Spectral analysis

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Chapter 1

Bounded operators

1.1 The spectrum of bounded operators

Let H_1 and H_2 be separable Hilbert spaces. $\mathcal{L}(H_1, H_2)$ denote the space of all bounded linear operators from H_1 to H_2 endowed with the topology generated by the operator norm

$$\|A\| = \sup\{\|Au\| \,:\, \|u\| \leq 1\}.$$

In this way $\mathcal{L}(H_1, H_2)$ becomes a Banach space. We write $\mathcal{L}(H)$ for $\mathcal{L}(H, H)$.

Definition 1.1. The spectrum $\sigma(T)$ of an operator $T \in \mathcal{L}(H)$ is the set of all $\lambda \in \mathbb{C}$, such that $\lambda I - T$ has no inverse in $\mathcal{L}(H)$. The complement $\rho(T) = \mathbb{C} \setminus \sigma(T)$ is called the resolvent set. If $\lambda \in \rho(T)$ the operator $(\lambda I - T)^{-1} \in \mathcal{L}(H)$ is called the resolvent of T at λ and is denoted by $R_T(\lambda)$. We have an operator-valued function

$$R_T : \rho(T) \longrightarrow \mathcal{L}(H).$$

An operator $T \in \mathcal{L}(H)$ is called normal if $TT^* = T^*T$.

Proposition 1.2. *Let $T \in \mathcal{L}(H)$. Then the spectrum $\sigma(T)$ of T is a compact subset of $\mathbb{C}$ and $|\lambda| \leq \|T\|$, for every $\lambda \in \sigma(T)$.*

Proof. First we show that $\rho(T)$ is open. Let $\lambda \in \rho(T)$. Then

$$\alpha = \|R_T(\lambda)\|^{-1} > 0.$$

Let $\mu \in \mathbb{C}$ with $|\mu| < \alpha$. We will show that $(\lambda + \mu)I - T$ has a bounded inverse. Then we proved that $\rho(T)$ is open. We have

$$\begin{aligned} (\lambda + \mu)I - T &= \lambda I - T + \mu I \\ &= (\lambda I - T)[I + \mu(\lambda I - T)^{-1}] \\ &= (\lambda I - T)(I + \mu R_T(\lambda)). \end{aligned}$$

Formally

$$(I + \mu R_T(\lambda))^{-1} = I + \sum_{k=1}^{\infty} (-1)^k (\mu R_T(\lambda))^k,$$

but as $|\mu| < \alpha$ we have

$$\| \sum_{k=1}^{\infty} (-1)^k (\mu R_T(\lambda))^k \| \leq \sum_{k=1}^{\infty} |\mu|^k \, \|R_T(\lambda)\|^k = \sum_{k=1}^{\infty} (|\mu|/\alpha)^k < \infty,$$

therefore the partial sums of $\sum_{k=1}^{\infty} (-1)^k (\mu R_T(\lambda))^k$ form a Cauchy sequence in $\mathcal{L}(H)$. Since $\mathcal{L}(H)$ is complete, we obtain that

$$\sum_{k=1}^{\infty} (-1)^k (\mu R_T(\lambda))^k \in \mathcal{L}(H),$$

and $\rho(T)$ is open.

If $\eta \in \mathbb{C}$ with $|\eta| > \|T\|$, then $I - T/\eta$ has a bounded inverse, since

$$(I - T/\eta)^{-1} = I + \sum_{k=1}^{\infty} (T/\eta)^k.$$

This implies that $\eta I - T$ has a bounded inverse. Hence, if $\lambda \in \sigma(T)$, then $|\lambda| \leq \|T\|$, and $\sigma(T)$ is a bounded set. □

The resolvent has the following properties:

Lemma 1.3. *If $\lambda, \mu \in \rho(T)$, then*

$$R_T(\lambda) - R_T(\mu) = (\mu - \lambda) R_T(\lambda) R_T(\mu). \tag{1.1}$$

If $\lambda \in \rho(T)$ and $|\lambda - \mu| < \|R_T(\lambda)\|^{-1}$, then

$$R_T(\mu) = \sum_{k=0}^{\infty} (\lambda - \mu)^k [R_T(\lambda)]^{k+1}, \tag{1.2}$$

therefore one says that R_T is a holomorphic operator valued function.

Proof. (1.1) follows from

$$\begin{aligned} R_T(\lambda) = R_T(\lambda)(\mu I - T) R_T(\mu) &= R_T(\lambda)[(\lambda I - T) + (\mu - \lambda) I] R_T(\mu) \\ &= R_T(\mu) + (\mu - \lambda) R_T(\lambda) R_T(\mu). \end{aligned}$$

(1.2) follows immediately from the proof of Proposition 1.2.

□

1.2 Compact operators

Let H_1 and H_2 be separable Hilbert spaces and $A : H_1 \longrightarrow H_2$ a bounded linear operator. The operator A is compact, if the image $A(B_1)$ of the unit ball B_1 in H_1 is a relatively compact subset of H_2, since H_2 is complete this is equivalent to the concept of a totally bounded set, i.e. for each $\epsilon > 0$ there exists a finite number of elements $v_1, \dots, v_m \in H_2$ such that

$$A(B_1) \subset \bigcup_{j=1}^{m} B(v_j, \epsilon),$$

where $B(v_j, \epsilon) = \{v \in H_2 : \|v - v_j\| < \epsilon\}$.

Another equivalent definition of compactness is : for each bounded sequence $(u_k)_k$ in H_1 the image sequence $(A(u_k))_k$ has a convergent subsequence in H_2.

Let $\mathcal{K}(H_1, H_2)$ denote the subspace of all compact operators from H_1 to H_2.

The following characterization of compactness is useful for the special operators in the text, see for instance [2]:

Proposition 1.4. *Let H_1 and H_2 be Hilbert spaces, and assume that $S\colon H_1 \to H_2$ is a bounded linear operator. The following three statements are equivalent:*

- *S is compact.*
- *For every $\epsilon > 0$ there is a $C = C_\epsilon > 0$ and a compact operator $T = T_\epsilon\colon H_1 \to H_2$ such that*
$$\|Sv\| \le C\,\|Tv\| + \epsilon\,\|v\|\,. \tag{1.3}$$
- *For every $\epsilon > 0$ there is a $C = C_\epsilon > 0$ and a compact operator $T = T_\epsilon\colon H_1 \to H_2$ such that*
$$\|Sv\|^2 \le C\,\|Tv\|^2 + \epsilon\,\|v\|^2\,. \tag{1.4}$$

Proposition 1.5. *$\mathcal{K}(H_1, H_2)$ is a closed subspace of $\mathcal{L}(H_1, H_2)$ endowed with the operator norm.*

Proof. Let $A \in \mathcal{L}(H_1, H_2)$. Suppose, for each $\epsilon > 0$, there is a compact operator A_ϵ such that $\|A - A_\epsilon\| \le \epsilon$. Then for each $u \in H_1$ we have

$$\|Au - A_\epsilon u\| \le \epsilon\|u\|.$$

Now we get

$$\begin{aligned} \|Au\| &= \|Au - A_\epsilon u + A_\epsilon u\| \\ &\le \|Au - A_\epsilon u\| + \|A_\epsilon u\| \\ &\le \epsilon\|u\| + \|A_\epsilon u\|. \end{aligned}$$

Proposition 1.4 implies that A is compact. $\square$

Proposition 1.6. *Suppose that $A \in \mathcal{K}(H_1, H_2)$, and that $S \in \mathcal{L}(H_1, H_1)$ and $T \in \mathcal{L}(H_2, H_2)$ is a bounded operator on H_2. Then both AS and TA are compact.*

Proof. If $(u_k)_k$ is a bounded sequence in H_1, then $(S(u_k))_k$ is also bounded, because S is a bounded operator. A is compact, so $(A(S(u_k)))_k$ has a convergent subsequence. Thus AS is compact.

To show that TA is compact we use Proposition 1.4:

$$\begin{aligned} \|TAu\| &\leq \|T\| \, \|Au\| \leq \|T\|(\epsilon\|u\| + C\|Au\|) \\ &\leq \epsilon \, \|T\| \, \|u\| + C \, \|T\| \, \|Au\|. \end{aligned}$$

□

Corollary 1.7. *Let H be a Hilbert space. $\mathcal{K}(H, H)$ forms a two-sided, closed ideal in $\mathcal{L}(H, H)$.*

Theorem 1.8. *Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator.*

The following properties are equivalent:

(i) A is compact;

(ii) the adjoint operator $A^* : H_2 \longrightarrow H_1$ is compact;

(iii) $A^*A : H_1 \longrightarrow H_1$ is compact.

The following characterization of compactness uses the uniform boundedness principle and the concept of weak convergence.

Definition 1.9. A sequence $(x_k)_k$ in a Hilbert space H is a weak null-sequence, if $(x_k, x) \to 0$ for each $x \in H$. A sequence $(x_k)_k$ converges weakly to x_0, if $(x_k - x_0)_k$ is a weak null-sequence.

Remark 1.10. A weakly convergent sequence $(x_k)_k$ in a Hilbert space is always bounded: we have

$$\sup_k |(x_k - x_0, x)| < \infty,$$

for all $x \in H$, then, by the uniform boundedness principle,

$$\sup_k \|x_k - x_0\| = \sup_k \sup_{\|x\| \leq 1} |(x_k - x_0, x)| < \infty$$

and therefore $\|x_k\| \leq \|x_k - x_0\| + \|x_0\| < \infty$, for all $k \in \mathbb{N}$.

In the same way we can show that each weak Cauchy sequence is bounded.

If $A \in \mathcal{L}(H_1, H_2)$ and $(x_k)_k$ is a weakly convergent sequence in H_1, then $(Ax_k)_k$ converges weakly in H_2, which follows from

$$(Ax_k - Ax_0, y)_2 = (x_k - x_0, A^*y)_1,$$

where $y \in H_2$.

Proposition 1.11. *Let $A \in \mathcal{L}(H_1, H_2)$ be a bounded operator between Hilbert spaces. A is compact if and only if $(Ax_k)_k$ converges to 0 in H_2 for each weak null-sequence $(x_k)_k$ in H_1.*

The following theorem is the spectral theorem for compact, self adjoint operators:

Theorem 1.12. *Let $A : H \longrightarrow H$ be a compact, self-adjoint operator on a separable Hilbert space H. Then there exists a real zero-sequence $(\mu_n)_n$ and an orthonormal system $(e_n)_n$ in H such that for $x \in H$*

$$Ax = \sum_{n=0}^{\infty} \mu_n (x, e_n) e_n,$$

where the sum converges in the operator norm, i.e.

$$\sup_{\|x\| \leq 1} \|Ax - \sum_{n=0}^{N} \mu_n (x, e_n) e_n\| \to 0,$$

as $N \to \infty$.

Proposition 1.13. *A bounded operator $A : H \longrightarrow H$ is compact if and only if there exists a sequence $(A_k)_k$ of linear operators with finite-dimensional range such that $\|A - A_k\| \to 0$ as $k \to \infty$.*

Proposition 1.14. *Let A be a compact self-adjoint operator on a separable Hilbert space H of infinite dimension. Let $(\lambda_k)_{k \geq 1}$ denote the eigenvalues of A. Then the spectrum $\sigma(A)$ of A has the form*

$$\sigma(A) = \{\lambda_k \,:\, k \in \mathbb{N}\} \cup \{0\}.$$

Proof. We have $\lambda_k \in \sigma(A)$, for all $k \in \mathbb{N}$; if $0 \notin \sigma(A)$, then A^{-1} exists and is continuous and $I = AA^{-1}$ is a compact operator. Hence the unit ball in H is compact and H must be of finite dimension, which is a contradiction.

Now let $\mu \neq \lambda_k, 0$, $\forall k$ and

$$Ax = \sum_{k=1}^{\infty} \lambda_k (x, x_k) x_k,$$

and $(y_j)_j$ the supplementation of $(x_k)_k$ to a complete orthonormal system of H. Then we have

$$Ix = \sum_{k=1}^{\infty} (x, x_k) x_k + \sum_{j} (x, y_j) y_j,$$

and the operator

$$Bx = \sum_{k=1}^{\infty} \frac{1}{\lambda_k - \mu} (x, x_k)x_k - \frac{1}{\mu} \sum_j (x, y_j)y_j$$

has norm $\|B\| = \sup\{1/|\lambda_k - \mu|, 1/|\mu|\}$, and is therefore a continuous operator. In addition, B is the inverse of $A - \mu I$, because for each k and j we have $B(A - \mu I)x_k = B(\lambda_k - \mu)x_k = x_k$ and $B(A - \mu I)y_j = -B\mu y_j = y_j$. Hence $B = (A - \mu I)^{-1}$ and $\mu \notin \sigma(A)$.

□

Now we drop the assumption of self-adjointness and obtain

Proposition 1.15. *Let $A : H_1 \longrightarrow H_2$ be a compact operator. There exists a decreasing zero-sequence $(s_n)_n$ in $\mathbb{R}^+$ and orthonormal systems $(e_n)_{n\geq 0}$ in H_1 and $(f_n)_{n\geq 0}$ in H_2, such that*

$$Ax = \sum_{n=0}^{\infty} s_n(x, e_n)f_n \ , \ \forall x \in H_1,$$

where the sum converges again in the operator norm.

