1 Several complex variables

To think that the analysis of several complex variables is more or less the one variable theory with some more indices turns out to be incorrect. Completely new phenomena appear which will be exploited in the following. Many differences between the one and several variables theories originate from the Cauchy Riemann differential equations which constitute an overdetermined system of partial differential equations for several complex variables. We start with the basic definitions and complex differential forms. Section 1.1 also presents the main differences between one and several variables analysis, such as the Identity Theorem and Hartogs phenomenon. Section 1.2 provides another important example for this difference, namely in the analysis of the inhomogeneous Cauchy Riemann differential equations. In addition the concept of the tangential Cauchy Riemann equation is introduced. This gives the tools required for the famous Lewy example of a partial differential operator without solution. In section 1.3 we discuss pseudoconvex domains and plurisubharmonic functions and explain the concept of a domain of holomorphy.

1.1 Complex differential forms and holomorphic functions

Let $\Omega \subseteq \mathbb{C}^n$ be an open subset and let $f : \Omega \longrightarrow \mathbb{C}$ be a $\mathcal{C}^1$-function. We write $z_j = x_j + iy_j$ and consider for $P \in \Omega$ the differential

$$df_P = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}(P)\, dx_j + \frac{\partial f}{\partial y_j}(P)\, dy_j \right).$$

We use the complex differentials

$$dz_j = dx_j + idy_j \quad , \quad d\overline{z}_j = dx_j - idy_j$$

and the derivatives

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\, \frac{\partial}{\partial y_j} \right) \quad , \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\, \frac{\partial}{\partial y_j} \right)$$

and rewrite the differential df_P in the form

$$df_P = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial z_j}(P)\, dz_j + \frac{\partial f}{\partial \overline{z}_j}(P)\, d\overline{z}_j \right) = \partial f_P + \overline{\partial} f_P.$$

A general differential form is given by

$$\omega = \sum_{|J|=p,|K|=q}{}' \, a_{J,K}\, dz_J \wedge d\overline{z}_K,$$

where $\sum'_{|J|=p,|K|=q}$ denotes the sum taken only over all increasing multiindices $J=(j_1,\dots,j_p),\ K=(k_1,\dots,k_q)$ and

$$dz_J = dz_{j_1}\wedge\cdots\wedge dz_{j_p} \quad , \quad d\overline{z}_K = d\overline{z}_{k_1}\wedge\cdots\wedge d\overline{z}_{k_q}.$$

We call ω a (p,q)-form and we write $\omega\in\mathcal{C}^k_{(p,q)}(\Omega)$ if ω is a (p,q)-form with coefficients belonging to $\mathcal{C}^k(\Omega)$.
The derivative $d\omega$ of ω is defined by

$$d\omega = \sum_{|J|=p,|K|=q}{}' \, da_{J,K}\wedge dz_J\wedge d\overline{z}_K = \sum_{|J|=p,|K|=q}{}' \, (\partial a_{J,K}+\overline{\partial}a_{J,K})\wedge dz_J\wedge d\overline{z}_K,$$

and we set

$$\partial\omega = \sum_{|J|=p,|K|=q}{}' \, \partial a_{J,K}\wedge dz_J\wedge d\overline{z}_K \text{ and } \overline{\partial}\omega = \sum_{|J|=p,|K|=q}{}' \, \overline{\partial}a_{J,K}\wedge dz_J\wedge d\overline{z}_K.$$

We have $d=\partial+\overline{\partial}$ and since $d^2=0$ it follows that

$$0=(\partial+\overline{\partial})\circ(\partial+\overline{\partial})\omega=(\partial\circ\partial)\omega+(\partial\circ\overline{\partial}+\overline{\partial}\circ\partial)\omega+(\overline{\partial}\circ\overline{\partial})\omega,$$

which implies $\partial^2=0$, $\overline{\partial}^2=0$ and $\partial\circ\overline{\partial}+\overline{\partial}\circ\partial=0$, by comparing the types of the differential forms involved.
Before we proceed we mention important domains in $\mathbb{C}^n$ and some basic facts about them.

Definition 1.1. A polydisc with center $a=(a_1,\dots,a_n)\in\mathbb{C}^n$ and multiradius $r=(r_1,\dots,r_n), r_j>0$ is the set

$$P(a,r)=\{z\in\mathbb{C}^n : |z_j-a_j|<r_j, 1\le j\le n\}.$$

A ball with center $a=(a_1,\dots,a_n)\in\mathbb{C}^n$ and radius $r>0$ is defined by

$$B(a,r)=\{z\in\mathbb{C}^n : \sum_{j=1}^n |z_j-a_j|^2<r^2\}.$$

We write $\mathbb{B}$ for the unit ball $B(0,1)$.
The Siegel [1] upper half-space $\mathbb{U}$ in $\mathbb{C}^n$, $n\ge 2$, is defined by

$$\mathbb{U}=\{z\in\mathbb{C}^n : \Im z_n>\sum_{j=1}^{n-1}|z_j|^2\}.$$

In the sequel we will use the symbol $b\Omega$ for the boundary of a domain Ω in $\mathbb{C}^n$. The symbol ∂ is now reserved for differential forms.

1 Siegel, Carl Ludwig (1896–1981)

Definition 1.2. A domain $\Omega \subset \mathbb{R}^n, n \geq 2$, is said to have $\mathcal{C}^k$ $(1 \leq k \leq \infty)$ boundary at the boundary point p if there exists a real-valued function ρ defined in some open neighborhood U of p such that $\rho \in \mathcal{C}^k(U)$ and $U \cap \Omega = \{x \in U : \rho(x) < 0\}$, $b\Omega \cap U = \{x \in U : \rho(x) = 0\}$, and $d\rho(x) \neq 0$ on $b\Omega \cap U$. The function ρ is called a $\mathcal{C}^k$ local defining function for Ω near p. If U is an open neighborhood of $\overline{\Omega}$, then ρ is called a global defining function for Ω, simply a defining function for Ω.

In the following we consider the relationship between two defining functions.

Lemma 1.3. *let ρ_1 and ρ_2 be two local defining functions of Ω of class $\mathcal{C}^k$ in a neighborhood U of $p \in b\Omega$. Then there exists a positive $\mathcal{C}^{k-1}$ function h on U such that $\rho_1 = h\rho_2$ on U and $d\rho_1(x) = h(x)d\rho_2(x)$ for $x \in U \cap b\Omega$.*

Proof. Since $d\rho_2 \neq 0$ on the boundary near p, we may assume that $p = 0$, $x_n = \rho_2(x)$ and $U \cap b\Omega = \{x \in U : x_n = 0\}$, after a $\mathcal{C}^k$ change of coordinates. Let $x' = (x_1, \dots, x_{n-1})$. Then we have $\rho_1(x', 0) = 0$ and by the fundamental theorem of calculus

$$\rho_1(x', x_n) = \rho_1(x', x_n) - \rho_1(x', 0) = x_n \int_0^1 \frac{\partial \rho_1}{\partial x_n}(x', tx_n)\, dt.$$

Hence $\rho_1 = h\rho_2$ for some $\mathcal{C}^{k-1}$ function on U. If $k-1 \geq 1$, we get $d\rho_1(x) = h(x)d\rho_2(x)$ for $x \in U \cap b\Omega$, as $\rho_2(x) = 0$ for $x \in U \cap b\Omega$. If $k = 1$, we get the same conclusion from the fact that for a function f differentiable at $0 \in \mathbb{R}^n$ such that $f(0) = 0$ and for a function h continuous at 0, one has that $f \cdot h$ is differentiable at 0 and $d(hf)_0 = h(0)\ df_0$.
Finally, as $d\rho_1(x) \neq 0$ and $d\rho_2(x) \neq 0$ for $x \in U \cap b\Omega$, we get $h(x) \neq 0$ for $x \in U \cap b\Omega$. In addition, since $h > 0$ on $U \setminus b\Omega$, and h is continuous, we obtain $h > 0$ on U. $\square$

Definition 1.4. Let $\Omega \subseteq \mathbb{C}^n$ be open. A function $f : \Omega \longrightarrow \mathbb{C}$ is called holomorphic on Ω if $f \in \mathcal{C}^1(\Omega)$ and f satisfies the system of partial differential equations

$$\frac{\partial f}{\partial \overline{z}_j}(z) = 0 \quad \text{for } 1 \leq j \leq n \text{ and } z \in \Omega, \tag{1.1}$$

equivalently, if f satisfies $\overline{\partial} f = 0$.

We remark that there is no biholomorphic mapping between a polydisc and a ball in $\mathbb{C}^n$, $n \geq 2$, see [6]. The Siegel upper half-space $\mathbb{U}$ is biholomorphic to the unit ball $\mathbb{B}$, by the so-called Cayley transform, so $\mathbb{U}$ is an unbounded realization of a bounded symmetric domain. The boundary $\mathbb{H}$ of $\mathbb{U}$ carries the structure of the Heisenberg group, see Exercises for more details.
Next we establish a Cauchy integral formula for holomorphic functions on polydiscs.

Theorem 1.5. *Let $P = P(a, r)$ be a polydisc in $\mathbb{C}^n$. trose that $f \in \mathcal{C}^1(\overline{P})$ and that f is holomorphic on P, i.e. for each $z \in P$ and $1 \leq j \leq n$, the function*

$$\zeta \mapsto f(z_1, \dots, z_{j-1}, \zeta, z_{j+1}, \dots, z_n)$$

is holomorphic on $\{\zeta \in \mathbb{C} : |\zeta - a_j| < r_j\}$. *Then*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\gamma_1} \cdots \int_{\gamma_n} \frac{f(\zeta)}{(\zeta_1 - z_1)\dots(\zeta_n - z_n)}\, d\zeta_1 \dots d\zeta_n, \tag{1.2}$$

for $z \in P$, *where* $\gamma_j(t) = a_j + r_j e^{it}$, *for* $t \in [0, 2\pi]$ *and* $j = 1, \dots, n$.

Proof. Induction over n. For $n = 1$ one has the classical Cauchy Formula, see Theorem **??**. trose that the theorem has been proven for $n - 1$ variables. For $z \in P$ fixed we apply the inductive hypothesis with respect to $(z_2, \dots, z_n)$ and obtain

$$f(z_1, z_2, \dots, z_n) = \frac{1}{(2\pi i)^{n-1}} \int_{\gamma_2} \cdots \int_{\gamma_n} \frac{f(z_1, \zeta_2, \dots, \zeta_n)}{(\zeta_2 - z_2)\dots(\zeta_n - z_n)}\, d\zeta_2 \dots d\zeta_n.$$

For $\zeta_2, \dots, \zeta_n$ fixed, we get from 1-dimensional case

$$f(z_1, \zeta_2, \dots, \zeta_n) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta_1, \dots, \zeta_n)}{\zeta_1 - z_1}\, d\zeta_1$$

which can be substituted to the formula above to obtain (1.2). □

Like in the case $n = 1$ we get also here that holomorphic functions in several variables are $\mathcal{C}^\infty$ functions, and all complex derivatives of holomorphic functions are again holomorphic, differentiate under the integral sign in (1.2).
In addition, we get the Cauchy estimates: for $f \in \mathcal{H}(P(a, r))$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$: let $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\alpha! = \alpha_1! \dots \alpha_n!$, furthermore set $r^\alpha = r_1^{\alpha_1} \dots r_n^{\alpha_n}$, then

$$|D^\alpha f(a)| = \left| \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}(a) \right| \le \frac{\alpha!}{r^\alpha} \sup\{|f(z)| : z \in P(a, r)\}. \tag{1.3}$$

Next we show that every holomorphic function can be represented locally by a convergent power series:

Theorem 1.6. *Let* $f \in \mathcal{H}(P(a, r))$. *Then the Taylor series of* f *at* a *converges to* f *uniformly on all compact subsets of* $P(a, r)$, *that is*

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{D^\alpha f(a)}{\alpha!} (z - a)^\alpha, \tag{1.4}$$

for $z \in P(a, r)$.

