Proseminar Theorie der partiellen Differentialgleichungen

14.10.2013: L^p-space, completeness, $\mathcal{C}_0^\infty(\Omega)$ is dense in $L^p(\Omega)$.

21.10.2013: Hölder and Minkowski inequalities, duality in L^p-spaces.

28.10.2013: generalized Hölder and Minkowski inequalities:

$$\left[\int \left(\int |F(x',x)|\,d\lambda(x')\right)^p d\lambda(x)\right]^{1/p} \leq \int \left(\int |F(x',x)|^p\,d\lambda(x)\right)^{1/p} d\lambda(x')$$

04.11.2013:
Exercises 1: Let $T_1 : \mathrm{dom}(T_1) \longrightarrow H_2$ be a densely defined operator and $T_2 : H_2 \longrightarrow H_3$ be a bounded operator. Then $(T_2\,T_1)^* = T_1^*\,T_2^*$, which includes that $\mathrm{dom}((T_2\,T_1)^*) = \mathrm{dom}(T_1^*\,T_2^*)$.

2: Let T be a densely defined operator on H and let S be a bounded operator on H. Then $(T+S)^* = T^* + S^*$.

3: Let $\Omega = \mathbb{B}$ be the open unit ball in $\mathbb{R}^n$, and

$$u(x) = |x|^{-\alpha} \ \ , \ x \in \mathbb{B}, x \neq 0.$$

Show that $u \in W^1(\mathbb{B})$ if and only if $\alpha < \frac{n-2}{2}$.

19.11.2013: Exercises 4: Let $u, v \in W^k(\Omega), |\alpha| \leq k$. Show that (i) $D^\alpha u \in W^{k-|\alpha|}(\Omega)$ and for multiindices α, β with $|\alpha| + |\beta| \leq k$ we have

$$D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u.$$

(ii) If $\phi \in \mathcal{C}_0^\infty(\Omega)$, then $\phi\, u \in W^k(\Omega)$ and

$$D^\alpha(\phi\, u) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta\phi\, D^{\alpha-\beta}u,$$

where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

5: Let E, F, G denote finite dimensional vector spaces over $\mathbb{C}$ with inner product. We consider an exact sequence of linear maps

$$E \xrightarrow{S} F \xrightarrow{T} G,$$

which means that $\mathrm{Im}S = \mathrm{Ker}T$, hence $TS = 0$. Given $f \in \mathrm{Im}S = \mathrm{Ker}T$, we want to solve $Su = f$ with $u \perp \mathrm{Ker}S$, then u will be called the canonical solution. Show that $SS^* + T^*T : F \longrightarrow F$ is bijective.
Let $N = (SS^* + T^*T)^{-1}$. Show that $u = S^*Nf$ is the canonical solution to $Su = f$.

25.11.2013:
Facts from Fourier analysis (see for instance W. Rudin: Real and complex analysis)

Let $f \in L^1(\mathbb{R}^n)$ and consider the Fouriertransform

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)\, e^{-ix\xi}\, d\lambda(x)\ ,\ \xi \in \mathbb{R}^n.$$

It follows that $\hat{f} \in C_0(\mathbb{R}^n)$, where $C_0(\mathbb{R}^n)$ is the space of all continuous functions g such that for each $\epsilon > 0$ there is a compact subset $K \subset \mathbb{R}^n$ such that

$$\sup\{|g(x)| : x \in \mathbb{R}^n \setminus K\} \le \epsilon.$$

If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, then $f \in C_0(\mathbb{R}^n)$ and $f(x) = \hat{\hat{f}}(-x)$.

