will show that the closedness of $T(H_1)$ implies, that $(\ker T)^{\perp} = \operatorname{im} T^*$; since $(\ker T)^{\perp}$ is closed, we will be finish.

Let $x \in \operatorname{im} T^*$. Then there exists $y \in H_2$ with $x = T^*y$. Now we get for $x' \in \ker T$ that

$$(x, x') = (T^*y, x') = (y, Tx') = 0,$$

hence $\operatorname{im} T^* \subseteq (\ker T)^{\perp}$.

For $x' \in (\ker T)^{\perp}$ we define a linear functional

$$\lambda(Tx) = (x, x')$$

on the closed subspace $T(H_1)$ of H_2. We remark that λ is well-defined, since $Tx = T\tilde{x}$ implies that $x - \tilde{x} \in \ker T$, hence $(x - \tilde{x}, x') = 0$ and $(x, x') = (\tilde{x}, x')$. The operator $T : H_1 \longrightarrow T(H_1)$ is continuous and surjective. Since $T(H_1)$ is closed, the open-mapping theorem implies $\|v\| \leq C\|Tv\|$, for all $v \in (\ker T)^{\perp}$ where $C > 0$ is a constant. Set $y = Tx$ and write $x = u + v$, where $u \in \ker T$ and $v \in (\ker T)^{\perp}$. Then we obtain

$$\begin{aligned} |\lambda(y)| = |(x, x')| &= |(v, x')| \\ &\leq \|v\|\|x'\| \end{aligned}$$

$$\begin{aligned} &\le C\|Tv\|\|x'\| \\ &= C\|Tx\|\|x'\| \\ &= C\|y\|\|x'\|. \end{aligned}$$

Hence λ is continuous on im$T$. By the Riesz representation theorem, there exists a uniquely determined element $z \in \text{im}T$ with

$$\lambda(y) = (y, z)_2 = (x, x')_1.$$

This implies $(y, z)_2 = (Tx, z)_2 = (x, T^*z)_1 = (x, x')_1$, for all $x \in H_1$, and hence $x' = T^*z \in \text{im}T^*$. □

Lemma 1.12. *Let $T : H_1 \longrightarrow H_2$ be a densely defined closed operator. im$T$ is closed in H_2 if and only if $T|_{dom(T)\cap(kerT)^\perp}$ is bounded from below, i.e.*

$$\|Tf\| \ge C\|f\| \quad , \ \forall f \in dom(T) \cap (kerT)^\perp.$$

Proof. On the graph $\mathcal{G}(T)$ we define the operator $\tilde{T}(\{f, Tf\}) = Tf$ and get a bounded linear operator

$$\tilde{T} : \mathcal{G}(T) \longrightarrow H_2,$$

since

$$\|\tilde{T}(\{f, Tf\})\| = \|Tf\| \le (\|f\|^2 + \|Tf\|^2)^{1/2} = \|\{f, Tf\}\|;$$

and im$\tilde{T}$ = im$T$.

By Lemma 1.10, im$T$ is closed if and only if $\tilde{T}\big|_{(\ker\tilde{T})^\perp}$ is bounded from below.

We have $\ker\tilde{T} = \ker T \oplus \{0\}$, and it remains to show that $\tilde{T}\big|_{(\ker\tilde{T})^\perp}$ is bounded from below, if and only if $T|_{\text{dom}(T)\cap(\ker T)^\perp}$ is bounded from below. But this follows from

$$\|\tilde{T}(\{f, Tf\})\| = \|Tf\| \ge C(\|f\|^2 + \|Tf\|^2)^{1/2},$$

and hence, for $0 < C < 1$,

$$\|Tf\|^2 \ge \frac{C^2}{1 - C^2}\,\|f\|^2.$$

□

Lemma 1.13. *Let $P, Q : H \longrightarrow H$ be orthogonal projections on the Hilbert space H. then the following assertions are equivalent*

(i) im(PQ) is closed;

(ii) im(QP) is closed;

(iii) im$(I - P)(I - Q)$ is closed;

(iv) $P(H) + (I - Q)(H)$ is closed.

Proof. (i) and (ii) are equivalent, since $QP = Q^*P^* = (PQ)^*$ and Lemma 1.11.

Suppose (ii) holds and let $(f_n)_n$ and $(g_n)_n$ be sequences in H with $Pf_n + (I - Q)g_n \to h$. Then

$$Q(Pf_n + (I - Q)g_n) = QPf_n \to Qh.$$

By assumption, $\mathrm{im}(QP)$ is closed, hence there exists $f \in H$ with $QPf = Qh$; it follows that $Qh = Pf - (I - Q)(Pf)$ and

$$\begin{aligned} h = Qh + (I - Q)h &= Pf - (I - Q)(Pf) + (I - Q)h \\ &= Pf + (I - Q)(h - Pf) \in P(H) + (I - Q)(H), \end{aligned}$$

which yields (iv).

If (iv) holds and $(f_n)_n$ is a sequence in H with $QPf_n \to h$, we get

$$QPf_n = Pf_n - (I - Q)Pf_n \in P(H) + (I - Q)(H).$$

Hence there exist $f, g \in H$ with $h = Pf + (I - Q)g$; and it follows that

$$Qh = Q(\lim_{n\to\infty} QPf_n) = \lim_{n\to\infty} Q^2Pf_n = h,$$

and

$$h = Pf + (I - Q)g = Qh = QPf \in \mathrm{im}(QP),$$

therefore (ii) holds.

Finally, replace P by $I-P$ and Q by $I-Q$. Then, using the assertions proved so far, we obtain the equivalence

$$\mathrm{im}(I - P)(I - Q) \text{ closed} \Leftrightarrow (I - P)(H) + Q(H) \text{ closed},$$

which proves the remaining assertions. □

At this point, we are able to prove Lemma 1.11 for densely defined closed operators.

Proposition 1.14. *Let $T : H_1 \longrightarrow H_2$ be a densely defined closed operator. $\mathrm{im}T$ is closed if and only if $\mathrm{im}T^*$ is closed.*

Proof. Let $P : H_1 \times H_2 \longrightarrow \mathcal{G}(T)$ be the orthogonal projection of $H_1 \times H_2$ on the closed subspace $\mathcal{G}(T)$ of $H_1 \times H_2$, and let $Q : H_1 \times H_2 \longrightarrow \{0\} \times H_2$ be the canonical orthogonal projection. Then $\mathrm{im}T \cong \mathrm{im}QP$ and since $I - Q : H_1 \times H_2 \longrightarrow H_1 \times \{0\}$ and

$$I - P : H_1 \times H_2 \longrightarrow \mathcal{G}(T)^\perp = V(\mathcal{G}(T^*)) \cong \mathcal{G}(T^*)$$

we obtain the desired result from Lemma 1.13. □

Proposition 1.15. *Let $T : H_1 \longrightarrow H_2$ be a densely defined closed operator and G a closed subspace of H_2 with $G \supseteq imT$. Suppose that $T^*|_{dom(T^*)\cap G}$ is bounded from below, i.e. $\|f\| \leq C\|T^*f\|$ for all $f \in dom(T^*) \cap G$, where $C > 0$ is a constant. Then $G = imT$.*

Proof. We have $\ker T^* = (\mathrm{im}T)^\perp$. Since $\mathrm{im}T \subseteq G$, it follows that $\ker T^* \supseteq G^\perp$. If $G^\perp$ is a proper subspace of $\ker T^*$, then $G \cap \ker T^* \neq \{0\}$, which is a contradiction to the assumption that $T^*|_{\mathrm{dom}(T^*)\cap G}$ is bounded from below. Hence $\ker T^* = G^\perp$ and

$$G = G^{\perp\perp} = (\ker T^*)^\perp = \mathrm{im}T^{\perp\perp} = \overline{(\mathrm{im}T)}.$$

In addition we have

$$T^*|_{\mathrm{dom}(T^*)\cap G} = T^*|_{\mathrm{dom}(T^*)\cap(\ker T^*)^\perp}$$

and, by Lemma 1.12 we obtain, that $\mathrm{im}T^*$ is closed. By Proposition 1.14 , $\mathrm{im}T$ is also closed and we get that $G = \mathrm{im}T$. □

Remark 1.16. The last proposition also holds in the other direction: if $T : H_1 \longrightarrow H_2$ is a densely defined closed operator and G is a closed subspace of H_2 with $G = \mathrm{im}T$, then $T^*|_{\mathrm{dom}(T^*)\cap G}$ is bounded from below. Since in this case $G = \mathrm{im}T$, we have that $\mathrm{im}T$ is closed and hence, by Lemma 1.14, $\mathrm{im}T^*$ is also closed. Therefore, Lemma 1.12 and the fact that $G = (\ker T^*)^\perp$ implies that $T^*|_{\mathrm{dom}(T^*)\cap G}$ is bounded from below.

Proposition 1.17. *Let $T : H_1 \longrightarrow H_2$ be a densely defined closed operator and let G be a closed subspace of H_2 with $G \supseteq imT$. Suppose that $T^*|_{dom(T^*)\cap G}$ is bounded from below. Then for each $v \in H_1$ with $v \perp kerT$ there exists $f \in dom(T^*) \cap G$ with $T^*f = v$ and $\|f\| \leq C\|v\|$.*

Proof. We have $\ker T = (\mathrm{im}T^*)^\perp$, hence $v \in (\ker T)^\perp = \overline{\mathrm{im}T^*}$. In addition $G^\perp \subseteq (\mathrm{im}T)^\perp = \ker T^*$ and therefore

$$\mathrm{im}T^*|_{\mathrm{dom}(T^*)\cap G} = \mathrm{im}T^*,$$

this means that $\mathrm{im}T^*$ is closed and that for $v \in (\ker)^\perp = \mathrm{im}T^*$ there exists $f \in \mathrm{dom}(T^*) \cap G$ with $T^*f = v$. The desired norm-inequality follows from the assumption that $T^*|_{\mathrm{dom}(T^*)\cap G}$ is bounded from below. □

In the following we introduce the fundamental concept of an unbounded self-adjoint operator, which will be crucial for both spectral theory and its applications to complex analysis.

Definition 1.18. Let $T : \mathrm{dom}(T) \longrightarrow H$ be a densely defined linear operator. T is symmetric if $(Tx, y) = (x, Ty)$ for all $x, y \in \mathrm{dom}(T)$. We say that T is self-adjoint if T is symmetric and $\mathrm{dom}(T) = \mathrm{dom}(T^*)$. This is equivalent to requiring that $T = T^*$ and implies that T is closed. We say that T is essentially self-adjoint if it is symmetric and its closure $\overline{T}$ is self-adjoint.

Remark 1.19. (a) If T is a symmetric operator, there are two natural closed extensions. We have $\mathrm{dom}(T) \subseteq \mathrm{dom}(T^*)$ and $T^* = T$ on $\mathrm{dom}(T)$. Since T^* is closed (Lemma 1.6), T^* is a closed extension of T, it is the maximal self-adjoint extension. T is also closable, by Lemma 1.2, therefore $\overline{T}$ exists, it is the minimal closed extension.

(b) If T is essentially self-adjoint, then its self-adjoint extension is unique. To prove this, let S be a self-adjoint extension of T. Then S is closed and, being an extension of T, it is also an extension of its smallest extension $\overline{T}$. Hence

$$\overline{T} \subset S = S^* \subset (\overline{T})^* = \overline{T},$$

and $S = \overline{T}$.

Lemma 1.20. *Let T be a densely defined, symmetric operator.*

(i) If dom$(T) = H$, then T is self-adjoint and T is bounded.

(ii) If T is self-adjoint and injective, then im(T) is dense in H, and T^{-1} is self-adjoint.

(iii) If im(T) is dense in H, then T is injective.

(iv) If im$(T) = H$, then T is self-adjoint, and T^{-1} is bounded.

Proof. (i) By assumption $\mathrm{dom}(T) \subseteq \mathrm{dom}(T^*)$. If $\mathrm{dom}(T) = H$, it follows that T is self-adjoint, therefore also closed (Lemma 1.5) and continuous by the closed graph theorem.

(ii) Suppose $y \perp \mathrm{Im}(T)$. Then $x \mapsto (Tx, y) = 0$ is continuous on $\mathrm{dom}(T)$, hence $y \in \mathrm{dom}(T^*) = \mathrm{dom}(T)$, and $(x, Ty) = (Tx, y) = 0$ for all $x \in \mathrm{dom}(T)$. Thus $Ty = 0$ and since T is assumed to be injective, it follows that $y = 0$. This proves that $\mathrm{Im}(T)$ in dense in H.

T^{-1} is therefore densely defined, with $\mathrm{dom}(T^{-1}) = \mathrm{im}(T)$, and $(T^{-1})^*$ exists. Now let $U : H \times H \longrightarrow H \times H$ be defined by $U((x, y)) = (-y, x)$. It easily follows that $U^2 = -I$ and $U^2(M) = M$ for any subspace M of $H \times H$, and we get $\mathcal{G}(T^{-1}) = U(\mathcal{G}(-T))$ and $U(\mathcal{G}(T^{-1})) = \mathcal{G}(-T))$. Being self-adjoint, T is closed; hence $-T$ is closed and T^{-1} is closed. By Lemma 1.6 applied to T^{-1} and to $-T$ we get the orthogonal decompositions

$$H \times H = U(\mathcal{G}(T^{-1})) \oplus \mathcal{G}((T^{-1})^*)$$

and

$$H \times H = U(\mathcal{G}(-T)) \oplus \mathcal{G}(-T)) = \mathcal{G}(T^{-1}) \oplus U(\mathcal{G}(T^{-1})).$$

Consequently

$$\mathcal{G}((T^{-1})^*) = [U(\mathcal{G}(T^{-1}))]^\perp = \mathcal{G}(T^{-1}),$$

which shows that $(T^{-1})^* = T^{-1}$.

(iii) Suppose $Tx = 0$. Then $(x, Ty) = (Tx, y) = 0$ for each $y \in \mathrm{dom}(T)$. Thus $x \perp \mathrm{im}(T)$, and therefore $x = 0$.

(iv) Since $\mathrm{im}(T) = H$, (iii) implies that T is injective, $\mathrm{dom}(T^{-1}) = H$. If $x, y \in H$, then $x = Tz$ and $y = Tw$, for some $z \in \mathrm{dom}(T)$ and $w \in \mathrm{dom}(T)$, so that

$$(T^{-1}x, y) = (z, Tw) = (Tz, w) = (x, T^{-1}y).$$

Hence T^{-1} is symmetric. (i) implies that T^{-1} is self-adjoint (and bounded), and now it follows from (ii) that $T = (T^{-1})^{-1}$ is also self-adjoint. □

Lemma 1.21. *Let T be a densely defined closed operator, dom$(T) \subseteq H_1$ and T : dom$(T) \longrightarrow H_2$. Then $B = (I + T^*T)^{-1}$ and $C = T(I + T^*T)^{-1}$ are everywhere defined and bounded, $\|B\| \leq 1$, $\|C\| \leq 1$; in addition B is self-adjoint and positive.*

Proof. Let $h \in H_1$ be an arbitrary element and consider $(h, 0) \in H_1 \times H_2$. Form the proof of Lemma 1.7 we get

$$H_1 \times H_2 = \mathcal{G}(T) \oplus V^{-1}(\mathcal{G}(T^*)), \tag{1.1}$$

which implies that $(h, 0)$ can be written in a unique way as

$$(h, 0) = (f, Tf) + (-T^*(-g), -g),$$

for $f \in \mathrm{dom}(T)$ and $g \in \mathrm{dom}(T^*)$, which gives $h = f + T^*g$ and $0 = Tf - g$. We set $Bh := f$ and $Ch := g$. In this way we get two linear operators B and C everywhere defined on H_1. The two equations from above can now be written as

$$I = B + T^*C, \quad 0 = TB - C,$$

which gives

$$C = TB \quad \text{and} \quad I = B + T^*TB = (I + T^*T)B. \tag{1.2}$$

The decomposition in (1.1) is orthogonal, therefore we obtain

$$\|h\|^2 = \|(h, 0)\|^2 = \|(f, Tf)\|^2 + \|(T^*g, -g)\|^2 = \|f\|^2 + \|Tf\|^2 + \|T^*g\|^2 + \|g\|^2,$$

and hence

$$\|Bh\|^2 + \|Ch\|^2 = \|f\|^2 + \|g\|^2 \leq \|h\|^2,$$

which implies $\|B\| \leq 1$ and $\|C\| \leq 1$.

For each $u \in \mathrm{dom}(T^*T)$ we get

$$((I + T^*T)u, u) = (u, u) + (Tu, Tu) \geq (u, u)$$

hence, if $(I + T^*T)u = 0$ we get $u = 0$. Therefore $(I + T^*T)^{-1}$ exists and (1.2) implies that $(I + T^*T)^{-1}$ is defined everywhere and $B = (I + T^*T)^{-1}$. Finally let $u, v \in H_1$. Then

$$\begin{aligned}(Bu, v) &= (Bu, (I + T^*T)Bv) = (Bu, Bv) + (Bu, T^*TBv) \\ &= (Bu, Bv) + (T^*TBu, Bv) = ((I + T^*T)Bu, Bv) = (u, Bv)\end{aligned}$$

and

$$(Bu, u) = (Bu, (I + T^*T)Bu) = (Bu, Bu) + (TBu, TBu) \geq 0,$$

which proves the lemma. □

Finally we describe a general method to construct self-adjoint operators associated with Hermitian sesquilinear forms. This leads to a self-adjoint extension of an unbounded operator, which is known as the Friedrichs extension.

Definition 1.22. Let $(\mathcal{V}, \|.\|_{\mathcal{V}})$ and $(H, \|.\|_H)$ be Hilbert spaces such that

$$\mathcal{V} \subset H, \tag{1.3}$$

and suppose that there exists a constant $C > 0$ such that for all $u \in \mathcal{V}$ we have

$$\|u\|_H \leq C\, \|u\|_{\mathcal{V}}. \tag{1.4}$$

We also assume that $\mathcal{V}$ is dense in H.

In this situation the space H can be imbedded into the dual space $\mathcal{V}'$: for $h \in H$ the mapping

$$L(u) = (u, h)_H\ ,\ \text{for } u \in \mathcal{V}$$

is continuous on $\mathcal{V}$, this follows from (1.4):

$$|L(u)| \leq \|u\|_H \|h\|_H \leq C\|h\|_H \|u\|_{\mathcal{V}}.$$

Hence there exists a uniquely determined $v_h \in \mathcal{V}'$ such that

$$v_h(u) = (u, h)_H\ ,\ \text{for } u \in \mathcal{V},$$

and the mapping $h \mapsto v_h$ is injective, as $\mathcal{V}$ is dense in H.

Definition 1.23. A form $a : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{C}$ is sesquilinear, if it is linear in the first component and anti linear in the second component. The form a is continuous if there exists a constant $C > 0$ such that

$$|a(u, v)| \leq C\|u\|_{\mathcal{V}}\, \|v\|_{\mathcal{V}} \tag{1.5}$$

for all $u, v \in \mathcal{V}$ and it is Hermitian if

$$a(u, v) = \overline{a(v, u)}$$

for all $u, v \in \mathcal{V}$.

The form a is called $\mathcal{V}$-elliptic if there exists a constant $\alpha > 0$ such that

$$|a(u, u)| \geq \alpha \|u\|_{\mathcal{V}}^2 \tag{1.6}$$

for all $u \in \mathcal{V}$.

Proposition 1.24. *Let a be a continuous, $\mathcal{V}$-elliptic form on $\mathcal{V} \times \mathcal{V}$. Using* (1.5) *and the Riesz representation theorem we can define a linear operator*

$$A : \mathcal{V} \longrightarrow \mathcal{V}$$

such that

$$a(u, v) = (Au, v)_{\mathcal{V}}. \tag{1.7}$$

This operator A is a topological isomorphism from $\mathcal{V}$ onto $\mathcal{V}$.

Proof. First we show that A is injective: (1.7) and (1.6) imply that for $u \in \mathcal{V}$ we have

$$\|Au\|_{\mathcal{V}} \|u\|_{\mathcal{V}} \geq |(Au, u)_{\mathcal{V}}| \geq \alpha \|u\|_{\mathcal{V}}^2,$$

hence

$$\|Au\|_{\mathcal{V}} \geq \alpha \|u\|_{\mathcal{V}}, \tag{1.8}$$

which implies that A is injective.

Now we claim that $A(\mathcal{V})$ is dense in $\mathcal{V}$. Let $u \in \mathcal{V}$ be such that $(Av, u)_{\mathcal{V}} = 0$ for each $v \in \mathcal{V}$. Taking $v = u$ we get $a(u, u) = 0$ and, by (1.6), $u = 0$, which proves the claim.