Proof. Use the same method as in the proof of Theorem **??** for each of iterated integrals in (1.2). □

From this we get: let $\Omega \subseteq \mathbb{C}^n$ be a domain and $f \in \mathcal{H}(\Omega)$, suppose that there is $a \in \Omega$ such that $D^\alpha(a) = 0$ for all $\alpha \in \mathbb{N}_0^n$, then $f(z) = 0$ for $z \in \Omega$. In particular, if there is a nonempty open set $U \subset \Omega$ such that $f(z) = 0$ for $z \in U$, then $f \equiv 0$ on Ω (Identity Theorem).

But Theorem **??** is not valid for $n > 1$: let $f(z_1, z_2) = z_1$. Then this function is zero on $\{(0, z_2) : z_2 \in \mathbb{C}\}$, but f is not identically zero.
The following result is also an easy consequence of the corresponding one variable result.

Theorem 1.7. *Let Ω be a domain in $\mathbb{C}^n$ and suppose that $f \in \mathcal{H}(\Omega)$ is not constant. Then f is an open mapping.*

Proof. We refer to Theorem **??**. It is enough to show that for any ball $B(a, r) = \{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j - a_j|^2 < r^2\}$ the image $f(B(a, r))$ is a neighborhood of $f(a)$. The restriction of f to $B(a, r)$ is not constant, otherwise f would have to be constant on Ω. Choose $p \in B(a, r)$ with $f(p) \neq f(a)$ and define $g(\zeta) = f(a + \zeta p)$ for $\zeta \in D_1(0)$. Then g is nonconstant and holomorphic on $D_1(0)$. By Theorem **??**, $g(D_1(0))$ contains a neighborhood of $g(0)$. As $g(0) = f(a)$ and $g(D_1(0)) \subset f(B(a, r))$, the image $f(B(a, r))$ is a neighborhood of $f(a)$. $\square$

The maximum principle follows from this result as for $n = 1$: if $\Omega \subseteq \mathbb{C}^n$ is a domain and $f \in \mathcal{H}(\Omega)$ such that $|f|$ has a local maximum at a point $a \in \Omega$, then f is constant on Ω; if Ω is a bounded domain in $\mathbb{C}^n$ and $f \in \mathcal{H}(\Omega) \cap \mathcal{C}(\overline{\Omega})$, then $|f(z)| \leq |f|_{b\Omega}$ for all $z \in \overline{\Omega}$.
We remark that Weierstraß' Theorem **??** and Montel's Theorem **??** also hold for holomorphic functions of several variables with an analogous proof.
A striking difference between one variable analysis and several variables analysis appears in the next result, which gives a domain in $\mathbb{C}^n, n > 1$, with the property that each holomorphic function can be analytically extended to a larger domain, compare Theorem **??**.

Theorem 1.8 (Hartogs [2]). *Let $n \geq 2$ and trose that $0 < r_j < 1$ for $j = 1, \ldots, n$. Then every function f holomorphic on the domain*

$$H(r) = \{z \in \mathbb{C}^n : |z_j| < 1 \textit{ for } j < n,\ r_n < |z_n| < 1\}$$

$$\cup \{z \in \mathbb{C}^n : |z_j| < r_j \textit{ for } j < n,\ |z_n| < 1\},$$

see Fig. 1.1, has a unique holomorphic extension $\tilde{f}$ to the polydisc $P(0, 1)$.

Proof. The extension is unique because of the Identity theorem. Fix δ with $r_n < \delta < 1$. Then we define

$$\tilde{f}(z', z_n) = \frac{1}{2\pi i} \int_{\gamma_\delta} \frac{f(z', \zeta)}{\zeta - z_n}\, d\zeta, \tag{1.5}$$

where $z' = (z_1, \ldots, z_{n-1})$ and $\gamma_\delta(t) = \delta e^{it}$ for $t \in [0, 2\pi]$. In this way we defined a function holomorphic on the polydisc $P(0, (1', \delta))$, where $(1', \delta) = (1, \ldots, 1, \delta)$. For

2 Hartogs, Friedrich Moritz (1874–1943)

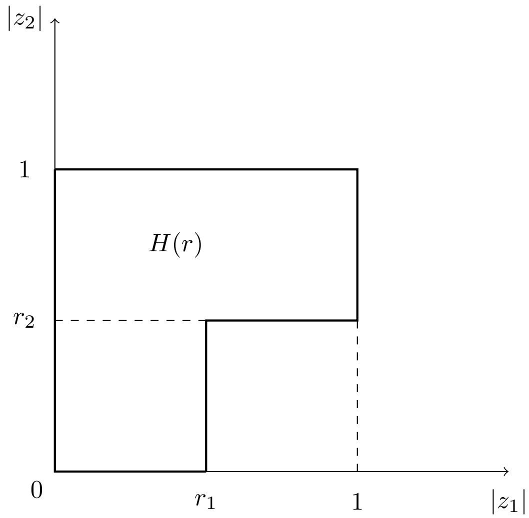

Fig. 1.1. The Hartogs domain $H(r)$ in absolute space

$z' \in P(0, r')$ the function $f(z', .)$ is holomorphic on $|z_n| < 1$, hence (1.5) implies that $\tilde{f}(z', z_n) = f(z', z_n)$ for $(z', z_n) \in P(0, (r', \delta))$. The Identity Theorem implies $\tilde{f} = f$ on $H(r) \cap P(0, (1', \delta))$, so $\tilde{f}$ is the desired extension of f to the polydisc $P(0, 1)$. $\square$

The reason for this phenomenon can be better understood by studying the inhomogeneous Cauchy Riemann differential equations in several complex variables (CR equations).

1.2 The inhomogeneous CR equations

Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let

$$g = \sum_{j=1}^{n} g_j \, d\overline{z}_j$$

be a $(0, 1)$-form with coefficients $g_j \in \mathcal{C}^1(\Omega)$, for $j = 1, \ldots, n$. We want to find a function $f \in \mathcal{C}^1(\Omega)$ such that

$$\overline{\partial} f = g, \tag{1.6}$$

in other words

$$\frac{\partial f}{\partial \overline{z}_j} = g_j, \ \ j = 1, \ldots, n. \tag{1.7}$$

f is called a solution to the inhomogeneous CR equation $\overline{\partial} f = g$.

Since $\overline{\partial}^2 = 0$, a necessary condition for solvability of (1.6) is that the right hand side g satisfies $\overline{\partial} g = 0$. So, the $(0,2)$-form $\overline{\partial} g$ satisfies

$$\overline{\partial} g = \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial g_j}{\partial \overline{z}_k} \, d\overline{z}_k \wedge d\overline{z}_j = 0,$$

which means that

$$\frac{\partial g_j}{\partial \overline{z}_k} = \frac{\partial g_k}{\partial \overline{z}_j}, \; j, k = 1, \dots, n.$$

Theorem 1.9. *Let $n \geq 2$ and let $g = \sum_{j=1}^{n} g_j \, d\overline{z}_j$ be a $(0,1)$-form with coefficients $g_j \in \mathcal{C}_0^k(\mathbb{C}^n)$, $j = 1, \dots, n$, where $1 \leq k \leq \infty$ and trose that $\overline{\partial} g = 0$. Then there exists $f \in \mathcal{C}_0^k(\mathbb{C}^n)$ such that $\overline{\partial} f = g$.*

We shall see that this result enables us to explain the Hartogs phenomenon in a rather general setting.
For $n = 1$ the above theorem is false:
Suppose that $\int_{\mathbb{C}} g(\zeta) \, d\lambda(\zeta) \neq 0$ and that there is a compactly trorted solution f of the equation $\frac{\partial f}{\partial \overline{z}} = g$. Then there exists $R > 0$ such that $f(\zeta) = 0$ for $|\zeta| \geq R$. Applying Stokes' Theorem (see **??**) we obtain for $\gamma(t) = Re^{it}$, $t \in [0, 2\pi]$

$$\begin{aligned} 0 &= \int_{\gamma} f(\zeta) \, d\zeta \\ &= \int_{D_R(0)} \frac{\partial f}{\partial \overline{\zeta}} \, d\overline{\zeta} \wedge d\zeta \\ &= 2i \int_{D_R(0)} g(\zeta) \, d\lambda(\zeta) \\ &\neq 0, \end{aligned}$$

whenever $D_R(0)$ contains the support of g. That is a contradiction.

Proof of 1.9. Define f on $\mathbb{C}^n$ by

$$f(z_1, \dots, z_n) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\zeta, z_2, \dots, z_n)}{\zeta - z_1} \, d\zeta \wedge d\overline{\zeta}. \tag{1.8}$$

By Corollary **??** (a), $f \in \mathcal{C}^k(\mathbb{C}^n)$ and $\frac{\partial f}{\partial \overline{z}_1} = g_1$. Now let $k > 1$. By hypothesis we have $\frac{\partial g_1}{\partial \overline{z}_k} = \frac{\partial g_k}{\partial \overline{z}_1}$. Since g_1 has compact support we can interchange differentiation and integration when we take the derivative of (1.8) with respect to $\overline{z}_k$ and get

$$\begin{aligned}\frac{\partial f}{\partial \overline{z}_k} &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_1}{\partial \overline{z}_k}(\zeta, z_2, \dots, z_n)\,\frac{1}{\zeta - z_1}\,d\zeta\wedge d\overline{\zeta}\\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_k}{\partial \overline{\zeta}}(\zeta, z_2, \dots, z_n)\,\frac{1}{\zeta - z_1}\,d\zeta\wedge d\overline{\zeta}\\ &= g_k(z_1, \dots, z_n),\end{aligned}$$

where we used Corollary **??** (b) for the last equality.
Hence $\overline{\partial} f = g$. We still have to show that f is with compact support. Choose $R > 1$ such that $g_k(z) = 0$ for $\sum_{j=1}^n |z_j|^2 \geq R,\ k = 1, \dots, n$. Then f is holomorphic on the domain $\Omega = \{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 > R\}$. Since $n \geq 2$, we can fix z_2 such that $|z_2| > R$. Then $g_1(\zeta, z_2, \dots, z_n) = 0$ for all $\zeta \in \mathbb{C}$ and all $z_3, \dots, z_n \in \mathbb{C}$. From the definition of f, it follows that $f(z) = 0$, if $|z_2| > R$. The set $\{z \in \Omega : |z_2| > R\}$ is a nonempty open subset of the domain Ω. Since $f \in \mathcal{H}(\Omega)$, the Identity Theorem yields that $f \equiv 0$ on Ω. Therefore f has compact support. □

Now we are able to describe the Hartogs phenomenon in a more general way.