Theorem (Plancherel): There exists a unitary operator

$$\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

such that $\mathcal{F}(f) = \hat{f}$ for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.
Definition: For $s \in [0, \infty)$ let

$$H^s = \{f \in L^2(\mathbb{R}^n) : (1 + |\xi|^2)^{s/2}\, \mathcal{F}f(\xi) \in L^2(\mathbb{R}^n)\}.$$

We endow H^s with the norm

$$\|f\|_s = (\int_{\mathbb{R}^n} (1 + |\xi|^2)^s\, |\mathcal{F}f(\xi)|^2\, d\lambda(\xi))^{1/2}.$$

For $1 \le j \le n$ and $f \in H^s$ we define the operator

$$\mathcal{D}_j f = \mathcal{F}^{-1} \xi_j \mathcal{F} f,$$

where ξ_j denotes the multiplication with the varaible ξ_j; more general for a multi-index α and $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ we define

$$\mathcal{D}^\alpha f = \mathcal{F}^{-1} \xi^\alpha \mathcal{F} f.$$

Exercise 6: Show that

$$\mathcal{D}^\alpha : H^s \longrightarrow H^{s-|\alpha|}$$

is a continuous linear operator, where $|\alpha| = \sum_{j=1}^n \alpha_j$.

Exercise 7: Let $f \in H^s,\ s > 1$ and $1 \le j \le n$. Show that

$$\mathcal{D}_j f(x) = \frac{1}{i} \lim_{h \to 0} \frac{1}{h} (f(x + he_j) - f(x)),$$

in the topology of H^{s-1}, where $e_j = (\delta_{j,k})_{k=1}^n$.
Let $f \in H^s \cap \mathcal{C}^1(\mathbb{R}^n)$. Show that

$$\mathcal{D}_j f = \frac{1}{i} \frac{\partial f}{\partial x_j}$$

almost everywhere.

02.12.2013
Exercise 8: Let

$$C_0^k(\mathbb{R}^n) := \{f \in \mathcal{C}^k(\mathbb{R}^n) : D^\alpha f \in C_0(\mathbb{R}^n),\ \forall |\alpha| \le k\},$$

endowed with the norm

$$\|f\|_k := \sup\{|D^\alpha f(x)| : x \in \mathbb{R}^n,\ |\alpha| \le k\}.$$

Show that H^s is continuously imbedded into $C_0^k(\mathbb{R}^n)$, if $s - k > \frac{n}{2}$.

09.12.2013

Exercise 9: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in L^2_{loc}(\Omega)$ and suppose that $V \subset\subset \Omega$.

The j^{th}-difference quotient of size h is

$$D_j^h u(x) = \frac{u(x+he_j) - u(x)}{h},$$

for $j = 1, \dots, n$ where $x \in V$ and $h \in \mathbb{R}$, $0 < |h| < \operatorname{dist}(V, b\Omega)$.

Further we define

$$D^h u := (D_1^h u, \dots, D_n^h u).$$

Let $v, w \in L^2_{loc}(\Omega)$ and $\ell \in \{1, \dots, n\}$. Show that

$$\int_\Omega v D_\ell^{-h} w \, d\lambda = -\int_\Omega w D_\ell^h v \, d\lambda$$

and

$$D_\ell^h(vw) = v^h D_\ell^h w + w D_\ell^h v$$

for $v^h(x) := v(x + he_\ell)$.

Show that , if $u \in H^1(\Omega)$, we have

$$(D_j^h u)_{x_k} = D_j^h(u_{x_k}).$$

Exercise 10: Let L be an elliptic second order partial differential operator and $f \in L^2(\Omega)$. Suppose that $u \in H^1(\Omega)$ is a weak solution of $Lu = f$. Show that $Lu = f$ almost everywhere in Ω. (Use the fact that, by inner regularity, one even has that $u \in H^2_{\text{loc}}(\Omega)$.)

16.12.2013 and 13.01.2014

Exercise 11: Let L be an elliptic second order partial differential operator and $f \in L^2(\Omega)$. Suppose that $u \in H^1(\Omega)$ is a weak solution of $Lu = f$. Show that for each open $V \subset\subset \Omega$ we have

$$\|u\|_{H^2(V)} \leq \tilde{C}(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}),$$

where $\tilde{C} > 0$ is a constant only depending on V, Ω, and the coefficients of L.