Proof. In order to show this one applies the spectral theorem for the self-adjoint, compact operator $A^*A : H_1 \longrightarrow H_1$ and gets

$$A^*Ax = \sum_{n=0}^{\infty} s_n^2(x, e_n)e_n, \tag{1.5}$$

where s_n^2 are the eigenvalues of A^*A. If $s_n > 0$, we set $f_n = s_n^{-1}Ae_n$ and get

$$(f_n, f_m) = \frac{1}{s_n s_m}(Ae_n, Ae_m) = \frac{1}{s_n s_m}(A^*Ae_n, e_m) = \frac{s_n^2}{s_n s_m}(e_n, e_m) = \delta_{n,m}.$$

For $y \in H_1$ with $y \perp e_n$ for each $n \in \mathbb{N}_0$ we have by (1.5) that

$$\|Ay\|^2 = (Ay, Ay) = (A^*Ay, y) = 0.$$

Hence we have

$$Ax = A\left(x - \sum_{n=0}^{\infty}(x, e_n)e_n\right) + A\left(\sum_{n=0}^{\infty}(x, e_n)e_n\right)$$

$$= \sum_{n=0}^{\infty}(x, e_n)Ae_n = \sum_{n=0}^{\infty} s_n(x, e_n)f_n.$$

Similar, as in the last theorem, we get

$$\|Ax - \sum_{k=1}^{n} s_k(x, e_k) f_k\|^2 = \sum_{k=n+1}^{\infty} |s_k(x, e_k)|^2 \leq (\|x\| \sup_{k>n} |s_k|)^2,$$

which implies that the series converges in the operator norm. □

The numbers s_n are called the s-numbers of A. They are uniquely determined by the operator A, since they are the square roots of the eigenvalues of A^*A.

Let $0 < p < \infty$. The operator A belongs to the Schatten-class $\mathbf{S}_p$, if its sequence $(s_n)_n$ of s-numbers belongs to l^p. The elements of the Schatten class $\mathbf{S}_2$ are called Hilbert-Schmidt operators.

Proposition 1.16. *Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator between Hilbert spaces. The following conditions are equivalent:*
(i) there is an orthonormal basis $(e_i)_{i \in I}$ of H_1, such that $\sum_{i \in I} \|Ae_i\|^2 < \infty$;
(ii) for each orthonormal basis $(f_j)_{j \in J}$ of H_1 one has $\sum_{j \in J} \|Af_j\|^2 < \infty$;
(iii) A is a Hilbert-Schmidt operator.

1.3 Bergman spaces and $\overline{\partial}$

In the following we describe the spectral representation of a compact operator closely related to the Cauchy-Riemann equations. To investigate the solution to the inhomogeneous $\overline{\partial}$-equation $\overline{\partial} u = g$, we will first consider the case where the right hand side g is a holomorphic function. Therefore we need an appropriate Hilbert space of holomorphic functions - the Bergman space. We will use standard basic facts about Hilbert spaces, such as the Riesz representation theorem for continuous linear functionals, facts about orthogonal projections, and complete orthonormal basis.

Let $\Omega \subseteq \mathbb{C}^n$ be a domain and the Bergman space

$$A^2(\Omega) = \{f : \Omega \longrightarrow \mathbb{C} \text{ holomorphic } : \|f\|^2 = \int_\Omega |f(z)|^2 \, d\lambda(z) < \infty\},$$

where λ is the Lebesgue measure of $\mathbb{C}^n$. The inner product is given by

$$(f, g) = \int_\Omega f(z) \, \overline{g(z)} \, d\lambda(z),$$

for $f, g \in A^2(\Omega)$.

For sake of simplicity we first restrict to domains $\Omega \subseteq \mathbb{C}$. We consider special continuous linear functionals on $A^2(\Omega)$: the point evaluations. Let $f \in A^2(\Omega)$ and fix $z \in \Omega$. By Cauchy's integral theorem we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

where $\gamma_s(t) = z + se^{it},\ t \in [0, 2\pi], 0 < s \le r$ and $D(z,r) = \{w\ :\ |w-z| < r\} \subset \Omega$. Using polar coordinates and integrating the above equality with respect to s between 0 and r we get

$$f(z) = \frac{1}{\pi r^2} \int_{D(z,r)} f(w)\, d\lambda(w). \tag{1.6}$$

Then, by Cauchy-Schwarz,

$$\begin{aligned}
|f(z)| &\le \frac{1}{\pi r^2} \int_{D(z,r)} 1\ .\ |f(w)|\, d\lambda(w) \\
&\le \frac{1}{\pi r^2} \left(\int_{D(z,r)} 1^2\, d\lambda(w) \right)^{1/2} \left(\int_{D(z,r)} |f(w)|^2\, d\lambda(w) \right)^{1/2} \\
&\le \frac{1}{\pi^{1/2} r} \left(\int_{\Omega} |f(w)|^2\, d\lambda(w) \right)^{1/2} \\
&\le \frac{1}{\pi^{1/2} r} \|f\|.
\end{aligned}$$

If K is a compact subset of Ω, there is an $r(K) > 0$ such that for any $z \in K$ we have $D(z, r(K)) \subset \Omega$ and we get

$$\sup_{z \in K} |f(z)| \le \frac{1}{\pi^{1/2} r(K)} \|f\|.$$

If $K \subset \Omega \subset \mathbb{C}^n$ we can find a polycylinder

$$P(z, r(K)) = \{w \in \mathbb{C}^n\ :\ |w_j - z_j| < r(K),\ j = 1, \ldots, n\}$$

such that for any $z \in K$ we have $P(z, r(K)) \subset \Omega$. Hence by iterating the above Cauchy integrals we get

Proposition 1.17. *Let $K \subset \Omega$ be a compact set. Then there exists a constant $C(K)$, only depending on K such that*

$$\sup_{z \in K} |f(z)| \le C(K)\, \|f\|, \tag{1.7}$$

for any $f \in A^2(\Omega)$.

Proposition 1.18. *$A^2(\Omega)$ is a Hilbert space.*

Proof. If $(f_k)_k$ is a Cauchy sequence in $A^2(\Omega)$, by (1.7), it is also a Cauchy sequence with respect to uniform convergence on compact subsets of Ω. Hence

The sequence $(f_k)_k$ has a holomorphic limit f with respect to uniform convergence on compact subsets of Ω. On the other hand, the original L^2-Cauchy sequence has a subsequence, which converges pointwise almost everywhere to the L^2-limit of the original L^2-Cauchy sequence (see for instance [8]), and so the L^2-limit coincides with the holomorphic function f. Therefore $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ and itself a Hilbert space. $\square$

For fixed $z \in \Omega$, (1.7) also implies that the point evaluation $f \mapsto f(z)$ is a continuous linear functional on $A^2(\Omega)$, hence, by the Riesz representation theorem, there is a uniquely determined function $k_z \in A^2(\Omega)$ such that

$$f(z) = (f, k_z) = \int_\Omega f(w)\,\overline{k_z(w)}\,d\lambda(w). \tag{1.8}$$

We set $K(z,w) = \overline{k_z(w)}$. Then $w \mapsto \overline{K(z,w)} = k_z(w)$ is an element of $A^2(\Omega)$, hence the function $w \mapsto K(z,w)$ is anti-holomorphic on Ω and we have

$$f(z) = \int_\Omega K(z,w) f(w)\,d\lambda(w)\ ,\ f \in A^2(\Omega).$$

The function of two complex variables $(z,w) \mapsto K(z,w)$ is called Bergman kernel of Ω and the above identity represents the reproducing property of the Bergman kernel.

Now we use the reproducing property for the holomorphic function $z \mapsto k_u(z)$, where $u \in \Omega$ is fixed:

$$\begin{aligned} k_u(z) &= \int_\Omega K(z,w) k_u(w)\,d\lambda(w) = \int_\Omega \overline{k_z(w)}\,\overline{K(u,w)}\,d\lambda(w) \\ &= \left(\int_\Omega K(u,w) k_z(w)\,d\lambda(w) \right)^{-} = \overline{k_z(u)}, \end{aligned}$$

hence we have $k_u(z) = \overline{k_z(u)}$, or $K(z,u) = \overline{K(u,z)}$.

It follows that the Bergman kernel is holomorphic in the first variable and anti-holomorphic in the second variable.

Proposition 1.19. *The Bergman kernel is uniquely determined by the properties that it is an element of $A^2(\Omega)$ in z and that it is conjugate symmetric and reproduces $A^2(\Omega)$.*

Proposition 1.20. *Let $K \subset \Omega$ be a compact subset and $\{\phi_j\}$ be an orthonormal basis of $A^2(\Omega)$. Then the series*

$$\sum_{j=1}^{\infty} \phi_j(z)\,\overline{\phi_j(w)}$$

sums uniformly on $K \times K$ to the Bergman kernel $K(z,w)$.

Now let $\phi \in L^2(\Omega)$. Since $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ there exists a uniquely determined orthogonal projection $P : L^2(\Omega) \longrightarrow A^2(\Omega)$.

For the function $P\phi \in A^2(\Omega)$ we use the reproducing property and obtain

$$P\phi(z) = \int_\Omega K(z,w)P\phi(w)\,d\lambda(w) = (P\phi, k_z) = (\phi, Pk_z) = (\phi, k_z); \tag{1.9}$$

where we still have used that P is a self-adjoint operator and that $Pk_z = k_z$. Hence

$$P\phi(z) = \int_\Omega K(z,w)\phi(w)\,d\lambda(w). \tag{1.10}$$

P is called the Bergman projection.

Example. The functions $\phi_n(z) = \sqrt{\frac{n+1}{\pi}}\, z^n$, $n = 0, 1, 2, \dots$ constitute an orthonormal basis in $A^2(\mathbb{D})$, $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$.

This follows from

$$\int_{\mathbb{D}} z^n\, \overline{z^m}\, d\lambda(z) = \int_0^{2\pi}\int_0^1 r^n e^{in\theta}\, r^m e^{-im\theta}\, r\, dr\, d\theta = \frac{2\pi}{n+m+2}\,\delta_{n,m}.$$

For each $f \in A^2(\mathbb{D})$ with Taylor series expansion $f(z) = \sum_{n=0}^\infty a_n z^n$ we get

$$(f, z^n) = \int_{\mathbb{D}} f(z)\overline{z^n}\, d\lambda(z) = \int_0^1\int_0^{2\pi} f(re^{i\theta}) r^n e^{-in\theta} r\, dr\, d\theta$$

$$= \int_0^1\int_0^{2\pi} \frac{f(re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}} re^{i\theta}\, d\theta\, r^{2n+1}\, dr = 2\pi a_n \int_0^1 r^{2n+1}\, dr = \pi\,\frac{a_n}{n+1},$$

where we used the fact that

$$a_n = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z^{n+1}}\, dz,$$

for $\gamma_r(\theta) = re^{i\theta}$. Hence, by the uniqueness of the Taylor series expansion, we obtain that $(f, \phi_n) = 0$, for each $n = 0, 1, 2, \dots$ implies $f \equiv 0$. This means that $(\phi_n)_{n=0}^\infty$ constitutes an orthonormal basis for $A^2(\mathbb{D})$ and we get

$$\|f\|^2 = \sum_{n=0}^\infty |(f, \phi_n)|^2,$$

which is equivalent to

$$\|f\|^2 = \pi \sum_{n=0}^\infty \frac{|a_n|^2}{n+1}\ ,\ f(z) = \sum_{n=0}^\infty a_n z^n.$$

Hence each $f \in A^2(\mathbb{D})$ can be written in the form $f = \sum_{n=0}^\infty c_n\, \phi_n$, where the sum converges in $A^2(\mathbb{D})$, but also uniformly on compact subsets of $\mathbb{D}$. For the coefficients c_n we have : $c_n = (f, \phi_n)$.

Now we compute an explicit formula for the Bergman kernel $K(z,w)$ of $\mathbb{D}$. The function $z \mapsto K(z,w)$, with $w \in \mathbb{D}$ fixed, belongs to $A^2(\mathbb{D})$. Hence we get from the above formula that

$$K(z,w) = \sum_{n=0}^{\infty} c_n \, \phi_n(z),$$

where $c_n = (K(.,w), \phi_n)$, in other words

$$\overline{c_n} = (\phi_n, K(.,w)) = \int_{\mathbb{D}} \phi_n(z) K(w,z) \, d\lambda(z) = \phi_n(w),$$

by the reproducing property of the Bergman kernel. This implies that the Bergman kernel is of the form

$$K(z,w) = \sum_{n=0}^{\infty} \phi_n(z) \, \overline{\phi_n(w)}, \tag{1.11}$$

where the sum converges uniformly in z on all compact subsets of $\mathbb{D}$. (This is true for any complete orthonormal system, as is shown above.) A simple computation now gives

$$K(z,w) = \sum_{n=0}^{\infty} \phi_n(z) \, \overline{\phi_n(w)} = \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1)(z\overline{w})^n = \frac{1}{\pi} \frac{1}{(1-z\overline{w})^2}. \tag{1.12}$$

Hence for each $f \in A^2(\mathbb{D})$ we have

$$f(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{1}{(1-z\overline{w})^2} \, f(w) \, d\lambda(w).$$

If we fix $z \in \mathbb{D}$ and set $f(w) = 1/(1-w\overline{z})^2$, then we get the interesting formula

$$\frac{1}{\pi} \int_{\mathbb{D}} \frac{1}{|1-z\overline{w}|^4} \, d\lambda(w) = \frac{1}{(1-|z|^2)^2}.$$

We will use properties of the Bergman kernel to solve the inhomogeneous Cauchy-Riemann equation

$$\frac{\partial u}{\partial \overline{z}} = g \quad \text{or} \quad \overline{\partial} u = g,$$

where

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \ z = x + iy \tag{1.13}$$

and $g \in A^2(\mathbb{D})$.