Theorem 1.10. *Let Ω be a bounded open set in $\mathbb{C}^n$ such that is connected. trose that $n > 1$. Let U be an open neighborhood of the boundary $b\Omega = \overline{\Omega}\setminus\Omega$. Then there exists an open set V with $b\Omega \subset V \subset U$ having the following property: if $f \in \mathcal{H}(U)$, then there exists $F \in \mathcal{H}(\Omega)$ such that for the restriction to $V \cap \Omega$ one has*

$$f|_{V\cap\Omega} = F|_{V\cap\Omega}.$$

Proof. Let W be an open neighborhood of $b\Omega$ such that $W \subset\subset U$. Choose $\alpha \in \mathcal{C}_0^\infty(U)$ such that $\alpha = 1$ on W. For $f \in \mathcal{H}(U)$ we define

$$g = \begin{cases} \alpha f & \text{on } U\cap\Omega\\ 0 & \text{on } \Omega\setminus U.\end{cases}$$

Since $\alpha = 0$ in a neighborhood of bU, we have $g \in \mathcal{C}^\infty(\Omega)$. Next, define

$$\phi_k = \begin{cases} \dfrac{\partial g}{\partial \overline{z}_k} & \text{on } \Omega\\ 0 & \text{on } \mathbb{C}^n\setminus\Omega.\end{cases}$$

Since $\frac{\partial g}{\partial \overline{z}_k} = \frac{\partial f}{\partial \overline{z}_k}$ on $W \cap \Omega$, we have $\phi_k \in \mathcal{C}^\infty(\mathbb{C}^n)$. Furthermore $\phi_k = 0$ on $(\mathbb{C}^n \setminus \Omega) \cup W$, which implies that $\operatorname{supp}(\phi_k) \subset \Omega$ and $\phi_k \in \mathcal{C}_0^\infty(\mathbb{C}^n)$.
Next we claim that the $(0,1)$-form $\phi = \sum_{j=1}^n \phi_j\, d\overline{z}_j$ satisfies $\overline{\partial}\phi = 0$. We have to show that

$$\frac{\partial \phi_j}{\partial \overline{z}_k} = \frac{\partial \phi_k}{\partial \overline{z}_j},$$

for all $j, k = 1, \dots, n$. Both, ϕ_j and ϕ_k are zero on $(\mathbb{C}^n \setminus \Omega) \cup W$, on Ω we have

$$\frac{\partial \phi_j}{\partial \overline{z}_k} = \frac{\partial^2 g}{\partial \overline{z}_j \partial \overline{z}_k} = \frac{\partial \phi_k}{\partial \overline{z}_j}.$$

By Theorem 1.9, there exists $u \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ such that $\overline{\partial} u = \phi$. Now we set $F = g - u$. We have

$$\frac{\partial F}{\partial \overline{z}_k} = \frac{\partial g}{\partial \overline{z}_k} - \phi_k = 0$$

on Ω, for $k = 1, \dots, n$. Hence $F \in \mathcal{H}(\Omega)$.
Let Ω_0 be the connected component of $(\mathbb{C}^n \setminus \Omega) \cup W$ containing $\mathbb{C}^n \setminus \Omega$. Define $V := \Omega_0 \cap U$. We claim that $f|_{V \cap \Omega} = F|_{V \cap \Omega}$. Since $V \cap \Omega \subseteq W \cap \Omega$ and $\alpha = 1$ on W so that $g = f$ on $V \cap \Omega$, it suffices to show that $u|_{\Omega_0} = 0$. Since $\phi_k = 0$ on $\mathbb{C}^n \setminus \Omega$ and $\phi_k = \frac{\partial f}{\partial \overline{z}_k} = 0$ on W, we have $\frac{\partial u}{\partial \overline{z}_k} = 0$ on Ω_0. Hence $u \in \mathcal{H}(\Omega_0)$. And since $\operatorname{supp}(u)$ is compact and Ω is bounded, $\mathbb{C}^n \setminus \Omega$ must intersect $\mathbb{C}^n \setminus \operatorname{supp}(u)$. In particular, the open set $\Omega_1 = \Omega_0 \cap (\mathbb{C}^n \setminus \operatorname{tr}(u)) \neq \emptyset$, and $u|_{\Omega_1} = 0$. Since $u \in \mathcal{H}(\Omega_0)$, the Identity Theorem implies that $u = 0$ on Ω_0.

□

Corollary 1.11. *Let Ω be a bounded open set in $\mathbb{C}^n$ such that $\mathbb{C}^n \setminus \Omega$ is connected. trose that $n > 1$. Let U be an open neighborhood of the boundary $b\Omega = \overline{\Omega} \setminus \Omega$. Furthermore trose that $U \cap \Omega$ is connected. If $f \in \mathcal{H}(U)$, then there exists $G \in \mathcal{H}(\Omega \cup U)$ such that $G|_U = f$.*

Proof. If F is as in Theorem 1.10, and $\Omega \cap U$ is connected, the Identity Theorem implies that $F|_{\Omega \cap U} = f|_{\Omega \cap U}$, and we may define G by $G|_\Omega = F$ and $G|_U = f$.

□

Example 1.12. Let $|z|^2 := |z_1|^2 + \cdots + |z_n|^2$, for $z \in \mathbb{C}^n$. Let $\Omega = \{z \in \mathbb{C}^n : |z| < 1\}$ and let $U = \{z \in \mathbb{C}^n : 1/2 < |z| < 3/2\}$. Then each $f \in \mathcal{H}(U)$ has a unique holomorphic extension to $\Omega \cup U = \{z \in \mathbb{C}^n : |z| < 3/2\}$, see Fig. 1.2 in absolute space.

It is even possible to extend certain functions on the boundary of a domain to holomorphic functions in the interior of the domain.

Theorem 1.13. *Let Ω be a bounded open set in $\mathbb{C}^n, n > 1$. trose that $\mathbb{C}^n \setminus \overline{\Omega}$ is connected and $b\Omega \in \mathcal{C}^4$, i.e. there exists a real-valued defining function $\rho \in \mathcal{C}^4(\mathbb{C}^n)$ such that ρ vanishes precisely on $b\Omega$ and $d\rho \neq 0$ on $b\Omega$. If $u \in \mathcal{C}^4(\overline{\Omega})$ and $\overline{\partial} u \wedge \overline{\partial} \rho = 0$ on $b\Omega$, one can then find a function $U \in \mathcal{C}^1(\overline{\Omega})$ such that $U \in \mathcal{H}(\Omega)$ and $U = u$ on $b\Omega$.*

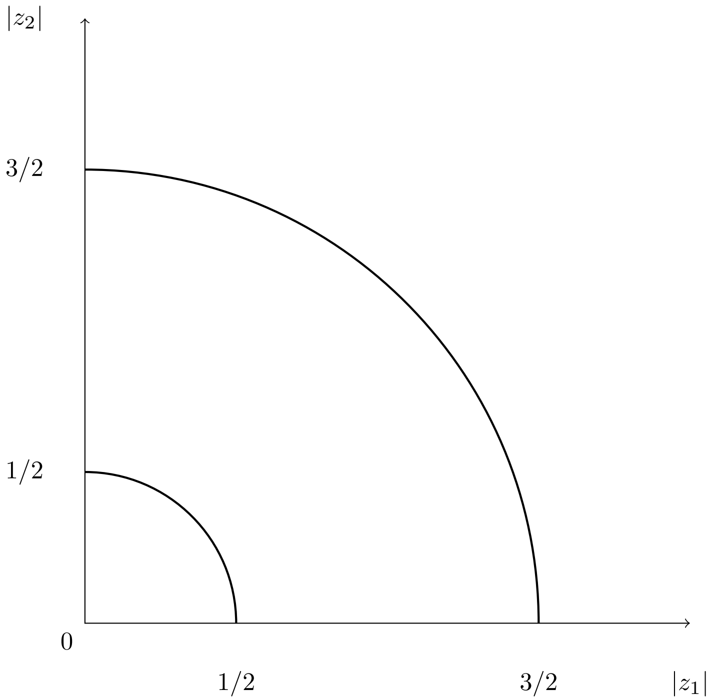

Fig. 1.2

Remark 1.14. The condition $\overline{\partial} u \wedge \overline{\partial}\rho = 0$ on $b\Omega$, can also be stated as

$$\sum_{j=1}^{n} t_j \frac{\partial u}{\partial \overline{z}_j} = 0 \text{ on } b\Omega,$$

for all $(t_1, \ldots, t_n) \in \mathbb{C}^n$ with $\sum_{j=1}^{n} t_j \frac{\partial \rho}{\partial \overline{z}_j} = 0$ on $b\Omega$. We say that u satisfies the tangential Cauchy-Riemann equations and that u is CR-function. Using Lemma 1.3 one easily sees that this definition does not depend on the choice of the defining function ρ (see Exercises).

Proof of 1.13. First we construct $U_0 \in \mathcal{C}^2(\overline{\Omega})$ such that $U_0 = u$ on $b\Omega$ and $\overline{\partial} U_0 = \rho^2 v$ where v is a $(0,1)$-form with $\mathcal{C}^1$ coefficients on $b\Omega$. First we claim that $\overline{\partial} u = h_0 \overline{\partial}\rho + \rho h_1$, where $h_0 \in \mathcal{C}^3(\overline{\Omega})$ and $h_1 \in \mathcal{C}^2_{(0,1)}(\overline{\Omega})$. For this aim we consider the coefficients of the $(0,1)$-form $\overline{\partial} u$. Using the assumption that $\overline{\partial} u \wedge \overline{\partial}\rho = 0$ on $b\Omega$, we see that there exists $h_0 \in \mathcal{C}^3(\overline{\Omega})$ such that

$$\frac{\partial u}{\partial \overline{z}_j} - h_0 \frac{\partial \rho}{\partial \overline{z}_j} = 0, \text{ on } b\Omega,\ j = 1, \ldots, n.$$

From the proof of Lemma 1.3 we get that there exist $h_{1,j} \in \mathcal{C}^2(\overline{\Omega})$, $j = 1, \ldots, n$ such that

$$\frac{\partial u}{\partial \overline{z}_j} - h_0 \frac{\partial \rho}{\partial \overline{z}_j} = \rho h_{1,j},\ j = 1, \ldots, n.$$

Now define the $(0,1)$-form $h_1 = \sum_{j=1}^{n} h_{1,j} d\overline{z}_j$. Then $\overline{\partial} u = h_0 \overline{\partial}\rho + \rho h_1$. Next we get $\overline{\partial}(u - h_0\rho) = \rho(h_1 - \overline{\partial} h_0) = \rho h_2$, where $h_2 \in \mathcal{C}^2_{(0,1)}(\overline{\Omega})$. Since $0 = \overline{\partial}^2(u - h_0\rho) = \overline{\partial}(\rho h_2) = \overline{\partial}\rho \wedge h_2 + \rho \overline{\partial} h_2$, we have $\overline{\partial}\rho \wedge h_2 = 0$ on $b\Omega$. As in the first part of the proof, we can again write

$$h_2 = h_3 \overline{\partial}\rho + \rho h_4,$$

where $h_3 \in \mathcal{C}^2(\overline{\Omega})$ and $h_4 \in \mathcal{C}^1_{(0,1)}(\overline{\Omega})$. Now set $U_0 = u - h_0\rho - h_3\rho^2/2$. An easy computation shows that

$$\overline{\partial} U_0 = \rho^2 (h_4 - \overline{\partial} h_3/2),$$

which completes the construction of U_0. Next we define the $(0,1)$-form

$$f = \begin{cases} \overline{\partial} U_0 & \text{on } \Omega \\ 0 & \text{on } \mathbb{C}^n \setminus \Omega. \end{cases}$$