Next we observe that (1.7) implies $a(u, Au) = \|Au\|_{\mathcal{V}}^2$, therefore, using (1.5), we obtain $\|A(u)\|_{\mathcal{V}} \leq C\|u\|_{\mathcal{V}}$, hence $A \in \mathcal{L}(\mathcal{V})$. If $(v_n)_n$ is a Cauchy sequence in $A(\mathcal{V})$ and $Au_n = v_n$, we derive from (1.8) that $(u_n)_n$ is also a Cauchy sequence. Let $u = \lim_{n\to\infty} u_n$. We know already that A is continuous, therefore $\lim_{n\to\infty} Au_n = Au$, which shows that $\lim_{n\to\infty} v_n = v = Au$ and $A(\mathcal{V})$ is closed. As we have already shown that $A(\mathcal{V})$ is dense in $\mathcal{V}$, we conclude that A is surjective.

Finally (1.8) yields that A^{-1} is continuous.

□

Proposition 1.25. *Let a be a Hermitian, continuous, $\mathcal{V}$-elliptic form on $\mathcal{V} \times \mathcal{V}$ and suppose that* (1.3) *and* (1.4) *hold. Let $dom(S)$ be the set of all $u \in \mathcal{V}$ such that the mapping $v \mapsto a(u, v)$ is continuous on $\mathcal{V}$ for the topology induced by H.*

For each $u \in dom(S)$ there exists a uniquely determined element $Su \in H$ such that

$$a(u,v) = (Su, v)_H \tag{1.9}$$

for each $v \in \mathcal{V}$ (by the Riesz representation theorem).

Then $S : dom(S) \longrightarrow H$ is a bijective densely defined self-adjoint operator and $S^{-1} \in \mathcal{L}(H)$. Moreover, $dom(S)$ is also dense in $\mathcal{V}$.

Proof. First we show that S is injective. For each $u \in \mathrm{dom}(S)$ we get from (1.6) and (1.4) that

$$\begin{aligned} \alpha\|u\|_H^2 &\leq C\alpha\|u\|_{\mathcal{V}}^2 \leq C|a(u,u)| \\ &= C|(Su,u)_H| \leq C\|Su\|_H\|u\|_H, \end{aligned}$$

which implies that

$$\alpha\|u\|_H \leq C\|Su\|_H, \tag{1.10}$$

for all $u \in \mathrm{dom}(S)$, therefore S is injective.

Now let $h \in H$ and consider the mapping $v \mapsto (h,v)_H$ for $v \in \mathcal{V}$. Then, by (1.4), we obtain

$$|(h,v)_H| \leq \|h\|_H\|v\|_H \leq C\|h\|_H\|v\|_{\mathcal{V}},$$

which implies that there exists a uniquely determined $w \in \mathcal{V}$ such that $(h,v)_H = (w,v)_{\mathcal{V}}$ for all $v \in \mathcal{V}$. Now we apply Proposition 1.24 and get from (1.7) that $a(u,v) = (w,v)_{\mathcal{V}}$, where $u = A^{-1}w$. Since $a(u,v) = (h,v)_H$ for each $v \in \mathcal{V}$, we conclude that $u \in \mathrm{dom}(S)$ and that $Su = h$, which shows that S is surjective.

Suppose that $(u,h)_H = 0$ for each $u \in \mathrm{dom}(S)$. As S is surjective, there is $v \in \mathrm{dom}(S)$ such that $Sv = h$ and we get that $(u,Sv)_H = 0$ for each $u \in \mathrm{dom}(S)$. Using the $\mathcal{V}$-ellipticity (1.6) we get for $u = v$ that

$$0 = (Sv,v)_H = a(v,v) \geq \alpha\|v\|_{\mathcal{V}}^2,$$

which implies that $v = 0$ and consequently $h = 0$. Therefore we have shown that $\mathrm{dom}(S)$ is dense in H.

As $a(u,v)$ is Hermitian, we get for $u, v \in \mathrm{dom}(S)$ that

$$(Su,v)_H = a(u,v) = \overline{a(v,u)} = \overline{(Sv,u)_H} = (u,Sv)_H.$$

Hence S is symmetric and $\mathrm{dom}(S) \subset \mathrm{dom}(S^*)$. Let $v \in \mathrm{dom}(S^*)$. Since S is surjective, there exists $v_0 \in \mathrm{dom}(S)$ such that $Sv_0 = S^*v$. This implies

$$(Su,v_0)_H = (u,Sv_0)_H = (u,S^*v)_H = (Su,v)_H,$$

for all $u \in \mathrm{dom}(S)$. Using again the surjectivity of S, we derive that $v = v_0 \in \mathrm{dom}(S)$. This implies that $\mathrm{dom}(S) = \mathrm{dom}(S^*)$ and that S is self-adjoint.

Finally we show that dom(S) is dense in $\mathcal{V}$. Let $h \in \mathcal{V}$ be such that $(u, h)_{\mathcal{V}} = 0$, for all $u \in$ dom(S). By Proposition 1.24 there exists $f \in \mathcal{V}$ such that $Af = h$. Then

$$\begin{aligned} 0 = (u, h)_{\mathcal{V}} &= (u, Af)_{\mathcal{V}} = \overline{(Af, u)_{\mathcal{V}}} \\ &= \overline{a(f, u)} = a(u, f) = (Su, f)_H. \end{aligned}$$

S is surjective, therefore we obtain $f = 0$ and $h = Af = 0$. □

Chapter 2

Distributions and Sobolev spaces

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^n$ an open subset and $\mathcal{D}(\Omega) = \mathcal{C}_0^\infty(\Omega)$ the space of $\mathcal{C}^\infty$-functions with compact support (test functions).

A sequence $(\phi_j)_j$ tends to 0 in $\mathcal{D}(\Omega)$ if there exists a compact set $K \subset \Omega$ such that $\mathrm{supp}\phi_j \subset K$ for every j and

$$\frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \to 0$$

uniformly on K for each $\alpha = (\alpha_1, \dots, \alpha_n)$.

A distribution is a linear functional u on $\mathcal{D}(\Omega)$ such that for every compact subset $K \subset \Omega$ there exists $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$and a constant $C > 0$ with

$$|u(\phi)| \leq C \sum_{|\alpha| \leq k} \sup_{x \in K} \left| \frac{\partial^{|\alpha|}\phi(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right|,$$

for each $\phi \in \mathcal{D}(\Omega)$ with support in K. We denote the space of distributions on Ω by $\mathcal{D}'(\Omega)$.

It is easily seen that $u \in \mathcal{D}'(\Omega)$ if and only if $u(\phi_j) \to 0$ for every sequence $(\phi_j)_j$ in $\mathcal{D}(\Omega)$ converging to 0 in $\mathcal{D}(\Omega)$.

Example 2.2. 1. Let $f \in L^1_{loc}(\Omega)$, where

$$L^1_{loc}(\Omega) = \{f : \Omega \longrightarrow \mathbb{C}\,\text{measurable} : f\,|_K \in L^1(K)\,\forall K \subset \Omega,\ K\,\text{compact}\}.$$

The mapping $T_f(\phi) = \int_\Omega f(x)\phi(x)\,d\lambda(x)$, $\phi \in \mathcal{D}(\Omega)$, is a distribution.

2. Let $a \in \Omega$ and $\delta_a(\phi) := \phi(a)$, which is the point evaluation in a. The distribution δ_a is called Dirac Delta distribution.

In the sequel, certain operations for ordinary functions, such as multiplication of functions and differentiation, is generalized to distributions.

Definition 2.3. Let $f \in \mathcal{C}^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$. The multiplication of u with f is defined by $(fu)(\phi) := u(f\phi)$ for $\phi \in \mathcal{D}(\Omega)$. Notice that $f\phi \in \mathcal{D}(\Omega)$.

For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $f \in \mathcal{D}(\mathbb{R}^n)$ the convolution of u and f is defined by

$$(u * f)(x) := u(y \mapsto f(x - y)),$$

which is a $\mathcal{C}^\infty$-function. If $u = T_g$ for some locally integrable function g it is the usual convolution of functions

$$(T_g * f)(x) = \int_\Omega g(y) f(x-y)\, d\lambda(y) = (g * f)(x).$$

Let

$$D_k = \frac{\partial}{\partial x_k} \quad \text{and} \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index. The partial derivative of a distribution $u \in \mathcal{D}'(\Omega)$ is defined by

$$(D_k u)(\phi) := -u(D_k \phi), \ \phi \in \mathcal{D}(\Omega);$$

higher order mixed derivatives are defined as

$$(D^\alpha u)(\phi) := (-1)^{|\alpha|}\, u(D^\alpha \phi), \ \phi \in \mathcal{D}(\Omega).$$

This definition stems from integrating by parts:

$$\int_\Omega (D_k f)\phi \, d\lambda = -\int_\Omega f(D_k \phi)\, d\lambda,$$

where $f \in \mathcal{C}^1(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$.

For an appropriate description of the appearing phenomena we will use further Hilbert spaces of differentiable functions - the Sobolev spaces.

Definition 2.4. If Ω is a bounded open set in $\mathbb{R}^n$, and k is a nonnegative integer we define the Sobolev space

$$W^k(\Omega) = \{f \in L^2(\Omega) : \partial^\alpha f \in L^2(\Omega), \, |\alpha| \leq k\},$$

where the derivatives are taken in the sense of distributions and endow the space with the norm

$$\|f\|_{k,\Omega} = \left[\sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha f|^2 \, d\lambda \right]^{1/2}, \tag{2.1}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex , $|\alpha| = \sum_{j=1}^n \alpha_j$ and

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

$W_0^k(\Omega)$ denotes the completion of $\mathcal{C}_0^\infty(\Omega)$ under $W^k(\Omega)$-norm. Since $\mathcal{C}_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, it follows that $W_0^0(\Omega) = W^0(\Omega) = L^2(\Omega)$. Using the Fourier transform it is also possible to introduce Sobolev spaces of non-integer exponent. (See [1, 5].)

In general a function can belong to a Sobolev space, and yet be discontinuous and unbounded.

Example 2.5. Take $\Omega = \mathbb{B}$ the open unit ball in $\mathbb{R}^n$, and

$$u(x) = |x|^{-\alpha} \quad , \ x \in \mathbb{B}, x \neq 0.$$

We claim that $u \in W^1(\mathbb{B})$ if and only if $\alpha < \frac{n-2}{2}$.

First note that u is smooth away from 0, and that

$$u_{x_j}(x) = \frac{-\alpha x_j}{|x|^{\alpha+2}} \quad , \ x \neq 0.$$

Hence

$$|\nabla u(x)| = \frac{|\alpha|}{|x|^{\alpha+1}} \quad , \ x \neq 0.$$

Now, recall the Gauß-Green -theorem: for a smoothly bounded $\omega \subseteq \mathbb{R}^n$ we have

$$\int_\omega \nabla \,.\, F(x)\, d\lambda(x) = \int_{b\omega} (F(x), \nu(x))\, d\sigma(x),$$

where $\nu(x) = \nabla r(x)$ is the normal to $b\omega$ at x, and F is a $\mathcal{C}^1$ vector field on $\overline{\omega}$, and

$$\nabla \,.\, F(x) = \sum_{j=1}^n \frac{\partial F_j}{\partial x_j}.$$

(see [4])

Let $\phi \in \mathcal{C}_0^\infty(\mathbb{B})$ and let $\mathbb{B}_\epsilon$ be the open ball around 0 with radius $\epsilon > 0$. Take $\omega = \mathbb{B} \setminus \mathbb{B}_\epsilon$ and

$$F(x) = (0, \ldots, 0, u\phi, 0, \ldots, 0),$$

where $u\phi$ appears at the j-th component. Then

$$\int_{\mathbb{B}\setminus\mathbb{B}_\epsilon} u(x)\, \phi_{x_j}(x)\, d\lambda(x) = -\int_{\mathbb{B}\setminus\mathbb{B}_\epsilon} u_{x_j}(x)\, \phi_{x_j}(x)\, d\lambda(x) + \int_{b\mathbb{B}_\epsilon} u(x)\phi(x)\nu_j(x)\, d\sigma(x),$$

where $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ denotes the inward pointing normal on $b\mathbb{B}_\epsilon$. If $\alpha < n-1$, then $|\nabla u(x)| \in L^1(\mathbb{B})$, and we obtain

$$\begin{aligned} \left| \int_{b\mathbb{B}_\epsilon} u(x)\phi(x)\nu_j(x)\, d\sigma(x) \right| &\leq \|\phi\|_\infty \int_{b\mathbb{B}_\epsilon} \epsilon^{-\alpha}\, d\sigma(x) \\ &\leq C\, \epsilon^{n-1-\alpha} \to 0, \end{aligned}$$

as $\epsilon \to 0$. Thus

$$\int_{\mathbb{B}} u(x)\, \phi_{x_j}(x)\, d\lambda(x) = -\int_{\mathbb{B}} u_{x_j}(x)\, \phi(x)\, d\lambda(x)$$

for all $\phi \in \mathcal{C}_0^\infty(\mathbb{B})$. As

$$|\nabla u(x)| = \frac{\alpha}{|x|^{\alpha+1}} \in L^2(\mathbb{B})$$

if and only if $2(\alpha+1) < n$ we get that $u \in W^1(\mathbb{B})$ if and only if $\alpha < \frac{n-2}{2}$.

Before we proceed we verify properties of weak derivatives, which are obviously true for smooth functions. As functions in Sobolev spaces are not necessarily smooth, we must always rely upon the definition of weak derivatives.

Proposition 2.6. *Assume $u, v \in W^k(\Omega)$, $|\alpha| \le k$. Then*
(i) $D^\alpha u \in W^{k-|\alpha|}(\Omega)$ and for multiindices α, β with $|\alpha| + |\beta| \le k$ we have

$$D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u.$$

(ii) If ω is an open subset of Ω, then $u \in W^k(\omega)$.
(iii) If $\phi \in \mathcal{C}_0^\infty(\Omega)$, then $\phi u \in W^k(\Omega)$ and

$$D^\alpha(\phi\, u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \phi\, D^{\alpha-\beta} u,$$

where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

Proof. To prove (i), fix $\phi \in \mathcal{C}_0^\infty(\Omega)$. Then $D^\beta \phi \in \mathcal{C}_0^\infty(\Omega)$, and

$$\begin{aligned} \int_\Omega D^\alpha u(x)\, D^\beta \phi(x)\, d\lambda(x) &= (-1)^{|\alpha|} \int_\Omega u(x)\, D^{\alpha+\beta}\phi(x)\, d\lambda(x) \\ &= (-1)^{|\alpha|}(-1)^{|\alpha+\beta|} \int_\Omega D^{\alpha+\beta} u(x)\, \phi(x)\, d\lambda(x) \\ &= (-1)^{|\beta|} \int_\Omega D^{\alpha+\beta} u(x)\, \phi(x)\, d\lambda(x). \end{aligned}$$

Hence $D^\beta(D^\alpha u) = D^{\alpha+\beta} u$ in the weak sense.

We omit the easy proof of (ii).

For (iii) we use induction on $|\alpha|$. Suppose first that $|\alpha| = 1$. Take any $\psi \in \mathcal{C}_0^\infty(\Omega)$. Then

$$\int_\Omega \phi(x) u(x) D^\alpha \psi(x)\, d\lambda(x) = \int_\Omega (u(x) D^\alpha(\phi(x)\psi(x)) - u(x)(D^\alpha \phi(x))\psi(x))\, d\lambda(x)$$

$$= -\int_\Omega (\phi(x) D^\alpha u(x) + u(x) D^\alpha \phi(x))\psi(x)\, d\lambda(x).$$

Therefore $D^\alpha(\phi u) = \phi D^\alpha u + u D^\alpha \phi$, as required. The induction step is carried out in a similar way.

□

Proposition 2.7. *Let $k \in \mathbb{N}$. Then $W^k(\Omega)$ is a Hilbert space.*

Proof. It is clear that the norm of $W^k(\Omega)$ stems from an inner product. To prove the completeness, let $(u_m)_m$ be a Cauchy sequence in $W^k(\Omega)$. Then for each multiindex α with $|\alpha| \leq k$, the sequence $(D^\alpha u_m)_m$ is a Cauchy sequence in $L^2(\Omega)$. Since $L^2(\Omega)$ is complete, there exist functions $u_\alpha \in L^2(\Omega)$ such that

$$D^\alpha u_m \to u_\alpha \ \text{ in } L^2(\Omega).$$

In particular, $u_m \to u_{(0,\dots,0)} := u$ in $L^2(\Omega)$.

Now we claim that $u \in W^k(\Omega)$ and $D^\alpha u = u_\alpha$ for $|\alpha| \leq k$. Fix $\phi \in \mathcal{C}_0^\infty(\Omega)$. Then, by Cauchy-Schwarz,

$$|\int_\Omega (u(x) - u_m(x))D^\alpha \phi(x)\, d\lambda(x)| \leq \|u - u_m\|_2 \, \|D^\alpha \phi\|_2,$$

where $\| \,.\, \|_2$ denotes the norm in $L^2(\Omega)$. Hence

$$\begin{aligned} \int_\Omega u(x) D^\alpha \phi(x)\, d\lambda(x) &= \lim_{m\to\infty} \int_\Omega u_m(x) D^\alpha \phi(x)\, d\lambda(x) \\ &= \lim_{m\to\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha u_m(x) \phi(x)\, d\lambda(x) \\ &= (-1)^{|\alpha|} \int_\Omega u_\alpha(x)\, \phi(x)\, d\lambda(x), \end{aligned}$$

which proves the claim. Since $D^\alpha u_m \to D^\alpha u$ in $L^2(\Omega)$ for all $|\alpha| \leq k$, we see that $u_m \to u$ in $W^k(\Omega)$. □

In the following we discuss two important examples: the Cauchy-Riemann equations and the Laplace equation:

Definition 2.8. Let $\Omega \subseteq \mathbb{C}^n$ be a domain.

$$L^2_{(0,1)}(\Omega) := \{u = \sum_{j=1}^n u_j \, d\overline{z}_j \ : \ u_j \in L^2(\Omega) \ j = 1, \dots, n\}$$

is the space of $(0,1)$- forms with coefficients in L^2, for $u, v \in L^2_{(0,1)}(\Omega)$ we define the inner product by

$$(u, v) = \sum_{j=1}^n (u_j, v_j).$$

In this way $L^2_{(0,1)}(\Omega)$ becomes a Hilbert space. $(0,1)$ forms with compactly supported $\mathcal{C}^\infty$ coefficients are dense in $L^2_{(0,1)}(\Omega)$.

Definition 2.9. Let $f \in \mathcal{C}_0^\infty(\Omega)$ and set

$$\overline{\partial} f := \sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j}\, d\overline{z}_j,$$

then

$$\overline{\partial} : \mathcal{C}_0^\infty(\Omega) \longrightarrow L^2_{(0,1)}(\Omega).$$

$\overline{\partial}$ is a densely defined unbounded operator on $L^2(\Omega)$. It does not have closed graph.

Definition 2.10. The domain $\mathrm{dom}(\overline{\partial})$ of $\overline{\partial}$ consists of all functions $f \in L^2(\Omega)$ such that $\overline{\partial} f$, in the sense of distributions, belongs to $L^2_{(0,1)}(\Omega)$, i.e. $\overline{\partial} f = g = \sum_{j=1}^n g_j\, d\overline{z}_j$, and for each $\phi \in \mathcal{C}_0^\infty(\Omega)$ we have

$$\int_\Omega f \left(\frac{\partial \phi}{\partial z_j} \right)^- d\lambda = - \int_\Omega g_j\, \overline{\phi}\, d\lambda\ ,\ j = 1, \dots, n. \tag{2.2}$$

It is clear that $\mathcal{C}_0^\infty(\Omega) \subseteq \mathrm{dom}(\overline{\partial})$, hence $\mathrm{dom}(\overline{\partial})$ is dense in $L^2(\Omega)$. Since differentiation is a continuous operation in distribution theory we have

Lemma 2.11. *$\overline{\partial} : dom(\overline{\partial}) \longrightarrow L^2_{(0,1)}(\Omega)$ has closed graph and $Ker\overline{\partial}$ is a closed subspace of $L^2(\Omega)$.*

Proof. We use the arguments of the proof of Proposition 2.7: let $(f_k)_k$ be a sequence in $\mathrm{dom}(\overline{\partial})$ such that $f_k \to f$ in $L^2(\Omega)$ and $\overline{\partial} f_k \to g$ in $L^2_{(0,1)}(\Omega)$. We have to show that $\overline{\partial} f = g$. From the proof of Proposition 2.7 we know that $\overline{\partial} f_k \to \overline{\partial} f$ as distributions. As $\overline{\partial} f_k \to g$ in $L^2_{(0,1)}(\Omega)$, it follows that $f \in \mathrm{dom}(\overline{\partial})$ and $\overline{\partial} f = g$.