Let

$$S(g)(z) = \int_{\mathbb{D}} K(z,w)g(w)(z-w)^{-} d\lambda(w). \tag{1.14}$$

Using the Bergman projection

$$P : L^2(\mathbb{D}) \longrightarrow A^2(\mathbb{D})$$

we get

$$S(g)(z) = \overline{z}g(z) - P(\tilde{g})(z),$$

where $\tilde{g}(w) = \overline{w}g(w)$. We claim that $S(g)$ is a solution of the inhomogeneous Cauchy-Riemann equation:

$$\frac{\partial}{\partial \overline{z}} S(g)(z) = \frac{\partial \overline{z}}{\partial \overline{z}} g(z) + \overline{z}\,\frac{\partial g}{\partial \overline{z}} + \frac{\partial P(\tilde{g})}{\partial \overline{z}} = g(z),$$

because g and $P(\tilde{g})$ are holomorphic functions, therefore $\overline{\partial} S(g) = g$. In addition we have $S(g) \perp A^2(\mathbb{D})$, because for arbitrary $f \in A^2(\mathbb{D})$ we get

$$(Sg, f) = (\tilde{g} - P(\tilde{g}), f) = (\tilde{g}, f) - (P(\tilde{g}), f) = (\tilde{g}, f) - (\tilde{g}, Pf) = (\tilde{g}, f) - (\tilde{g}, f) = 0.$$

The operator $S : A^2(\mathbb{D}) \longrightarrow L^2(\mathbb{D})$ is called the canonical solution operator to $\overline{\partial}$.

Now we want to show that S is a compact operator. For this purpose we consider the adjoint operator S^* and prove that S^*S is compact, which implies that S is compact (Theorem 1.8).

For $g \in A^2(\mathbb{D})$ and $f \in L^2(\mathbb{D})$ we have

$$(Sg, f) = \int_{\mathbb{D}} \left(\int_{\mathbb{D}} K(z,w)g(w)(z-w)^{-}\, d\lambda(w) \right) \overline{f(z)}\, d\lambda(z)$$

$$= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} K(w,z)(z-w)f(z)\, d\lambda(z) \right)^{-} g(w)\, d\lambda(w) = (g, S^*f),$$

hence

$$S^*(f)(w) = \int_{\mathbb{D}} K(w,z)(z-w)f(z)\, d\lambda(z). \tag{1.15}$$

Now set

$$c_n^2 = \int_{\mathbb{D}} |z|^{2n}\, d\lambda(z) = \frac{\pi}{n+1},$$

and $\phi_n(z) = z^n/c_n$, $n \in \mathbb{N}_0$, then the Bergman kernel $K(z,w)$ can be expressed in the form

$$K(z,w) = \sum_{k=0}^{\infty} \frac{z^k \overline{w}^k}{c_k^2}.$$

Next we compute

$$P(\tilde{\phi}_n)(z) = \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{z^k \overline{w}^k}{c_k^2}\, \overline{w}\, \frac{w^n}{c_n}\, d\lambda(w) = \sum_{k=1}^{\infty} \frac{z^{k-1}}{c_{k-1}^2} \int_{\mathbb{D}} \frac{\overline{w}^k w^n}{c_n}\, d\lambda(w) = \frac{c_n z^{n-1}}{c_{n-1}^2},$$

hence we have

$$S(\phi_n)(z) = \overline{z}\, \phi_n(z) - \frac{c_n z^{n-1}}{c_{n-1}^2} \quad ,\ n \in \mathbb{N}.$$

Now we apply S^* and get

$$S^* S(\phi_n)(w) = \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} (z - w) \left(\frac{\overline{z} z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2} \right) d\lambda(z).$$

The last integral is computed in two steps: first the multiplication by z

$$\int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} \left(\frac{\overline{z} z^{n+1}}{c_n} - \frac{c_n z^n}{c_{n-1}^2} \right) d\lambda(z)$$

$$\begin{aligned}
&= \int_{\mathbb{D}} \frac{z^{n+1}}{c_n} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^{k+1}}{c_k^2}\, d\lambda(z) - \frac{c_n}{c_{n-1}^2} \int_{\mathbb{D}} z^n \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2}\, d\lambda(z) \\
&= \frac{w^n}{c_n^3} \int_{\mathbb{D}} |z|^{2n+2}\, d\lambda(z) - \frac{w^n}{c_{n-1}^2 c_n} \int_{\mathbb{D}} |z|^{2n}\, d\lambda(z) \\
&= \left(\frac{c_{n+1}^2}{c_n^3} - \frac{c_n}{c_{n-1}^2} \right) w^n.
\end{aligned}$$

Next the multiplication by w

$$w \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} \left(\frac{\overline{z} z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2} \right) d\lambda(z)$$

$$\begin{aligned}
&= w \int_{\mathbb{D}} \frac{z^n}{c_n} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^{k+1}}{c_k^2}\, d\lambda(z) - w \int_{\mathbb{D}} \frac{c_n z^{n-1}}{c_{n-1}^2} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2}\, d\lambda(z) \\
&= w \left(\frac{c_n w^{n-1}}{c_{n-1}^2} - \frac{c_n w^{n-1}}{c_{n-1}^2} \right) \\
&= 0,
\end{aligned}$$

it follows that

$$S^* S(\phi_n)(w) = \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} \right) \phi_n(w) \quad , n = 1, 2, \ldots,$$

for $n = 0$ an analogous computation shows

$$S^*S(\phi_0)(w) = \frac{c_1^2}{c_0^2}\,\phi_0(w).$$

Finally we get

Proposition 1.21. *Let $S : A^2(\mathbb{D}) \longrightarrow L^2(\mathbb{D})$ be the canonical solution operator for $\overline{\partial}$ and $(\phi_k)_k$ the normalized monomials. Then*

$$S^*S\phi = \frac{c_1^2}{c_0^2}\,(\phi,\phi_0)\phi_0 + \sum_{n=1}^{\infty}\left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}\right)(\phi,\phi_n)\phi_n \tag{1.16}$$

for each $\phi \in A^2(\mathbb{D})$.

Since

$$\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} = \frac{1}{(n+2)(n+1)} \to 0 \ \text{ as } n \to \infty,$$

it follows that S^*S is compact and S too.

We have also shown that the s-numbers of S are $\left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}\right)^{1/2}$ and since

$$\sum_{n=1}^{\infty}\left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}\right) < \infty$$

it follows that S is Hilbert-Schmidt.

1.4 Resolutions of the identity

We start with some definitions and basic facts.

Definition 1.22. Let Ω be a subset of $\mathbb{C}$ and $\mathbf{M}$ be a σ-algebra in Ω and let H be a Hilbert space. A resolution of the identity is a mapping

$$E : \mathbf{M} \longrightarrow \mathcal{L}(H)$$

of $\mathbf{M}$ to the algebra $\mathcal{L}(H)$ of bounded linear operators on H with the following properties

(a) $E(\emptyset) = 0$ (zero-operator), $E(\Omega) = I$ (identity on H).
(b) For each $\omega \in \mathbf{M}$ the image $E(\omega)$ is an orthogonal projection on H.
(c) $E(\omega' \cap \omega'') = E(\omega')E(\omega'')$.
(d) If $\omega' \cap \omega'' = \emptyset$, then $E(\omega' \cup \omega'') = E(\omega') + E(\omega'')$.
(e) For every $x \in H$ and $y \in H$, the set function $E_{x,y}$ defined by

$$E_{x,y}(\omega) = (E(\omega)x, y)$$

is a complex measure on $\mathbf{M}$.

We collect some immediate consequences of these properties.

Since each $E(\omega)$ is an orthogonal (i.e. self-adjoint) projection, we have for $x \in H$

$$E_{x,x}(\omega) = (E(\omega)x, x) = (E(\omega)^2 x, x) = (E(\omega)x, E(\omega)x) = \|E(\omega)x\|^2, \quad (1.17)$$

hence each $E_{x,x}$ is a positive measure on $\mathbf{M}$ with total variation

$$E_{x,x}(\Omega) = \|x\|^2. \quad (1.18)$$

By (c), any two of the projections $E(\omega)$ commute with each other; if $\omega \cap \omega' = \emptyset$, (a) and (c) show that $\operatorname{im}(E(\omega)) \perp \operatorname{im}(E(\omega'))$, which follows form

$$(E(\omega)x, E(\omega')y) = (E(\omega)^2 x, E(\omega')y) = (E(\omega)x, E(\omega)E(\omega')y) = 0.$$

By (d), E is finitely additive. Concerning countable additivity we have the following result

Proposition 1.23. *If E is a resolution of the identity, and if $x \in H$, then*

$$\omega \mapsto E(\omega)x$$

is a countably additive H-valued measure on $\mathbf{M}$.

If $\omega_n \in \mathbf{M}$ and $E(\omega_n) = 0$ for $n \in \mathbb{N}$, and if $\omega = \bigcup_{n=1}^{\infty} \omega_n$, then $E(\omega) = 0$.

Proof. By (d), $\omega \mapsto (E(\omega)x, y)$ is a complex measure, hence

$$\sum_{n=1}^{\infty} (E(\omega_n)x, y) = (E(\omega)x, y), \quad (1.19)$$

for every $y \in H$.

For $n \neq m$ we have $E(\omega_n)x \perp E(\omega_m)x$. Let

$$\Lambda_N(y) = \sum_{n=1}^{N} (y, E(\omega_n)x).$$

By (1.18), the sequence $(\Lambda_N(y))_N$ converges for every $y \in H$. The uniform boundedness principle implies that $(\|\Lambda_N\|)_N$ is bounded, where

$$\|\Lambda_N\| = \|E(\omega_1)x + \cdots + E(\omega_N)x\| = (\|E(\omega_1)x\|^2 + \cdots + \|E(\omega_N)x\|^2)^{1/2},$$

hence, using the orthogonality, the partial sums

$$\sum_{n=1}^{N} E(\omega_n)x$$

form a Cauchy sequence in H, so

$$\sum_{n=1}^{\infty} E(\omega_n)x = E(\omega)x$$

and $\omega \mapsto E(\omega)x$ is countably additive and therefore a complex measure on $\mathbf{M}$.

For the second claim, observe that $E(\omega_n) = 0$ implies $E_{x,x}(\omega_n) = 0$ for every $x \in H$. Since $E_{x,x}$ is countably additive, it follows that $E_{x,x}(\omega) = 0$. But $\|E(\omega)x\|^2 = E_{x,x}(\omega)$. Hence $E(\omega) = 0$. □

Definition 1.24. Let E be a resolution of the identity on $\mathbf{M}$ and let f be a complex $\mathbf{M}$-measurable function on $\mathbf{\Omega}$. There is a countable family $(D_k)_k$ of open discs forming a base for the topology of $\mathbb{C}$. Let V be the union of those D_k for which $E(f^{-1}(D_k)) = 0$. By Proposition 1.23, $E(f^{-1}(V)) = 0$. Also, V is the largest open subset of $\mathbb{C}$ with this property. The essential range of f is, by definition, the complement of V. It is the smallest closed subset of $\mathbb{C}$ that contains $f(z)$ for all $z \in \mathbf{\Omega}$ except those that lie in some set $\omega \in \mathbf{M}$ with $E(\omega) = 0$. We say f is essentially bounded if its essential range is bounded, hence compact. The largest value of $|\lambda|$, as λ runs through the essential range of f, is called the essential supremum $\|f\|_\infty$ of f.

Let B be the algebra of all bounded complex $\mathbf{M}$-measurable functions on $\mathbf{\Omega}$ with the norm

$$\|f\| = \sup_{z \in \mathbf{\Omega}} |f(z)|,$$

and let

$$N = \{f \in B : \|f\|_\infty = 0\},$$

which, by Propositon 1.23, is a closed ideal. Hence B/N is a Banach algebra, which is denoted by $L^\infty(E)$. The norm of a coset $[f] = f + N$ is $\|f\|_\infty$, and the spectrum $\sigma([f])$ is the essential range of f, the spectrum of an element g in a Banach algebra is the set of all complex numbers λ such that $\lambda e - g$ is not invertible.

In the next step we describe that a resolution of the identity induces an isometric isomorphism of the Banach algebra $L^\infty(E)$ onto a closed normal subalgebra $\mathcal{A}$ of $\mathcal{L}(H)$, the algebra of all bounded linear operators from H to H, a normal subalgebra is a commutative one which contains T^* for every $T \in \mathcal{A}$.