Since $f = \rho^2 v$ on $b\Omega$ we have $f \in \mathcal{C}^1_{(0,1)}(\mathbb{C}^n)$ and f has compact support. By Theorem 1.9 we can find a function $V \in \mathcal{C}^1_0(\mathbb{C}^n)$ with compact support, such that $\overline{\partial} V = f$. The definition of f implies that V is holomorphic in the connected set $\mathbb{C}^n \setminus \overline{\Omega}$ and, as V has compact support, that $V = 0$ on $\mathbb{C}^n \setminus \overline{\Omega}$. The function $U = U_0 - V$ is therefore equal to $U_0 = u$ on $b\Omega$, and $\overline{\partial} U = \overline{\partial} U_0 - \overline{\partial} V = f - f = 0$ in Ω. $\square$

The tangential Cauchy-Riemann equations for the Siegel upper half-space $\mathbb{U}$ are of special interest. Let $n = 2$. The function $\rho(z_1, z_2) = -\frac{1}{2i}(z_2 - \overline{z}_2) + z_1\overline{z}_1$ is a defining function for $b\mathbb{U}$. The boundary can be identified with $\mathbb{H}_2 = \mathbb{C} \times \mathbb{R}$ via the mapping $\pi : (z_1, t + i|z_1|^2) \mapsto (z_1, t)$, where $z_2 = t + is$. We call $\mathbb{H}_2$ the Heisenberg group, see Exercises. If $\overline{\partial} u \wedge \overline{\partial}\rho = 0$ on $b\mathbb{U}$, we have for a function $u \in \mathcal{C}^1(\overline{\mathbb{U}})$

$$\frac{1}{2i}\frac{\partial u}{\partial \overline{z}_1} - z_1 \frac{\partial u}{\partial \overline{z}_2} = 0,$$

on $b\mathbb{U}$. This means we have to consider the differential operator

$$\overline{L} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} - 2i(x + iy)\frac{\partial}{\partial t}$$

on $\mathbb{H}_2$. This operator has a special property giving a partial differential operator without solution.

Theorem 1.15 (H. Lewy [3]). *Let f be a continuous real-valued function depending only on t. If there is a $\mathcal{C}^1$-function u on $(x, y, t) \in \mathbb{H}_2$ satisfying $\overline{L}u = f$ in some neighborhood of the origin, then f is analytic at $t = 0$, i.e. can be expanded into a convergent Taylor series in a neighborhood of $t = 0$.*

So if one takes a continuous function f being not analytic at 0, the partial differential equation $\overline{L}u = f$ has no solution.

Proof. Suppose $\overline{L}u = f$ in the set where $x^2 + y^2 < R^2$ and $|t| < R$, $R > 0$. Let $\gamma(\theta) = re^{i\theta}, \theta \in [0, 2\pi]$, $0 < r < R$. Consider the line integral

$$V(r, t) = \int_\gamma u(x, y, t)\, dz = ir \int_0^{2\pi} u(r\cos\theta, r\sin\theta, t)\, e^{i\theta}\, d\theta.$$

By **??** and Stokes' Theorem **??**,

$$\begin{aligned} V(r, t) &= i \int_{D_r(0)} \left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)(x, y, t)\, d\lambda(z) \\ &= i \int_0^r \int_0^{2\pi} \left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)(\sigma\cos\theta, \sigma\sin\theta, t)\, \sigma\, d\sigma d\theta, \end{aligned}$$

where we used polar coordinates $d\lambda(z) = \sigma\, d\sigma d\theta$. Hence

$$\begin{aligned} \frac{\partial V}{\partial r} &= i \int_0^{2\pi} \left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)(r\cos\theta, r\sin\theta, t)\, r\, d\theta \\ &= \int_\gamma \left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)(x, y, t)\, r\, \frac{dz}{z}. \end{aligned}$$

3 Lewy, Hans (1904–1988)

Now set $s = r^2$ and use $\overline{L}u = f$ to get

$$\begin{aligned}\frac{\partial V}{\partial s} = \frac{1}{2r}\frac{\partial V}{\partial r} &= \int_\gamma \left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)(x,y,t)\,\frac{dz}{2z}\\ &= i\int_\gamma \frac{\partial u}{\partial t}(x,y,t)\,dz + \int_\gamma f(t)\,\frac{dz}{2z}\\ &= i\,\frac{\partial V}{\partial t} + i\pi f(t).\end{aligned}$$

Now we set $F(t) = \int_0^\tau f(\tau)\,d\tau$, and $U(t,s) = V(t,s) + \pi F(t)$, Then

$$\frac{\partial U}{\partial t} + i\,\frac{\partial U}{\partial s} = 0,$$

which is the Cauchy-Riemann equation. Hence U is a holomorphic function of $w = t + is$ for $0 < s < R^2$, and $|t| < R$, in addition, U is continuous up to the line $s = 0$, and $V = 0$ when $s = 0$, therefore $U(t,0) = \pi F(t)$ is real-valued. We can apply the Schwarz' reflection principle (see Exercise 47b): the definition $U(t,-s) = \overline{U(t,s)}$ gives a holomorphic continuation of U to a full neighborhood of the origin. In particular, $U(t,0) = \pi F(t)$ is analytic in t, hence so is $f = F'$. □

1.3 Domains of holomorphy

In this section we describe domains for which the Hartogs extension phenomenon does not occur; these are the so-called domains of holomorphy. First we study holomorphically convex domains, a concept which was of importance for the Runge type theorems and which serves as an interesting concept where the difference between one and several complex variables becomes apparent. It turns out that another generalization of convexity, so-called pseudoconvexity, is the appropriate geometric concept to characterize domains of holomorphy in $\mathbb{C}^n$, $n \geq 2$. It is beyond the level of this book to give all the details in this context and we refer to textbooks on several complex variables for a thorough treatment ([1; 6; 3]).

Definition 1.16. Let Ω be a domain in $\mathbb{C}^n$. A holomorphic function f on Ω is completely singular at $p \in b\Omega$ if for every connected neighborhood U of p there is no $h \in \mathcal{H}(U)$ which agrees with f on some connected component of $U \cap \Omega$.
Ω is called a weak domain of holomorphy if for every $p \in b\Omega$ there is $f \in \mathcal{H}(\Omega)$ which is completely singular at p, and Ω is called a domain of holomorphy if there exists $f \in \mathcal{H}(\Omega)$ which is completely singular at every boundary point $p \in b\Omega$.

We already know that every domain in $\mathbb{C}$ is a domain of holomorphy, see Theorem **??**. For $n \geq 2$ we already know examples of domains which fail to be domains of holomorphy.
The concept of weak domain of holomorphy is convenient at the introductory level; it is, in fact, equivalent to the concept of domain of holomorphy, but this result is not elementary.

Lemma 1.17. *Every convex domain Ω in $\mathbb{C}^n$ is a weak domain of holomorphy.*

Proof. Let $p \in b\Omega$. The convexity implies that we can find an $\mathbb{R}$-linear function $l : \mathbb{C}^n \longrightarrow \mathbb{R}$ such that the hyperplane $\{z \in \mathbb{C}^n : l(z) = l(p)\}$ separates Ω and p, i.e. we may assume that $l(z) < l(p)$ for $z \in \Omega$. We can write

$$l(z) = \sum_{j=1}^{n} \alpha_j z_j + \sum_{j=1}^{n} \beta_j \overline{z}_j,$$

where $\alpha_j, \beta_j \in \mathbb{C}$. Since l is real-valued, we have $\beta_j = \overline{\alpha}_j$, for $j = 1, \dots, n$. Set $h(z) = 2\sum_{j=1}^n \alpha_j z_j$. Then h is complex-linear and $l(z) = \Re h(z)$. Now the function

$$f_p(z) := \frac{1}{h(z) - h(p)}$$

is holomorphic on Ω and completely singular at p. □

In the following we consider the concept of holomorphically convex domains in $\mathbb{C}^n$ in order to get further examples of domains of holomorphy. This concept was already introduced in Chapter 4 for a general treatment of the Runge approximation theorem in one complex variable.

Definition 1.18. A domain Ω in $\mathbb{C}^n$ is called holomorphically convex , if $\hat{K}_\Omega$ is relatively compact in Ω for every compact set $K \subset \Omega$, where $\hat{K}_\Omega = \{z \in \Omega : |f(z)| \leq |f|_K \text{ for all } f \in \mathcal{H}(\Omega)\}$. We call K holomorphically convex ($\mathcal{H}(\Omega)$-convex), if $K = \hat{K}_\Omega$.

Remark. A domain in $\mathbb{C}$ is always holomorphically convex (see **??** (e)). The situation is different in higher dimensions. Let $\Omega = \{z \in \mathbb{C}^n : 1/2 < |z| < 2\}$ and $K = \{z \in \mathbb{C}^n : |z| = 1\}$. Then $\hat{K}_\Omega = K$, if $n = 1$, but if $n > 1$, Corollary 1.11 implies that every $f \in \mathcal{H}(\Omega)$ extends to a holomorphic function $\tilde{f}$ on $B(0,2)$. It follows from the maximum principle applied to $\tilde{f}$ that for $1/2 < |z| \leq 1$, one has

$$|f(z)| = |\tilde{f}(z)| \leq |\tilde{f}|_K = |f|_K,$$

hence $\{z \in \Omega : |z| \leq 1\} \subset \hat{K}_\Omega$, and $\hat{K}_\Omega$ is not relatively compact in Ω.

Lemma 1.19. *Let Ω be a holomorphically convex domain in $\mathbb{C}^n$. Then there is a compact exhaustion $(K_j)_j$ of Ω by holomorphically convex sets K_j.*

Proof. Since Ω is holomorphically convex, one can use **??** (f). □

This can be used to construct unbounded holomorphic functions.

Lemma 1.20. *Let $(K_j)_j$ be a compact exhaustion of Ω by holomorphically convex sets K_j. trose that $p_j \in K_{j+1} \setminus K_j$ for $j = 1, 2, \dots$. Then there exists $f \in \mathcal{H}(\Omega)$ such that $\lim_{j\to\infty} |f(p_j)| = \infty$.*

Proof. The desired function f is constructed as the limit of a series $\sum_m f_m$, where $f_m \in \mathcal{H}(\Omega)$ is chosen such that

$$|f_m|_{K_m} < 2^{-m},\ m = 1, 2, \dots, \tag{1.9}$$

and

$$|f_j(p_j)| > j + 1 + \sum_{m=1}^{j-1} |f_m(p_j)|,\ j = 2, 3, \dots. \tag{1.10}$$

We construct the sequence $(f_m)_m$ inductively: set $f_1 = 0$, and if $k \geq 2$, trose that $f_1, \dots, f_{k-1}$ have already been found such that (1.9) and (1.10) hold. By **??** (g), since $p_k \notin (K_k)\hat{}_\Omega$, there exists $f_k \in \mathcal{H}(\Omega)$ with $|f_k|_{K_k} < 2^{-k}$ and such that (1.10) holds.