Now we can apply Lemma 1.9 and get that $\mathrm{Ker}\overline{\partial}$ is a closed subspace of $L^2(\Omega)$. □

Example 2.12. For the Laplace operator

$$-\triangle = - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$$

we extend its domain as

$$\mathrm{dom}(-\triangle) = \{ f \in L^2(\mathbb{R}^n) : D^\alpha f \in L^2(\mathbb{R}^n)\,, |\alpha| \le 2 \} = W^2(\mathbb{R}^2),$$

and obtain, by a similar reasoning as before, a closed operator from $\mathrm{dom}(-\triangle)$ to $L^2(\mathbb{R}^n)$, which is in addition symmetric and positive, since we have

$$(-\triangle u, u) = \sum_{j=1}^n (D_j u, D_j u),$$

for $u \in \mathrm{dom}(-\triangle)$.

Next we approximate solutions of a first order differential operator by regularization using convolutions. For this purpose the following generalization of Minkowski's inequality is useful.

Lemma 2.13. *Let $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a nonnegative measurable function. Then*

$$[\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} F(x,y)\, d\lambda(y))^2\, d\lambda(x)]^{1/2} \leq \int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} F(x,y)^2\, d\lambda(x))^{1/2}\, d\lambda(y), \quad (2.3)$$

where we suppose that the right side is finite.

Proof. We use the duality for L^2-spaces:

$$\|f\|_2 = \sup\{|\int_{\mathbb{R}^n} f(x)\,\overline{g(x)}\, d\lambda(x)| \, : \, \|g\|_2 = 1\}, \quad (2.4)$$

where $f \in L^2(\mathbb{R}^n)$.

Let

$$f(x) = \int_{\mathbb{R}^n} F(x,y)\, d\lambda(y).$$

Then

$$\begin{aligned} \|f\|_2 &= [\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} F(x,y)\, d\lambda(y))^2\, d\lambda(x)]^{1/2} \\ &= \sup\{|\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(x,y)\overline{g(x)}\, d\lambda(y)\, d\lambda(x)| : \|g\|_2\} \\ &= \sup\{|\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(x,y)\overline{g(x)}\, d\lambda(x)\, d\lambda(y)| : \|g\|_2\} \\ &\leq \int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} F(x,y)^2\, d\lambda(x))^{1/2}\, d\lambda(y), \end{aligned}$$

where we used the Cauchy-Schwarz inequality in the last step.

□

To begin with we define for a function f on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ the function f_x to be $f_x(y) = f(x+y)$.

Lemma 2.14. *If $f \in L^2(\mathbb{R}^n)$, then $\lim_{x \to 0} \|f_x - f\|_2 = 0$.*

Proof. If g is continuous with compact support, then g is uniformly continuous, so $g_x \to g$ uniformly as $x \to 0$. Since g_x and g are supported in a common compact set for $|x| \leq 1$, it follows that $\|g_x - g\|_2 \to 0$. Given $f \in L^2(\mathbb{R}^n)$

and $\epsilon > 0$, choose a continuous function g with compact support such that $\|f - g\|_2 < \epsilon/3$. Then also $\|f_x - g_x\|_2 < \epsilon/3$, so

$$\|f_x - f\|_2 \le \|f_x - g_x\|_2 + \|g_x - g\|_2 + \|g - f\|_2 < \|g_x - g\|_2 + 2\epsilon/3.$$

For $|x|$ sufficiently small, $\|g_x - g\|_2 < \epsilon/3$, hence $\|f_x - f\|_2 < \epsilon$.

□

Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a function with support in the unit ball such that $\chi \ge 0$ and

$$\int_{\mathbb{R}^n} \chi(x)\, d\lambda(x) = 1.$$

We define $\chi_\epsilon(x) = \epsilon^{-n}\chi(x/\epsilon)$ for $\epsilon > 0$. Let $f \in L^2(\mathbb{R}^n)$ and define for $x \in \mathbb{R}^n$

$$\begin{aligned} f_\epsilon(x) = (f * \chi_\epsilon)(x) &= \int_{\mathbb{R}^n} f(x')\chi_\epsilon(x - x')\, d\lambda(x') \\ &= \int_{\mathbb{R}^n} f(x - x')\chi_\epsilon(x')\, d\lambda(x') \\ &= \int_{\mathbb{R}^n} f(x - \epsilon x')\chi(x')\, d\lambda(x'). \end{aligned}$$

In the first integral we can differentiate under the integral sign to show that $f_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^n)$.

The family of functions $(\chi_\epsilon)_\epsilon$ is called an approximation to the identity.

Lemma 2.15. $\|f_\epsilon - f\|_2 \to 0$ *as* $\epsilon \to 0$.

Proof.

$$f_\epsilon(x) - f(x) = \int_{\mathbb{R}^n} \left[f(x - \epsilon x') - f(x)\right] \chi(x')\, d\lambda(x').$$

We use Minkowski's inequality (2.3) to get

$$\|f_\epsilon - f\|_2 \le \int_{\mathbb{R}^n} \|f_{-\epsilon x'} - f\|_2\, |\chi(x')|\, d\lambda(x').$$

But $\|f_{-\epsilon x'} - f\|_2$ is bounded by $2\|f\|_2$ and tends to 0 as $\epsilon \to 0$ by Lemma 2.15. Now set

$$F_\epsilon(x') = |f_{-\epsilon x'} - f\|_2\, \chi(x').$$

Then $F_\epsilon(x') \to 0$ as $\epsilon \to 0$ and

$$|F_\epsilon(x')| \le 2\|f\|_2\, \chi(x'),$$

and we can apply the dominated convergence theorem to get the desired result.

□

If $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we have

$$D_j(u * \chi_\epsilon) = (D_j u) * \chi_\epsilon,$$

where $D_j = \partial/\partial x_j$. This also true, if $u \in L^2(\mathbb{R}^n)$ and $D_j u$ is defined in the sense of distributions. We will show even more using these methods for approximating a function in a Sobolev space by smooth functions.

Let $\Omega \subseteq \mathbb{R}^n$ be an open subset and let

$$\Omega_\epsilon = \{x \in \Omega : \operatorname{dist}(x, b\Omega) > \epsilon\}.$$

Lemma 2.16. *Let $u \in W^k(\Omega)$ and set $u_\epsilon = u * \chi_\epsilon$ in Ω_ϵ. Then*
(i) $u_\epsilon \in \mathcal{C}^\infty(\Omega_\epsilon)$, for each $\epsilon > 0$,
(ii)$D^\alpha u_\epsilon = D^\alpha u * \chi_\epsilon$ in Ω_ϵ, for $|\alpha| \leq k$.

Proof. (i) has already been shown.

(ii) means that the ordinary α^{th}-partial derivative of the smooth functions u_ϵ is the ϵ-mollification of the α^{th}-weak partial derivative of u. To see this, we take $x \in \Omega_\epsilon$ and compute

$$\begin{aligned} D^\alpha u_\epsilon(x) &= D^\alpha \int_\Omega u(y)\chi_\epsilon(x-y)\, d\lambda(y) \\ &= \int_\Omega D_x^\alpha \chi_\epsilon(x-y) u(y)\, d\lambda(y) \\ &= (-1)^{|\alpha|} \int_\Omega D_y^\alpha \chi_\epsilon(x-y) u(y)\, d\lambda(y). \end{aligned}$$

For a fixed $x \in \Omega_\epsilon$ the function $\phi(y) := \chi_\epsilon(x-y)$ belongs to $\mathcal{C}^\infty(\Omega)$. The definition of the α^{th}-weak partial derivative implies

$$\int_\Omega D_y^\alpha \chi_\epsilon(x-y) u(y)\, d\lambda(y) = (-1)^{|\alpha|} \int_\Omega \chi_\epsilon(x-y)\, D^\alpha u(y)\, d\lambda(y).$$

Thus

$$\begin{aligned} D^\alpha u_\epsilon(x) &= (-1)^{|\alpha|+|\alpha|} \int_\Omega \chi_\epsilon(x-y)\, D^\alpha u(y)\, d\lambda(y) \\ &= (D^\alpha u * \chi_\epsilon)(x), \end{aligned}$$

which proves (ii).

□

We are now ready to prove

Lemma 2.17 (Friedrichs' Lemma). *If $v \in L^2(\mathbb{R}^n)$ with compact support and a is a $\mathcal{C}^1$-function in a neighborhood of the support of v, it follows that*

$$\|aD_j(v * \chi_\epsilon) - (aD_jv) * \chi_\epsilon\|_2 \to 0 \quad \text{as } \epsilon \to 0,$$

where $D_j = \partial/\partial x_j$ and aD_jv is defined in the sense of distributions.

Proof. If $v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we have

$$D_j(v * \chi_\epsilon) = (D_jv) * \chi_\epsilon \to D_jv \quad , \quad (aD_jv) * \chi_\epsilon \to aD_jv,$$

with uniform convergence. We claim that

$$\|aD_j(v * \chi_\epsilon) - (aD_jv) * \chi_\epsilon\|_2 \leq C\|v\|_2, \tag{2.5}$$

where $v \in L^2(\mathbb{R}^n)$ and C is some positive constant independent of ϵ and v. Since $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, the lemma will follow like in the proof of Lemma 2.15 from (2.5) and the dominated convergence theorem.

To show (2.5) we may assume that $a \in \mathcal{C}_0^1(\mathbb{R}^n)$, since v has compact support. We have for $v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
& a(x)D_j(v * \chi_\epsilon)(x) - ((aD_jv) * \chi_\epsilon)(x) \\
= \;& a(x)\, D_j \int v(x-y)\chi_\epsilon(y)\, d\lambda(y) - \int a(x-y)\frac{\partial v}{\partial x_j}(x-y)\chi_\epsilon(y)\, d\lambda(y) \\
= \;& \int (a(x) - a(x-y))\, \frac{\partial v}{\partial x_j}(x-y)\chi_\epsilon(y)\, d\lambda(y) \\
= \;& -\int (a(x) - a(x-y))\, \frac{\partial v}{\partial y_j}(x-y)\chi_\epsilon(y)\, d\lambda(y) \\
= \;& \int (a(x) - a(x-y))v(x-y)\, \frac{\partial}{\partial y_j}\chi_\epsilon(y)\, d\lambda(y) \\
- \;& \int \left(\frac{\partial}{\partial y_j}a(x-y)\right) v(x-y)\chi_\epsilon(y)\, d\lambda(y).
\end{aligned}$$

Let M be the Lipschitz constant for a such that $|a(x) - a(x-y)| \leq M|y|$, for all $x, y \in \mathbb{R}^n$. Then

$$\begin{aligned}
&|a(x)D_j(v * \chi_\epsilon)(x) - ((aD_jv) * \chi_\epsilon)(x)| \\
&\leq M \int |v(x-y)|(\chi_\epsilon(y) + |y\frac{\partial}{\partial y_j}\chi_\epsilon(y)|)\, d\lambda(y).
\end{aligned}$$

By Minkowski's inequality (2.3) we obtain

$$\|aD_j(v * \chi_\epsilon) - (aD_jv) * \chi_\epsilon\|_2 \quad \leq \quad M\, \|v\|_2 \int (\chi_\epsilon(y) + |y\frac{\partial}{\partial y_j}\chi_\epsilon(y)|)\, dy$$

$$= \quad M(1+m_j)\|v\|_2,$$

where

$$m_j = \int |y \frac{\partial}{\partial y_j} \chi_\epsilon(y)|\, dy = \int |y \frac{\partial}{\partial y_j} \chi(y)|\, d\lambda(y).$$

This shows (2.5) when $v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Snce $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we have proved (2.5) and the lemma. □

Lemma 2.18. *Let*

$$L = \sum_{j=1}^n a_j D_j + a_0$$

be a first order differential operator with variable coefficients where $a_j \in \mathcal{C}^1(\mathbb{R}^n)$ and $a_0 \in \mathcal{C}(\mathbb{R}^n)$. If $v \in L^2(\mathbb{R}^n)$ with compact support and $Lv = f \in L^2(\mathbb{R}^n)$ where Lv is defined in the distribution sense, the convolution $v_\epsilon = v * \chi_\epsilon$ is in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ and $v_\epsilon \to v$, $Lv_\epsilon \to f$ in $L^2(\mathbb{R}^n)$ as $\epsilon \to 0$.

Proof. Since $a_0 v \in L^2(\mathbb{R}^n)$, we have

$$\lim_{\epsilon\to 0} a_0(v * \chi_\epsilon) = \lim_{\epsilon \to 0}(a_0 v * \chi_\epsilon) = a_0 v$$

in $L^2(\mathbb{R}^n)$. Using Friedrichs' Lemma 2.17, we have

$$Lv_\epsilon - Lv * \chi_\epsilon = Lv_\epsilon - f * \chi_\epsilon \to 0$$

in $L^2(\mathbb{R}^n)$ as $\epsilon \to 0$. The lemma follows easily since $f * \chi_\epsilon \to f$ in $L^2(\mathbb{R}^n)$. □

Before we proceed with results about Sobolev spaces we prove an important inequality for the sgn-function.

Let $z \in \mathbb{C}$. Define

$$\text{sgn} z = \begin{cases} \overline{z}/|z| & z \neq 0 \\ 0 & z = 0. \end{cases}$$

Proposition 2.19. *Suppose that $f \in L^1_{loc}(\mathbb{R}^n)$ with $\nabla f \in L^1_{loc}(\mathbb{R}^n)$. Then*

$$\nabla |f| \in L^1_{loc}(\mathbb{R}^n)$$

and

$$\nabla |f|(x) = \Re[sgn(f(x))\, \nabla f(x)] \tag{2.6}$$

almost everywhere. In particular, we have

$$|\nabla |f|| \leq |\nabla f|, \tag{2.7}$$

almost everywhere.

Proof. Let $z \in \mathbb{C}$ and $\epsilon > 0$. We define

$$|z|_\epsilon := \sqrt{|z|^2 + \epsilon^2} - \epsilon$$

and observe that

$$0 \le |z|_\epsilon \le |z| \text{ and } \lim_{\epsilon \to 0} |z|_\epsilon = |z|.$$

If $u \in \mathcal{C}^\infty(\mathbb{R}^n)$, then $|u|_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^n)$ and as $|u|^2 = u\,\overline{u}$ we get

$$\nabla |u|_\epsilon = \frac{\Re(u\,\nabla u)}{\sqrt{|u|^2 + \epsilon^2}}. \tag{2.8}$$

Now let f be as assumed, take an approximation to the identity $(\chi_\delta)_\delta$ and define

$$f_\delta = f * \chi_\delta.$$

By Lemma 2.14, Lemma 2.15 and Lemma 2.16, we obtain that $f_\delta \to f$, $|f_\delta| \to |f|$, and $\nabla f_\delta \to \nabla f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $\delta \to 0$.

Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a test function. There exists a subsequence $\delta_k \to 0$ such that $f_{\delta_k}(x) \to f(x)$ for almost every $x \in \operatorname{supp}\phi$. For simplicity we omit the index k now. Using the dominated convergence theorem and (2.8) we get

$$\begin{aligned} \int (\nabla\phi)\,|f|\,d\lambda &= \lim_{\epsilon \to 0} \int (\nabla\phi)\,|f|_\epsilon\,d\lambda \\ &= \lim_{\epsilon \to 0}\lim_{\delta \to 0} \int (\nabla\phi)\,|f_\delta|_\epsilon\,d\lambda \\ &= -\lim_{\epsilon \to 0}\lim_{\delta \to 0} \int \phi\,\frac{\Re(\overline{f}_\delta \nabla f_\delta)}{\sqrt{|f_\delta|^2 + \epsilon^2}}\,d\lambda. \end{aligned}$$

Since $\nabla f_\delta \to \nabla f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we get taking the limit $\delta \to 0$ that

$$\int (\nabla\phi)\,|f|\,d\lambda = -\lim_{\epsilon \to 0} \int \phi\,\frac{\Re(\overline{f}\nabla f)}{\sqrt{|f|^2 + \epsilon^2}}\,d\lambda,$$

and since $\phi\nabla f \in L^1(\mathbb{R}^n$ and $\overline{f}/\sqrt{|f|^2 + \epsilon^2} \to \operatorname{sgn} f$ as $\epsilon \to 0$ we get the desired result by applying once more dominated convergence. □

In the sequel we still use the methods from above for approximating a function in a Sobolev space by smooth functions. In a similar way as in the last lemma one gets

Lemma 2.20. *If $u \in W^k(\Omega)$, then $u_\epsilon \to u$ in $W^k_{loc}(\Omega)$, as $\epsilon \to 0$, this means that this happens in each space $W^k(\omega)$, where ω is an open subset with $\omega \subset\subset \Omega$.*

Using a smooth partition of unity we still show that one can find smooth functions which approximate in the $W^k(\Omega)$, and not just in $W^k_{loc}(\Omega)$.

Lemma 2.21. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $u \in W^k(\Omega)$. Then there exist functions $u_m \in \mathcal{C}^\infty(\Omega) \cap W^k(\Omega)$ such that $u_m \to u$ in $W^k(\Omega)$.*

Note that we do not assert that $u_m \in \mathcal{C}^\infty(\overline{\Omega})$.

Proof. We write $\Omega = \bigcup_{j=1}^\infty \omega_j$, where

$$\omega_j := \{x \in \Omega : \mathrm{dist}(x, b\Omega) > 1/j\} \ \ , \ j = 1, 2, \ldots .$$

Set $U_j := \omega_{j+3} \setminus \overline{\omega}_{j+1}$, and choose any open set $U_0 \subset\subset \Omega$ so that $\Omega = \bigcup_{j=0}^\infty U_j$. Let $(\phi_j)_j$ be a smooth partition of unity subordinate to the open sets $(U_j)_j$: that is $0 \le \phi_j \le 1$, $\phi_j \in \mathcal{C}_0^\infty(U_j)$ and $\sum_{j=0}^\infty \phi_j = 1$ on Ω.

According to Proposition 2.6 $\phi_j u \in W^k(\Omega)$ and the support of $\phi_j u$ is contained in U_j.

Now we use Lemma 2.20: fix $\epsilon > 0$ and choose $\epsilon_j > 0$ so small that $u_j := (\phi_j u) * \chi_{\epsilon_j}$ satisfies

$$\|u_j - \phi_j u\|_{W^k(\Omega)} \le \frac{\epsilon}{2^{j+1}} \ \ , \ j = 0, 1, \ldots ,$$

and u_j has support in $V_j := \omega_{j+4} \setminus \overline{\omega}_j \supset U_j$ for $j = 1, 2, \ldots$.

Now define $v := \sum_{j=0}^\infty u_j$. This function belongs to $\mathcal{C}^\infty(\Omega)$, since for each open set $\omega \subset\subset \Omega$ there are at most finitely many nonzero terms in the sum. Since $u = \sum_{j=0}^\infty \phi_j u$, we have for each $\omega \subset\subset \Omega$

$$\begin{aligned} \|v - u\|_{W^k(\omega)} &\le \sum_{j=0}^\infty \|u_j - \phi_j u\|_{W^k(\Omega)} \\ &\le \epsilon \sum_{j=0}^\infty \frac{1}{2^{j+1}} \\ &= \epsilon. \end{aligned}$$

Finally, take the supremum over all sets $\omega \subset\subset \Omega$, to conclude that

$$\|v - u\|_{W^k(\Omega)} \le \epsilon.$$

□

Before we proceed to prove the density result for the $\overline{\partial}$-setting we show that a function $u \in W^k(\Omega)$ can be approximated by functions in $\mathcal{C}^\infty(\overline{\Omega})$, where all derivatives extend continuously to $\overline{\Omega}$. This of course requires some conditions on the boundary $b\Omega$.

Proposition 2.22. *Let Ω be a bounded open set in $\mathbb{R}^n$ and assume that $b\Omega$ is $\mathcal{C}^1$. Let $u \in W^k(\Omega)$. Then there exist functions $u_m \in \mathcal{C}^\infty(\overline{\Omega})$ such that $u_m \to u$ in $W^k(\Omega)$.*

Proof. Let $x_0 \in b\Omega$. As $b\Omega$ is $\mathcal{C}^1$, there exists a radius $r > 0$ and a $\mathcal{C}^1$-function $\gamma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ such that

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > \gamma(x_1, \dots, x_{n-1})\}.$$

We set $V := \Omega \cap B(x_0, r/2)$ and define the shifted point $x_\epsilon := x + \mu\epsilon e_n$ for $x \in V$ and $\epsilon > 0$. We see that for some fixed, sufficiently large number $\mu > 0$ the ball $B(x_\epsilon, \epsilon)$ lies in $\Omega \cap B(x_0, r)$ for all $x \in V$ and all small $\epsilon > 0$.

Now we define $u_\epsilon(x) := u(x_\epsilon)$ for $x \in V$; this is the function u translated a distance $\mu\epsilon$ in the e_n-direction. Next we write $v_\epsilon = u_\epsilon * \chi_\epsilon$. The idea is that we have moved up enough so that there is room to mollify within Ω. We have $v_\epsilon \in \mathcal{C}^\infty(\overline{V})$.