For this purpose let $\{\omega_1, \ldots, \omega_n\}$ be a partition of $\mathbf{\Omega}$, with $\omega_j \in \mathbf{M}$ and let s be a simple function, such that $s = \alpha_j$ on ω_j. Define $\Psi(s) \in \mathcal{L}(H)$ by

$$\Psi(s) = \sum_{j=1}^{n} \alpha_j E(\omega_j).$$

Since each $E(\omega_j)$ is self-adjoint, $\Psi(s)^* = \Psi(\overline{s})$. If t is another simple function and $\alpha, \beta \in \mathbb{C}$, we have

$$\Psi(s)\Psi(t) = \Psi(st) \text{ and } \Psi(\alpha s + \beta t) = \alpha\Psi(s) + \beta\Psi(t).$$

For $x, y \in H$ we get

$$(\Psi(s)x, y) = \sum_{j=1}^{n} \alpha_j (E(\omega_j)x, y) = \sum_{j=1}^{n} \alpha_j E_{x,y}(\omega_j) = \int_\Omega s\, dE_{x,y}.$$

In addition we have

$$\Psi(s)^*\Psi(s) = \Psi(|s|^2) \text{ and } \|\Psi(s)x\|^2 = \int_\Omega |s|^2\, dE_{x,x}.$$

By (1.18) this implies

$$\|\Psi(s)x\| \le \|s\|_\infty \|x\|, \tag{1.20}$$

and if $x \in \operatorname{im}(E(\omega_j))$, then

$$\Psi(s)x = \alpha_j E(\omega_j)x = \alpha_j x,$$

since the projections $E(\omega_j)$ have mutually orthogonal ranges. If j is chosen so that $|\alpha_j| = \|s\|_\infty$ it follows by (1.20) that

$$\|\Psi(s)\| = \sup_{\|x\|\le 1} \|\Psi(s)x\| = \|s\|_\infty. \tag{1.21}$$

Now suppose that $f \in L^\infty(E)$. There is a sequence of simple measurable functions s_k that converges to f in the norm of $L^\infty(E)$. By (1.21), the corresponding operators $\Psi(s_k)$ form a Cauchy sequence in $\mathcal{L}(H)$, which is therefore norm-convergent to an operator that we call $\Psi(f)$. By (1.21), we get

$$\|\Psi(f)\| = \|f\|_\infty. \tag{1.22}$$

Thus Ψ is an isometric isomorphism of $L^\infty(E)$ into $\mathcal{L}(H)$. Since $L^\infty(E)$ is complete, $\mathcal{A} = \Psi(L^\infty(E))$ is closed in $\mathcal{L}(H)$. In addition we have

$$(\Psi(f)x, y) = \int_\Omega f\, dE_{x,y} \quad \text{and} \quad \|\Psi(f)x\|^2 = \int_\Omega |f|^2\, dE_{x,x},$$

which justifies the notation

$$\Psi(f) = \int_\Omega f\, dE.$$

The spectral theorem indicates that every bounded normal operator T on a Hilbert space induces a resolution E of the identity on the Borel subsets of its spectrum $\sigma(T)$ and that T can be reconstructed from E by an integral of the type discussed before.

Using Banach algebra techniques such as the Gelfand transform (see [9]) one obtains the spectral decomposition for a single normal operator

Proposition 1.25. *If $T \in \mathcal{L}(H)$ and T is normal, then there exists a uniquely determined resolution of the identity E on the Borel subsets of the spectrum $\sigma(T)$ which satisfies*

$$T = \int_{\sigma(T)} \lambda \, dE(\lambda) \text{ and } (Tx, y) = \int_{\sigma(T)} \lambda \, dE_{x,y}(\lambda). \tag{1.23}$$

Furthermore, every projection $E(\omega)$ commutes with every $S \in \mathcal{L}(H)$ which commutes with T.

We shall refer to this E as the spectral decomposition of T.

We list a few consequences of the spectral decomposition.

If $\omega \subseteq \sigma(T)$ is a nonempty open set, then $E(\omega) \neq 0$.

If f is a bounded Borel function on $\sigma(T)$, it is customary to denote the operator

$$\Psi(f) = \int_{\sigma(T)} f \, dE$$

by $f(T)$.

The mapping $f \mapsto f(T)$ establishes a homomorphism of the algebra of all bounded Borel functions on $\sigma(T)$ into $\mathcal{L}(H)$, which carries the function 1 to I and the identity function on $\sigma(T)$ to T, and satisfies

$$\overline{f}(T) = f(T)^* \quad \text{and} \quad \|f(T)\| \leq \sup\{|f(\lambda)| : \lambda \in \sigma(T)\}.$$

The procedure explained above is also called symbolic calculus.

If $f \in \mathcal{C}(\sigma(T))$, then $f \mapsto f(T)$ is an isomorphism on $\mathcal{C}(\sigma(T))$ satisfying

$$\|f(T)x\|^2 = \int_{\sigma(T)} |f|^2 \, dE_{x,x}.$$

The eigenvalues of a normal operator can be characterized in terms of the spectral decomposition. For this purpose we mention the following applications of the symbolic calculus.

Proposition 1.26. *Let $T \in \mathcal{L}(H)$ be a normal operator and E its spectral decomposition. If $f \in \mathcal{C}(\sigma(T))$ and $\omega_0 = f^{-1}(0)$, then*

$$\mathit{ker}(f(T)) = \mathit{im}(E(\omega_0)).$$

Proof. We set $h(\lambda) = 1$ on ω_0 and $h(\lambda) = 0$ on $\tilde{\omega} = \sigma(T) \setminus \omega_0$. Then $fh = 0$ and by the symbolic calculus $f(T)h(T) = 0$. Since $h(T) = E(\omega_0)$, it follows that

$$\operatorname{im}(E(\omega_0)) \subseteq \ker(f(T)).$$

For the opposite inclusion we define for $n \in \mathbb{N}$ the set

$$\omega_n = \{\lambda \in \sigma(T) : 1/n \leq |f(\lambda)| < 1/(n-1)\}.$$

Then $\tilde{\omega}$ is the union of the disjoint Borel sets ω_n. We define $f_n(\lambda) = 1/f(\lambda)$ on ω_n and $f_n(\lambda) = 0$ on $\sigma(T) \setminus \omega_n$. Then each f_n is a bounded Borel function on $\sigma(T)$ and

$$f_n(T)f(T) = E(\omega_n), \ n \in \mathbb{N}.$$

If $f(T)x = 0$, it follows that $E(\omega_n)x = 0$. Since the mapping $\omega \mapsto E(\omega)x$ is countably additive (Proposition 1.23), we obtain $E(\tilde{\omega})x = 0$. But we also have that

$$E(\tilde{\omega}) + E(\omega_0) = I.$$

Hence $E(\omega_0)x = x$ and therefore

$$\ker(f(T)) \subseteq \mathrm{im}(E(\omega_0)).$$

□

Proposition 1.27. *Let $T \in \mathcal{L}(H)$ be a normal operator and E its spectral decomposition. Let $\lambda_0 \in \sigma(T)$ and $E_0 = E(\{\lambda_0\})$. Then*

(a) $\ker(T - \lambda_0 I) = \mathrm{im}(E_0)$,

(b) λ_0 is an eigenvalue of T if and only if $E_0 \neq 0$,

(c) every isolated point of $\sigma(T)$ is an eigenvalue of T,

(d) if $\sigma(T) = \{\lambda_1, \lambda_2, \dots\}$ is a countable set, then every $x \in H$ has a unique expansion of the form

$$x = \sum_{j=1}^{\infty} x_j,$$

where $Tx_j = \lambda_j x_j$, and $x_j \perp x_k$ for $j \neq k$.

(e) If $\sigma(T)$ has no limit point except possibly 0 and if $\dim \ker(T - \lambda I) < \infty$, for $\lambda \neq 0$, then T is compact (compare with Section 2.1).

Proof. (a) Is an immediate consequence of Proposition 1.26 with $f(\lambda) = \lambda - \lambda_0$.

(b) follows from (a).

(c) If λ_0 is an isolated point of $\sigma(T)$, then $\{\lambda_0\}$ is a nonempty open subset of $\sigma(T)$ and, by the properties of the spectral decomposition listed above, we get $E_0 \neq 0$, therefore (c) follows from (b).

(d) Let $E_j = E(\{\lambda_j\})$ for $j \in \mathbb{N}$. The projections E_j have pairwise orthogonal ranges and the mapping $\omega \mapsto E(\omega)x$ is countably additive (Proposition 1.23), hence for each $x \in H$ we have

$$\sum_{j=1}^{\infty} E_j x = E(\sigma(T))x = x,$$

and the series converges in the norm of H. Now set $x_j = E_j x$ and observe that uniqueness follows from the orthogonality of the vectors x_j and that $Tx_j = \lambda_j x_j$ follows from (a).

(e) Let $\{\lambda_j\}$ be an enumeration of the nonzero points of $\sigma(T)$ such that $|\lambda_1| \geq |\lambda_2| \geq \dots$ and define $f_n(\lambda) = \lambda$ if $\lambda = \lambda_j$ and $j \leq n$, and put $f_n(\lambda) = 0$ at the other points of $\sigma(T)$. We set again $E_j = E(\{\lambda_j\})$ and obtain

$$f_n(T) = \lambda_1 E_1 + \cdots + \lambda_n E_n.$$

Since $\dim \operatorname{im}(E_j) = \dim \ker(T - \lambda_j I) < \infty$, each $f_n(T)$ is a compact operator. We have $|\lambda - f_n(\lambda)| \leq |\lambda_n|$ for all $\lambda \in \sigma(T)$ and, by the symbolic calculus,

$$\|T - f_n(T)\| \leq |\lambda_n| \to 0 \text{ as } n \to \infty.$$

By Proposition 1.5, T is compact.

□

The symbolic calculus is powerful tool in operator theory. Finally we mention important applications to positive operators:

Definition 1.28. An operator $T \in \mathcal{L}(H)$ is called positive if $(Tx, x) \geq 0$ for every $x \in H$. We write $T \geq 0$.

Proposition 1.29. *(a) $T \in \mathcal{L}(H)$ is positive if and only if $T = T^*$ and $\sigma(T) \subset [0, \infty)$.*

(b) Every positive $T \in \mathcal{L}(H)$ has a unique positive square root $S \in \mathcal{L}(H)$, i.e. $S^2 = T$.

(c) If $T \in \mathcal{L}(H)$, then T^*T is positive and the positive square root P of T^*T is the only positive operator in $\mathcal{L}(H)$ which satisfies $\|Px\| = \|Tx\|$ for every $x \in H$.

(d) If $T \in \mathcal{L}(H)$ is normal, then T has a polar decomposition $T = UP$, where U is unitary and P is positive.

Proof. (a) (Tx, x) and (x, Tx) are complex conjugates of each other. If T is positive, (Tx, x) is real, so that

$$(x, T^*x) = (Tx, x) = (x, Tx),$$

for every $x \in H$. Hence $T = T^*$ (by the proof of Theorem 1.12). Let $\lambda = \alpha + i\beta \in \sigma(T)$ and put $T_\lambda = T - \lambda I$. Then

$$\|T_\lambda x\|^2 = \|Tx - \alpha x\|^2 + \beta^2 \|x\|^2,$$

so that $\|T_\lambda x\| \geq |\beta| \, \|x\|$. If $\beta \neq 0$, it follows that T_λ is invertible, which means $\lambda \notin \sigma(T)$. So we get that $\sigma(T)$ lies in the real axis. If $\lambda > 0$, we obtain

$$\lambda \|x\|^2 = (\lambda x, x) \leq ((T + \lambda I)x, x) \leq \|(T + \lambda I)x\| \, \|x\|,$$

so that $\|(T+\lambda I)x\| \geq \lambda\|x\|$, which implies that $T+\lambda I$ is invertible in $\mathcal{L}(H)$, and $-\lambda \notin \sigma(T)$, hence $\sigma(T) \subset [0,\infty)$.

Now assume that $T = T^*$ and $\sigma(T) \subset [0,\infty)$. Let E be the spectral decomposition of T. We have

$$(Tx,x) = \int_{\sigma(T)} \lambda \, dE_{x,x}(\lambda).$$

Since $E_{x,x}$ is a positive measure and $\lambda \geq 0$ on $\sigma(T)$, we obtain that T is positive.

(b) By (a), Proposition 1.2 and the symbolic calculus, we have $\sigma(T)$ is a compact subset of $\mathbb{R}^+$ and there exists a uniquely determined spectral measure E such that

$$T = \int_{\sigma(T)} \lambda \, dE(\lambda).$$

Define

$$S = \int_{\sigma(T)} \lambda^{1/2} \, dE(\lambda).$$

Then S is a positive self-adjoint operator with $S^2 = T$. In addition there is a sequence of polynomials p_n such that $p_n(\lambda) \to \lambda^{1/2}$ uniformly on $\sigma(T)$ (Stone-Weierstraß) and

$$\lim_{n\to\infty} \|p_n(T) - S\| = 0.$$

Let $\tilde{S}$ be an arbitrary positive self-adjoint operator with $\tilde{S}^2 = T$. Since $T\tilde{S} = \tilde{S}^3$ and $\tilde{S}T = \tilde{S}^3$, the operator $\tilde{S}$ commutes with T and so with polynomials of T. Hence also with $S = \lim_{n\to\infty} p_n(T)$. Let $x \in H$ and put $y = (S-\tilde{S})x$. Using that $\tilde{S}S = S\tilde{S}$ and $S^2 = \tilde{S}^2$, we obtain

$$(Sy,y) + (\tilde{S}y,y) = ((S+\tilde{S})(S-\tilde{S})x,y) = ((S^2-\tilde{S}^2)x,y) = 0.$$

Since S and $\tilde{S}$ are positive, $(Sy,y) = (\tilde{S}y,y) = 0$. Hence $Sy = \tilde{S}y = 0$, because $(S.,.)$ is a positive semidefinite sesquilinearform, for which the Cauchy-Schwarz inequality applies

$$|(Sy,z)|^2 \leq (Sy,y)(Sz,z),$$

for all $z \in H$. Now we get

$$\|(S-\tilde{S})x\|^2 = ((S-\tilde{S})^2x,x) = ((S-\tilde{S})y,x) = 0,$$

which yields $Sx = \tilde{S}x$ and $S = \tilde{S}$.