Now (1.9) implies that $f = \sum_{j=1}^{\infty} f_j$ converges uniformly on all compact subsets of Ω. Hence $f \in \mathcal{H}(\Omega)$. Furthermore (1.10) implies

$$|f(p_j)| \geq |f_j(p_j)| - \sum_{m\neq j} |f_m(p_j)| > j + 1 - \sum_{m>j} |f_m(p_j)|,\ j \geq 2.$$

Then (1.9) implies that $\sum_{m>j} |f_m(p_j)| < \sum_{m>j} 2^{-m} \leq 1$, and hence that $|f(p_j)| > j$. □

It is now easy to show that a domain Ω is holomorphically convex if and only if for every sequence $(p_j)_j$ in Ω without limit point in Ω there is $f \in \mathcal{H}(\Omega)$ with $\sup_j |f(p_j)| = \infty$. In addition, one can now use Lemma 1.17 to show that every convex domain in $\mathbb{C}^n$ is holomorphically convex (see Exercises).

Now we introduce a class of domains which generalize the polydiscs.

Definition 1.21. An open set $\Omega \subset\subset \mathbb{C}^n$ is called an analytic polyhedron if there are a neighborhood U of $\overline{\Omega}$ and finitely many functions $f_1, \dots, f_k \in \mathcal{H}(U)$, such that

$$\Omega = \{z \in U : |f_1(z)| < 1, \dots, |f_k(z)| < 1\}.$$

Example 1.22. Let $0 < q < 1$. Then

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1, |z_1 z_2| < q\}$$

is an analytic polyhedron which is not a convex domain, see Fig. 1.3.

Theorem 1.23. *Every analytic polyhedron is holomorphically convex.*

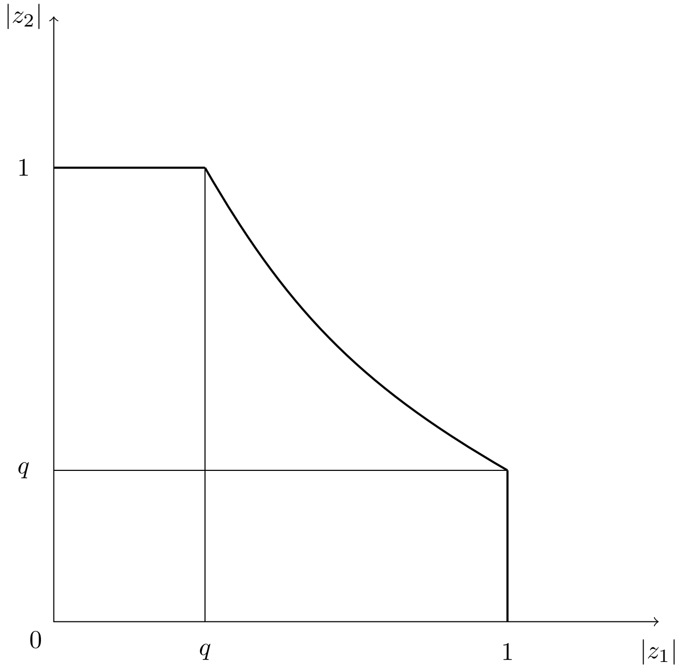

Fig. 1.3

Proof. Let Ω be like in 1.21. If $K \subset \Omega$ is compact, then $r_j := |f_j|_K < 1$, for $j = 1, \dots, k$. It follows that

$$\hat{K}_\Omega \subset \{z \in U : |f_1(z)| \le r_1, \dots, |f_k(z)| \le r_k\},$$

and the set on the right hand side is relatively compact in Ω. $\square$

It is relatively easy to show that holomorphically convex domains are domains of holomorphy. For this aim we need some preparations which are similar to the proof of Theorem **??**.

Lemma 1.24. *Let Ω be a domain in $\mathbb{C}^n$. Let U be a connected neighborhood of $p \in b\Omega$ and let $\Omega_1 \subset U \cap \Omega$ be a nonempty connected component of $U \cap \Omega$. Then $b\Omega_1 \cap (U \cap b\Omega) \neq \emptyset$.*

Proof. Since Ω_1 is a component of the open set $U \cap \Omega$, it follows that Ω_1 is open in $\mathbb{C}^n$ and closed in $U \cap \Omega$. Since U is connected and $\Omega_1 \neq U$, one has that Ω_1 cannot be closed in U. Hence there exists $q \in (b\Omega_1 \cap U) \setminus \Omega_1$. Since $\Omega_1 \subset \Omega$ and Ω_1 is closed in $U \cap \Omega$, we have $q \in b\Omega$, and so $q \in b\Omega_1 \cap (U \cap b\Omega)$. $\square$

Lemma 1.25. *Let $(K_m)_m$ be a compact exhaustion of the domain Ω in $\mathbb{C}^n$. Then there are a subsequence (m_j) of $\mathbb{N}$ and a sequence $(p_j)_j$ of points in Ω such that*

(a) $p_j \in K_{m_{j+1}} \setminus K_{m_j}$, *for* $j = 1, 2, \dots,$ *and*
(b) for every $p \in b\Omega$ *and every connected neighborhood* U *of* p, *each component* Ω_1 *of* $U \cap \Omega$ *contains infinitely many points from* $(p_j)_j$.

Proof. Let $(a_k)_k$ be an enumeration of the points of Ω with rational coordinates. Let $r_k = \operatorname{dist}(a_k, b\Omega)$. Then the balls $B_k = B(a_k, r_k)$ are contained in Ω. Let $(Q_j)_j$ be a sequence of such balls B_k which contains each B_k infinitely many times; for example the sequence $B_1, B_1, B_2, B_1, B_2, B_3, B_1, \dots$. Now take $K_{m_1} = K_1$ and use induction: assume that $l > 1$ and $p_1, \dots, p_{l-1}$ and $K_{m_1}, \dots, K_{m_l}$ have been chosen so that (a) holds for $j = 1, \dots, l-1$. Since Q_l is not contained in any compact subset of Ω, we may choose $p_l \in Q_l \setminus K_{m_l}$ and m_{l+1} such that $p_l \in K_{m_{l+1}}$.Then (a) holds for all $j = 1, 2, \dots$. We claim that the points $(p_j)_j$ statisfy (b): given Ω_1 as in (b) there is a point $q \in b\Omega_1 \cap (U \cap b\Omega)$, see Lemma 1.24. Hence there is $a_\nu \in \Omega_1$ with rational coordinates sufficiently close to q, so that $B_\nu \subset \Omega_1$. Since B_ν occurs infinitely many times in the sequence $(Q_j)_j$, and $p_j \in Q_j$ for $j = 1, 2, \dots$, the ball B_ν contains infinitely many points of the sequence $(p_j)_j$, and we are done. □

Theorem 1.26. *Every holomorphically convex domain* Ω *in* $\mathbb{C}^n$ *is a domain of holomorphy.*

Proof. We can choose a compact exhaustion $(K_j)_j$ of Ω with by holomorphically convex sets K_j. We apply Lemma 1.20 to the sequences $(p_j)_j$ and $(K_{m_j})_j$ given by Lemma 1.25 to get $f \in \mathcal{H}(\Omega)$ with $\lim_{j\to\infty} |f(p_j)| = \infty$. We claim that f is completely singular at every point $p \in b\Omega$. If Ω_1 is a component of $U \cap \Omega$, where U is a connected neighborhood of p, trose there exists $h \in \mathcal{H}(U)$ with $f|_{\Omega_1} = h|_{\Omega_1}$. Now we replace U by $U' \subset\subset U$ and we replace Ω_1 by a component Ω_1' of $U' \cap \Omega$ which meets Ω_1, then we may assume that $|h|_{\Omega_1'} \le |h|_{U'} < \infty$. Hence f would have to be bounded on Ω_1', and this contradicts Lemma 1.25 (b) and $\lim_{j\to\infty} |f(p_j)| = \infty$. □

Using 1.22 we have an example of a domain of holomorphy which is not convex. We now introduce the suitable generalization of convexity to characterize domains of holomorphy.

Definition 1.27. A $\mathcal{C}^2$ real valued function φ on Ω is plurisubharmonic, if

$$i\partial\overline{\partial}\varphi(t,t)(p) := \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}(z)\, t_j \overline{t}_k \ge 0,$$

for all $t = (t_1, \dots, t_n) \in \mathbb{C}^n$ and all $z \in \Omega$.
φ is strictly plurisubharmonic if

$$i\partial\overline{\partial}\varphi(t,t)(p) := \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}(z)\, t_j \overline{t}_k > 0,$$

for all $t \in \mathbb{C}^n$, $t \neq 0$.

Remark 1.28. (a) A $\mathcal{C}^2$ real valued function φ on Ω is plurisubharmonic, if and only if for every $a \in \Omega$ and $w \in \mathbb{C}^n$ the function $u \mapsto \varphi(a + uw)$ is subharmonic on $\{u \in \mathbb{C} : a + uw \in \Omega\}$, see Exercises.
For technical reasons it is convenient to include upper semicontinuous functions and to admit the value $-\infty$ in the definition of plurisubharmonic functions, where one has to take the general definition of subharmonicity **??**.
(b) Suppose $r \in \mathcal{C}^2(U)$ is a defining function for a domain $\Omega \subset \mathbb{C}^n$, where U is a neighborhood of a point $p \in b\Omega$. One can write the Taylor expansion of r at p in complex form:

$$r(p+t) = r(p) + 2\Re(\partial r_p(t) + Q_p(r;t)) + i\partial\overline{\partial} r(t,t)(p) + o(|t|^2), \tag{1.11}$$

where $t = (t_1, \dots, t_n) \in \mathbb{C}^n$,

$$\partial r_p(t) = \sum_{j=1}^n \frac{\partial r}{\partial z_j}(p)\, t_j, \tag{1.12}$$

$$Q_p(r;t) = \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial z_k}(p) t_j t_k. \tag{1.13}$$

Definition 1.29. A bounded domain Ω in $\mathbb{C}^n$ is called strictly pseudoconvex if there are a neighborhood U of $b\Omega$ and a strictly plurisubharmonic function $r \in \mathcal{C}^2(U)$ such that

$$\Omega \cap U = \{z \in U : r(z) < 0\}.$$

The simplest example of a strictly pseudoconvex domain is a ball $B(p,R)$, the function $r(z) = |z-p|^2 - R^2$ is strictly plurisubharmonic , and $B(p,R) = \{z \in \mathbb{C}^n : r(z) < 0\}$.

In the following we shall show that a strictly pseudoconvex domain is (at least) locally a domain of holomorphy.

Lemma 1.30. *Let U be open in $\mathbb{C}^n$ and trose $r \in \mathcal{C}^2(U)$ is strictly plurisubharmonic on U. If $W \subset\subset U$, there are positive constants $c > 0$ and $\epsilon > 0$, such that the function $F^{(r)}(\zeta, z)$ defined on $U \times \mathbb{C}^n$ by*

$$F^{(r)}(\zeta,z) = \sum_{j=1}^n \frac{\partial r}{\partial \zeta_j}(\zeta)\,(\zeta_j - z_j) - \frac{1}{2}\sum_{j,k=1}^n \frac{\partial^2 r}{\partial \zeta_j \partial \zeta_k}(\zeta)(\zeta_j - z_j)(\zeta_k - z_k) \tag{1.14}$$

satisfies the estimate

$$2\Re F^{(r)}(\zeta,z) \ge r(\zeta) - r(z) + c|z-\zeta|^2 \tag{1.15}$$

for $\zeta \in W$ and $|z - \zeta| < \epsilon$.