We now claim that $v_\epsilon \to u$ in $W^k(V)$ as $\epsilon \to 0$. Let α be a multiindex with $|\alpha| \le k$. Then

$$\|D^\alpha v_\epsilon - D^\alpha u\|_{L^2(V)} \le \|D^\alpha v_\epsilon - D^\alpha u_\epsilon\|_{L^2(V)} + \|D^\alpha u_\epsilon - D^\alpha u\|_{L^2(V)}.$$

The second term on the right hand side goes to zero with ϵ, since, by Lemma 2.14, translation is continuous in the L^2-norm. The first term also vanishes as $\epsilon \to 0$, by a similar reasoning as in Lemma 2.18.

Let $\delta > 0$. Since $b\Omega$ is compact, one can find finitely many points $x_j \in b\Omega$, radii $r_j > 0$, corresponding sets $V_j = \Omega \cap B(x_j, r_j/2)$, and functions $v_j \in \mathcal{C}^\infty(\overline{V}_j)$, $j = 1, \dots, N$ such that

$$b\Omega \subset \bigcup_{j=1}^{N} B(x_j, r_j/2) \quad \text{and} \quad \|v_j - u\|_{W^k(V_j)} \le \delta. \tag{2.9}$$

Now we take an open set $V_0 \subset\subset \Omega$ such that

$$\Omega \subset \bigcup_{j=0}^{N} V_j$$

and select, using Lemma 2.20, a function $v_0 \in \mathcal{C}^\infty(\overline{V}_0)$ satisfying

$$\|v_0 - u\|_{W^k(V_0)} \le \delta. \tag{2.10}$$

Finally we take a smooth partition $(\phi_j)_j$ of unity subordinate to the open sets $(V_j)_j$ in Ω for $j = 0, \dots, N$. Define $v := \sum_{j=0}^N \phi_j v_j$. Then $v \in \mathcal{C}^\infty(\overline{\Omega})$. Since $u = \sum_{j=0}^N \phi_j u$ we see that for each $|\alpha| \le k$:

$$\|D^\alpha v - D^\alpha u\|_{L^2(\Omega)} \quad \le \quad \sum_{j=0}^{N} \|D^\alpha(\phi_j v_j) - D^\alpha(\phi_j u)\|_{L^2(V_j)}$$

$$\begin{aligned} &\leq \sum_{j=0}^{N} \|v_j - u\|_{W^k(V_j)} \\ &= C(N+1)\delta, \end{aligned}$$

where we used (2.9) and (2.10).

□

A set A is precompact (i.e. $\overline{A}$ is compact) in a Banach space X if and only if for every positive number ϵ there is a finite subset N_ϵ of points of X such that $A \subset \bigcup_{y \in N_\epsilon} B_\epsilon(y)$. A set N_ϵ with this property is called a finite ϵ-net for A.

We recall the Arzela-Ascoli theorem: Let Ω be a bounded domain in $\mathbb{R}^n$. A subset K of $\mathcal{C}(\overline{\Omega})$ is precompact in $\mathcal{C}(\overline{\Omega})$ if the following two conditions hold:

(i) There exists a constant M such that $|\phi(x)| \leq M$ holds for every $\phi \in K$ and $x \in \Omega$. (Boundedness)

(ii) For every $\epsilon > 0$ there exists $\delta > 0$ such that if $\phi \in K,\ x, y \in \Omega$, and $|x - y| < \delta$, then $|\phi(x) - \phi(y)| < \epsilon$. (Equicontinuity)

Let $(\chi_\epsilon)_\epsilon$ be an approximation to the identity (see Chapter 5.1) Recall that $u * \chi_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^n)$, if $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ (Lemma 2.15).

In a similar way one proves the following result: If Ω is a domain in $\mathbb{R}^n$ and $u \in L^2(\Omega)$, then $u * \chi_\epsilon \in L^2(\Omega)$ and

$$\|u * \chi_\epsilon\|_2 \leq \|u\|_2 \quad , \quad \lim_{\epsilon \to 0+} \|u * \chi_\epsilon - u\|_2 = 0.$$

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and u a complex-valued function on Ω. Let

$$\tilde{u}(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

Theorem 2.23. *A bounded subset $\mathcal{A}$ of $L^2(\Omega)$ is precompact in $L^2(\Omega)$ if and only if for every $\epsilon > 0$ there exists a number $\delta > 0$ and a subset $\omega \subset\subset \Omega$ such that for every $u \in \mathcal{A}$ and $h \in \mathbb{R}^n$ with $|h| < \delta$ both of the following inequalities hold:*

$$\int_\Omega |\tilde{u}(x+h) - \tilde{u}(x)|^2 \, d\lambda(x) < \epsilon^2 \quad , \quad \int_{\Omega \setminus \overline{\omega}} |u(x)|^2 \, d\lambda(x) < \epsilon^2. \tag{2.11}$$

Proof. Let $\tau_h u(x) = u(x+h)$ denote the translate of u by h. First assume that $\mathcal{A}$ is precompact. Since $\mathcal{A}$ has a finite $\epsilon/6$- net, and since $\mathcal{C}_0(\Omega)$ is dense in $L^2(\Omega)$, there exists a finite set $S \subset \mathcal{C}_0(\Omega)$, such that for each $u \in \mathcal{A}$ there exists $\phi \in S$ satisfying $\|u - \phi\|_2 < \epsilon/3$. Let ω be the union of the supports of the finitely many functions in S. Then $\omega \subset\subset \Omega$ and the second inequality follows immediately. To prove the first inequality choose a closed ball B_r of radius r centered at the origin and containing ω. Note that $(\tau_h\phi-\phi)(x) = \phi(x+h)-\phi(x)$ is uniformly continuous and vanishes outside B_{r+1} provided $|h| < 1$. Hence

$$\lim_{|h|\to 0}\int_{\mathbb{R}^n} |\tau_h\phi(x) - \phi(x)|^2\, d\lambda(x) = 0,$$

the convergence being uniform for $\phi \in S$. For $|h|$ sufficiently small, we have $\|\tau_h\phi-\phi\|_2 < \epsilon/3$. If $\phi \in S$ satisfies $\|u-\phi\|_2 < \epsilon/3$, then also $\|\tau_h\tilde{u}-\tau_h\phi\|_2 < \epsilon/3$. Hence we have for $|h|$ sufficiently small (independent of $u \in \mathcal{A}$),

$$\|\tau_h\tilde{u} - \tilde{u}\|_2 \le \|\tau_h\tilde{u} - \tau_h\phi\|_2 + \|\tau_h\phi - \phi\|_2 + \|\phi - u\|_2 < \epsilon$$

and the first inequality follows.

It is sufficient to prove the converse for the special case $\Omega = \mathbb{R}^n$, as it follows for general Ω from its application in this special case to the set $\tilde{\mathcal{A}} = \{\tilde{u} : u \in \mathcal{A}\}$.

Let $\epsilon > 0$ be given and choose $\omega \subset\subset \mathbb{R}^n$ such that for all $u \in \mathcal{A}$

$$\int_{\mathbb{R}^n\setminus\overline{\omega}} |u(x)|^2\, d\lambda(x) < \frac{\epsilon}{3}.$$

For any $\eta > 0$ the function $u * \chi_\eta \in \mathcal{C}^\infty(\mathbb{R}^n)$ and in particular it belongs to $\mathcal{C}(\overline{\omega})$. If $\phi \in \mathcal{C}_0(\mathbb{R}^n)$, then by Hölder's inequality

$$\begin{aligned}|\chi_\eta * \phi(x) - \phi(x)|^2 &= \left|\int_{\mathbb{R}^n} \chi_\eta(y)(\phi(x-y) - \phi(x))\, d\lambda(y)\right|^2\\ &\le \int_{B_\eta} \chi_\eta(y)|\tau_{-y}\phi(x) - \phi(x)|^2\, d\lambda(y)\end{aligned}$$

Hence

$$\|\chi_\eta * \phi - \phi\|_2 \le \sup_{h\in B_\eta} \|\tau_h\phi - \phi\|_2.$$

If $u \in L^2(\mathbb{R}^n)$, let $(\phi_j)_j$ be a sequence in $\mathcal{C}_0(\mathbb{R}^n)$ converging to u in L^2 norm. Then $(\chi_\eta * \phi_j)_j$ is a sequence converging to $\chi_\eta * u$ in $L^2(\mathbb{R}^n)$. Since also $\tau_h\phi_j \to \tau_h u$ in $L^2(\mathbb{R}^n)$, we have

$$\|\chi_\eta * u - u\|_2 \le \sup_{h\in B_\eta} \|\tau_h u - u\|_2.$$

From the first inequality in our assumption we derive that $\lim_{|h|\to 0}\|\tau_h u-u\|_2 = 0$ uniformly for $u \in \mathcal{A}$. Hence $\lim_{\eta\to 0}\|\chi_\eta * u - u\|_2 = 0$ uniformly for $u \in \mathcal{A}$. Fix $\eta > 0$ so that

$$\int_{\overline{\omega}} |\chi_\eta * u(x) - u(x)|^2\, d\lambda(x) < \frac{\epsilon}{6}$$

for all $u \in \mathcal{A}$.

We show that $\{\chi_\eta * u : u \in \mathcal{A}\}$ satisfies the conditions of the Arzela-Ascoli theorem on $\overline{\omega}$ and hence is precompact in $\mathcal{C}(\overline{\omega})$. We have

$$|\chi_\eta * u(x)| \le \left(\sup_{y\in\mathbb{R}^n} \chi_\eta^2(y)\right)^{1/2} \|u\|_2,$$

which is bounded uniformly for $x \in \mathbb{R}^n$ and $u \in \mathcal{A}$ since $\mathcal{A}$ is bounded in $L^2(\mathbb{R}^n)$ and η is fixed. Similarly

$$|\chi_\eta * u(x+h) - \chi_\eta * u(x)| \le \left(\sup_{y\in\mathbb{R}^n} \chi_\eta^2(y)\right)^{1/2} \|\tau_h u - u\|_2$$

and so $\lim_{|h|\to 0} \chi_\eta * u(x+h) = \chi_\eta * u(x)$ uniformly for $x \in \mathbb{R}^n$ and $u \in \mathcal{A}$. Thus $\{\chi_\eta * u : u \in \mathcal{A}\}$ is precompact in $\mathcal{C}(\overline{\omega})$ and there exists a finite set $\{\psi_1, \dots, \psi_m\}$ of functions in $\mathcal{C}(\overline{\omega})$ such that if $u \in \mathcal{A}$, then for some j, $1 \le j \le m$, and all $x \in \overline{\omega}$ we have

$$|\psi_j(x) - \chi_\eta * u(x)| < \sqrt{\frac{\epsilon}{6|\overline{\omega}|}}.$$

This together with the inequality $(|a| + |b|)^2 \le 2(|a|^2 + |b|^2)$ implies that

$$\int_{\mathbb{R}^n} |u(x) - \tilde{\psi}_j(x)|^2 \, d\lambda(x) = \int_{\mathbb{R}^n\setminus\overline{\omega}} |u(x)|^2 \, d\lambda(x) + \int_{\overline{\omega}} |u(x) - \psi_j(x)|^2 \, dx$$

$$< \frac{\epsilon}{3} + 2 \int_{\overline{\omega}} (|u(x) - \chi_\eta * u(x)|^2 + |\chi_\eta * u(x) - \psi_j(x)|^2) \, d\lambda(x)$$

$$< \frac{\epsilon}{3} + 2\left(\frac{\epsilon}{6} + \frac{\epsilon}{6.|\overline{\omega}|} |\overline{\omega}|\right) = \epsilon.$$

Hence $\mathcal{A}$ has a finite ϵ-net in $L^2(\mathbb{R}^n)$ and is therefore precompact in $L^2(\mathbb{R}^n)$. □

Remark 2.24. (a) With the same proof one gets:

A bounded subset $\mathcal{A}$ of $L^2(\Omega)$ is precompact in $L^2(\Omega)$ if and only if the following two conditions are satisfied:

(i) for every $\epsilon > 0$ and for each $\omega \subset\subset \Omega$ there exists a number $\delta > 0$ such that for every $u \in \mathcal{A}$ and $h \in \mathbb{R}^n$ with $|h| < \delta$ the following inequality holds:

$$\int_\omega |\tilde{u}(x+h) - \tilde{u}(x)|^2 \, d\lambda(x) < \epsilon^2; \tag{2.12}$$

(ii) for every $\epsilon > 0$ there exists $\omega \subset\subset \Omega$ such that for every $u \in \mathcal{A}$

$$\int_{\Omega\setminus\overline{\omega}} |u(x)|^2 \, d\lambda(x) < \epsilon^2. \tag{2.13}$$

Our next aim is to prove the classical Rellich Lemma, which states that the embedding of $W^1(\Omega)$ into $L^2(\Omega)$ is compact, provided that Ω is a bounded domain with a $\mathcal{C}^1$-boundary. In the first step we show that functions in the Sobolev space $W^1(\Omega)$ can be continuously extended to functions in $W^1(\mathbb{R})$, provided that Ω is a bounded domain with a $\mathcal{C}^1$-boundary.

Proposition 2.25. *Assume that Ω is a bounded domain with a $\mathcal{C}^1$-boundary. Select a bounded open set V such that $\Omega \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^1(\Omega) \longrightarrow W^1(\mathbb{R}^n)$$

such that for each $u \in W^1(\Omega)$:
(i) $Eu = u$ almost everywhere in Ω,
(ii) Eu has support within V,
(iii) there exists a constant C depending only on Ω and V such that

$$\|Eu\|_{W^1(\mathbb{R}^n)} \leq C\|u\|_{W^1(\Omega)}.$$

Proof. We use the method of a higher-order reflection for the extension. Let $x_0 \in b\Omega$ and suppose first that $b\Omega$ is flat near x_0, lying in the plane $\{x_n = 0\}$. Then we may assume there exists an open ball B centered in x_0 with radius r such that

$$B^+ := B \cap \{x_n \geq 0\} \subset \overline{\Omega}\ , \quad B^- := B \cap \{x_n \leq 0\} \subset \mathbb{R}^n \setminus \Omega.$$

Temporarily we suppose that $u \in \mathcal{C}^\infty(\overline{\Omega})$. We define then

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in B^+ \\ -3u(x_1, \dots, x_{n-1}, -x_n) + 4u(x_1, \dots, x_{n-1}, -\frac{x_n}{2}) & \text{if } x \in B^-. \end{cases}$$

This is called a higher-order reflection of u from B^+ to B^-.

First we show: $\tilde{u} \in \mathcal{C}^1(B)$. To check this we write

$$u^- := \tilde{u}\,|_{B^-} \ \text{ and } u^+ := \tilde{u}\,|_{B^+}\ .$$

By definition, we have

$$\frac{\partial u^-}{\partial x_n}(x) = 3\frac{\partial u}{\partial x_n}(x_1, \dots, x_{n-1}, -x_n) - 2\frac{\partial u}{\partial x_n}(x_1, \dots, x_{n-1}, -\frac{x_n}{2})$$

and so

$$u^-_{x_n}\,|_{\{x_n=0\}} = u^+_{x_n}\,|_{\{x_n=0\}}\ .$$

Now since $u^+ = u^-$ on $\{x_n = 0\}$, we see that also

$$u^-_{x_j}\,|_{\{x_n=0\}} = u^+_{x_j}\,|_{\{x_n=0\}}$$

for $j = 1, \dots, n-1$. Hence we have

$$D^\alpha u^- \mid_{\{x_n=0\}} = D^\alpha u^+ \mid_{\{x_n=0\}},$$

for each $|\alpha| \leq 1$, which implies $\tilde{u} \in \mathcal{C}^1(B)$.

Using these computations one readily sees that

$$\|\tilde{u}\|_{W^1(B)} \leq C\|u\|_{W^1(B^+)}, \tag{2.14}$$

for some constant $C > 0$ which does not depend on u.

If $b\Omega$ is not flat near x_0, we can find a $\mathcal{C}^1$-mapping $\mathbf{\Phi}$, with inverse $\mathbf{\Psi}$, which straightens out $b\Omega$ near x_0. We write $y = \mathbf{\Phi}(x)$ and $x = \mathbf{\Psi}(y)$ and define $u^*(y) := u(\mathbf{\Psi}(y))$. We choose a small ball B and use the same reasoning as before to extend u^* from B^+ to a function $\tilde{u}^*$ defined on all of B, such that $\tilde{u}^* \in \mathcal{C}^1(B)$ and as in 2.14 we get

$$\|\tilde{u}^*\|_{W^1(B)} \leq C\|u^*\|_{W^1(B^+)}. \tag{2.15}$$

Let $W := \mathbf{\Psi}(B)$. Then converting back to the x-variables, we obtain an extension $\tilde{u}$ of u to W, with

$$\|\tilde{u}\|_{W^1(W)} \leq C\|u\|_{W^1(\Omega)}. \tag{2.16}$$

Since $b\Omega$ is compact, there exist finitely many points $x_0^j \in b\Omega$, open sets W_j, and extensions $\tilde{u}_j$ of u to W_j for $j = 1, \dots, N$, such that $b\Omega \subset \bigcup_{j=1}^N W_j$. Take $W_0 \subset\subset \Omega$ with $\Omega \subset \bigcup_{j=0}^N W_j$, and let $(\phi_j)_j$ be an associated partition of unity.

Write

$$\tilde{u} = \sum_{j=0}^N \phi_j \tilde{u}_j,$$

where $\tilde{u}_0 = u$. Then, by (2.16), we obtain the estimate

$$\|\tilde{u}\|_{W^1(\mathbb{R}^n)} \leq C\|u\|_{W^1(\Omega)} \tag{2.17}$$

for some constant $C > 0$ independent of u. In addition we arrange for the support of $\tilde{u}$ to lie within $V \supset\supset \Omega$.

We define $Eu := \tilde{u}$ and observe that the mapping $u \mapsto Eu$ is linear. So far we have assumed that $u \in \mathcal{C}^\infty(\overline{\Omega})$. Now take $u \in W^1(\Omega)$, and choose a sequence $u_m \in \mathcal{C}^\infty(\overline{\Omega})$ converging to u in $W^1(\Omega)$(see Proposition 2.22). Estimate (2.17) implies

$$\|Eu_m - Eu_\ell\|_{W^1(\mathbb{R}^n)} \leq C\|u_m - u_\ell\|_{W^1(\Omega)}.$$

Hence $(Eu_m)_m$ is a Cauchy sequence and so converges to $\tilde{u} =: Eu$. This extension does not depend on the particular choice of the approximating sequence $(u_m)_m$. □

In a similar way we treat the problem how to assign boundary values along $b\Omega$ to a function $u \in W^1(\Omega)$, assuming that $b\Omega$ is $\mathcal{C}^1$.

Proposition 2.26. *Let Ω be a bounded domain with $\mathcal{C}^1$- boundary. Then there exists a bounded linear operator*

$$T : W^1(\Omega) \longrightarrow L^2(b\Omega)$$

such that

(i) $Tu = u\,|_{b\Omega}$, if $u \in W^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$;

(ii) and

$$\|Tu\|_{L^2(b\Omega)} \leq C\|u\|_{W^1(\Omega)},$$

for each $u \in W^1(\Omega)$, with the constant C depending only on Ω.

We call Tu the trace of u on $b\Omega$.

Proof. First we assume that $u \in \mathcal{C}^1(\overline{\Omega})$ and proceed as in the proof of Proposition 2.25. We suppose that $x_0 \in b\Omega$ and that $b\Omega$ is flat near x_0, lying in the plane $\{x_n = 0\}$. We choose an open ball B as in the previous proof and let $\tilde{B}$ denote the concentric ball of radius $r/2$. Select a function $\chi \in \mathcal{C}_0^\infty(B)$ with $\chi \geq 0$ in B and $\chi = 1$ on $\tilde{B}$. Denote Γ the portion of $b\Omega$ within $\tilde{B}$. Set $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} = \{x_n = 0\}$. Then we have

$$\begin{aligned}
\int_\Gamma |u|^2 \, d\lambda(x') &\leq \int_{\{x_n=0\}} \chi |u|^2 \, d\lambda(x') = -\int_{B^+} (\chi |u|^2)_{x_n} \, d\lambda(x) \\
&= -\int_{B^+} (|u|^2 \chi_{x_n} + |u| (\mathrm{sgn} u) \, u_{x_n} \chi) \, d\lambda(x) \\
&\leq C \int_{B^+} (|u|^2 + |\nabla u|^2) \, d\lambda(x),
\end{aligned}$$

where we used Proposition 2.19 and the inequality $ab \leq a^2/2 + b^2/2$, for $a, b \geq 0$.

After this we straighten out the boundary near x_0 to get the estimate

$$\int_\Gamma |u|^2 \, d\lambda(x') \leq C \int_\Omega (|u|^2 + |\nabla u|^2) \, d\lambda(x),$$

where Γ is some open subset of $b\Omega$ containing x_0.

Since $b\Omega$ is compact, there exist finitely many points $x_{0,k} \in b\Omega$ and open subsets $\Gamma_k \subset b\Omega$ $(k = 1, \dots, N)$ such that

$$b\Omega = \bigcup_{k=1}^N \Gamma_k$$

and

$$\|u\|_{L^2(\Gamma_k)} \leq C\|u\|_{W^1(\Omega)},$$

for $k = 1, \dots, N$. Hence, if we write

$$Tu := u\,|_{b\Omega},$$

we get

$$\|Tu\|_{L^2(b\Omega)} \leq C\|u\|_{W^1(\Omega)}, \tag{2.18}$$

for some constant C, which does not depend on u.