(c) Note first that

$$(T^*Tx,x) = (Tx,Tx) = \|Tx\|^2 \geq 0, \text{ for } x \in H,$$

so that T^*T is positive. If $P \in \mathcal{L}(H)$ and $P^* = P$, then

$$(P^2x,x) = (Px,Px) = \|Px\|^2.$$

Then, by the proof of Theorem 1.12 , it follows that $\|Px\| = \|Tx\|$ for every $x \in H$ if and only if $P^2 = T^*T$.

(d) Put $p(\lambda) = |\lambda|$ and $u(\lambda) = \lambda/|\lambda|$ for $\lambda \neq 0$ and $u(0) = 1$. Then p and u are bounded Borel functions on $\sigma(T)$. Put $P = p(T)$ and $U = u(T)$. As $p \geq 0$ we get from (a) that $P \geq 0$. Since $u\overline{u} = 1$, we get $UU^* = U^*U = I$, and since $\lambda = u(\lambda)p(\lambda)$, the relation $T = UP$ follows from the symbolic calculus.

□

Chapter 2

Spectral analysis of unbounded operators

2.1 Closed operators

Definition 2.1. Let H_1, H_2 be Hilbert spaces and $T : \mathrm{dom}(T) \longrightarrow H_2$ be a densely defined linear operator, i.e. $\mathrm{dom}(T)$ is a dense linear subspace of H_1. Let $\mathrm{dom}(T^*)$ be the space of all $y \in H_2$ such that $x \mapsto (Tx, y)_2$ defines a continuous linear functional on $\mathrm{dom}(T)$. Since $\mathrm{dom}(T)$ is dense in H_1 there exists a uniquely determined element $T^*y \in H_1$ such that $(Tx, y)_2 = (x, T^*y)_1$ (Riesz representation theorem). The map $y \mapsto T^*y$ is linear and $T^* : \mathrm{dom}(T^*) \longrightarrow H_1$ is the adjoint operator to T.

T is called a closed operator, if the graph

$$\mathcal{G}(T) = \{(f, Tf) \in H_1 \times H_2 \, : \, f \in \mathrm{dom}(T)\}$$

is a closed subspace of $H_1 \times H_2$.

The inner product in $H_1 \times H_2$ is

$$((x, y), (u, v)) = (x, u)_1 + (y, v)_2.$$

If $\tilde{V}$ is a linear subspace of H_1 which contains $\mathrm{dom}(T)$ and $\tilde{T}x = Tx$ for all $x \in \mathrm{dom}(T)$ then we say that $\tilde{T}$ is an extension of T.

An operator T with domain $\mathrm{dom}(T)$ is said to be closable if it has a closed extension $\tilde{T}$.

Lemma 2.2. *Let T be a densely defined closable operator. Then there is a closed extension $\overline{T}$, called its closure, whose domain is smallest among all closed extensions.*

Proof. Let $\mathcal{V}$ be the set of $x \in H_1$ for which there exist $x_k \in \mathrm{dom}(T)$ and $y \in H_2$ such that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} Tx_k = y$. Since $\tilde{T}$ is a closed extension of T it follows that $x \in \mathrm{dom}(\tilde{T})$ and $\tilde{T}x = y$. Therefore y is uniquely determined by x. We define $\overline{T}x = y$ with $\mathrm{dom}(\overline{T}) = \mathcal{V}$. Then $\overline{T}$ is an extension of T and every closed extension of T is also an extension of $\overline{T}$. The graph of $\overline{T}$ is the closure of the graph of T in $H_1 \times H_2$. Hence $\overline{T}$ is a closed operator. $\square$

Lemma 2.3. *Let $T_1 : \mathrm{dom}(T_1) \longrightarrow H_2$ be a densely defined operator and $T_2 : H_2 \longrightarrow H_3$ be a bounded operator. Then $(T_2\,T_1)^* = T_1^*\,T_2^*$, which includes that $\mathrm{dom}((T_2\,T_1)^*) = \mathrm{dom}(T_1^*\,T_2^*)$.*

Proof. Note that

$$\operatorname{dom}(T_1^* T_2^*) = \{f \in \operatorname{dom}(T_2^*) \,:\, T_2^*(f) \in \operatorname{dom}(T_1^*)\}.$$

Let $f \in \operatorname{dom}(T_1^* T_2^*)$ and $g \in \operatorname{dom}(T_2 T_1)$. Then

$$(T_1^* T_2^* f, g) = (T_2^* f, T_1 g) = (f, T_2 T_1 g),$$

hence $\operatorname{dom}(T_1^* T_2^*) \subseteq \operatorname{dom}((T_2 T_1)^*)$.

Now let $f \in \operatorname{dom}((T_2 T_1)^*)$. As T_2^* is bounded and everywhere defined on H_3, and for all $g \in \operatorname{dom}(T_2 T_1) = \operatorname{dom}(T_1)$ we have

$$((T_2 T_1)^* f, g) = (f, T_2 T_1 g) = (T_2^* f, T_1 g).$$

Hence $T_2^* f \in \operatorname{dom}(T_1^*)$ and $f \in \operatorname{dom}(T_1^* T_2^*)$. □

Lemma 2.4. *Let T be a densely defined operator on H and let S be a bounded operator on H. Then $(T + S)^* = T^* + S^*$.*

Proof. Let $f \in \operatorname{dom}(T^* + S^*) = \operatorname{dom}(T^*)$. Then for all $g \in \operatorname{dom}(T + S) = \operatorname{dom}(T)$ we have

$$((T^* + S^*)f, g) = (T^* f, g) + (S^* f, g) = (f, Tg) + (f, Sg) = (f, (T + S)g),$$

hence $f \in \operatorname{dom}((T + S)^*)$ and $(T + S)^* f = T^* f + S^* f$.

If $f \in \operatorname{dom}((T + S)^*)$, then for all $g \in \operatorname{dom}(T + S) = \operatorname{dom}(T)$ we have

$$([(T + S)^* - S^*]f, g) = (f, (T + S)g) - (f, Sg) = (f, Tg),$$

therefore $f \in \operatorname{dom}(T^*)$ and $\operatorname{dom}((T + S)^*) = \operatorname{dom}(T^* + S^*) = \operatorname{dom}(T^*)$. □

Lemma 2.5. *Let $T : \operatorname{dom}(T) \longrightarrow H_2$ be a densely defined linear operator and define $V : H_1 \times H_2 \longrightarrow H_2 \times H_1$ by $V((x, y)) = (y, -x)$. Then*

$$\mathcal{G}(T^*) = [V(\mathcal{G}(T))]^\perp = V(\mathcal{G}(T)^\perp);$$

in particular T^* is always closed.

Proof. $(y, z) \in \mathcal{G}(T^*) \Leftrightarrow (Tx, y)_2 = (x, z)_1$ for each $x \in \operatorname{dom}(T)$ $\Leftrightarrow$ $((x, Tx), (-z, y)) = 0$ for each $x \in \operatorname{dom}(T)$ $\Leftrightarrow$ $V^{-1}((y, z)) = (-z, y) \in \mathcal{G}(T)^\perp$. Hence $\mathcal{G}(T^*) = V(\mathcal{G}(T)^\perp)$ and since V is unitary we have $V^* = V^{-1}$ and $[V(\mathcal{G}(T))]^\perp = V(\mathcal{G}(T)^\perp)$.

□

Lemma 2.6. *Let $T : \operatorname{dom}(T) \longrightarrow H_2$ be a densely defined, closed linear operator. Then*

$$H_2 \times H_1 = V(\mathcal{G}(T)) \oplus \mathcal{G}(T^*).$$

Proof. $\mathcal{G}(T)$ is closed, therefore, by Lemma 2.5: $\mathcal{G}(T^*)^\perp = V(\mathcal{G}(T))$. □

Lemma 2.7. *Let $T : dom(T) \longrightarrow H_2$ be a densely defined, closed linear operator. Then $dom(T^*)$ is dense in H_2 and $T^{**} = T$.*

Proof. Let $z \perp \mathrm{dom}(T^*)$. Hence $(z, y)_2 = 0$ for each $y \in \mathrm{dom}(T^*)$. We have

$$V^{-1} : H_2 \times H_1 \longrightarrow H_1 \times H_2$$

where $V^{-1}((y, x)) = (-x, y)$, and $V^{-1}V = \mathrm{Id}$. Now, by Lemma 2.6, we have

$$H_1 \times H_2 \cong V^{-1}(H_2 \times H_1) = V^{-1}(V(\mathcal{G}(T)) \oplus \mathcal{G}(T^*)) \cong \mathcal{G}(T) \oplus V^{-1}(\mathcal{G}(T^*)).$$

Hence $(z, y)_2 = 0 \Leftrightarrow ((0, z), (-T^*y, y)) = 0$ for each $y \in \mathrm{dom}(T^*)$ implies $(0, z) \in \mathcal{G}(T)$ and therefore $z = T(0) = 0$, which means that $\mathrm{dom}(T^*)$ is dense in H_2.

Since T and T^* are densely defined and closed we have by Lemma 2.5

$$\mathcal{G}(T) = \mathcal{G}(T)^{\perp\perp} = [V^{-1}\mathcal{G}(T^*)]^\perp = \mathcal{G}(T^{**}),$$

where $-V^{-1}$ corresponds to V in considering operators from H_2 to H_1. □

Lemma 2.8. *Let $T : dom(T) \longrightarrow H_2$ be a densely defined linear operator. Then $kerT^* = (imT)^\perp$, which means that $kerT^*$ is closed.*

Proof. Let $v \in \ker T^*$ and $y \in \mathrm{im}T$, which means that there exists $u \in \mathrm{dom}(T)$ such that $Tu = y$. Hence

$$(v, y)_2 = (v, Tu)_2 = (T^*v, u)_1 = 0,$$

and $\ker T^* \subseteq (\mathrm{im}T)^\perp$.

And if $y \in (\mathrm{im}T)^\perp$, then $(y, Tu)_2 = 0$ for each $u \in \mathrm{dom}(T)$, which implies that $y \in \mathrm{dom}(T^*)$ and $(y, Tu)_2 = (T^*y, u)_1$ for each $u \in \mathrm{dom}(T)$. Since each $\mathrm{dom}(T)$ is dense in H_1 we obtain $T^*y = 0$ and $(\mathrm{im}T)^\perp \subseteq \ker T^*$. □

Lemma 2.9. *Let $T : dom(T) \longrightarrow H_2$ be a densely defined, closed linear operator. Then $kerT$ is a closed linear subspace of H_1.*

Proof. We use Lemma 2.8 for T^* and get $\ker T^{**} = (\mathrm{im}T^*)^\perp$. Since, by Lemma 2.7, $T^{**} = T$ we obtain $\ker T = (\mathrm{im}T^*)^\perp$ and that $\ker T$ is a closed linear subspace of H_1.

□

2.2 Self-adjoint operators

In the following we introduce the fundamental concept of an unbounded self-adjoint operator, which will be crucial for both spectral theory and its applications to complex analysis.

Definition 2.10. Let $T : \mathrm{dom}(T) \longrightarrow H$ be a densely defined linear operator. T is symmetric if $(Tx, y) = (x, Ty)$ for all $x, y \in \mathrm{dom}(T)$. We say that T is self-adjoint if T is symmetric and $\mathrm{dom}(T) = \mathrm{dom}(T^*)$. This is equivalent to requiring that $T = T^*$ and implies that T is closed. We say that T is essentially self-adjoint if it is symmetric and its closure $\overline{T}$ is self-adjoint.

Remark 2.11. (a) If T is a symmetric operator, there are two natural closed extensions. We have $\mathrm{dom}(T) \subseteq \mathrm{dom}(T^*)$ and $T^* = T$ on $\mathrm{dom}(T)$. Since T^* is closed (Lemma 2.6), T^* is a closed extension of T, it is the maximal self-adjoint extension. T is also closable, by Lemma 2.2, therefore $\overline{T}$ exists, it is the minimal closed extension.

(b) If T is essentially self-adjoint, then its self-adjoint extension is unique. To prove this, let S be a self-adjoint extension of T. Then S is closed and, being an extension of T, it is also an extension of its smallest extension $\overline{T}$. Hence

$$\overline{T} \subset S = S^* \subset (\overline{T})^* = \overline{T},$$

and $S = \overline{T}$.

Lemma 2.12. *Let T be a densely defined, symmetric operator.*

(i) If $\mathrm{dom}(T) = H$, then T is self-adjoint and T is bounded.

(ii) If T is self-adjoint and injective, then $\mathrm{im}(T)$ is dense in H, and T^{-1} is self-adjoint.

(iii) If $\mathrm{im}(T)$ is dense in H, then T is injective.

(iv) If $\mathrm{im}(T) = H$, then T is self-adjoint, and T^{-1} is bounded.

Proof. (i) By assumption $\mathrm{dom}(T) \subseteq \mathrm{dom}(T^*)$. If $\mathrm{dom}(T) = H$, it follows that T is self-adjoint, therefore also closed (Lemma 2.5) and continuous by the closed graph theorem.