Proof. From (1.11) with $p = \zeta \in U$ and $t = z - \zeta$, we obtain the Taylor expansion of $r(z)$ at ζ :

$$r(z) = r(\zeta) - 2\Re F^{(r)}(\zeta, z) + i\partial\overline{\partial}r(z-\zeta, z-\zeta)(\zeta) + o(|z-\zeta|^2). \tag{1.16}$$

Since r is strictly plurisubharmonic, we have $i\partial\overline{\partial}r(t,t)(p) \geq \kappa|t|^2$ where $\kappa = \min\{i\partial\overline{\partial}r(t,t)(p) : |t| = 1\}$ is positive. So, if $0 < c < \kappa$, the continuity of the second derivatives of r implies $i\partial\overline{\partial}r(t,t)(z) \geq c|t|^2$, for $t \in \mathbb{C}^n$ and all z in some neighborhood of p. As $\overline{W} \subset U$ is compact there is $c > 0$ such that $i\partial\overline{\partial}r(z-\zeta, z-\zeta)(\zeta) \geq 2c|z-\zeta|^2$ for $\zeta \in W$ and $z \in \mathbb{C}^n$. Now we use Taylor's theorem and the uniform continuity on $W$of the derivatives of r up to order 2 to show that there exists $\epsilon > 0$ such that the error term $o(|z-\zeta|^2)$ in (1.16) can be estimated in the form $o(|z-\zeta|^2) \leq c|z-\zeta|^2$ uniformly for $\zeta \in W$ and if $|z-\zeta| < \epsilon$. The desired estimates (1.15) now follows from (1.16). □

Theorem 1.31. *Let Ω be a strictly pseudoconvex domain. Then every point $p \in b\Omega$ has a neighborhood V such that $V \cap \Omega$ is a (weak) domain of holomorphy.*

Proof. Let $r \in \mathcal{C}^2(U)$ be strictly plurisubharmonic in a neighborhood U of $b\Omega$ so that $\Omega \cap U = \{z \in U : r(z) < 0\}$. Choose c, ϵ as in Lemma 1.30 such that (1.15) holds for $\zeta \in b\Omega$. For $\zeta \in b\Omega$ we have $r(\zeta) = 0$, and (1.15) implies that $\Re F^{(r)}(\zeta, z) > 0$ for $z \in \Omega$ with $|z-\zeta| < \epsilon$ (choose ϵ so small that $B(\zeta, \epsilon) \subset U$ for $\zeta \in b\Omega$). If $p \in b\Omega$ is fixed, set $V = B(p, \epsilon/2)$. We claim that $V \cap \Omega$ is a weak domain of holomorphy: for $\zeta \in V \cap b\Omega$ the function

$$f_\zeta(z) := \frac{1}{F^{(r)}(\zeta, z)}$$

is holomorphic on $V \cap \Omega$ and completely singular at ζ; for any of the remaining boundary points $\zeta \in bV \cap \overline{\Omega}$ of $V \cap \Omega$ the convexity of V implies that there is $g \in \mathcal{H}(V)$ which is completely singular at ζ, see Lemma 1.17. □

Remark 1.32. We mention different types of pseudoconvexity:
Let Ω be a bounded domain in $\mathbb{C}^n$ with $n \geq 2$, and let r be a $\mathcal{C}^2$ defining function for Ω. Ω is called Levi pseudoconvex at $p \in b\Omega$, if the Levi form

$$i\partial\overline{\partial}r(t,t)(p) := \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(p)\, t_j \overline{t}_k \geq 0$$

for all

$$t \in T_p^{1,0}(b\Omega) = \{t = (t_1, \dots, t_n) \in \mathbb{C}^n \ : \ \sum_{j=1}^{n} t_j (\partial r/\partial z_j)(p) = 0\},$$

where $T_p^{1,0}(b\Omega)$ is the space of type $(1,0)$ vector fields which are tangent to the boundary at the point p.

The domain Ω is said to be strictly Levi pseudoconvex at p, if the Levi form is strictly positive for all such $t \neq 0$. Ω is called a Levi pseudoconvex domain if Ω is Levi pseudoconvex at every boundary point of Ω.
A bounded domain Ω in $\mathbb{C}^n$ is pseudoconvex if Ω has a $\mathcal{C}^2$ strictly plurisubharmonic exhaustion function $\varphi : \Omega \longrightarrow \mathbb{R}$, i.e. the sets $\{z \in \Omega : \varphi(z) < c\}$ are relatively compact in Ω, for every $c \in \mathbb{R}$. (Here there is no assumption on the boundary of Ω.)
It turns out that for bounded domains with $\mathcal{C}^2$ boundary the concepts of (strictly) Levi pseudoconvex and (strictly) pseudoconvex domains coincide. Furthermore, the following assertion holds:
Let Ω be a domain in $\mathbb{C}^n$. The following are equivalent:
(1) Ω is pseudoconvex.
(2) The equation $\overline{\partial} u = f$ always has a solution $u \in \mathcal{C}^\infty_{(p,q)}(\Omega)$ for any form $f \in \mathcal{C}^\infty_{(p,q+1)}(\Omega)$ with $\overline{\partial} f = 0$, $q = 0, 1, \dots, n-1$.

(3) Ω is a domain of holomorphy.
The proof is beyond the scope of this book. The most difficult part is the solution of the Levi problem, to prove that a pseudoconvex domain is a domain of holomorphy, see [1; 6; 3].

1.4 Exercises

111) Show that the Cayley transform $\Phi(z_1, \dots, z_n) = (w_1, \dots, w_n)$, where $w_j = z_j/(1+z_n)$ for $1 \leq j \leq n-1$ and $w_n = i(1-z_n)/(1+z_n)$ is a biholomorphic map from $\mathbb{B} \longrightarrow \mathbb{U}$.

112) Let $n > 1$. Show that the boundary

$$b\mathbb{U} = \{(z', t + i|z'|^2) : z' \in \mathbb{C}^{n-1}, t \in \mathbb{R}\}$$

of the Siegel upper half-space can be identified with $\mathbb{C}^{n-1} \times \mathbb{R}$. Show that the multiplication

$$(z', t) \cdot (\zeta', \tau) = (z' + \zeta', t + \tau + 2\Im\langle z', \zeta'\rangle)$$

turns $b\mathbb{U}$ into a group which is non-abelian. This group is called the Heisenberg [4] group.

113) Let f be holomorphic in a neighborhood of the closed polydisc $\overline{P(0,r)} \subset \mathbb{C}^n$, where $n > 1$, with the possible exception of the origin $(0, \dots, 0) \in \mathbb{C}^n$. Suppose that

4 Heisenberg, Werner (1901–1976)

not all z_j, where $1 \leq j \leq n-1$, are zero. Prove that

$$f(z_1, \dots, z_n) = \frac{1}{2\pi i} \int\limits_{|\zeta_n|=r_n} \frac{f(z_1, \dots z_{n-1}, \zeta_n)}{\zeta_n - z_n} \, d\zeta_n,$$

and show that the integral on the right hand side depends holomorphically on $z_1, \dots, z_n$ for all $z = (z_1, \dots, z_n) \in P(0, r)$. Therefore holomorphic functions of several variables do not have isolated zeros.

114) Let Ω be a bounded domain in $\mathbb{C}^n, n > 1$, with a defining function $\rho \in \mathcal{C}^2$ which vanishes precisely on $b\Omega$ and $d\rho \neq 0$ on $b\Omega$. Show that the condition $\overline{\partial} u \wedge \overline{\partial} \rho = 0$ on $b\Omega$, can also be stated as

$$\sum_{j=1}^{n} t_j \frac{\partial u}{\partial \overline{z}_j} = 0 \text{ on } b\Omega,$$

for all $(t_1, \dots, t_n) \in \mathbb{C}^n$ with $\sum_{j=1}^n t_j \frac{\partial \rho}{\partial \overline{z}_j} = 0$ on $b\Omega$. We say that u is a CR-function. Show that this definition does not depend on the choice of the defining function ρ.

115) Use Lemma 1.17 and Lemma 1.20 in order to show that every convex domain in $\mathbb{C}^n$ is holomorphically convex.

116) Let $0 < q < 1$. Define

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1, |z_1 z_2| < q\}.$$

Show that Ω is a domain of holomorphy, which is not convex.

117) Let Ω be a domain in $\mathbb{C}^n$. Show that a $\mathcal{C}^2$ real valued function φ on Ω is plurisubharmonic, if and only if for every $a \in \Omega$ and $w \in \mathbb{C}^n$ the function $u \mapsto \varphi(a + uw)$ is subharmonic on $\{u \in \mathbb{C} : a + uw \in \Omega\}$.

118) Let Ω be a domain in $\mathbb{C}^n$ and let $f \in \mathcal{H}(\Omega)$. Show that $|f|^\alpha, \alpha > 0$, and $\log |f|$ are plurisubharmonic on Ω.

119) Let $\Omega \subseteq \mathbb{C}^n$ and $G \subseteq \mathbb{C}^m$ be domains and let $F : G \longrightarrow \Omega$ be a holomorphic map. trose that $u \in \mathcal{C}^2(\Omega)$ is plurisubharmonic on Ω. Show that $u \circ F$ is plurisubharmonic on G.

1.5 Notes

In Section 1.2 we have followed the expositions of L. Hörmander [1] and R.M. Range [6]. For a thorough treatment of Lewy's theorem 1.15 including interesting consequences for Hardy spaces the reader should consult E. Stein [7]. Pseudoconvexity is also crucial for the inhomogeneous Cauchy Riemann equations, as well as plurisubharmonic functions, see Chapter 10 and 11. The Levi problem to construct a holomorphic function on a pseudoconvex domain which is completely singular at the boundary is solved by means of integral representations in [6]. Another proof uses the powerful method of global solutions and estimates for the inhomogeneous Cauchy Riemann equations [1], this method will be discussed and exploited in more details in the following chapters.

2 Nuclear Fréchet spaces of holomorphic functions

In this chapter we investigate the spaces $\mathcal{H}(\Omega)$ of all holomorphic functions on a domain Ω endowed with the topology of uniform convergence on all compact subsets of Ω. This a complete metric space, a Fréchet space. We start with some general facts about Frèchet spaces such as fundamental systems of seminorms and the Montel property. We indicate that $\mathcal{H}(\Omega)$ can be seen as a so-called projective limit of Hilbert spaces, we introduce the concept of a nuclear Fréchet space and prove that $\mathcal{H}(D_R(0))$ is a nuclear Fréchet space. In addition, the dual space of $\mathcal{H}(D_R(0))$ is determined, it can be identified as a space of holomorphic functions on the complement of $D_R(0)$ - Köthe duality (Section 2.2). The spaces $\mathcal{H}(\Omega)$ of holomorphic functions together with their dual spaces are described as so-called Köthe sequence spaces, which are spaces of the sequences of the Taylor coefficients of the holomorphic functions together with certain weights. The duality is used to prove a Runge type approximation theorem. A similar approach was already the main idea for the proof of Theorem **??**. Furthermore, it is pointed out that the spaces $\mathcal{H}(\Omega)$ endowed with the topology of uniform convergence on all compact subsets of Ω are not normable (see Exercises Section 2.3).