Inequality (2.18) holds for $u \in \mathcal{C}^1(\overline{\Omega})$. Assume now that $u \in W^1(\Omega)$. Then, by Proposition 2.22 there exist functions $u_m \in \mathcal{C}^\infty(\overline{\Omega})$ converging to u in $W^1(\Omega)$. By (2.18) we have

$$\|Tu_m - Tu_\ell\|_{L^2(b\Omega)} \leq C\|u_m - u_\ell\|_{W^1(\Omega)}, \tag{2.19}$$

hence $(Tu_m)_m$ is a Cauchy sequence in $L^2(b\Omega)$. Set

$$Tu := \lim_{m\to\infty} Tu_m,$$

where the limit is taken in $L^2(b\Omega)$. By (2.19), this definition does not depend on the particular choice of the smooth functions approximating u.

If $u \in W^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$, one can use the fact that the functions $u_m \in \mathcal{C}^\infty(\overline{\Omega})$ constructed in the proof of Proposition 2.22 converge uniformly to u on $\overline{\Omega}$. This implies $Tu = u\,|_{b\Omega}$. □

Remark 2.27. One can actually show, that under the same conditions as before, $u \in H_0^1(\Omega)$, if and only if $Tu = 0$ on $b\Omega$.

Finally we now investigate the embedding of $W^1(\Omega)$ into $L^2(\Omega)$ in more detail.

Lemma 2.28. (Rellich-Kondrachov) *Let Ω be a bounded domain with a $\mathcal{C}^1$ boundary. Then the embedding $j : W^1(\Omega) \longrightarrow L^2(\Omega)$ is compact.*

Proof. We have to show that the unit ball in $W^1(\Omega)$ is precompact in $L^2(\Omega)$.

For this purpose we apply Proposition 2.25 and consider the extension of elements of the unit ball in $W^1(\Omega)$ to $\mathbb{R}^n$. Let $\mathcal{F}$ denote the set of all these extensions. Then, by Proposition 2.25 (iii), $\mathcal{F}$ is a bounded set in $L^2(\mathbb{R}^n)$. By Lemma 2.15 we know that for each $\epsilon > 0$ there exists a number $N > 0$ such that

$$\|\chi_{1/k} * f - f\|_{L^2(\mathbb{R}^n)} \leq \epsilon, \tag{2.20}$$

for each $f \in \mathcal{F}$ and for each $k > N$.

By Hölder's inequality we have

$$\|\chi_{1/k} * f\|_{L^\infty(\mathbb{R}^n)} \le C_k \|f\|_{L^2(\mathbb{R}^n)}, \tag{2.21}$$

for all $f \in \mathcal{F}$, where $C_k = \|\chi_{1/k}\|_{L^2(\mathbb{R}^n)}$.

Hence we can now verify the second condition in Theorem 2.23: given $\epsilon > 0$ there exists $\omega \subset\subset \Omega$ such that

$$\|f\|_{L^2(\Omega\setminus\omega)} < \epsilon,$$

for each f in the unit ball of $W^1(\Omega)$: indeed, we consider the extensions to $\mathbb{R}^n$ and write

$$\|f\|_{L^2(\Omega\setminus\omega)} \le \|f - \chi_{1/k} * f\|_{L^2(\mathbb{R}^n} + \|\chi_{1/k} * f\|_{L^2(\Omega\setminus\omega)},$$

we use Proposition 2.25 (iii) and (2.20), and, in view of (2.21) we have to choose ω such that $|\Omega \setminus \omega|$ is small enough.

We are left to verify the first condition of Theorem 2.23: let $\omega \subset\subset \Omega$ and $\epsilon > 0$ and consider first a function $u \in \mathcal{C}^\infty(\overline{\Omega})$. Let $h \in \mathbb{R}^n$ such that $|h| < \mathrm{dist}(\omega, b\Omega)$ and set

$$v(t) := u(x + th)\ ,\ t \in [0, 1].$$

Then $v'(t) = h \cdot \nabla u(x + th)$ and

$$u(x + h) - u(x) = v(1) - v(0) = \int_0^1 v'(t)\, dt = \int_0^1 h \cdot \nabla u(x + th)\, dt. \tag{2.22}$$

Hence we obtain

$$|u(x + h) - u(x)|^2 \le |h|^2 \int_0^1 |\nabla u(x + th)|^2\, dt$$

and

$$\begin{aligned}\int_\omega |u(x + h) - u(x)|^2\, d\lambda(x) &\le |h|^2 \int_\omega \int_0^1 |\nabla u(x + th)|^2\, dt\, d\lambda(x)\\ &= |h|^2 \int_0^1 \int_\omega |\nabla u(x + th)|^2\, d\lambda(x)\, dt\\ &= |h|^2 \int_0^1 \int_{\omega+th} |\nabla u(x)|^2\, d\lambda(x)\, dt.\end{aligned}$$

If $|h| < \mathrm{dist}(\omega, b\Omega)$, there exists $\omega' \subset\subset \Omega$ such that $\omega + th \subset \omega'$ for each $t \in [0, 1]$. Therefore we get the estimate

$$\|\tau_h u - u\|^2_{L^2(\omega)} \le |h|^2 \int_{\omega'} |\nabla u(x)|^2\, d\lambda(x). \tag{2.23}$$

If u belongs to the unit ball in $W^1(\Omega)$, we approximate u by functions in $\mathcal{C}^\infty(\overline{\Omega})$ (Proposition 2.22), apply (2.23) and pass to the limit getting

$$\|\tau_h u - u\|^2_{L^2(\omega)} \le |h|^2 \int_{\omega'} |\nabla u(x)|^2 \, d\lambda(x) \le |h|^2,$$

which shows that the first condition of Theorem 2.23 holds.

□

Remark 2.29. If $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is a bounded domain with a $\mathcal{C}^1$ boundary, it even follows that

$$W^1(\Omega) \subset L^q(\Omega) \ , \ q \in [1, 2n/(n-2))$$

and that the imbedding is also compact (see for instance [2]).

In order to apply Sobolev space theory, we are forced to study difference quotient approximations to weak derivatives.

Definition 2.30. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in L^2_{loc}(\Omega)$ and suppose that $V \subset\subset \Omega$.

The j^{th}-difference quotient of size h is

$$D^h_j u(x) = \frac{u(x + he_j) - u(x)}{h},$$

for $j = 1, \dots, n$ where $x \in V$ and $h \in \mathbb{R}$, $0 < |h| < \operatorname{dist}(V, b\Omega)$.

Further we define

$$D^h u := (D^h_1 u, \dots, D^h_n u).$$

Proposition 2.31. *(i) Let $u \in W^1(\Omega)$. Then for each $V \subset\subset \Omega$ we have*

$$\|D^h u\|_{L^2(V)} \le C \|\nabla u\|_{L^2(\Omega)} \tag{2.24}$$

for some constant $C > 0$ and all h with $0 < |h| < \frac{1}{2}\, dist(V, b\Omega)$.

(ii) Assume that $u \in L^2(V)$, and that there exists a constant $C > 0$ such that

$$\|D^h u\|_{L^2(V)} \le C \tag{2.25}$$

for all h with $0 < |h| < \frac{1}{2}\, dist(V, b\Omega)$. Then $u \in W^1(V)$ and

$$\|\nabla u\|_{L^2(V)} \le C. \tag{2.26}$$

Proof. Suppose first that $u \in \mathcal{C}^1(\Omega)$. Then for each $x \in V$, $j = 1, \ldots, n$ and $0 < |h| < \frac{1}{2}\operatorname{dist}(V, b\Omega)$, we have

$$u(x + he_j) - u(x) = h \int_0^1 u_{x_j}(x + the_j)\, dt,$$

and hence

$$|u(x + he_j) - u(x)| \leq |h| \int_0^1 |\nabla u(x + the_j)|\, dt.$$

So we obtain

$$\begin{aligned} \int_V |D^h u|^2\, d\lambda &\leq C \sum_{j=1}^n \int_V \int_0^1 |\nabla u(x + the_j)|^2\, dt\, d\lambda(x) \\ &= C \sum_{j=1}^n \int_0^1 \int_V |\nabla u(x + the_j)|^2\, d\lambda(x)\, dt. \end{aligned}$$

Thus

$$\int_V |D^h u|^2\, d\lambda \leq C \int_\Omega |\nabla u|^2\, d\lambda(x).$$

This estimate this true for a smooth u, and by Lemma 2.21 it is valid for arbitrary $u \in W^1(\Omega)$, hence we have shown (i).

Now suppose that (2.25) holds. We choose $j = 1, \ldots, n$ and $\phi \in \mathcal{C}_0^\infty(V)$ and note that for small enough h we have

$$\int_V u(x) \left(\frac{\phi(x + he_j) - \phi(x)}{h} \right) d\lambda(x) = -\int_V \left(\frac{u(x) - u(x - he_j)}{h} \right) \phi(x)\, d\lambda(x),$$

this means

$$\int_V u\, D_j^h \phi\, d\lambda = -\int_V (D_j^{-h} u)\, \phi\, d\lambda. \tag{2.27}$$

Hence (2.25) implies that

$$\sup_h \|D_j^{-h} u\|_{L^2(V)} < \infty.$$

Using the fact that for each bounded sequence in a Hilbert space there exists a weakly convergent subsequence, we conclude that there exists a function $v_j \in L^2(V)$ and a subsequence $h_k \to 0$ such that $D_j^{-h_k} u \to v_j$ weakly in $L^2(V)$. Then we have

$$\begin{aligned} \int_V u\phi_{x_j}\, d\lambda &= \int_\Omega u\phi_{x_j}\, d\lambda = \lim_{h_k \to 0} \int_\Omega u D_j^{h_k} \phi\, d\lambda \\ &= -\lim_{h_k \to 0} \int_V D_j^{-h_k} u\, \phi\, d\lambda \end{aligned}$$

$$= \quad -\int_V v_j \phi \, d\lambda = -\int_\Omega v_j \phi \, d\lambda.$$

Hence $v_j = u_{x_j}$ in the sense of distributions, and so $\nabla u \in L^2(V)$. As $u \in L^2(V)$, we get $u \in W^1(V)$.

□

We prove a basic result concerning elliptic partial differential equations of order 2 with variable coefficients and the smoothness of their weak solutions.

Definition 2.32. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and $a_{jk} \in \mathcal{C}^1(\Omega)$, $b_j, c \in L^\infty(\Omega)$ for $j, k = 1, \ldots, n$. Define the partial differential operator L by

$$Lu = -\sum_{j,k=1}^{n} (a_{jk}(x) u_{x_j})_{x_k} + \sum_{j=1}^{n} b_j(x) u_{x_j} + c(x) u, \tag{2.28}$$

with $a_{jk} = a_{kj}$ for $j, k = 1, \ldots, n$. We

We say that the partial differential operator L is elliptic if there exists a constant $C > 0$ such that

$$\sum_{j,k=1}^{n} a_{jk}(x) t_j t_k \geq C|t|^2 \tag{2.29}$$

for almost every $x \in \Omega$ and all $t \in \mathbb{R}^n$.

Ellipticity means that the symmetric $(n \times n)$ matrix $A(x) = (a_{jk}(x))_{j,k=1}^n$ is positive definite, with smallest eigenvalue greater than or equal to C.

If $a_{jk} = \delta_{jk}, \; b_j = 0, \; c = 0$, then $L = -\triangle$.

Definition 2.33. Let $f \in L^2(\Omega)$. We say that a function $u \in H^1(\Omega)$ is a weak solution to the elliptic partial differential equation

$$Lu = f \text{ in } \Omega,$$

if for the bilinear form

$$a(u, v) = \int_\Omega \Big(\sum_{j,k=1}^{n} a_{jk} u_{x_j} v_{x_k} + \sum_{j=1}^{n} b_j u_{x_j} v + cuv\Big) \, d\lambda$$

we have

$$a(u, v) = (f, v)$$

for all $v \in H_0^1(\Omega)$, where $(.,.)$ denotes the inner product in $L^2(\Omega)$.

In the next proposition we show what is called the interior H^2-regularity of the operator L.

Proposition 2.34. *Let L be as in the above definition and $f \in L^2(\Omega)$. Suppose that $u \in H^1(\Omega)$ is a weak solution of $Lu = f$. Then $u \in H^2_{loc}(\Omega)$; and for each open $V \subset\subset \Omega$ we have*

$$\|u\|_{H^2(V)} \leq \tilde{C}(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}), \tag{2.30}$$

where $\tilde{C} > 0$ is a constant only depending on V, Ω, and the coefficients of L.

Proof. Choose an open set W such that $V \subset\subset W \subset\subset \Omega$. Next select a smooth cutoff function ψ with $0 \leq \psi \leq 1$, $\psi = 1$ on V, and $\psi = 0$ on $\mathbb{R}^n \setminus W$.

Since u is a weak solution of $Lu = f$, we have $a(u, v) = (f, v)$ for all $v \in H^1_0(\Omega)$ and hence

$$\sum_{j,k=1}^{n} \int_\Omega a_{jk} u_{x_j} v_{x_k} \, d\lambda = \int_\Omega \tilde{f} v \, d\lambda, \tag{2.31}$$

where

$$\tilde{f} := f - \sum_{j=1}^{n} b_j u_{x_j} - cu. \tag{2.32}$$

Let $\ell \in \{1, \dots, n\}$ and $h \in \mathbb{R}$ such that $|h| > 0$ is small. We substitute

$$v = -D_\ell^{-h}(\psi^2 D_\ell^h u) \tag{2.33}$$

into (2.31), where $D_\ell^h u$ denotes the difference quotient (Definition 2.30). For the left hand side of (2.31) we get

$$-\sum_{j,k=1}^{n} \int_\Omega a_{jk} u_{x_j} [D_\ell^{-h}(\psi^2 D_\ell^h u)]_{x_k} \, d\lambda = \sum_{j,k=1}^{n} \int_\Omega D_\ell^h(a_{jk} u_{x_j})(\psi^2 D_\ell^h u)_{x_k} \, d\lambda$$

$$= \sum_{j,k=1}^{n} \int_\Omega [a_{jk}^h (D_\ell^h u_{x_j})(\psi^2 D_\ell^h u)_{x_k} + (D_\ell^h a_{jk}) u_{x_j} (\psi^2 D_\ell^h u)_{x_k}] \, d\lambda,$$

where we used the formulas

$$\int_\Omega v D_\ell^{-h} w \, d\lambda = -\int_\Omega w D_\ell^h v \, d\lambda$$

and

$$D_\ell^h(vw) = v^h D_\ell^h w + w D_\ell^h v,$$

for $v^h(x) := v(x + he_\ell)$. We continue the computation of the left hand side of (2.31) and obtain

$$\sum_{j,k=1}^{n} \int_\Omega a_{jk}^h (D_\ell^h u_{x_j})(D_\ell^h u_{x_k}) \psi^2 \, d\lambda +$$

$$\sum_{j,k=1}^{n} \int_{\Omega} [a_{jk}^h (D_\ell^h u_{x_j})(D_\ell^h u) 2\psi\psi_{x_k} +$$

$$(D_\ell^h a_{jk}) u_{x_j} (D_\ell^h u_{x_k}) \psi^2 + (D_\ell^h a_{jk}) u_{x_j} (D_\ell^h u) 2\psi\psi_{x_k}] \, d\lambda =$$

$$T_1 + T_2.$$

The first term can estimated from below using ellipticity (2.29):

$$T_1 \geq C \int_{\Omega} \psi^2 |D_\ell^h \nabla u|^2 \, d\lambda. \tag{2.34}$$

For the second term T_2 we have by the assumptions on a_{jk}, b_j and c that there exists a constant $C' > 0$ such that

$$|T_2| \leq C' \int_{\Omega} [\psi |D_\ell^h \nabla u| \, |D_\ell^h u| + \psi |D_\ell^h \nabla u| \, |\nabla u| + \psi |D_\ell^h u| \, |\nabla u|] \, d\lambda.$$

Take into account that $\psi = 0$ on $\mathbb{R}^n \setminus W$ and use the small constant–large constant trick to get

$$|T_2| \leq \epsilon \int_{\Omega} \psi^2 |D_\ell^h \nabla u|^2 \, d\lambda + \frac{C'}{\epsilon} \int_W [|D_\ell^h u|^2 + |\nabla u|^2] \, d\lambda,$$

now choose $\epsilon = C/2$ and the estimate (2.24) to obtain

$$|T_2| \leq \frac{C}{2} \int_{\Omega} \psi^2 |D_\ell^h \nabla u|^2 \, d\lambda + C'' \int_{\Omega} |\nabla u|^2 \, d\lambda.$$

Hence, by (2.34), we see that the left hand side of (2.31) can be estimated from below by

$$\frac{C}{2} \int_{\Omega} \psi^2 |D_\ell^h \nabla u|^2 \, d\lambda - C'' \int_{\Omega} |\nabla u|^2 \, d\lambda. \tag{2.35}$$

The absolute value of the right hand side of (2.31) is certainly less than

$$C'' \int_{\Omega} (|f| + |\nabla u| + |u|) |v| \, d\lambda. \tag{2.36}$$

Using Proposition 2.31 (i) we derive that

$$\begin{aligned} \int_{\Omega} |v|^2 \, d\lambda &\leq C'' \int_{\Omega} |\nabla(\psi^2 D_\ell^h u)|^2 \, d\lambda \\ &\leq C'' \int_W (|D_\ell^h u|^2 + \psi^2 |D_\ell^h \nabla u|^2) \, d\lambda \\ &\leq C'' \int_{\Omega} (|\nabla u|^2 + \psi^2 |D_\ell^h \nabla u|^2) \, d\lambda. \end{aligned}$$

Again by the small constant–large constant trick and by (2.36) we obtain now that the absolute value of the right hand side of (2.31) can be estimated form above by

$$\epsilon \int_\Omega \psi^2 |D_\ell^h \nabla u|^2)\, d\lambda + \frac{C'}{\epsilon} \int_\Omega (|f|^2 + |u|^2 + |\nabla u|^2)\, d\lambda.$$

Let $\epsilon = C/4$. Then the absolute value of the right hand side of (2.31) can be estimated form above by

$$\frac{C}{4} \int_\Omega \psi^2 |D_\ell^h \nabla u|^2)\, d\lambda + C''' \int_\Omega (|f|^2 + |u|^2 + |\nabla u|^2)\, d\lambda. \tag{2.37}$$

Finally, combine (2.31), (2.35) and (2.37) to see that

$$\int_V |D_\ell^h \nabla u|^2\, d\lambda \leq \int_\Omega \psi^2 |D_\ell^h \nabla u|^2\, d\lambda \leq \tilde{C} \int_\Omega (|f|^2 + |u|^2 + |\nabla u|^2)\, d\lambda,$$

for some constant $\tilde{C} > 0$, for $\ell = 1, \dots, n$ and all sufficiently small $|h| \neq 0$.

Using Proposition 2.31 (ii) we derive that $\nabla u \in H^1_{\text{loc}}(\Omega)$, and hence that $u \in H^2_{\text{loc}}(\Omega)$, with the estimate

$$\|u\|_{H^2(V)} \leq \tilde{C}(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}).$$

□

Remark 2.35. (a) It is not difficult to show that even

$$\|u\|_{H^2(V)} \leq \tilde{C}(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})$$

holds in Proposition 2.34.

(b) The result that $u \in H^2_{\text{loc}}(\Omega)$ implies that $Lu = f$ almost everywhere in Ω. To see this, note that for each $v \in \mathcal{C}_0^\infty(\Omega)$, we have

$$a(u, v) = (f, v),$$

and since $u \in H^2_{\text{loc}}(\Omega)$, we can integrate by parts and obtain

$$a(u, v) = (Lu, v).$$

Thus $(Lu - f, v) = 0$ for all $v \in \mathcal{C}_0^\infty(\Omega)$, and so $Lu = f$ almost everywhere in Ω.

In the sequel we will use the dual space of $H_0^1(\Omega)$, which is denoted by $H^{-1}(\Omega)$.

We will describe $H_0^{-1}(\Omega)$ as a certain space of distributions, which will be helpful later on. Recall that the dual-norm is given by

$$\|f\|_{H_0^{-1}(\Omega)} = \sup\{|f(u)| : u \in H_0^1(\Omega)\ ,\ \|u\|_{H_0^1(\Omega)} \leq 1\}.$$

Proposition 2.36. *Let $f \in H_0^{-1}(\Omega)$. Then there exist functions $f_0, f_1, \ldots, f_n$ in $L^2(\Omega)$ such that*

$$f(v) = \int_\Omega (f_0 v + \sum_{j=1}^{n} f_j\, v_{x_j})\, d\lambda\,, \tag{2.38}$$

$$\|f\|_{H_0^{-1}(\Omega)} = \inf\left\{ \left(\int_\Omega \sum_{j=0}^{n} |f_j^2|\, d\lambda \right)^{1/2} : f \textit{ satisfies } (2.38) \right\}. \tag{2.39}$$

We write

$$f = f_0 - \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j},$$

whenever (2.38) holds.