(ii) Suppose $y \perp \mathrm{Im}(T)$. Then $x \mapsto (Tx, y) = 0$ is continuous on $\mathrm{dom}(T)$, hence $y \in \mathrm{dom}(T^*) = \mathrm{dom}(T)$, and $(x, Ty) = (Tx, y) = 0$ for all $x \in \mathrm{dom}(T)$. Thus $Ty = 0$ and since T is assumed to be injective, it follows that $y = 0$. This proves that $\mathrm{Im}(T)$ in dense in H.

T^{-1} is therefore densely defined, with $\mathrm{dom}(T^{-1}) = \mathrm{im}(T)$, and $(T^{-1})^*$ exists. Now let $U : H \times H \longrightarrow H \times H$ be defined by $U((x, y)) = (-y, x)$. It easily follows that $U^2 = -I$ and $U^2(M) = M$ for any subspace M of $H \times H$, and we get $\mathcal{G}(T^{-1}) = U(\mathcal{G}(-T))$ and $U(\mathcal{G}(T^{-1})) = \mathcal{G}(-T))$. Being self-adjoint, T

is closed; hence $-T$ is closed and T^{-1} is closed. By Lemma 2.6 applied to T^{-1} and to $-T$ we get the orthogonal decompositions

$$H \times H = U(\mathcal{G}(T^{-1})) \oplus \mathcal{G}((T^{-1})^*)$$

and

$$H \times H = U(\mathcal{G}(-T)) \oplus \mathcal{G}(-T)) = \mathcal{G}(T^{-1}) \oplus U(\mathcal{G}(T^{-1})).$$

Consequently

$$\mathcal{G}((T^{-1})^*) = [U(\mathcal{G}(T^{-1}))]^{\perp} = \mathcal{G}(T^{-1}),$$

which shows that $(T^{-1})^* = T^{-1}$.

(iii) Suppose $Tx = 0$. Then $(x, Ty) = (Tx, y) = 0$ for each $y \in \mathrm{dom}(T)$. Thus $x \perp \mathrm{im}(T)$, and therefore $x = 0$.

(iv) Since $\mathrm{im}(T) = H$, (iii) implies that T is injective, $\mathrm{dom}(T^{-1}) = H$. If $x, y \in H$, then $x = Tz$ and $y = Tw$, for some $z \in \mathrm{dom}(T)$ and $w \in \mathrm{dom}(T)$, so that

$$(T^{-1}x, y) = (z, Tw) = (Tz, w) = (x, T^{-1}y).$$

Hence T^{-1} is symmetric. (i) implies that T^{-1} is self-adjoint (and bounded), and now it follows from (ii) that $T = (T^{-1})^{-1}$ is also self-adjoint. □

Lemma 2.13. *Let T be a densely defined closed operator, dom$(T) \subseteq H_1$ and T : dom$(T) \longrightarrow H_2$. Then $B = (I + T^*T)^{-1}$ and $C = T(I + T^*T)^{-1}$ are everywhere defined and bounded,* $\|B\| \leq 1, \quad \|C\| \leq 1$; *in addition B is self-adjoint and positive.*

Proof. Let $h \in H_1$ be an arbitrary element and consider $(h, 0) \in H_1 \times H_2$. Form the proof of Lemma 2.7 we get

$$H_1 \times H_2 = \mathcal{G}(T) \oplus V^{-1}(\mathcal{G}(T^*)), \tag{2.1}$$

which implies that $(h, 0)$ can be written in a unique way as

$$(h, 0) = (f, Tf) + (-T^*(-g), -g),$$

for $f \in \mathrm{dom}(T)$ and $g \in \mathrm{dom}(T^*)$, which gives $h = f + T^*g$ and $0 = Tf - g$. We set $Bh := f$ and $Ch := g$. In this way we get two linear operators B and C everywhere defined on H_1. The two equations from above can now be written as

$$I = B + T^*C, \quad 0 = TB - C,$$

which gives

$$C = TB \quad \text{and} \quad I = B + T^*TB = (I + T^*T)B. \tag{2.2}$$

The decomposition in (2.1) is orthogonal, therefore we obtain

$$\|h\|^2 = \|(h, 0)\|^2 = \|(f, Tf)\|^2 + \|(T^*g, -g)\|^2 = \|f\|^2 + \|Tf\|^2 + \|T^*g\|^2 + \|g\|^2,$$

and hence

$$\|Bh\|^2 + \|Ch\|^2 = \|f\|^2 + \|g\|^2 \leq \|h\|^2,$$

which implies $\|B\| \leq 1$ and $\|C\| \leq 1$.

For each $u \in \text{dom}(T^*T)$ we get

$$((I + T^*T)u, u) = (u, u) + (Tu, Tu) \geq (u, u)$$

hence, if $(I + T^*T)u = 0$ we get $u = 0$. Therefore $(I + T^*T)^{-1}$ exists and (2.2) implies that $(I + T^*T)^{-1}$ is defined everywhere and $B = (I + T^*T)^{-1}$. Finally let $u, v \in H_1$. Then

$$\begin{aligned}(Bu, v) &= (Bu, (I + T^*T)Bv) = (Bu, Bv) + (Bu, T^*TBv) \\ &= (Bu, Bv) + (T^*TBu, Bv) = ((I + T^*T)Bu, Bv) = (u, Bv)\end{aligned}$$

and

$$(Bu, u) = (Bu, (I + T^*T)Bu) = (Bu, Bu) + (TBu, TBu) \geq 0,$$

which proves the lemma. □

At this point we can describe the concept of the core of an operator, which will be very useful later for spectral analysis.

Definition 2.14. Let T be a closable operator with domain $\text{dom}(T)$. A subspace $D \subset \text{dom}(T)$ is called a core of the operator T if the closure of the restriction $T\,|_D$ is an extension of T.

Remark 2.15. If T is a closed operator, then $\overline{T\,|_D} = T$.

Lemma 2.16. *Let T be a densely defined closed operator, $dom(T) \subseteq H_1$ and $T : dom(T) \longrightarrow H_2$. Then $dom(T^*T)$ is a core of the operator T.*

Proof. We have to show that $\mathcal{G}(T) = \overline{\mathcal{G}(T\,|_{\text{dom}(T^*T)})}$. For this purpose we consider elements (x, Tx) in the graph of T. We suppose that $(x, Tx) \perp (y, Ty)$ for each $y \in \text{dom}(T^*T)$. Then

$$(x, (I + T^*T)y) = (x, y) + (Tx, Ty) = ((x, Tx), (y, Ty)) = 0,$$

and as $\text{im}(I + T^*T) = H_1$ (Lemma 2.13) we conclude that $x = 0$, which means that $\mathcal{G}(T\,|_{\text{dom}(T^*T)})$ is dense in $\mathcal{G}(T)$. □

2.3 The $\overline{\partial}$-Neumann operator

Let $\Omega \subseteq \mathbb{C}^n$ be an open subset and $f : \Omega \longrightarrow \mathbb{C}$ be a $\mathcal{C}^1$-function. We write $z_j = x_j + iy_j$ and consider for $P \in \Omega$ the differential

$$df_P = \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j}(P)\, dx_j + \frac{\partial f}{\partial y_j}(P)\, dy_j \right).$$

We use the complex differentials

$$dz_j = dx_j + idy_j \quad , \quad d\overline{z}_j = dx_j - idy_j$$

and the derivatives

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j} \right) \quad , \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\,\frac{\partial}{\partial y_j} \right)$$

and rewrite the differential df_p in the form

$$df_P = \sum_{j=1}^n \left(\frac{\partial f}{\partial z_j}(P)\, dz_j + \frac{\partial f}{\partial \overline{z}_j}(P)\, d\overline{z}_j \right) = \partial f_P + \overline{\partial} f_P.$$

A general differential form is given by

$$\omega = \sum_{|J|=p,|K|=q}{}' \, a_{J,K}\, dz_J \wedge d\overline{z}_K,$$

where $\sum_{|J|=p,|K|=q}'$ denotes the sum taken only over all increasing multiindices $J = (j_1, \dots, j_p),\ K = (k_1, \dots, k_q)$ and

$$dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_p} \quad , \quad d\overline{z}_K = d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q}.$$

The derivative $d\omega$ of ω is defined by

$$d\omega = \sum_{|J|=p,|K|=q}{}' \, da_{J,K} \wedge dz_J \wedge d\overline{z}_K = \sum_{|J|=p,|K|=q}{}' \, (\partial a_{J,K} + \overline{\partial} a_{J,K}) \wedge dz_J \wedge d\overline{z}_K,$$

and we set

$$\partial\omega = \sum_{|J|=p,|K|=q}{}' \, \partial a_{J,K} \wedge dz_J \wedge d\overline{z}_K \text{ and } \overline{\partial}\omega = \sum_{|J|=p,|K|=q}{}' \, \overline{\partial} a_{J,K} \wedge dz_J \wedge d\overline{z}_K.$$

We have $d = \partial + \overline{\partial}$ and since $d^2 = 0$ it follows that

$$0 = (\partial + \overline{\partial}) \circ (\partial + \overline{\partial})\omega = (\partial \circ \partial)\omega + (\partial \circ \overline{\partial} + \overline{\partial} \circ \partial)\omega + (\overline{\partial} \circ \overline{\partial})\omega,$$

which implies $\partial^2 = 0$, $\overline{\partial}^2 = 0$ and $\partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0$, by comparing the types of the differential forms involved.

Definition 2.17. Let

$$L^2_{(0,1)}(\Omega) := \{u = \sum_{j=1}^{n} u_j \, d\overline{z}_j \ : \ u_j \in L^2(\Omega), \ j = 1, \dots, n\}$$

be the space of $(0,1)$- forms with coefficients in $L^2(\Omega)$. For $u, v \in L^2_{(0,1)}(\Omega)$ we define the inner product by

$$(u, v) = \sum_{j=1}^{n} (u_j, v_j).$$

In this way $L^2_{(0,1)}(\Omega)$ becomes a Hilbert space. $(0,1)$ forms with compactly supported $\mathcal{C}^\infty$ coefficients are dense in $L^2_{(0,1)}(\Omega)$.

Definition 2.18. Let $f \in \mathcal{C}_0^\infty(\Omega)$ and set

$$\overline{\partial} f := \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} \, d\overline{z}_j,$$

then

$$\overline{\partial} : \mathcal{C}_0^\infty(\Omega) \longrightarrow L^2_{(0,1)}(\Omega).$$

$\overline{\partial}$ is a densely defined unbounded operator on $L^2(\Omega)$. It does not have closed graph.

Definition 2.19. The domain $\mathrm{dom}(\overline{\partial})$ of $\overline{\partial}$ consists of all functions $f \in L^2(\Omega)$ such that $\overline{\partial} f$, in the sense of distributions, belongs to $L^2_{(0,1)}(\Omega)$, i.e. $\overline{\partial} f = g = \sum_{j=1}^{n} g_j \, d\overline{z}_j$, and for each $\phi \in \mathcal{C}_0^\infty(\Omega)$ we have

$$\int_\Omega f \left(\frac{\partial \phi}{\partial z_j} \right)^{-} d\lambda = - \int_\Omega g_j \, \overline{\phi} \, d\lambda \ , \ j = 1, \dots, n. \tag{2.3}$$

It is clear that $\mathcal{C}_0^\infty(\Omega) \subseteq \mathrm{dom}(\overline{\partial})$, hence $\mathrm{dom}(\overline{\partial})$ is dense in $L^2(\Omega)$. Since differentiation is a continuous operation in distribution theory we have

Lemma 2.20. *$\overline{\partial}$: dom($\overline{\partial}$) $\longrightarrow L^2_{(0,1)}(\Omega)$ has closed graph and Ker$\overline{\partial}$ is a closed subspace of $L^2(\Omega)$.*

Proof. Let $(f_k)_k$ be a sequence in $\mathrm{dom}(\overline{\partial})$ such that $f_k \to f$ in $L^2(\Omega)$ and $\overline{\partial} f_k \to g$ in $L^2_{(0,1)}(\Omega)$. We have to show that $\overline{\partial} f = g$. We know that $\overline{\partial} f_k \to \overline{\partial} f$ as distributions. As $\overline{\partial} f_k \to g$ in $L^2_{(0,1)}(\Omega)$, it follows that $f \in \mathrm{dom}(\overline{\partial})$ and $\overline{\partial} f = g$.

Now we can apply Lemma 2.9 and get that $\mathrm{Ker}\overline{\partial}$ is a closed subspace of $L^2(\Omega)$. □

Now we consider the $\overline{\partial}$-complex

$$L^2(\Omega) \xrightarrow{\overline{\partial}} L^2_{(0,1)}(\Omega) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} L^2_{(0,n)}(\Omega) \xrightarrow{\overline{\partial}} 0\,, \tag{2.4}$$

where $L^2_{(0,q)}(\Omega)$ denotes the space of $(0,q)$-forms on Ω with coefficients in $L^2(\Omega)$. The $\overline{\partial}$-operator on $(0,q)$-forms is given by

$$\overline{\partial}\left(\sum_J{}' a_J\, d\overline{z}_J\right) = \sum_{j=1}^n \sum_J{}' \frac{\partial a_J}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_J, \tag{2.5}$$

where $\sum'$ means that the sum is only taken over strictly increasing multi-indices $J = (j_1, \dots, j_q)$.

The derivatives are taken in the sense of distributions, and the domain of $\overline{\partial}$ consists of those $(0,q)$-forms for which the right hand side belongs to $L^2_{(0,q+1)}(\Omega)$. So $\overline{\partial}$ is a densely defined closed operator, and therefore has an adjoint operator from $L^2_{(0,q+1)}(\Omega)$ into $L^2_{(0,q)}(\Omega)$ denoted by $\overline{\partial}^*$.