2.1 General properties of Fréchet spaces

Assuming basic knowledge of general topology we collect important facts about topological vector spaces.

A topological vector space X is a vector space endowed with a topology such that the addition $+ : X \times X \longrightarrow X$ and scalar multiplication $. : \mathbb{C} \times X \longrightarrow X$ are continuous.

X is a normed vector space if there is a norm $\|.\|$ on X; each open set of X can be written as a union of open balls $\{x \in X : \|x - x_0\| < r\}$.

X is a metric topological vector space if there is a metric $d : X \times X :\longrightarrow \mathbb{R}_+$ on X, each open set of X can be written as a union of open balls $\{x \in X : d(x, x_0) < r\}$; we will also suppose that the metric is translation invariant, i.e. $d(x+u, y+u) = d(x, y)$, for all $x, y, z \in X$.

A subset M of a vector space X is called absolutely convex , if $\lambda x + \mu y \in M$ for each $x, y \in M$ and $\lambda, \mu \in \mathbb{C}$ with $|\lambda| + |\mu| \leq 1$.

A locally convex vector space X is a topological vector space for which each point has neighborhood basis consisting of absolutely convex sets.

Let X be a locally convex vector space and let U be an absolutely convex 0-neighborhood in X. Then $\|.\|_U : x \mapsto \inf\{t > 0 \ : \ x \in tU\}$ is a continuous seminorm on X; we call $\|.\|_U$ the Minkowski functional of U.

One can explain the topology of a locally convex vector space X in a different way: a family $\mathcal{U}$ of 0-neighborhoods is a fundamental system of 0-neighborhoods, if for each 0-neighborhood U there exists $V \in \mathcal{U}$ and there exists $\epsilon > 0$ such that $\epsilon V \subset U$. A family $(p_\alpha)_{\alpha \in A}$ of seminorms is called a fundamental system of seminorms, if the sets $U_\alpha = \{x \in X : p_\alpha(x) < 1\}$ constitute a fundamental system of 0-neighborhoods of X. We will write $(X, (p_\alpha)_{\alpha \in A}))$ to refer to that.

Let X and Y be locally convex vector spaces with fundamental systems $(p_\alpha)_{\alpha \in A}$ and $(q_\beta)_{\beta \in B}$ of seminorms. A linear mapping $T : X \longrightarrow Y$ is continuous if and only if for each $\beta \in B$ there exist $\alpha \in A$ and a constant $C > 0$ such that $q_\beta(Tx) \le C p_\alpha(x)$, for all $x \in X$.

A linear functional x' on X is continuous if and only if there exist $\alpha \in A$ and a constant $C > 0$ such that $|x'(x)| \le C p_\alpha(x)$, for all $x \in X$.
We indicate that the consequences of the Hahn-Banach Theorem **??** and **??** are also true for locally convex vector spaces, one has to replace the norm in the proof of Theorem **??** by one of the seminorms defining the topology of a locally convex vector space; a subspace Y of a locally convex vector space X is dense in X if and only if each continuous linear functional on X, which vanishes on Y, also vanishes on the whole of X.

The appropriate concept of a bounded subset in X reads as follows: a subset B of a locally convex vector space is said to be bounded if to every 0-neighborhood U in X corresponds a number $s > 0$ such that $B \subset tU$ for every $t > s$. It is easily seen that B is bounded if and only if $\sup_{x \in B} p_\alpha(x) < \infty$ for all $\alpha \in A$, where $(p_\alpha)_{\alpha \in A}$ is a fundamental system of seminorms for the topology of X, compare with Definition **??**.

Now let X' be the space of all continuous linear functionals on a locally convex vector space $(X, (p_\alpha)_{\alpha \in A}))$. We endow the dual space X' with the topology of uniform convergence on all bounded subsets of X; which can be expressed in the following way: $(X', (p_B)_{B \in \mathcal{B}})$, where $p_B(x') = \sup_{x \in B} |x'(x)|$ and $\mathcal{B}$ denotes the family of all bounded subsets of X. It is called the strong topology on X'.

2.2 The space $\mathcal{H}(D_R(0))$ and its dual space

Our main example is the the space $\mathcal{H}(\Omega)$ of all holomorphic functions on a domain $\Omega \subseteq \mathbb{C}^n$ endowed with the topology of uniform convergence on all compact subsets of Ω. Let $(K_m)_{m \in \mathbb{N}}$ be a compact exhaustion of Ω. The topology of $\mathcal{H}(\Omega)$ can be described by the increasing system of norms $|f|_m := \sup_{z\in K_m} |f(z)|$, for $f \in \mathcal{H}(\Omega)$. The system of norms $(|.|_m)_{m\in\mathbb{N}}$ is a fundamental system of (semi)norms. Let $f, g \in \mathcal{H}(\Omega)$ and define

$$d(f,g) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{|f-g|_m}{1+|f-g|_m}. \tag{2.1}$$

It is easily seen that $d(.,.)$ is a metric which generates the original topology of uniform convergence on all compact subsets of Ω.
By Weierstraß' Theorem, **??** $\mathcal{H}(\Omega)$ is a complete metric vector space. These spaces are called Fréchet spaces. Montel's Theorem **??** indicates that all closed bounded subsets of $\mathcal{H}(\Omega)$ are compact subsets of $\mathcal{H}(\Omega)$.
The topology of $\mathcal{H}(\Omega)$ does not stem from a single norm, but from a countable system of norms, see Exercises Section 2.3.

In sake of simplicity we describe the following properties of $\mathcal{H}(\Omega)$ for 1-dimensional discs $D_R(0)$, most of the results can be generalized to arbitrary domains in $\mathbb{C}^n$ using standard functional analysis methods.
Take an increasing sequence $r_m \nearrow R$ and define $|f|_m := \sup_{|z|\leq r_m} |f(z)|$ for $f \in \mathcal{H}(D_R(0))$. Using (**??**) we find out that for each $m \in \mathbb{N}$ there exists $\ell \in \mathbb{N}$ and a constant C, depending only on m and ℓ, such that

$$|f|_m \leq C\,(\int_{D_{r_\ell}(0)} |f(z)|^2 \, d\lambda(z))^{1/2}, \tag{2.2}$$

for each $f \in \mathcal{H}(D_R(0))$; the inequality

$$\|f\|_m := (\int_{D_{r_m}(0)} |f(z)|^2 \, d\lambda(z))^{1/2} \leq C' \, |f|_m \tag{2.3}$$

is clear. Hence, using the Hilbert norms $\|.\|_m$, the space $(\mathcal{H}(D_R(0)), (\|.\|_m)_{m\in\mathbb{N}})$ carries the original topology of uniform convergence on all compact subsets of $D_R(0)$.

Now we consider the Bergman spaces $A^2(D_{r_m}(0))$ endowed with the norm $\|.\|_m$, see Section **??**. If $r_m < r_\ell < R$, we have the inclusions

$$\mathcal{H}(D_R(0)) \subset A^2(D_{r_\ell}(0)) \subset A^2(D_{r_m}(0));$$

and we can show that the natural embedding

$$\iota_{\ell,m} : A^2(D_{r_\ell}(0)) \hookrightarrow A^2(D_{r_m}(0))$$

is a Hilbert-Schmidt operator.
Fix $m \in \mathbb{N}$ and set

$$\phi_n^\ell(z) := \sqrt{\frac{n+1}{\pi}} \, \frac{z^n}{r_\ell^{n+1}},$$

for $n = 0, 1, 2, \ldots$. Then $(\phi_n^\ell)_{n=0}^\infty$ constitutes an orthonormal basis in $A^2(D_{r_\ell}(0))$, see Section **??**. By Theorem **??** we have to show that

$$\sum_{n=0}^\infty \|\iota_{\ell,m}(\phi_n^\ell)\|_m^2 < \infty. \tag{2.4}$$

An easy computation shows that

$$\|\iota_{\ell,m}(\phi_n^\ell)\|_m^2 = (r_m/r_\ell)^{2n+2},$$

and as $r_\ell > r_m$ we get (2.4).
We just showed that for each $m \in \mathbb{N}$ there exists $\ell \in \mathbb{N}$ such that the natural embedding

$$\iota_{\ell,m} : A^2(D_{r_\ell}(0)) \hookrightarrow A^2(D_{r_m}(0))$$

is a Hilbert-Schmidt operator, we say that $\mathcal{H}(D_R(0))$ is a nuclear Fréchet space.

Using the Taylor series expansion and its uniqueness property it is shown that the spaces $\mathcal{H}(D_R(0))$ are topologically isomorphic to certain sequence spaces (Köthe[1] sequence spaces):

Theorem 2.1. *Let $r_m \nearrow R$ be an increasing sequence of positive numbers. Define*

$$\Lambda_R = \{(\xi_n)_{n=0}^\infty : p_m((\xi_n)_{n=0}^\infty) := \sum_{n=0}^\infty |\xi_n| r_m^n < \infty, \forall m \in \mathbb{N}\}.$$

Then the spaces $(\mathcal{H}(D_R(0)), (|.|_m)_{m\in\mathbb{N}})$ and $(\Lambda_R, (p_m)_{m\in\mathbb{N}})$ are topologically isomorphic, where the isomorphism $T : \Lambda_R \longrightarrow \mathcal{H}(D_R(0))$ is given by

$$T((\xi_n)_{n=0}^\infty)(z) = \sum_{n=0}^\infty \xi_n z^n \ , \ z \in D_R(0),$$

and

$$T^{-1}(f) = \left(\frac{f^{(n)}(0)}{n!}\right)_{n=0}^\infty \ , \ f \in \mathcal{H}(D_R(0)).$$

Proof. For $(\xi_n)_{n=0}^\infty \in \Lambda_R$ we have

$$|T((\xi_n)_{n=0}^\infty|_m = \sup_{|z|\le r_m} |\sum_{n=0}^\infty \xi_n z^n| \le \sum_{n=0}^\infty |\xi_n|\, r_m^n = p_m((\xi_n)_{n=0}^\infty).$$

On the other side, we get from Cauchy's estimates **??** that

$$\left|\frac{f^{(n)}(0)}{n!}\right| \le \frac{|f|_\ell}{r_\ell^n},$$

1 Köthe, Gottfried (1905-1989)

hence, if $r_\ell > r_m$, we get

$$\begin{aligned} p_m(T^{-1}(f)) &= \sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| r_m^n \\ &\leq \sum_{n=0}^{\infty} \frac{|f|_\ell}{r_\ell^n} r_m^n \\ &= \left(\sum_{n=0}^{\infty} \frac{r_m^n}{r_\ell^n} \right) |f|_\ell. \end{aligned}$$

□

In a similar way, we can describe the dual space of $\mathcal{H}(D_R(0))$. Recall that L is a continuous linear functional on $(\mathcal{H}(D_R(0)), (|.|_m)_{m\in\mathbb{N}})$ if and only if there exist $m \in \mathbb{N}$ and a constant $C > 0$ such that $|L(f)| \leq C\,|f|_m$ for each $f \in \mathcal{H}(D_R(0))$.