Proof. For $u, v \in H_0^1(\Omega)$, the inner product is given by

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v\, d\lambda + \int_\Omega uv\, d\lambda.$$

If $f \in H_0^{-1}(\Omega)$, the Riesz representation theorem implies that there exists a unique function $u \in H_0^1(\Omega)$, such that

$$f(v) = (u, v)\ ,\ \forall v \in H_0^1(\Omega),$$

hence

$$f(v) = \int_\Omega \nabla u \cdot \nabla v\, d\lambda + \int_\Omega uv\, d\lambda, \tag{2.40}$$

which gives (2.38), where $f_0 = u$ and $f_j = u_{x_j}$, $j = 1, \ldots, n$. By Cauchy-Schwarz we obtain

$$\|f\|_{H_0^{-1}(\Omega)} \leq \left(\int_\Omega \sum_{j=0}^{n} |f_j|^2\, d\lambda \right)^{1/2},$$

and setting $v = u/\|u\|_{H_0^1(\Omega)}$ in (2.40) we deduce

$$\|f\|_{H_0^{-1}(\Omega)} = \left(\int_\Omega \sum_{j=0}^{n} |f_j|^2\, d\lambda \right)^{1/2},$$

which gives (2.39).

□

Now we consider the boundary-value problem

$$\begin{cases} Lu = f_0 - \sum_{j=1}^n \frac{\partial f_j}{\partial x_j} & \text{in } \Omega \\ u = 0 & \text{on } b\Omega, \end{cases} \tag{2.41}$$

where L is defined by (2.28) and $f_j \in L^2(\Omega)$, for $j = 0, 1, \ldots, n$.

By Proposition 2.36 we see that the righthand term

$$f = f_0 - \sum_{j=1}^n \frac{\partial f_j}{\partial x_j}$$

belongs to $H_0^{-1}(\Omega)$.

Definition 2.37. A function $u \in H_0^1(\Omega)$ is a weak solution of problem (2.41) if

$$a(u, v) = \langle f, v \rangle$$

for all $v \in H_0^1(\Omega)$, where the bilinear form $a(u, v)$ is given in Definition 2.33 and where

$$\langle f, v \rangle = \int_\Omega [f_0 v + \sum_{j=1}^n f_j v_{x_j}]\, d\lambda$$

and where $\langle .,. \rangle$ is the pairing of $H_0^{-1}(\Omega)$ and $H_0^1(\Omega)$.

In the following we will prove estimates for elliptic partial differential operators which will enable us to apply the general functional analysis results from Chapter 1 to show existence and uniqueness of weak solutions.

Proposition 2.38. *[Energy estimates] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\mathcal{C}^1$-boundary. Let L be an elliptic partial differential operator of second order and $a(u, v)$ the corresponding bilinear form (see Definition 2.33). There exists constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that*

$$|a(u, v)| \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}, \tag{2.42}$$

and

$$\beta \|u\|^2_{H_0^1(\Omega)} \leq a(u, u) + \gamma \|u\|^2_{L^2(\Omega)} \tag{2.43}$$

for all $u, v \in H_0^1(\Omega)$.

Proof. It is easily seen that

$$|a(u, v)| \leq \sum_{j,k=1}^n \|a_{jk}\|_{L^\infty} \int_\Omega |\nabla u|\, |\nabla v|\, d\lambda$$

$$\begin{aligned} &+ \sum_{j=1}^{n} \|b_j\|_{L^\infty} \int_\Omega |\nabla u|\,|v|\,d\lambda + \|c\|_{L^\infty} \int_\Omega |u|\,|v|\,d\lambda \\ &\leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}, \end{aligned}$$

for some appropriate constant α.

Ellipticity of L implies that

$$\begin{aligned} C \int_\Omega |\nabla u|^2 \, d\lambda &\leq \int_\Omega \sum_{j,k=1}^{n} a_{jk} u_{x_j} u_{x_k} \, d\lambda \\ &= a(u,u) - \int_\Omega (\sum_{j=1}^{n} b_j u_{x_j} u + cu^2) \, d\lambda \\ &\leq a(u,u) + \sum_{j=1}^{n} \|b_j\|_{L^\infty} \int_\Omega |\nabla u|\,|u|\,d\lambda + \|c\|_{L^\infty} \int_\Omega |u|^2)\,d\lambda. \end{aligned}$$

The small constant - large constant trick gives

$$\int_\Omega |\nabla u|\,|u|\,d\lambda \leq \epsilon \int_\Omega |\nabla u|^2 \, d\lambda + \frac{1}{4\epsilon} \int_\Omega |u|^2) \, d\lambda,$$

taking $\epsilon > 0$ so small that

$$\epsilon \sum_{j=1}^{n} \|b_j\|_{L^\infty} < \frac{C}{2},$$

we obtain

$$\frac{C}{2} \int_\Omega |\nabla u|^2 \, d\lambda \leq a(u,u) + C' \int_\Omega |u|^2) \, d\lambda.$$

Now add $\frac{C}{2} \int_\Omega |u|^2 \, d\lambda$ on both sides. This gives the desired result

$$\beta \|u\|^2_{H_0^1(\Omega)} \leq a(u,u) + \gamma \|u\|^2_{L^2(\Omega)}.$$

□

Remark 2.39. (i) If $\gamma > 0$, we cannot directly use Proposition 1.24. The following existence result must confront this possibility.

(ii) For complex valued functions the corresponding sesquilinear forms are

$$a(u,v) = \int_\Omega (\sum_{j,k=1}^{n} a_{jk} u_{x_j} \overline{v_{x_k}} + \sum_{j=1}^{n} b_j u_{x_j} \overline{v} + cu\overline{v}) \, d\lambda,$$

ellipticity means

$$\sum_{j,k=1}^{n} a_{jk}(x) t_j \overline{t_k} \geq C|t|^2,$$

for each $t \in \mathbb{C}^n$.

The corresponding energy estimates now reads as

$$\beta \|u\|^2_{H^1_0(\Omega)} \leq \Re a(u,u) + \gamma \|u\|^2_{L^2(\Omega)}$$

for all $u, v \in H^1_0(\Omega)$. This inequality is also sometimes called Gåriding's inequality and the corresponding bilinear form is called coercive.

Proposition 2.40. *Let Ω and L as before. There is a number $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and for each function $f \in L^2(\Omega)$, there is a unique weak solution $u \in H^1_0(\Omega)$ of the boundary-value problem*

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega \\ u = 0 & \text{on } b\Omega. \end{cases} \tag{2.44}$$

Proof. Let $\mu \geq \gamma$, and define the bilinear form

$$a_\mu(u,v) = a(u,v) + \mu(u,v), \ u, v \in H^1_0(\Omega),$$

which corresponds to the operator $L_\mu u := Lu + \mu u$.

Now fix $f \in L^2(\Omega)$ and set $\langle f, v \rangle := (f,v)_{L^2(\Omega)}$. This is a bounded linear functional on $L^2(\Omega)$, and thus on $H^1_0(\Omega)$.

Then we can apply Proposition 1.24: we take $\mathcal{V} = H^1_0(\Omega)$. By Proposition 2.38, the bilinear form a_μ satisfies all assumptions of Proposition 1.24. Hence there exists a uniquely determined function $u \in H^1_0(\Omega)$ satisfying

$$a_\mu(u,v) = \langle f, v \rangle.$$

Actually the operator A in Proposition 1.24 coincides with $L + \mu I$, therefore this operator is an isomorphism between $H^1_0(\Omega)$ and $H^{-1}_0(\Omega)$. □

Example 2.41. If $Lu = -\triangle u$, so that

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\lambda$$

one can take $\gamma = 0$. A similar assertion holds for the general operator

$$Lu = -\sum_{j,k=1}^n (a_{jk} u_{x_j})_{x_k} + cu,$$

provided $c \geq 0$ in Ω.

To get more detailed information regarding the solvability of second order elliptic differential operators we will now use the Rellich-Kondrachov Lemma and the Fredholm alternative for compact operators.

Definition 2.42. The formal adjoint L^* of L is given by

$$L^*v = -\sum_{j,k=1}^{n}(a_{jk}v_{x_k})_{x_j} - \sum_{j=1}^{n} b_j v_{x_j} + (c - \sum_{j=1}^{n}\frac{\partial b_j}{\partial x_j})v,$$

for $b_j \in \mathcal{C}^1(\overline{\Omega})$, $j = 1, \ldots, n$.

The adjoint bilinear form a^* is defined by

$$a^*(v,u) = a(u,v),$$

for all $u, v \in H_0^1(\Omega)$.

We say that $v \in H_0^1(\Omega)$ is a weak solution of the adjoint problem

$$\begin{cases} L^*v = f & \text{in } \Omega \\ v = 0 & \text{on } b\Omega, \end{cases} \tag{2.45}$$

provided

$$a^*(v,u) = (f,u)$$

for all $u \in H_0^1(\Omega)$.

Proposition 2.43. *(i) Precisely one of the following statements holds:*

(a) for each $f \in L^2(\Omega)$ there exists a unique weak solution u of the boundary-value problem

$$\begin{cases} Lu = f & \textit{in } \Omega \\ u = 0 & \textit{on } b\Omega \end{cases} \tag{2.46}$$

or else

(b) there exists a weak solution $u \not\equiv 0$ of the homogeneous problem

$$\begin{cases} Lu = 0 & \textit{in } \Omega \\ u = 0 & \textit{on } b\Omega. \end{cases} \tag{2.47}$$

(ii) Furthermore, if (b) holds, the dimension of the subspace $N \subset H_0^1(\Omega)$ of weak solutions of (2.47) *is finite and equals to the dimension of the subspace $N^* \subset H_0^1(\Omega)$ of weak solutions of*

$$\begin{cases} L^*v = 0 & \textit{in } \Omega \\ v = 0 & \textit{on } b\Omega. \end{cases} \tag{2.48}$$

(iii) Finally, the boundary-value problem (2.46) *has a weak solution if and only if $(f,v) = 0$ for all $v \in N^*$.*

The dichotomy (a), (b) is the Fredholm alternative.

Proof. Choose $\mu = \gamma$ (here we suppose that $\gamma > 0$) and consider the bilinear form $a_\gamma(u, v)$ and the corresponding operator $L_\gamma = L + \gamma I$. By Proposition 2.40, we have that for each $g \in L^2(\Omega)$ there exists a uniquely determined $u \in H_0^1(\Omega)$ with

$$a_\gamma(u, v) = (g, v), \ \forall v \in H_0^1(\Omega). \tag{2.49}$$

We already know that L_γ is an isomorphism and we write

$$u = L_\gamma^{-1} g \tag{2.50}$$

whenever (2.49) holds.

Now we see that $u \in H_0^1(\Omega)$ is a weak solution of (2.46) if and only if

$$a_\gamma(u, v) = (\gamma u + f, v), \ \forall v \in H_0^1(\Omega); \tag{2.51}$$

this means if and only if

$$u = L_\gamma^{-1}(\gamma u + f). \tag{2.52}$$

We rewrite this as $u - Ku = h$,

$$Ku := \gamma L_\gamma^{-1} u \text{ and } h := L_\gamma^{-1} f. \tag{2.53}$$

Next we claim that $K : L^2(\Omega) \longrightarrow L^2(\Omega)$ is a compact operator. By the energy estimates (2.43) we have

$$\beta \|u\|^2_{H_0^1(\Omega)} \leq a_\gamma(u, u) = (g, u) \leq \|g\|_{L^2(\Omega)} \|u\|_{H_0^1(\Omega)},$$

which implies that for $g \in L^2(\Omega)$ we have

$$\|Kg\|_{H_0^1(\Omega)} \leq C' \|g\|_{L^2(\Omega)},$$

where $C' > 0$ is an appropriate constant.

Therefore $K : L^2(\Omega) \longrightarrow H_0^1(\Omega)$ is a continuous operator. As, by the Rellich-Kondrachov Lemma 2.28, the imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we derive that K as operator from $L^2(\Omega)$ to $L^2(\Omega)$ is compact.

Recall Fredholm's alternative: Let $A : H \longrightarrow H$ be a compact linear operator on the Hilbert space H. Then

(1) $\ker(I - A)$ is finite dimensional,
(2) $\operatorname{im}(I - A)$ is closed,
(3) $\operatorname{im}(I - A) = \ker(I - A^*)^\perp$,
(4) $\ker(I - A) = \{0\}$ if and only if $\operatorname{im}(I - A) = H$,
(5) $\dim \ker(I - A) = \dim \ker(I - A^*)$.

In particular we have: either (*) for each $f \in H$, the equation $u - Au = f$ has a unique solution or else (**) the homogeneous equation $u - Au = 0$ has solutions $u \neq 0$. In the second case, the space of solutions of the homogeneous problem is finite- dimensional, and the nonhomogeneous equation $u - Au = f$ has a solution if and only if $f \in \ker(I - A^*)^\perp$.

Now we see that if (*) holds, then there exists a unique weak solution to (2.46). On the other hand, if (**) is valid, then necessarily $\gamma \neq 0$ and the dimension of the space N is finite and equals to the dimension of the space N^* of solutions of

$$v - K^* v = 0. \tag{2.54}$$

So, $u - Ku = 0$ has nonzero solutions in $L^2(\Omega)$ if and only if u is a weak solution to (2.47) and (2.54) holds if and only if v is a weak solution of (2.48).

Finally, observe that $u - Ku = h$ has a unique solution if and only if $(h, v) = 0$ for all v solving (2.54), and we get from (2.53) and (2.54)

$$(h, v) = \frac{1}{\gamma}(Kf, v) = \frac{1}{\gamma}(f, K^* v) = \frac{1}{\gamma}(f, v).$$

Consequently, the boundary-value problem (2.46) has a solution if and only if $(f, v) = 0$ for all weak solutions v of (2.48). □

Remark 2.44. (Higher boundary regularity) Let $m \in \mathbb{N}$, and assume that $a_{jk}, b_j, c \in \mathcal{C}^{m+1}(\overline{\Omega}),\ j, k = 1, \dots, n$ and $f \in H^m(\Omega)$. Suppose that $u \in H_0^1(\Omega)$ is a weak solution of the boundary-value problem (2.46). Assume finally that $b\Omega$ is $\mathcal{C}^{m+2}$. Then $u \in H^{m+2}(\Omega)$ and

$$\|u\|_{H^{m+2}(\Omega)} \leq C'(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}).$$

(see [2, 3])

Finally we arrive at a situation which leads to the next chapter.

Proposition 2.45. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\mathcal{C}^1$-boundary. Let L be an elliptic second order partial differential operator.*

(i) There exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the boundary-value problem

$$\begin{cases} Lu = \mu u + f & \text{in } \Omega \\ u = 0 & \text{on } b\Omega \end{cases} \tag{2.55}$$

has a unique weak solution for each $f \in L^2(\Omega)$ if and only if $\mu \notin \Sigma$, and the solution operators are compact as operators from $L^2(\Omega)$ to $L^2(\Omega)$.

(ii) If Σ is infinite, then $\Sigma = (\mu_k)_{k=1}^\infty$, is a nondecreasing sequence with

$$\mu_k \to +\infty.$$

Definition 2.46. We call Σ the (real) spectrum of the operator L.

Note that in particular the boundary-value problem

$$\begin{cases} Lu = \mu u & \text{in } \Omega \\ u = 0 & \text{on } b\Omega \end{cases}$$

has a nontrivial solution $w \not\equiv 0$ if and only if $\mu \in \Sigma$, in which case μ is called an eigenvalue of L, and w a corresponding eigenfunction. The partial differential equation $Lu = \mu u$ for $L = -\triangle$ is called Helmholtz's equation.

Proof. Let γ be the constant from Proposition 2.38 and assume $\mu > -\gamma$. Assume also with no loss of generality that $\gamma > 0$.

According to Fredholm alternative, the boundary-value problem (2.55) has a unique weak solution for each $f \in L^2(\Omega)$ if and only if $u \equiv 0$ is the only solution of the homogeneous problem

$$\begin{cases} Lu = \mu u & \text{in } \Omega \\ u = 0 & \text{on } b\Omega. \end{cases}$$

This in turn is true if and only if $u \equiv 0$ is the only weak solution of

$$\begin{cases} Lu + \gamma u = (\gamma + \mu) u & \text{in } \Omega \\ u = 0 & \text{on } b\Omega. \end{cases} \tag{2.56}$$

Now (2.56) holds exactly when

$$u = L_\gamma^{-1}(\gamma + \mu) u = \frac{\gamma + \mu}{\gamma} Ku, \tag{2.57}$$

where, as in the proof of Proposition 2.43, we have set $Ku = \gamma L_\gamma^{-1} u$. Recall also that $K : L^2(\Omega) \longrightarrow L^2(\Omega)$ is a compact operator.

If $u \equiv 0$ is the only solution of (2.57), we see that $\frac{\gamma}{\gamma+\mu}$ is not an eigenvalue of K, and this is true if and only if (2.55) has a unique weak solution for each $f \in L^2(\Omega)$.

The collection of all eigenvalues of the compact operator K comprises either a finite set or else the values of a sequence converging to 0. In the second case we see, according to $\mu > -\gamma$ and (2.57), that (2.55) has a unique weak solution for all $f \in L^2(\Omega)$, except for a sequence $\mu_k \to +\infty$.

□

Before we concentrate on spectral analysis, we describe the variational formulation of elliptic boundary value problems. For this purpose we first give a different interpretation of Proposition 1.24. Let H be a Hilbert space over $\mathbb{R}$.

Lemma 2.47. *Let E be a non-empty, convex, closed subset of the Hilbert space H, i.e. for $x, y \in E$ one has $tx + (1-t)y \in E$, for each $t \in [0,1]$. Then E contains a uniquely determined element of minimal norm. For each $f \in H$ there exists a uniquely determined element $u \in E$ (we write $u = Pf$) such that*

$$\|f-u\| = \min_{v\in E} \|f-v\| = dist(f, E). \tag{2.58}$$

Moreover, u is characterized by the property

$$u \in E \text{ and } (f-u, v-u) \leq 0 \ , \forall v \in E. \tag{2.59}$$

Proof. The first statement is standard Hilbert space theory. The second statement follows from the first by taking $f+E$ instead of E.

Suppose that (2.59) holds for$u \in E$. Then for each $w \in E$ and for each $t \in [0,1]$ we have

$$v = (1-t)u + tw \in E,$$

hence

$$\|f-u\| \leq \|f - [(1-t)u + tw]\| = \|(f-u) - t(w-u)\|.$$

Therefore

$$\|f-u\|^2 \leq \|f-u\|^2 - 2t(f-u, w-u) + t^2\|w-u\|^2,$$

which implies that $2(f-u, w-u) \leq t\|w-u\|^2$, for each $t \in (0,1]$. Now let $t \to 0$. Then we get (2.59).

Conversely, assume that u satisfies (2.59). Then we have

$$\|u-f\|^2 - \|v-f\|^2 = 2(f-u, v-u) - \|u-v\|^2 \leq 0 \ , \forall v \in E,$$

which implies (2.58). □

Remark 2.48. If E is a closed linear subspace of H, the element u from (2.58) can be expressed by the orthogonal projection $P : H \longrightarrow E$ in the form $Pf = u$, and Pf is characterized by

$$(f - Pf, v) = 0 \ \forall v \in E. \tag{2.60}$$

By (2.59) we have $(f - Pf, v - Pf) \leq 0$, $\forall v \in E$ and thus $(f - Pf, tv - Pf) \leq 0 \ , \forall v \in E \ , t \in \mathbb{R}$. This implies (2.60). Conversely (2.60) implies $(f - Pf, v - Pf) = 0$, as $(f - Pf, Pf) = 0$, which means that (2.59) holds.

Lemma 2.49. *Let $a : H \times H \longrightarrow \mathbb{R}$ be a continuous H-elliptic bilinear form. Let E be a nonempty closed and convex subset of H. Then, given any $\varphi \in H'$, there exists a unique element $u \in E$ such that*

$$a(u, v-u) \geq \varphi(v-u) \ \forall v \in E. \tag{2.61}$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in E \text{ and } \frac{1}{2}\, a(u,u) - \varphi(u) = \min_{v\in E}\left[\frac{1}{2}\, a(v,v) - \varphi(v)\right]. \qquad (2.62)$$

Proof. By Proposition 1.24 there exists a unique element $Au \in H$ such that

$$a(u,v) = (Au, v)\ \forall v \in H.$$

So we have to find an element $u \in E$ such that

$$(Au, v-u) \geq (f, v-u)\ ,\forall v \in E,$$

where $f \in H$ represents $\varphi : \varphi(v) = (f,v)$.