We consider the $\overline{\partial}$-complex

$$L^2_{(0,q-1)}(\Omega) \underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q)}(\Omega) \underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q+1)}(\Omega), \tag{2.6}$$

for $1 \le q \le n-1$.

Proposition 2.21. *The complex Laplacian $\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial}$, defined on the domain dom$(\Box) = \{u \in L^2_{(0,q)}(\Omega) : u \in$ dom$(\overline{\partial}) \cap$ dom$(\overline{\partial}^*), \overline{\partial}u \in$ dom$(\overline{\partial}^*), \overline{\partial}^* u \in$ dom$(\overline{\partial})\}$ acts as an unbounded, densely defined, closed and self-adjoint operator on $L^2_{(0,q)}(\Omega)$, for $1 \le q \le n$, which means that $\Box = \Box^*$ and dom$(\Box) =$ dom$(\Box^*)$.*

Proof. dom$(\Box)$ contains all smooth forms with compact support, hence $\Box$ is densely defined. To show that $\Box$ is closed depends on the fact that both $\overline{\partial}$ and $\overline{\partial}^*$ are closed : note that

$$(\Box u, u) = (\overline{\partial}\,\overline{\partial}^* u + \overline{\partial}^*\,\overline{\partial} u, u) = \|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2, \tag{2.7}$$

for $u \in$ dom$(\Box)$. We have to prove that for every sequence $u_k \in$ dom$(\Box)$ such that $u_k \to u$ in $L^2_{(0,q)}(\Omega)$ and $\Box u_k$ converges, we have $u \in$ dom$(\Box)$ and $\Box u_k \to \Box u$. It follows from (2.7) that

$$(\Box(u_k - u_\ell), u_k - u_\ell) = \|\overline{\partial}(u_k - u_\ell)\|^2 + \|\overline{\partial}^*(u_k - u_\ell)\|^2,$$

which implies that $\overline{\partial} u_k$ converges in $L^2_{(0,q+1)}(\Omega)$ and that $\overline{\partial}^* u_k$ converges in $L^2_{(0,q-1)}(\Omega)$. Since $\overline{\partial}$ and $\overline{\partial}^*$ are closed operators, we get $u \in$ dom$(\overline{\partial}) \cap$ dom$(\overline{\partial}^*)$ and $\overline{\partial} u_k \to \overline{\partial} u$ in $L^2_{(0,q+1)}(\Omega)$ and $\overline{\partial}^* u_k \to \overline{\partial}^* u$ in $L^2_{(0,q-1)}(\Omega)$.

To show that $\overline{\partial}u \in \mathrm{dom}(\overline{\partial}^*)$ and $\overline{\partial}^* u \in \mathrm{dom}(\overline{\partial})$, we first notice that $\overline{\partial}\,\overline{\partial}^* u_k$ and $\overline{\partial}^*\,\overline{\partial}u_k$ are orthogonal which follows from

$$(\overline{\partial}\,\overline{\partial}^* u_k, \overline{\partial}^*\,\overline{\partial}u_k) = (\overline{\partial}^2\,\overline{\partial}^* u_k, \overline{\partial}u_k) = 0.$$

Therefore the convergence of $\Box u_k = \overline{\partial}\,\overline{\partial}^* u_k + \overline{\partial}^*\,\overline{\partial}u_k$ implies that both $\overline{\partial}\,\overline{\partial}^* u_k$ and $\overline{\partial}^*\,\overline{\partial}u_k$ converge. Now use again that $\overline{\partial}$ and $\overline{\partial}^*$ are closed operators to obtain that $\overline{\partial}\,\overline{\partial}^* u_k \to \overline{\partial}\,\overline{\partial}^* u$ and $\overline{\partial}^*\,\overline{\partial}u_k \to \overline{\partial}^*\,\overline{\partial}u$. This implies that $\Box u_k \to \Box u$. Hence $\Box$ is closed.

In order to show that $\Box$ is self-adjoint we use Lemma 2.13. Define

$$R = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial} + I$$

on $\mathrm{dom}(\Box)$. By Lemma 2.13 both $(I + \overline{\partial}\,\overline{\partial}^*)^{-1}$ and $(I + \overline{\partial}^*\,\overline{\partial})^{-1}$ are bounded, self-adjoint operators. Consider

$$L = (I + \overline{\partial}\,\overline{\partial}^*)^{-1} + (I + \overline{\partial}^*\,\overline{\partial})^{-1} - I.$$

Then L is bounded and self-adjoint. We claim that $L = R^{-1}$. Since

$$(I + \overline{\partial}\,\overline{\partial}^*)^{-1} - I = (I - (I + \overline{\partial}\,\overline{\partial}^*))(I + \overline{\partial}\,\overline{\partial}^*)^{-1} = -\overline{\partial}\,\overline{\partial}^*(I + \overline{\partial}\,\overline{\partial}^*)^{-1},$$

we have that the range of $(I + \overline{\partial}\,\overline{\partial}^*)^{-1}$ is contained in $\mathrm{dom}(\overline{\partial}\,\overline{\partial}^*)$. Similarly, we have that the range of $(I + \overline{\partial}^*\,\overline{\partial})^{-1}$ is contained in $\mathrm{dom}(\overline{\partial}^*\,\overline{\partial})$ and we get

$$L = (I + \overline{\partial}^*\,\overline{\partial})^{-1} - \overline{\partial}\,\overline{\partial}^*(I + \overline{\partial}\,\overline{\partial}^*)^{-1}.$$

Since $\overline{\partial}^2 = 0$, we have that the range of L is contained in $\mathrm{dom}(\overline{\partial}^*\,\overline{\partial})$ and

$$\overline{\partial}^*\,\overline{\partial}L = \overline{\partial}^*\,\overline{\partial}(I + \overline{\partial}^*\,\overline{\partial})^{-1}.$$

Similarly, we have that the range of L is contained in $\mathrm{dom}(\overline{\partial}\,\overline{\partial}^*)$ and

$$\overline{\partial}\,\overline{\partial}^* L = \overline{\partial}\,\overline{\partial}^*(I + \overline{\partial}\,\overline{\partial}^*)^{-1}.$$

This implies that the range of L is contained in $\mathrm{dom}(\Box)$. In addition we have

$$RL = \overline{\partial}\,\overline{\partial}^*(I + \overline{\partial}\,\overline{\partial}^*)^{-1} + \overline{\partial}^*\,\overline{\partial}(I + \overline{\partial}^*\,\overline{\partial})^{-1} + L = I.$$

If $Ru = 0$, we get $\Box u = -u$ and $0 \le (\Box u, u) = -(u, u)$, which implies that $u = 0$. Hence R is injective and we have that $L = R^{-1}$. By Lemma 2.13 we know that L is self-adjoint. Apply Lemma 2.12 to get that R is self-adjoint. Therefore $\Box = R - I$ is self-adjoint. $\square$

For the rest of this section we will now suppose that Ω is a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. It can be shown that

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2 \geq c\,\|u\|^2, \tag{2.8}$$

for each $u \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$, $c > 0$ (see for instance [10]).

The next result describes the implication of the basic estimates (2.8) for the $\Box$-operator.

Proposition 2.22. *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Then $\Box : dom(\Box) \longrightarrow L^2_{(0,q)}(\Omega)$ is bijective and has a bounded inverse*

$$N : L^2_{(0,q)}(\Omega) \longrightarrow dom(\Box).$$

N is called $\overline{\partial}$-Neumann operator. In addition

$$\|Nu\| \leq \frac{1}{c}\,\|u\|. \tag{2.9}$$

Proof. Since $(\Box u, u) = \|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2$, it follows that for a convergent sequence $(\Box u_n)_n$ we get

$$\|\Box u_n - \Box u_m\|\,\|u_n - u_m\| \geq (\Box(u_n - u_m), u_n - u_m) \geq c\|u_n - u_m\|^2,$$

which implies that $(u_n)_n$ is convergent and since $\Box$ is a closed operator we obtain that $\Box$ has closed range. If $\Box u = 0$, we get $\overline{\partial}u = 0$ and $\overline{\partial}^* u = 0$ and by (2.8) that $u = 0$, hence $\Box$ is injective. By Lemma 2.12 (ii) the range of $\Box$ is dense, therefore $\Box$ is surjective.

We showed that

$$\Box : \mathrm{dom}(\Box) \longrightarrow L^2_{(0,q)}(\Omega)$$

is bijective and therefore, by Lemma 2.12 (iv), has a bounded inverse

$$N : L^2_{(0,q)}(\Omega) \longrightarrow \mathrm{dom}(\Box).$$

For $u \in L^2_{(0,q)}(\Omega)$ we use (2.8) for Nu to obtain

$$\begin{aligned} c\|Nu\|^2 &\leq \|\overline{\partial}Nu\|^2 + \|\overline{\partial}^* Nu\|^2 \\ &= (\overline{\partial}^*\overline{\partial}Nu, Nu) + (\overline{\partial}\overline{\partial}^* Nu, Nu) \\ &= (u, Nu) \leq \|u\|\,\|Nu\|, \end{aligned}$$

which implies (2.9).

$\Box$

For $u \in L^2_{(0,q)}(\Omega)$ and $v \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ we get

$$(u,v) = (\Box Nu, v) = ((\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})Nu, v) = (\overline{\partial}^* Nu, \overline{\partial}^* v) + (\overline{\partial}Nu, \overline{\partial}v). \quad (2.10)$$

Now we discuss a different approach to the $\overline{\partial}$-Neumann operator, which is related to the quadratic form

$$Q(u,v) = (\overline{\partial}u, \overline{\partial}v) + (\overline{\partial}^* u, \overline{\partial}^* v).$$

For this purpose we consider the embedding

$$j : \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*) \longrightarrow L^2_{(0,q)}(\Omega),$$

where $\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ is endowed with the graph-norm

$$u \mapsto (\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}.$$

The graph-norm stems from the inner product

$$Q(u,v) = (u,v)_Q = (\Box u, v) = (\overline{\partial}u, \overline{\partial}v) + (\overline{\partial}^* u, \overline{\partial}^* v).$$

The basic estimates (2.8) imply that j is a bounded operator with operator norm

$$\|j\| \leq \frac{1}{\sqrt{c}}.$$

By (2.8) it follows in addition that $\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ endowed with the graph-norm $u \mapsto (\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$ is a Hilbert space.

Since $(u,v) = (u, jv)$, we have that $(u,v) = (j^*u, v)_Q$. Equation (2.10) suggests that as an operator to $\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$, N coincides with j^* and as an operator to $L^2_{(0,q)}(\Omega)$, N should be equal to $j \circ j^*$. For this purpose set

$$\tilde{N} = j \circ j^*, \quad (2.11)$$

and note that $\tilde{N}^* = (j \circ j^*)^* = j \circ j^* = \tilde{N}$, i.e. $\tilde{N}$ is self-adjoint (of course also bounded). We claim that the range of $\tilde{N}$ is contained in $\mathrm{dom}(\Box)$. To show this we use an approach due to F. Berger (see [1]): since $\Box$ is self-adjoint it suffices to show that $\tilde{N}u \in \mathrm{dom}(\Box^*)$ for all $u \in L^2_{(0,q)}(\Omega)$, which means to show that the functional $v \mapsto (\Box v, \tilde{N}u)$ is bounded on $\mathrm{dom}(\Box)$:

$$|(\Box v, \tilde{N}u)| = |((\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})v, \tilde{N}u)| = |(\overline{\partial}v, \overline{\partial}\tilde{N}u) + (\overline{\partial}^* v, \overline{\partial}^* \tilde{N}u)|$$

$$= |Q(v, j^*u)| = |(jv, u)| = |(v,u)| \leq \|v\|\,\|u\|.$$

For $v \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ we have

$$(\Box\tilde{N}u, v) = (\tilde{N}u, v)_Q = (j^*u, v)_Q = (u, jv) = (u,v),$$

hence $\square \tilde{N} u = u$, in a similar way we obtain for $u \in \mathrm{dom}(\square)$

$$(\tilde{N}\square u, v) = (\square u, \tilde{N} v) = (u, \tilde{N} v)_Q = (u, j^* v)_Q = (ju, v) = (u, v),$$

which implies that $\tilde{N}\square u = u$. Altogether we obtain that $N = \tilde{N}$.

Proposition 2.23. *The operators*

$$\overline{\partial} N : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q+1)}(\Omega) \text{ and } \overline{\partial}^* N : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q-1)}(\Omega)$$

are both bounded.

Proof. From the above considerations on N we get

$$\|\overline{\partial} N u\|^2 + \|\overline{\partial}^* N u\|^2 = (j^* u, j^* u)_Q \leq \|j^*\|^2 \, \|u\|^2,$$

for $u \in L^2_{(0,q)}(\Omega)$, which implies the result. □

Proposition 2.24. *Let N_q denote the $\overline{\partial}$-Neumann operator on $L^2_{(0,q)}(\Omega)$. Then*

$$N_{q+1}\overline{\partial} = \overline{\partial} N_q, \tag{2.12}$$

on $dom(\overline{\partial})$ and

$$N_{q-1}\overline{\partial}^* = \overline{\partial}^* N_q, \tag{2.13}$$

on $dom(\overline{\partial}^*)$.

In addition we have that $\overline{\partial}^* N_q$ is zero on $(ker\overline{\partial})^\perp$.