Exercise 12: (Fredholm alternative) Let $A : H \longrightarrow H$ be a compact linear operator on the Hilbert space H. Show that

(i) $\ker(I - A)$ is finite dimensional,

(ii) $\operatorname{im}(I - A)$ is closed,

(iii) $\operatorname{im}(I - A) = \ker(I - A^*)^\perp$,

(iv) $\ker(I - A) = \{0\}$ if and only if $\operatorname{im}(I - A) = H$,

(v) $\dim \ker(I - A) = \dim \ker(I - A^*)$.

In particular we have: either for each $f \in H$, the equation $u - Au = f$ has a unique solution or else the homogeneous equation $u - Au = 0$ has solutions $u \neq 0$. (Fredholm alternative)

In the second case, the space of solutions of the homogeneous problem is finite-dimensional, and the nonhomogeneous equation $u - Au = f$ has a solution if and only if $f \in \ker(I - A^*)^\perp$.

20.01.2014

Exercise 13: Let L be an elliptic second order partial differential operator and $a(u,v)$ its corresponding bilinear form. Suppose that Ω is a bounded open domain in $\mathbb{R}^n$ with $\mathcal{C}^1$-boundary. Consider the nonzero boundary-value problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } b\Omega, \end{cases}$$

where g is the trace of some $w \in H^1(\Omega)$. Set $\tilde{u} = u - w$ and $\tilde{f} = f - Lw$. Show that $\tilde{u} \in H_0^1(\Omega)$ and $\tilde{f} \in H_0^{-1}(\Omega)$, and prove that $\tilde{u}$ is a weak solution of

$$\begin{cases} L\tilde{u} = \tilde{f} & \text{in } \Omega \\ \tilde{u} = 0 & \text{on } b\Omega. \end{cases}$$

In this way the nonzero boundary-value problem can be transformed to the zero boundary-value problem.

Exercise 14:
Let $\Omega \subset \mathbb{R}^n$ be an open domain with $\mathcal{C}^1$-boundary. We are looking for a function $u : \overline{\Omega} \longrightarrow \mathbb{R}$ satifsying

$$\begin{cases} -\triangle u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } b\Omega \end{cases} \tag{1}$$

and f is a given function on Ω, where $\frac{\partial u}{\partial n}$ denotes the outward normal derivative of u. (u is called a solution to the Neumann problem)
A weak solution of (1) is a function $u \in H^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v \, d\lambda + \int_\Omega uv \, d\lambda = \int_\Omega fv \, d\lambda, \ \forall v \in H^1(\Omega).$$

Show that for every $f \in L^2(\Omega)$ there exists a unique weak solution $u \in H^1(\Omega)$ of (1) and show that u is given by

$$\min_{v \in H^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega (|\nabla v|^2 + v^2) \, d\lambda - \int_\Omega fv \, d\lambda \right\}.$$

Hint: use Green's formula:

$$\int_\Omega (\triangle g) h \, d\lambda = \int_{b\Omega} \frac{\partial g}{\partial n} h \, d\sigma - \int_\Omega \nabla g \cdot \nabla h \, d\lambda,$$

for each $g \in \mathcal{C}^2(\overline{\Omega})$ and $h \in \mathcal{C}^1(\overline{\Omega})$.

27.01.2014

Exercise 15: Gauge invariance of Schrödinger operators with magnetic fields.

Let $A, A' \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ be such that $dA = dA'$. Suppose that $V \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $V \geq 0$. Show that $\sigma(H(A,V)) = \sigma(H(A',V))$.

Hint: Show that $A = A' + dg$, where $g \in \mathcal{C}^1(\mathbb{R}^n)$ (Poincaré Lemma) and that $H(A,V)$ and $H(A',V)$ are unitarily equivalent.