Let $\rho > 0$ be a constant to be determined later. We see now that (2.61) is equivalent to

$$(\rho f - \rho Au + u - u, v-u) \leq 0\ ,\forall v \in E. \qquad (2.63)$$

Next we claim that

$$\|Pf_1 - Pf_2\| \leq \|f_1 - f_2\|,\ \forall f_1, f_2 \in H.$$

Let $u_j = Pf_j\,, j = 1,2$. Then, by Lemma 2.47,

$$(f_1 - u_1, v - u_1) \leq 0 \text{ and } (f_2 - u_2, v - u_2) \leq 0\ ,\forall v \in E.$$

Choose $v = u_2$ in the first and $v = u_1$ in the second inequality and add them. The result is

$$\|u_1 - u_2\|^2 \leq (f_1 - f_2, u_1 - u_2),$$

which proves the claim.

Now we set $Sv = P(\rho f - \rho Av + v)$, for $v \in E$. We claim that if $\rho > 0$ is properly chosen then S is a strict contraction. We have

$$\|Sv_1 - Sv_2\| \leq \|(v_1 - v_2) - \rho(Av_1 - Av_2)\|,$$

hence

$$\begin{aligned}\|Sv_1 - Sv_2\|^2 &\leq \|v_1 - v_2\|^2 - 2\rho(Av_1 - Av_2, v_1 - v_2) + \rho^2\|Av_1 - Av_2\|^2 \\ &\leq \|v_1 - v_2\|^2\,(1 - 2\rho\alpha + \rho^2 C^2),\end{aligned}$$

where α and C are as in Definition 1.23. Now choose $0 < \rho < 2\alpha/C^2$, then

$$1 - 2\rho\alpha + \rho^2 C^2 < 1,$$

and the mapping S has a unique fixed point (Banach fixed point Theorem). So there exists $u \in E$ such that $u = Su = P(\rho f - \rho Au + u)$, Now use (2.63) and Lemma 2.47 to get (2.61).

If a is also symmetric, then $a(u,v)$ defines a new inner product on H with corresponding norm $a(u,u)^{1/2}$ which is equivalent to the original norm $\|u\|$. It follows that H is also a Hilbert space for this new inner product. So, by the Riesz representation theorem, there exists $g \in H$ such that

$$a(g,v) = \varphi(v)$$

and (2.61) reads as

$$a(g-u, v-u) \leq 0 \ , \forall v \in E.$$

We know that u is simply the projection onto E of g for the new inner product and, by Lemma 2.47, u is the unique element in E that achieves

$$\min_{v \in E} a(g-v, g-v)^{1/2}.$$

This amounts to minimizing on E the function

$$v \mapsto a(g-v, g-v) = a(v,v) - 2a(g,v) + a(g,g) = a(v,v) - 2\varphi(v) + a(g,g),$$

or equivalently the function

$$v \mapsto \frac{1}{2}\, a(v,v) - \varphi(v).$$

□

Corollary 2.50. *Let $a : H \times H \longrightarrow \mathbb{R}$ be a continuous H-elliptic bilinear form. Then, given any $\varphi \in H'$, there exists a unique element $u \in H$ such that*

$$a(u,v) = \varphi(v) \ , \forall v \in H. \tag{2.64}$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in H \ \text{and}\ \frac{1}{2}\, a(u,u) - \varphi(u) = \min_{v \in H} \left[\frac{1}{2}\, a(v,v) - \varphi(v) \right]. \tag{2.65}$$

Proof. Take $E = H$ and proceed as in Remark 2.48. □

In the language of the calculus of variations one says that (2.64) is the Euler equation associated with the minimization problem (2.65).

Finally we discuss two important examples:

1. Dirichlet problem for the Laplacian

Let $\Omega \subset \mathbb{R}^n$ be an open domain with $\mathcal{C}^1$-boundary. We are looking for a function $u : \overline{\Omega} \longrightarrow \mathbb{R}$ satifsying

$$\begin{cases} -\triangle u + u = f & \text{in } \Omega \\ u = 0 & \text{on } b\Omega \end{cases} \tag{2.66}$$

and f is a given function on Ω.

A classical solution of (2.66) is a function $u \in \mathcal{C}^2(\overline{\Omega})$ satisfying (2.66) in the usual sense. A weak solution of (2.66) is a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v \, d\lambda + \int_\Omega uv \, d\lambda = \int_\Omega fv \, d\lambda, \ \forall v \in H_0^1(\Omega).$$

We claim that every classical solution is a weak solution: indeed, $u \in H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ and $u = 0$ on $b\Omega$, so that $u \in H_0^1(\Omega)$. If $v \in \mathcal{C}_0^1(\Omega)$ we have

$$\int_\Omega \nabla u \cdot \nabla v \, d\lambda + \int_\Omega uv \, d\lambda = \int_\Omega fv \, d\lambda,$$

and by density this remains true for all $v \in H_0^1(\Omega)$.

Proposition 2.51. *Given any $f \in L^2(\Omega)$, there exists a unique weak solution $u \in H_0^1(\Omega)$ of* (2.66). *Furthermore, u is obtained by*

$$\min_{v \in H_0^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega (|\nabla v|^2 + |v|^2) \, d\lambda - \int_\Omega fv \, d\lambda \right\}.$$

This is Dirichlet's principle.

Proof. Apply Proposition 1.24 for $H = H_0^1(\Omega)$ and the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, d\lambda + \int_\Omega uv \, d\lambda,$$

and apply Corollary 2.50. □

We indicate that, by Proposition 2.34, each solution $u \in H_0^1(\Omega)$ is at least in $H^2_{\text{loc}}(\Omega)$.

Finally we show how to recover a classical solution: assume that the weak solution $u \in H_0^1(\Omega)$ of (2.66) belongs to $\mathcal{C}^2(\overline{\Omega})$. Then $u = 0$ on $b\Omega$ and, by partial integration,

$$\int_\Omega (-\triangle u + u) v \, d\lambda = \int_\Omega fv \, d\lambda \ \forall v \in \mathcal{C}_0^1(\Omega)$$

and thus $-\triangle u + u = f$ almost everywhere on Ω. In fact, $-\triangle u + u = f$ everywhere on Ω, since $u \in \mathcal{C}^2(\Omega)$; thus u is a classical solution.

2. Neumann problem for the Laplacian

Let $\Omega \subset \mathbb{R}^n$ be an open domain with $\mathcal{C}^1$-boundary. We are looking for a function $u : \overline{\Omega} \longrightarrow \mathbb{R}$ satifsying

$$\begin{cases} -\triangle u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } b\Omega \end{cases} \tag{2.67}$$

and f is a given function on Ω, where $\frac{\partial u}{\partial n}$ denotes the outward normal derivative of u.

A classical solution of (2.67) is a function $u \in \mathcal{C}^2(\overline{\Omega})$ satisfying (2.67) in the usual sense. A weak solution of (2.67) is a function $u \in H^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v \, d\lambda + \int_\Omega uv \, d\lambda = \int_\Omega fv \, d\lambda, \ \forall v \in H^1(\Omega).$$

The solution u is obtained by

$$\min_{v \in H^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega (|\nabla v|^2 + |v|^2) \, d\lambda - \int_\Omega fv \, d\lambda \right\}.$$

Further details and various examples can be found in [3] and [2].

Remark 2.52. Let $\Omega \subset \mathbb{R}^n$ be an open domain with $\mathcal{C}^1$-boundary. The operator $T_0 = -\triangle$, defined on $\mathcal{C}_0^\infty(\Omega)$, has two different self-adjoint extensions. the Dirchlet and the Neumann tealization, hence fails to be essentially self-adjoint.

Taks $T = -\triangle + I$ and consider (2.66) and (2.67). Recall Proposition 1.25 with the corresponding bilinear forms

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\lambda + \int_\Omega uv \, d\lambda \ , \ u, v \in H_0^1(\Omega)$$

for (2.66), which yields the self-adjoint extension S_0, and

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\lambda + \int_\Omega uv \, d\lambda \ , \ u, v \in H^1(\Omega)$$

for (2.67), which yields the self-adjoint extension S_1. Finally consider the self-adjoint operators $S_0 - I$ and $S_1 - I$ (see also [6]).

Chapter 3

Spectral analysis

Definition 3.1. The resolvent set of a linear operator $T : \mathrm{dom}(T) \longrightarrow H$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is an injective mapping of $\mathrm{dom}(T)$ onto H whose inverse belongs to $\mathcal{L}(H)$. The spectrum $\sigma(T)$ of T is the complement of the resolvent set of T.

First we collect some informations about the spectrum of an unbounded operator.

Lemma 3.2. *If the spectrum $\sigma(T)$ of an operator T does not coincide with the whole of the complex plane $\mathbb{C}$ then T must be a closed operator. The spectrum of a linear operator is always closed. Moreover, if $\zeta \notin \sigma(T)$ and $c := \|R_T(\zeta)\| = \|(\zeta I - T)^{-1}\|$, then the spectrum $\sigma(T)$ does not intersect the ball $\{w \in \mathbb{C} : |\zeta - w| < c^{-1}\}$. The resolvent operator R_T is a holomorphic operator valued function.*

Proof. For $\zeta \notin \sigma(T)$ let $S = (\zeta I - T)^{-1}$ which is a bounded operator. Let $x_n \in \mathrm{dom}(T)$ with $x = \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Tx_n = y$ and set $u_n = (\zeta I - T)x_n$. Then

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (\zeta x_n - Tx_n) = \zeta x - y,$$

therefore

$$S(\zeta x - y) = \lim_{n\to\infty} Su_n = \lim_{n\to\infty} x_n = x.$$

This implies $x \in \mathrm{dom}(T)$ and $(\zeta I - T)x = \zeta x - y$, or $Tx = y$. Hence T is closed.
The remainder of the proof is similar to the case when T is bounded. □

Proposition 3.3. *The spectrum $\sigma(T)$ of any self-adjoint operator T is real and non-empty. If $\zeta \notin \sigma(T)$ then*

$$\|(\zeta I - T)^{-1}\| \leq |\Im\zeta|^{-1}. \tag{3.1}$$

Moreover,

$$(\overline{\zeta} I - T)^{-1} = ((\zeta I - T)^{-1})^*. \tag{3.2}$$

Proof. Let $\zeta = \xi + i\eta$ and $\eta \neq 0$ and set $K = \frac{1}{\eta}(T - \xi I)$. Using Lemma 1.4, it follows that $K^* = K$. Let $f \in \mathrm{dom}(K)$ such that $Kf = K^*f = if$, then

$i(f,f) = (Kf,f) = (f,Kf) = -i(f,f)$, which implies $f = 0$ and that $K - iI$ is injective. In a similar way one shows that $K + iI$ is injective.

The identity

$$\|Kx \pm ix\|^2 = \|Kx\|^2 + \|x\|^2 \ , \ x \in \mathrm{dom}(K)$$

implies that $(K \pm iI)x \leftrightarrow (x, Kx)$ is an isometric one-to-one correspondence between $\mathrm{im}(K \pm iI)$ and the graph $\mathcal{G}(K)$ of K. Hence $\mathrm{im}(K \pm iI)$ is closed. Now we obtain from Lemma 1.8 that $\mathrm{im}(K \pm iI)^\perp = \ker(K \pm iI) = \{0\}$. Therefore $(K \pm iI)^{-1}$ is defined on the whole of H. Since we have

$$\|Kx \pm ix\|^2 = \|Kx\|^2 + \|x\|^2 \ , \ x \in \mathrm{dom}(K),$$

we get

$$\|(K \pm iI)^{-1}y\| = \|(K \pm iI)^{-1}(K \pm iI)x\| = \|x\| \leq \|(K \pm iI)x\| = \|y\|,$$

for each $y \in H$, which implies that

$$\|(K \pm iI)^{-1}\| \leq 1. \tag{3.3}$$

Thus $\pm i \notin \sigma(K)$ and hence $\zeta \notin \sigma(T)$. In addition (3.3) implies (3.1).

Now let $x_1, x_2 \in \mathrm{dom}(T)$. Then

$$((T - \zeta I)x_1, x_2) = (x_1, (T - \overline{\zeta} I)x_2).$$

Putting $y_1 = (T - \zeta I)x_1$ and $y_2 = (T - \overline{\zeta} I)x_2$ and rewriting the last equation in terms of y_1 and y_2 yields (3.2).

Finally suppose that $\sigma(T) = \emptyset$. Then for any $x, y \in H$ the complex-valued function

$$f(\zeta) := ((\zeta I - T)^{-1}x, y)$$

is holomorphic on $\mathbb{C}$ and, by (3.1), vanishes as $|\zeta| \to \infty$. Liouville's theorem now implies that $f = 0$ identically. Since $x, y \in H$ are arbitrary, we obtain $(\zeta I - T)^{-1}$ is identically zero. This is false, hence $\sigma(T) \neq \emptyset$. □

Proposition 3.4. *Let T be a closed symmetric operator. Then the following statements are equivalent:*

(i) T is self-adjoint;

(ii) ker$(T^* + iI) = \{0\}$ and ker$(T^* - iI) = \{0\}$;

(iii) im$(T + iI) = H$ and im$(T - iI) = H$.

Proof. (i) implies (ii): this follows since $\pm i \notin \sigma(T)$.

(ii) implies (iii): Notice that $\ker(T^* \pm iI) = \{0\}$ if and only if $\mathrm{im}(T \mp iI)$ is dense in H. This follows easily from

$$(Tu \pm iu, v) = (u, T^* \mp iv),$$

for $u, v \in \mathrm{dom}(T)$. So it remains to show that $\mathrm{im}(T \mp iI)$ is closed. The symmetry of T implies that

$$\|(T \mp iI)u\|^2 = \|Tu\|^2 + \|u\|^2, \tag{3.4}$$

for $u \in \mathrm{dom}(T)$. Now, since T is closed, we easily obtain that $\mathrm{im}(T \mp iI)$ is closed.

(iii) implies (i): Let $u \in \mathrm{dom}(T^*)$. By (iii) there exists $v \in \mathrm{dom}(T)$ such that

$$(T - iI)v = (T^* - iI)u.$$

Since T is symmetric, we have also $(T^* - iI)(v - u) = 0$. But, if $(T + iI)$ is surjective, then $(T^* - iI)$ is injective (Lemma 1.8) and we obtain $u = v$. This proves that $u \in \mathrm{dom}(T)$ and that T is self-adjoint. □

We proved during the assertion (ii) implies (iii) that

Lemma 3.5. *If T is closed and symmetric, then im$(T \pm iI)$ is closed.*

In a similar way we obtain a characterization for essentially self-adjoint operators.

Proposition 3.6. *Let A be a symmetric operator. Then the following statements are equivalent:*

(i) A is essentially self-adjoint;

(ii) ker$(A^* + iI) = \{0\}$ and ker$(A^* - iI) = \{0\}$;

(iii) im$(A + iI)$ and im$(A - iI)$ are dense in H.

Proof. We apply Proposition 3.4 to $\overline{A}$ and notice that $\overline{A}$ is symmetric and that Lemma 1.5 implies that $A^* = (\overline{A})^*$. In addition we use Lemma 3.5.

□

If A is also a positive operator, we get

Proposition 3.7. *Let A be a positive, symmetric operator. Then the following statements are equivalent:*

(i) A is essentially self-adjoint;

(ii) ker$(A^* + bI) = \{0\}$ for some $b > 0$;

(iii) im$(A + bI)$ is dense in H.

Proof. We proceed in a similar way as before and notice that for a positive, symmetric operator A we have

$$((A + bI)u, u) \geq b\|u\|^2, \tag{3.5}$$

for $u \in \mathrm{dom}(A)$, which is a good substitute for (3.4).

By Lemma 1.8 (ii) and (iii) are equivalent. Since the closure of a positive, symmetric operator is again positive and symmetric, it remains to show that a closed, positive symmetric operator T is self-adjoint if and only if $\ker(T^*+bI) = \{0\}$ for some $b > 0$.

We can suppose that $b = 1$. If T is self-adjoint, then the spectrum $\sigma(T) \subseteq \mathbb{R}^+$, hence $\ker(T+I) = \ker(T^*+I) = \{0\}$.

For the converse, we first show that $\mathrm{im}(T+I)$ is closed: let $(y_k)_k \subset \mathrm{im}(T+I)$ be a convergent sequence. There exists a sequence $(x_k)_k \subset \mathrm{dom}(T)$ such that $y_k = (T+I)x_k$. Then

$$(x_k, y_k) = (x_k, Tx_k) + \|x_k\|^2 \geq \|x_k\|^2,$$

and, by Cauchy-Schwarz,

$$\|x_k\| \leq \|y_k\| \tag{3.6}$$

Since $(y_k)_k$ is convergent, $\sup_k \|y_k\| < \infty$, and, by (3.6), $\sup_k \|x_k\| < \infty$. Now, positivity implies

$$\begin{aligned} \|x_k - x_\ell\|^2 &\leq ((x_k - x_\ell, (T+I)(x_k - x_\ell)) \\ &\leq (\|x_k\| + \|x_\ell\|)\|y_k - y_\ell\| \\ &\leq C\|y_k - y_\ell\|. \end{aligned}$$

Hence $(x_k)_k$ is a Cauchy sequence. Since we supposed that T is closed, there exists $x \in \mathrm{dom}(T)$ such that $x = \lim_{k\to\infty} x_k$ and $(T+I)x = y = \lim_{k\to\infty} y_k$. Hence $\mathrm{im}(T+I)$ is closed.

The assumption $\ker(T^*+I) = \{0\}$ now gives $\mathrm{im}(T+I) = H$. In order to show that T is self-adjoint. it suffices to show that $\mathrm{dom}(T^*) \subseteq \mathrm{dom}(T)$. Let $x \in \mathrm{dom}(T^*)$. There exists $y \in \mathrm{dom}(T)$ such that

$$(T+I)y = (T^*+I)y = (T^*+I)x,$$

since $\mathrm{dom}(T) \subseteq \mathrm{dom}(T^*)$. This implies $(T^*+I)(x-y) = 0$, and hence $x = y \in \mathrm{dom}(T)$. □

Now we consider differential operators $H(A, V)$ of the form

$$H(A, V) = -\Delta_A + V, \tag{3.7}$$

where $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the electric potential and

$$A = \sum_{j=1}^{n} A_j dx_j, \; A_j : \mathbb{R}^n \longrightarrow \mathbb{R}, \; j = 1, \ldots, n$$

is a 1-form, and

$$\Delta_A = \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} - iA_j \right)^2 .$$

The 2-form

$$B = dA = \sum_{j<k} \left(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \right) dx_j \wedge dx_k$$

is the magnetic field, which is responsible for specific spectral properties of the operator $H(A, V)$, as will be seen later.

Under appropriate assumptions on A and V the operator $H(A, V)$ acts as an unbounded self-adjoint operator on $L^2(\mathbb{R}^n)$. In many aspects of the spectral theory of the Schrödinger operator with magnetic field $H(A, V)$, it is convenient to compare this operator with the ordinary Schrödinger operator

$$H(0, V) = -\Delta + V,$$

and then to employ well-known properties of $H(0, V)$.

Let $X_j = (-i\frac{\partial}{\partial x_j} - A_j)$ for $j = 1, \ldots, n$. Then

$$-\triangle_A = \sum_{j=1}^{n} X_j^2, \tag{3.8}$$

and for $u, v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we have $(X_j u, v) = (u, X_j v),\ j = 1, \ldots, n$ and

$$(-\triangle_A u, u) = \sum_{j=1}^{n} \|X_j u\|^2. \tag{3.9}$$

Proposition 3.8. *Let $A \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ and V be a continuous real-valued function on $\mathbb{R}^m$, such that*

$$V(x) \geq -C,\ \forall x \in \mathbb{R}^m,$$

where $C > 0$ is a positive constant. Let $dom(H(A, V)) = \mathcal{C}_0^\infty(\mathbb{R}^m)$. Then $H(A, V)$ is a symmetric, semibounded operator on $L^2(\mathbb{R}^n)$.

Proof. For $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned}(H(A, V)u, u) &= \int_{\mathbb{R}^n} (-\triangle_A u + Vu)\,\overline{u}\,d\lambda \\ &= \int_{\mathbb{R}^n} \sum_{j=1}^{n} |X_j u|^2\,d\lambda + \int_{\mathbb{R}^n} V|u|^2\,d\lambda \\ &\geq -C\,\|u\|^2.\end{aligned}$$

□

Using the Friedrichs extension 1.25 , we obtain

Proposition 3.9. *Let $H(A, V)$ be as in Proposition 3.8. Then $H(A, V)$ admits a self-adjoint extension.*

Proof. Define

$$a(u, v) := (H(A, V)u, v) + (C + 1)(u, v)$$

and define $\mathcal{V}$ to be the completion of $\mathcal{C}_0^\infty(\mathbb{R}^n)$ with respect to the inner product $a(u, v)$. Then one can apply Proposition 1.25 to get the desired result. □

Recall that a function $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ is the distributional derivative of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with respect to x_j (formally $g = \partial f/\partial x_j$), if

$$(g, \phi) = -\left(f, \frac{\partial \phi}{\partial x_j}\right),$$

for each $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

Let $f_k, f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We say that f_k converges to f in the distributional sense, if

$$(f_k, \phi) \to (f, \phi)$$

for each $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

Let $f, g \in L^1_{\text{loc}}(\mathbb{R}^n)$. We say that $f \geq g$ in the distributional sense, if

$$(f, \phi) \geq (g, \phi),$$

for all positive $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

A useful tool for spectral analysis of Schrödinger operators is Kato's inequality sometimes also called the diamagnetic inequality:

Proposition 3.10. *Let $A \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$. Then, for all $f \in L^1_{loc}(\mathbb{R}^n)$ with $(-i\nabla - A)^2 f \in L^2_{loc}(\mathbb{R}^n)$, we have*

$$\triangle|f| \geq -\Re(sgn(f)(-i\nabla - A)^2 f) = \Re(sgn(f)\triangle_A f), \tag{3.10}$$

in the distributional sense, where sgn is defined in Chapter 5.