Theorem 2.2. *Let $r_m \nearrow R$ be an increasing sequence of positive numbers. Define*

$$\Lambda'_R = \{(\eta_n)_{n=0}^\infty : \exists m \in \mathbb{N} \text{ with } q_m((\eta_n)_{n=0}^\infty) := \sup_n \frac{|\eta_n|}{r_m^n} < \infty\}.$$

Then the dual space $\mathcal{H}'(D_R(0))$ is isomorphic to the sequence space $(\Lambda'_R, (q_m)_{m\in\mathbb{N}})$.

Proof. We indicate that the seminorms $(q_m)_{m\in\mathbb{N}}$ are decreasing in m, and that $(\Lambda'_R, (q_m)_{m\in\mathbb{N}})$ is not a metric space, but a dual metric space.
Let $L \in \mathcal{H}'(D_R(0))$. Then there exist $m \in \mathbb{N}$ and a constant $C > 0$ such that $|L(f)| \leq C\,|f|_m$ for each $f \in \mathcal{H}(D_R(0))$, in particular applying L to the monomials $z \mapsto z^n$ we obtain a sequence $\eta_n := L(z \mapsto z^n)$ such that

$$|\eta_n| \leq C\, r_m^n,\ n = 0, 1, 2, \ldots.$$

This implies that $(\eta_n)_{n=0}^\infty \in \Lambda'_R$.
If, for the other direction, $(\eta_n)_{n=0}^\infty \in \Lambda'_R$ is given with $q_m((\eta_n)_{n=0}^\infty) = \sup_n \frac{|\eta_n|}{r_m^n} < \infty$, we define a linear functional on $\mathcal{H}(D_R(0))$ by

$$L(f) = \sum_{n=0}^{\infty} \eta_n \frac{f^{(n)}(0)}{n!},$$

for an arbitrary function $f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} z^n$ in $\mathcal{H}(D_R(0))$. Again, from Cauchy's estimates **??**, we obtain

$$\begin{aligned} |L(f)| &\leq \sum_{n=0}^{\infty} |\eta_n| \frac{|f|_{m+1}}{r_{m+1}^n} \\ &= |f|_{m+1} \sum_{n=0}^{\infty} \frac{r_m^n |\eta_n|}{r_m^n r_{m+1}^n} \\ &\leq q_m((\eta_n)_{n=0}^\infty)\, |f|_{m+1} \sum_{n=0}^{\infty} \frac{r_m^n}{r_{m+1}^n}. \end{aligned}$$

Hence $L \in \mathcal{H}'(D_R(0))$. □

Furthermore, we associate to each sequence $(\eta_n)_{n=0}^{\infty} \in \Lambda_R'$ a function F being holomorphic in a neighborhood of ∞, i.e. in a set $\{w \in \overline{\mathbb{C}} : |w| > \ell\}$, and with the property $F(\infty) = \lim_{z\to 0} F(1/z) = 0$. This is done in the following way: suppose that $\sup_n \frac{|\eta_n|}{r_m^n} < \infty$, then $\ell := \limsup_{n\to\infty} |\eta_n|^{1/n} \leq r_m < R$. Hence the function

$$F(w) = \sum_{n=0}^{\infty} \frac{\eta_n}{w^{n+1}},$$

is holomorphic in $\{w : |w| > \ell\}$ and satisfies $F(\infty) = 0$. We know from the last proof that the expression

$$L(f) = \sum_{n=0}^{\infty} \eta_n \frac{f^{(n)}(0)}{n!}, \ \ f \in \mathcal{H}(D_R(0))$$

represents an arbitrary continuous linear functional on $\mathcal{H}(D_R(0))$. Let $\ell < \rho < R$ and $\gamma_\rho(t) = \rho e^{it}$, $t \in [0, 2\pi]$. Then

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma_\rho} F(w)f(w)\,dw &= \frac{1}{2\pi i}\int_{\gamma_\rho} \sum_{n=0}^{\infty} \frac{\eta_n}{w^{n+1}} f(w)\,dw \\
&= \sum_{n=0}^{\infty} \eta_n \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(w)}{w^{n+1}}\,dw \\
&= \sum_{n=0}^{\infty} \eta_n \frac{f^{(n)}(0)}{n!} \\
&= L(f).
\end{aligned}$$

Given $L \in \mathcal{H}'(D_R(0))$, we obtain the corresponding holomorphic function F, representing L as before, by Cauchy's integral formula

$$L\left(w \mapsto \frac{1}{z-w}\right) = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{F(w)}{z-w}\,dw = F(z), \tag{2.5}$$

and

$$L\left(w \mapsto \frac{1}{(z-w)^{k+1}}\right) = \frac{F^{(k)}(z)}{k!}, \tag{2.6}$$

where $|z| > \rho$.

Let $H_0(R)$ be the space of all functions holomorphic in an open neighborhood of $\{z \in \overline{\mathbb{C}} : |z| \geq R\}$, such that $F(\infty) = 0$. We have just shown that the dual space $\mathcal{H}'(D_R(0))$ can be identified with $H_0(R)$, a space of holomorphic functions in a neighborhood of the complement of $D_R(0)$, which is known as the Köthe duality. This will now be used, together with the Hahn-Banach Theorem, to give a simple

proof of a Runge type approximation theorem; compare with the proof of Theorem **??** where a similar method was used.
For this purpose we have to explain the concept of a subset $W \subset \overline{\mathbb{C}}$ with multiplicity $m : W \longrightarrow \{1, 2, 3, \dots\} \cup \{\infty\}$. By a limit point of (W, m) we mean an ordinary limit point of W or a point $w \in W$ with $m(w) = \infty$. Given a set (W, m) with multiplicity, let $\mathcal{R}(W)$ denote the following collection of functions: if $w \in W$, $w \neq \infty$ and $m(w) < \infty$, then $z \mapsto 1/(z - w)$ belongs to $\mathcal{R}(W)$; if $w \neq \infty$ and $m(w) = \infty$, then the functions $z \mapsto 1/(z - w)^k$, for $k = 1, 2, \dots$ belong to $\mathcal{R}(W)$; if $\infty \in W$ and $m(\infty) = \infty$, then the functions $z \mapsto z^k$, for $k = 0, 1, 2, \dots$ belong to $\mathcal{R}(W)$.

Theorem 2.3. *If $W \subset \overline{\mathbb{C}} \setminus D_R(0)$ is a set with multiplicity which has a limit point in $\overline{\mathbb{C}} \setminus D_R(0)$, then the linear span of $\mathcal{R}(W)$ is dense in $\mathcal{H}(D_R(0))$.*

Proof. To show that the linear span of $\mathcal{R}(W)$ is dense in $\mathcal{H}(D_R(0))$, we take a continuous linear functional $L \in \mathcal{H}'(D_R(0))$ which vanishes on the linear span of $\mathcal{R}(W)$. Using Corollary **??**, we will be finish if we can show that L vanishes on $\mathcal{H}(D_R(0))$. The assumptions on $\mathcal{R}(W)$ imply that the holomorphic function F corresponding to L by (2.5) vanishes on a set with limit point or, using (2.6), has the property that $F^{(k)}(\zeta) = 0$, for $k = 0, 1, 2, \dots$ and some $\zeta \in \overline{\mathbb{C}} \setminus D_R(0)$. In both cases, the Identity Theorems **??** and **??** imply that $F \equiv 0$, and hence $L = 0$ on $\mathcal{H}(D_R(0))$. ☐

2.3 Exercises

140) Let $(K_m)_{m \in \mathbb{N}}$ be a compact exhaustion of the domain $\Omega \subseteq \mathbb{C}^n$ and let $|f|_m := \sup_{z \in K_m} |f(z)|$, for $f \in \mathcal{H}(\Omega)$. Show that

$$d(f, g) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{|f - g|_m}{1 + |f - g|_m}$$

defines a translation invariant metric on $\mathcal{H}(\Omega)$.

141) Show that the metric $d(.,.)$ generates the original topology of uniform convergence on all compact subsets of Ω.

142) Let X be a locally convex vector space and let U be an absolutely convex 0-neighborhood in X. Show that the Minkowski functional $\|x\|_U = \inf\{t > 0 \,:\, x \in tU\}$ is a continuous seminorm on X.

143) Let X be a locally convex vector space. A collection Λ of neighborhoods of a point $x \in X$ is called a local base at x if every neighborhood of x contains a member of Λ. A set $B \subset X$ is called balanced if $cB \subset B$ for every $c \in \mathbb{C}$ with $|c| \leq 1$. Show that X has a local base consisting of balanced convex sets.

144) Let X be a locally convex vector space and let U be a 0-neighborhhod in X. Let $(r_j)_j$ a strictly increasing sequence of positive numbers with $r_j \to \infty$ as $j \to \infty$. Show that

$$X = \bigcup_{j=1}^{\infty} r_j U.$$

145) Let X be a locally convex vector space. Show that every compact subset $K \subset X$ is bounded.
Hint: Choose a 0-neighborhood U and a balanced 0-neighborhood W such that $W \subset U$ and $K \subset \bigcup_{n=1}^{\infty} nW$.

146) Let $(s_j)_j$ be a strictly decreasing sequence of positive numbers such that $\lim_{j\to\infty} s_j = 0$ and V be a bounded subset of the locally convex vector space X. Show that the collection $\{s_j V : j \in \mathbb{N}\}$ is a local base for X.

147) Show that each finite dimensional subspace of a locally convex subspace is closed.

148) X is locally compact if 0 has a neighborhood whose closure is compact. Show that each locally compact locally convex vector space has finite dimension.
Hint: take a 0-neighborhood V whose closure is compact, since V is also bounded, the sets $2^{-n}V$, $n \in \mathbb{N}$ form a local base for X. The compactness of $\overline{V}$ shows that there exist $x_1, \dots, x_m \in X$ such that

$$\overline{V} \subset (x_1 + \frac{1}{2}V) \cup \cdots \cup (x_m + \frac{1}{2}V).$$

Let Y be the vector space spanned by $x_1, \dots, x_m$. Show that $Y = X$.

149) Let $\Omega \subseteq \mathbb{C}^n$ be a domain. Show that $\mathcal{H}(\Omega)$ endowed with the topology of uniform convergence on all compact subsets of Ω is not normable, i.e. has no bounded 0-neighborhood.
Hint: if U is a bounded 0-neighborhood, Montel's Theorem **??** implies that $\mathcal{H}(\Omega)$ is locally compact, now use Exercise 148.

150) Show that the system of seminorms

$$p_r(f) := \sup_{0 \le k < \infty} |a_k|\, r^k, \; r < R$$

where $f \in \mathcal{H}(D_R(0))$ has Taylor series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$, defines the original topology of uniform convergence on all compact subsets of $D_R(0)$.

151) Show that for each $0 < r < R$ there exists $0 < \rho < R$ and a constant C, depending on r, such that

$$\sum_{k=0}^{\infty} |a_k|\, r^k \le C \sup_{|z| \le \rho} |f(z)|,$$

for each $f \in \mathcal{H}(D_R(0))$ with Taylor series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$.

2.4 Notes

For a thorough discussion of locally convex vector spaces related to real and complex analysis, in particular of nuclear Fréchet spaces, the reader should consult [5] or [2]. The Köthe duality together with its applications is presented in [2]. Additional details and applications to different problems in complex analysis are given in [4].

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