Proof. Let $A_1, \ldots, A_n$ be the components of A. Notice that

$$-\triangle_A f = (-i\nabla - A)^2 f = \sum_{j=1}^{n} (-i\frac{\partial}{\partial x_j} - A_j)^2 f.$$

The assumption $(-i\nabla - A)^2 f \in L^2_{\text{loc}}(\mathbb{R}^n)$, and the regularity property of second-order elliptic operators (see Proposition 2.34) imply that $f \in W^2_{\text{loc}}(\mathbb{R}^n)$, in particular $\triangle f, \nabla f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

First suppose that u is smooth.Then, with $|u|_\epsilon = \sqrt{|u|^2+\epsilon^2} - \epsilon$, we get

$$\nabla |u|_\epsilon = \frac{\Re(\overline{u}\nabla u)}{\sqrt{|u|^2+\epsilon^2}} = \frac{\Re(\overline{u}(\nabla - iA)u)}{\sqrt{|u|^2+\epsilon^2}}. \tag{3.11}$$

A straightforward calculation shows that for a smooth function g we have

$$g\triangle g = \operatorname{div}(g\nabla g) - |\nabla g|^2.$$

Hence we obtain

$$\begin{aligned}
\sqrt{|u|^2+\epsilon^2}\,\triangle |u|_\epsilon &= \operatorname{div}(\sqrt{|u|^2+\epsilon^2}\;\nabla|u|_\epsilon) - |\nabla |u|_\epsilon|^2 \\
&= \Re\left[\overline{\nabla u}\cdot(\nabla - iA)u + \overline{u}\operatorname{div}((\nabla - iA)u)\right] - |\nabla|u|_\epsilon|^2 \\
&= \Re[\overline{(\nabla u - iAu)}\cdot(\nabla - iA)u \\
&\quad +(-iA\overline{u})\cdot(\nabla - iA)u + \overline{u}\operatorname{div}((\nabla - iA)u)] - |\nabla|u|_\epsilon|^2 \\
&= |(\nabla - iA)u|^2 - |\nabla|u|_\epsilon|^2 \\
&\quad +\Re\left[(-iA\overline{u})\cdot(\nabla - iA)u + \overline{u}\operatorname{div}((\nabla - iA)u)\right].
\end{aligned}$$

An easy calculation shows that

$$(-iA\overline{u})\cdot(\nabla - iA)u + \overline{u}\operatorname{div}((\nabla - iA)u) = \overline{u}\,(\nabla - iA)^2 u.$$

From (3.11) we get

$$|\nabla|u|_\epsilon|^2 \le \frac{|\overline{u}(\nabla - iA)u|^2}{|u|^2+\epsilon^2} = \frac{|u|^2\,|(\nabla - iA)u)|^2}{|u|^2+\epsilon^2} \le |(\nabla - iA)u)|^2.$$

So we finally see that

$$\triangle |u|_\epsilon \ge \Re\,\frac{\overline{u}\,(\nabla - iA)^2 u}{\sqrt{|u|^2+\epsilon^2}}. \tag{3.12}$$

The rest of the proof uses approximative units and follows the same lines as the proof of the Proposition 2.19. □

Using Kato's inequality and a criterion for essential self-adjointness we obtain

Proposition 3.11. *Let $A \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ and $V \in L^2_{loc}(\mathbb{R}^n)$ and $V \ge 0$. Then the Schrödinger operator $H(A,V) = -\triangle_A + V$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^n)$. In this case the Friedrichs extension is the uniquely determined self-adjoint extension (see Remark 1.19 (b) and Proposition 3.9).*

Proof. By Proposition 3.7, it is sufficient to show that

$$\ker(H(A,V)^* + I) = \{0\}.$$

Since $\mathrm{dom}(H(A,V)^*) \subseteq L^2(\mathbb{R}^n)$, the triviality of the kernel follows from the statement: if

$$-\triangle_A u + Vu + u = 0, \tag{3.13}$$

for $u \in L^2(\mathbb{R}^n)$, then $u = 0$.

If $u \in L^2(\mathbb{R}^n)$ and $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, one has $uV \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. In addition we have the inclusion

$$L^2(\mathbb{R}^n) \subset L^2_{\mathrm{loc}}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

which follows from the estimate

$$\int_K |u|\, d\lambda \leq |K|\, (\int_K |u|^2\, d\lambda)^{1/2}.$$

Hence we have $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and, by (3.13), that $\triangle u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where the derivative is taken in the sense of distributions.

From (3.10) and (3.13) we obtain

$$\begin{aligned} \triangle|u| &\geq \Re(\mathrm{sgn}(u)\, \triangle_A u) \\ &= \Re(\mathrm{sgn}(u)\, (V+1)u) \\ &= |u|\, (V+1) \geq 0. \end{aligned}$$

If $(\chi_\epsilon)_\epsilon$ is an approximate unit, we get

$$\triangle(\chi_\epsilon * |u|) = \chi_\epsilon * \triangle|u| \geq 0. \tag{3.14}$$

Since $\chi_\epsilon * |u| \in \mathrm{dom}(\triangle)$, we have

$$(\triangle(\chi_\epsilon * |u|), \chi_\epsilon * |u|) = -\|\nabla(\chi_\epsilon * |u|)\|^2 \leq 0. \tag{3.15}$$

By (3.14), the left side of (3.15) is nonnegative, so $\nabla(\chi_\epsilon * |u|) = 0$ and hence $\chi_\epsilon * |u| = c \geq 0$. But $|u| \in L^2(\mathbb{R}^n)$ and $\chi_\epsilon * |u| \to |u|$ in $L^2(\mathbb{R}^n)$, and so $c = 0$. Hence $\chi_\epsilon * |u| = 0$, so $|u| = 0$ and $u = 0$. □

For other interesting applications of spectral analysis see [7].

Chapter 4

$\overline{\partial}$

Finally we demonstrate some methods for the Cauchy-Riemann equations. We consider the $\overline{\partial}$-complex

$$L^2(\Omega) \xrightarrow{\overline{\partial}} L^2_{(0,1)}(\Omega) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} L^2_{(0,n)}(\Omega) \xrightarrow{\overline{\partial}} 0\,, \tag{4.1}$$

where $L^2_{(0,q)}(\Omega)$ denotes the space of $(0,q)$-forms on Ω with coefficients in $L^2(\Omega)$. The $\overline{\partial}$-operator on $(0,q)$-forms is given by

$$\overline{\partial}\left(\sum_J{}' a_J\, d\overline{z}_J\right) = \sum_{j=1}^n \sum_J{}' \frac{\partial a_J}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_J, \tag{4.2}$$

where $\sum'$ means that the sum is only taken over strictly increasing multi-indices J.

The derivatives are taken in the sense of distributions, and the domain of $\overline{\partial}$ consists of those $(0,q)$-forms for which the right hand side belongs to $L^2_{(0,q+1)}(\Omega)$. So $\overline{\partial}$ is a densely defined closed operator, and therefore has an adjoint operator from $L^2_{(0,q+1)}(\Omega)$ into $L^2_{(0,q)}(\Omega)$ denoted by $\overline{\partial}^*$.

We consider the $\overline{\partial}$-complex

$$L^2_{(0,q-1)}(\Omega) \underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q)}(\Omega) \underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q+1)}(\Omega), \tag{4.3}$$

for $1 \le q \le n-1$.

Proposition 4.1. *The complex Laplacian* $\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial}$, *defined on the domain* $dom(\Box) = \{u \in L^2_{(0,q)}(\Omega) : u \in dom(\overline{\partial}) \cap dom(\overline{\partial}^*), \overline{\partial} u \in dom(\overline{\partial}^*), \overline{\partial}^* u \in dom(\overline{\partial})\}$ *acts as an unbounded, densely defined, closed and self-adjoint operator on* $L^2_{(0,q)}(\Omega)$, *for* $1 \le q \le n$, *which means that* $\Box = \Box^*$ *and* $dom(\Box) = dom(\Box^*)$.

Proof. dom($\Box$) contains all smooth forms with compact support, hence $\Box$ is densely defined. To show that $\Box$ is closed depends on the fact that both $\overline{\partial}$ and $\overline{\partial}^*$ are closed : note that

$$(\Box u, u) = (\overline{\partial}\,\overline{\partial}^* u + \overline{\partial}^*\,\overline{\partial} u, u) = \|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2, \tag{4.4}$$

for $u \in \operatorname{dom}(\Box)$. We have to prove that for every sequence $u_k \in \operatorname{dom}(\Box)$ such that $u_k \to u$ in $L^2_{(0,q)}(\Omega)$ and $\Box u_k$ converges, we have $u \in \operatorname{dom}(\Box)$ and $\Box u_k \to \Box u$. It follows from (4.4) that

$$(\Box(u_k - u_\ell), u_k - u_\ell) = \|\overline{\partial}(u_k - u_\ell)\|^2 + \|\overline{\partial}^*(u_k - u_\ell)\|^2,$$

which implies that $\overline{\partial} u_k$ converges in $L^2_{(0,q+1)}(\Omega)$ and that $\overline{\partial}^* u_k$ converges in $L^2_{(0,q-1)}(\Omega)$. Since $\overline{\partial}$ and $\overline{\partial}^*$ are closed operators, we get $u \in \operatorname{dom}(\overline{\partial}) \cap \operatorname{dom}(\overline{\partial}^*)$ and $\overline{\partial} u_k \to \overline{\partial} u$ in $L^2_{(0,q+1)}(\Omega)$ and $\overline{\partial}^* u_k \to \overline{\partial}^* u$ in $L^2_{(0,q-1)}(\Omega)$.

To show that $\overline{\partial} u \in \operatorname{dom}(\overline{\partial}^*)$ and $\overline{\partial}^* u \in \operatorname{dom}(\overline{\partial})$, we first notice that $\overline{\partial}\,\overline{\partial}^* u_k$ and $\overline{\partial}^*\,\overline{\partial} u_k$ are orthogonal which follows from

$$(\overline{\partial}\,\overline{\partial}^* u_k, \overline{\partial}^*\,\overline{\partial} u_k) = (\overline{\partial}^2\,\overline{\partial}^* u_k, \overline{\partial} u_k) = 0.$$

Therefore the convergence of $\Box u_k = \overline{\partial}\,\overline{\partial}^* u_k + \overline{\partial}^*\,\overline{\partial} u_k$ implies that both $\overline{\partial}\,\overline{\partial}^* u_k$ and $\overline{\partial}^*\,\overline{\partial} u_k$ converge. Now use again that $\overline{\partial}$ and $\overline{\partial}^*$ are closed operators to obtain that $\overline{\partial}\,\overline{\partial}^* u_k \to \overline{\partial}\,\overline{\partial}^* u$ and $\overline{\partial}^*\,\overline{\partial} u_k \to \overline{\partial}^*\,\overline{\partial} u$. This implies that $\Box u_k \to \Box u$. Hence $\Box$ is closed.

In order to show that $\Box$ is self-adjoint we use Lemma 1.21. Define

$$R = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial} + I$$

on $\operatorname{dom}(\Box)$. By Lemma 1.21 both $(I + \overline{\partial}\,\overline{\partial}^*)^{-1}$ and $(I + \overline{\partial}^*\,\overline{\partial})^{-1}$ are bounded, self-adjoint operators. Consider

$$L = (I + \overline{\partial}\,\overline{\partial}^*)^{-1} + (I + \overline{\partial}^*\,\overline{\partial})^{-1} - I.$$

Then L is bounded and self-adjoint. We claim that $L = R^{-1}$. Since

$$(I + \overline{\partial}\,\overline{\partial}^*)^{-1} - I = (I - (I + \overline{\partial}\,\overline{\partial}^*))(I + \overline{\partial}\,\overline{\partial}^*)^{-1} = -\overline{\partial}\,\overline{\partial}^*(I + \overline{\partial}\,\overline{\partial}^*)^{-1},$$

we have that the range of $(I + \overline{\partial}\,\overline{\partial}^*)^{-1}$ is contained in $\operatorname{dom}(\overline{\partial}\,\overline{\partial}^*)$. Similarly, we have that the range of $(I + \overline{\partial}^*\,\overline{\partial})^{-1}$ is contained in $\operatorname{dom}(\overline{\partial}^*\,\overline{\partial})$ and we get

$$L = (I + \overline{\partial}^*\,\overline{\partial})^{-1} - \overline{\partial}\,\overline{\partial}^*(I + \overline{\partial}\,\overline{\partial}^*)^{-1}.$$

Since $\overline{\partial}^2 = 0$, we have that the range of L is contained in $\operatorname{dom}(\overline{\partial}^*\,\overline{\partial})$ and

$$\overline{\partial}^*\,\overline{\partial} L = \overline{\partial}^*\,\overline{\partial}(I + \overline{\partial}^*\,\overline{\partial})^{-1}.$$

Similarly, we have that the range of L is contained in $\operatorname{dom}(\overline{\partial}\,\overline{\partial}^*)$ and

$$\overline{\partial}\,\overline{\partial}^* L = \overline{\partial}\,\overline{\partial}^*(I + \overline{\partial}\,\overline{\partial}^*)^{-1}.$$

This implies that the range of L is contained in dom($\Box$). In addition we have

$$RL = \overline{\partial}\,\overline{\partial}^*(I + \overline{\partial}\,\overline{\partial}^*)^{-1} + \overline{\partial}^*\,\overline{\partial}(I + \overline{\partial}^*\,\overline{\partial})^{-1} + L = I.$$

If $Ru = 0$, we get $\Box u = -u$ and $0 \leq (\Box u, u) = -(u, u)$, which implies that $u = 0$. Hence R is injective and we have that $L = R^{-1}$. By Lemma 1.21 we know that L is self-adjoint. Apply Lemma 1.20 to get that R is self-adjoint. Therefore $\Box = R - I$ is self-adjoint. □

We will now suppose that Ω is a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. It can be shown that

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}^*u\|^2 \geq c\,\|u\|^2, \tag{4.5}$$

for each $u \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*),\ c > 0$.

First we will show that (4.5) implies that $\overline{\partial}$ and $\overline{\partial}^*$ have closed image.

Proposition 4.2. *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Then $\overline{\partial}$ and $\overline{\partial}^*$ have closed image.*

Proof. We notice that $\ker\overline{\partial} = (\mathrm{im}\overline{\partial}^*)^\perp$, which implies that

$$(\ker\overline{\partial})^\perp = \overline{\mathrm{im}\overline{\partial}^*} \subseteq \ker\overline{\partial}^*.$$

If $u \in \ker\overline{\partial} \cap \ker\overline{\partial}^*$, we have by (4.5) that $u = 0$. Hence

$$(\ker\overline{\partial})^\perp = \ker\overline{\partial}^*. \tag{4.6}$$

If $u \in \mathrm{dom}(\overline{\partial}) \cap (\ker\overline{\partial})^\perp$, then $u \in \ker\overline{\partial}^*$, and (4.5) implies

$$\|u\| \leq \frac{1}{c}\,\|\overline{\partial}u\|.$$

Now we can apply Lemma 1.12 to conclude that $\mathrm{im}\overline{\partial}$ is closed. Finally Proposition 1.14 gives that $\mathrm{im}\overline{\partial}^*$ is also closed.

□

The next result describes the implication of the basic estimates (4.5) for the $\Box$-operator.

Proposition 4.3. *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Then $\Box : dom(\Box) \longrightarrow L^2_{(0,q)}(\Omega)$ is bijective and has a bounded inverse*

$$N : L^2_{(0,q)}(\Omega) \longrightarrow dom(\Box).$$

N is called $\overline{\partial}$-Neumann operator. In addition

$$\|Nu\| \leq \frac{1}{c}\,\|u\|. \tag{4.7}$$

Proof. Since $(\Box u, u) = \|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2$, it follows that for a convergent sequence $(\Box u_n)_n$ we get

$$\|\Box u_n - \Box u_m\| \, \|u_n - u_m\| \geq (\Box(u_n - u_m), u_n - u_m) \geq c\|u_n - u_m\|^2,$$

which implies that $(u_n)_n$ is convergent and since $\Box$ is a closed operator we obtain that $\Box$ has closed range. If $\Box u = 0$, we get $\overline{\partial} u = 0$ and $\overline{\partial}^* u = 0$ and by (4.5) also that $u = 0$, hence $\Box$ is injective. By Lemma 1.20 (ii) the image of $\Box$ is dense, therefore $\Box$ is surjective.

We showed that

$$\Box : \operatorname{dom}(\Box) \longrightarrow L^2_{(0,q)}(\Omega)$$

is bijective and therefore, by Lemma 1.20 (iv), has a bounded inverse

$$N : L^2_{(0,q)}(\Omega) \longrightarrow \operatorname{dom}(\Box).$$

For $u \in L^2_{(0,q)}(\Omega)$ we use (4.5) for Nu to obtain

$$\begin{aligned} c\|Nu\|^2 &\leq \|\overline{\partial} Nu\|^2 + \|\overline{\partial}^* Nu\|^2 \\ &= (\overline{\partial}^* \overline{\partial} Nu, Nu) + (\overline{\partial}\overline{\partial}^* Nu, Nu) \\ &= (u, Nu) \leq \|u\| \, \|Nu\|, \end{aligned}$$

which implies (4.7).

$\Box$

Finally we get a nice formula for the canonical solution operator for the inhomogeneous Cauchy-Riemann equation.

Proposition 4.4. *Let $\alpha \in L^2_{(0,q)}(\Omega)$, with $\overline{\partial}\alpha = 0$. Then $u_0 = \overline{\partial}^* N_q \alpha$ is the canonical solution of $\overline{\partial} u = \alpha$, this means $\overline{\partial} u_0 = \alpha$ and $u_0 \perp \ker \overline{\partial}$, and*

$$\|\overline{\partial}^* N_q \alpha\| \leq c^{-1/2} \, \|\alpha\|. \tag{4.8}$$

Proof. For $\alpha \in L^2_{(0,q)}(\Omega)$ with $\overline{\partial}\alpha = 0$ we get

$$\alpha = \overline{\partial}\,\overline{\partial}^* N_q \alpha + \overline{\partial}^* \,\overline{\partial} N_q \alpha. \tag{4.9}$$

If we apply $\overline{\partial}$ to the last equality we obtain:

$$0 = \overline{\partial}\alpha = \overline{\partial}\overline{\partial}^* \overline{\partial} N_q \alpha,$$

since $\overline{\partial} N_q \alpha \in \operatorname{dom}(\overline{\partial}^*)$ we have

$$0 = (\overline{\partial}\,\overline{\partial}^* \overline{\partial} N_q \alpha, \overline{\partial} N_q \alpha) = (\overline{\partial}^* \overline{\partial} N_q \alpha, \overline{\partial}^* \overline{\partial} N_q \alpha) = \|\overline{\partial}^* \overline{\partial} N_q \alpha\|^2.$$

Finally we set $u_0 = \overline{\partial}^* N_q \alpha$ and derive from (4.9) that for $\overline{\partial}\alpha = 0$

$$\alpha = \overline{\partial} u_0,$$

and we see that $u_0 \perp \ker \overline{\partial}$, since for $h \in \ker \overline{\partial}$ we get

$$(u_0, h) = (\overline{\partial}^* N_q \alpha, h) = (N_q \alpha, \overline{\partial} h) = 0.$$

It follows that

$$\begin{aligned} \|\overline{\partial}^* N_q \alpha\|^2 &= (\overline{\partial}\, \overline{\partial}^* N_q \alpha, N_q \alpha) \\ &= (\overline{\partial}\, \overline{\partial}^* N_q \alpha, N_q \alpha) + (\overline{\partial}^* \overline{\partial} N_q \alpha, N_q \alpha) \\ &= (\alpha, N_q \alpha) \leq \|\alpha\| \, \|N_q \alpha\| \end{aligned}$$

and using (4.7) we obtain

$$\|\overline{\partial}^* N_q \alpha\| \leq c^{-1/2} \, \|\alpha\|, \tag{4.10}$$

□

For further details see [8].

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