Proof. For $u \in \mathrm{dom}(\overline{\partial})$ we have $\overline{\partial} u = \overline{\partial}\overline{\partial}^*\overline{\partial} N_q u$ and

$$N_{q+1}\overline{\partial} u = N_{q+1}\overline{\partial}\,\overline{\partial}^*\overline{\partial} N_q u = N_{q+1}(\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\overline{\partial} N_q u = \overline{\partial} N_q u,$$

which proves (2.12). In a similar way we get (2.13).

Now let $k \in (\ker\overline{\partial})^\perp$ and $u \in \mathrm{dom}(\overline{\partial})$, then

$$(\overline{\partial}^* N_q k, u) = (N_q k, \overline{\partial} u) = (k, N_q \overline{\partial} u) = (k, \overline{\partial} N_{q-1} u) = 0,$$

since $\overline{\partial} N_{q-1} u \in \ker(\overline{\partial})$, which gives $\overline{\partial}^* N_q k = 0$.

□

Since we already know that both operators $\overline{\partial} N_q$ and $\overline{\partial}^* N_q$ are bounded, we can extend both operators $N_{q+1}\overline{\partial}$ and $N_{q-1}\overline{\partial}^*$ to bounded operators on $L^2_{(0,q)}(\Omega)$.

Proposition 2.25. *Let $\alpha \in L^2_{(0,q)}(\Omega)$, with $\overline{\partial}\alpha = 0$. Then $u_0 = \overline{\partial}^* N_q \alpha$ is the canonical solution of $\overline{\partial} u = \alpha$, this means $\overline{\partial} u_0 = \alpha$ and $u_0 \perp ker\,\overline{\partial}$, and*

$$\|\overline{\partial}^* N_q \alpha\| \leq c^{-1/2} \, \|\alpha\|. \tag{2.14}$$

Proof. For $\alpha \in L^2_{(0,q)}(\Omega)$ with $\overline{\partial}\alpha = 0$ we get

$$\alpha = \overline{\partial}\,\overline{\partial}^* N_q\alpha + \overline{\partial}^*\,\overline{\partial}N_q\alpha. \tag{2.15}$$

If we apply $\overline{\partial}$ to the last equality we obtain:

$$0 = \overline{\partial}\alpha = \overline{\partial}\overline{\partial}^*\overline{\partial}N_q\alpha,$$

and since $\overline{\partial}N_q\alpha \in \mathrm{dom}(\overline{\partial}^*)$ we have

$$0 = (\overline{\partial}\,\overline{\partial}^*\overline{\partial}N_q\alpha, \overline{\partial}N_q\alpha) = (\overline{\partial}^*\overline{\partial}N_q\alpha, \overline{\partial}^*\overline{\partial}N_q\alpha) = \|\overline{\partial}^*\overline{\partial}N_q\alpha\|^2. \tag{2.16}$$

Finally we set $u_0 = \overline{\partial}^* N_q\alpha$ and derive from (2.15) and (2.16) that for $\overline{\partial}\alpha = 0$

$$\alpha = \overline{\partial}u_0,$$

and we see that $u_0 \perp \ker\overline{\partial}$, since for $h \in \ker\overline{\partial}$ we get

$$(u_0, h) = (\overline{\partial}^* N_q\alpha, h) = (N_q\alpha, \overline{\partial}h) = 0.$$

It follows that

$$\begin{aligned}
\|\overline{\partial}^* N_q\alpha\|^2 &= (\overline{\partial}\,\overline{\partial}^* N_q\alpha, N_q\alpha) \\
&= (\overline{\partial}\,\overline{\partial}^* N_q\alpha, N_q\alpha) + (\overline{\partial}^*\overline{\partial}N_q\alpha, N_q\alpha) \\
&= (\alpha, N_q\alpha) \leq \|\alpha\|\,\|N_q\alpha\|
\end{aligned}$$

and using (2.9) we obtain

$$\|\overline{\partial}^* N_q\alpha\| \leq c^{-1/2}\,\|\alpha\|.$$

□

We showed that the canonical solution operator S_q for $\overline{\partial}$ coincides with $\overline{\partial}^* N_q$ as operator on

$$L^2_{(0,q)}(\Omega) \cap \ker\overline{\partial}$$

and is a bounded operator.

The $\overline{\partial}$-Neumann operator N can be expressed in terms of the canonical solution operators:

Proposition 2.26.

$$N_q = S_q^*\,S_q + S_{q+1}\,S_{q+1}^*. \tag{2.17}$$

Proof. We use (2.12) and (2.13) to show that

$$\overline{\partial}^* N_q = \overline{\partial}^* N_q^* = (N_q \overline{\partial})^* \text{ and } (\overline{\partial}^* N_q)^* = N_q \overline{\partial},$$

and

$$\overline{\partial} N_q = \overline{\partial}^{**} N_q^* = (N_q \overline{\partial}^*)^* = (\overline{\partial}^* N_{q+1})^* \text{ and } \overline{\partial}^* N_{q+1} = (\overline{\partial} N_q)^* = N_q \overline{\partial}^*,$$

where we applied Lemma 2.3. Hence it follows that for $u \in L^2_{(0,q)}(\Omega)$ we have

$$\begin{aligned} N_q u &= N_q(\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^* \overline{\partial}) N_q u \\ &= (N_q \overline{\partial})(\overline{\partial}^* N_q) u + (N_q \overline{\partial}^*)(\overline{\partial} N_q) u \\ &= (\overline{\partial}^* N_q)^* (\overline{\partial}^* N_q) u + (\overline{\partial}^* N_{q+1})(\overline{\partial}^* N_{q+1})^* u \\ &= S_q^*\, S_q u + S_{q+1}\, S_{q+1}^* u. \end{aligned}$$

□

Proposition 2.27. *Let $P_q : L^2_{(0,q)}(\Omega) \longrightarrow \ker\overline{\partial}$ denote the orthogonal projection, which is the Bergman projection for $q = 0$. Then*

$$P_q = I - \overline{\partial}^* N_{q+1} \overline{\partial}, \tag{2.18}$$

on $dom(\overline{\partial})$.

Proof. First we show that the range of the right hand side of (2.18), which we denote by $\tilde{P}$, coincides with $\ker\overline{\partial}$: for $u \in \mathrm{dom}(\overline{\partial})$ we have

$$\overline{\partial} u - \overline{\partial}\,\overline{\partial}^* N_{q+1} \overline{\partial} u = \overline{\partial} u - \Box N_{q+1} \overline{\partial} u + \overline{\partial}^* \overline{\partial} N_{q+1} \overline{\partial} u = \overline{\partial} u - \overline{\partial} u = 0,$$

where we used (2.12) to show that $\overline{\partial} N_{q+1} \overline{\partial} u = N_{q+2} \overline{\partial}\,\overline{\partial} u = 0$, and since

$$u - \overline{\partial}^* N_{n+1} \overline{\partial} u = u$$

for $u \in \ker\overline{\partial}$, we have shown the first claim. Now we obtain

$$\tilde{P}^* = (I - \overline{\partial}^* N_{q+1} \overline{\partial})^* = I - \overline{\partial}^* N_{q+1} \overline{\partial}^{**} = \tilde{P},$$

and

$$\begin{aligned} \tilde{P}^2 u &= \tilde{P} u - \overline{\partial}^* N_{q+1} \overline{\partial} \tilde{P} u \\ &= \tilde{P} u - \overline{\partial}^* N_{q+1} \overline{\partial} u + \overline{\partial}^* N_{q+1} \overline{\partial}\,\overline{\partial}^* N_{q+1} \overline{\partial} u \\ &= \tilde{P} u - \overline{\partial}^* N_{q+1} \overline{\partial} u + \overline{\partial}^* N_{q+1} (\Box - \overline{\partial}^* \overline{\partial}) N_{q+1} \overline{\partial} u \\ &= \tilde{P} u. \end{aligned}$$

This means that $\tilde{P}$ coincides with P_q on $\mathrm{dom}(\overline{\partial})$. □

Finally we remark that $\tilde{P}$ can be extended to a unique bounded operator on $L^2_{(0,q)}(\Omega)$, with coincides with P_q : for $u \in \mathrm{dom}(\overline{\partial})$ we have by (2.12) that $\overline{\partial}^* N_{q+1}\overline{\partial} u = \overline{\partial}^*\overline{\partial} N_q u$ and $u = \Box N_q u = \overline{\partial}\,\overline{\partial}^* N_q u + \overline{\partial}^*\overline{\partial} N_q u$ is an orthogonal decomposition, which follows from

$$(\overline{\partial}\,\overline{\partial}^* N_q u, \overline{\partial}^*\overline{\partial} N_q u) = (\overline{\partial}\,\overline{\partial}\,\overline{\partial}^* N_q u, \overline{\partial} N_q u) = 0.$$

Hence

$$\|\overline{\partial}^* N_{q+1}\overline{\partial} u\| = \|\overline{\partial}^*\overline{\partial} N_q u\| \leq \|u\|, \ u \in \mathrm{dom}(\overline{\partial}),$$

which proves the claim since $\mathrm{dom}(\overline{\partial})$ is dense in $L^2_{(0,q)}(\Omega)$.

Remark 2.28. Using the symbolic calculus for bounded self-adjoint operators we are able to interpret the basic estimate (2.8) in the following way:

Suppose that

$$\Box : \mathrm{dom}(\Box) \longrightarrow L^2_{(0,1)}(\Omega)$$

is bijective and has a bounded inverse N, then the basic estimate

$$\|u\|^2 \leq C\,(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2), \ u \in \mathrm{dom}(\Box) \ (2.8)$$

must hold.

N is self-adjoint and bounded and, by Proposition 1.29, therefore has a bounded self-adjoint root $N^{1/2}$ which is again injective. By Lemma 2.12 $N^{1/2}$ has a self-adjoint inverse which will be denoted by $N^{-1/2}$. Let $u \in \mathrm{dom}(\Box)$. Then there exists $w \in L^2_{(0,1)}(\Omega)$ such that $Nw = u$. Hence we have $N^{1/2}v = u$, where $v = N^{1/2}w$ and $N^{-1/2}v = w = N^{-1/2}N^{-1/2}u$ is well defined. Now we get

$$\begin{aligned}
\|u\|^2 = \|N^{1/2}v\|^2 \leq C\|v\|^2 &= C\,(N^{-1/2}u, N^{-1/2}u) \\
&= C\,(N^{-1/2}N^{-1/2}u, u) = C\,(N^{-1/2}N^{-1/2}Nw, Nw) \\
&= C\,(w, Nw) = C\,(\Box u, u) \\
&\leq C\,(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2),
\end{aligned}$$

which is the basic estimate (2.8).

2.4 Spectral decomposition of unbounded self-adjoint operators

Let Ω be a subset of $\mathbb{C}$ and M be a σ-algebra in Ω and let H be a Hilbert space. Let

$$E : \mathrm{M} \longrightarrow \mathcal{L}(H)$$

be a resolution of the identity. The symbolic calculus associates to every $f \in L^\infty(E)$ an operator $\Psi(f) \in \mathcal{L}(H)$ by the formula

$$(\Psi(f)x, y) = \int_\Omega f\, dE_{x,y} \ ,\ x, y \in H.$$

Now we will extend this for unbounded measurable functions f.

Lemma 2.29. *Let $f : \Omega \longrightarrow \mathbb{C}$ be a measurable function. Put*

$$\mathcal{D}_f = \{x \in H : \int_\Omega |f|^2\, dE_{x,x} < \infty\}.$$

Then $\mathcal{D}_f$ is a dense subspace of H. If $x, y \in H$, then

$$\int_\Omega |f|\, d|E_{x,y}| \leq \|y\| \left[\int_\Omega |f|^2\, dE_{x,x}\right]^{1/2}. \tag{2.19}$$

If f is bounded and $u = \Psi(f)v$, for $v \in H$, then

$$dE_{x,u} = \overline{f}\, dE_{x,v} \ ,\ x \in H. \tag{2.20}$$

Proof. Let $z = x + y$ and $\omega \in \mathbf{M}$. Then

$$\|E(\omega)z\|^2 \leq (\|E(\omega)x\| + \|E(\omega)y\|)^2 \leq 2(\|E(\omega)x\|^2 + \|E(\omega)y\|^2).$$

Recall that $E_{x,x}(\omega) = (E(\omega)x, x) = (E(\omega)^2 x, x) = \|E(\omega)x\|^2$, so we get from above

$$E_{z,z}(\omega) \leq 2(E_{x,x}(\omega) + E_{y,y}(\omega)),$$

which implies that $\mathcal{D}_f$ is closed under addition. It is clear that $\mathcal{D}_f$ is also closed under scalar multiplication. Therefore $\mathcal{D}_f$ is a subspace of H.

For $n \in \mathbb{N}$, let ω_n be the subset of Ω where $|f| < n$. If $x \in \operatorname{im}(E(\omega_n))$, then

$$E(\omega)x = E(\omega)E(\omega_n)x = E(\omega \cap \omega_n)x \ ,\ \omega \in \mathbf{M}.$$

Hence

$$E_{x,x}(\omega) = E_{x,x}(\omega \cap \omega_n),$$

therefore

$$\int_\Omega |f|^2\, dE_{x,x} = \int_{\omega_n} |f|^2\, dE_{x,x} \leq n^2 \|x\|^2 < \infty.$$

Thus $\operatorname{im}(E(\omega_n)) \subset \mathcal{D}_f$. Since $\Omega = \bigcup_{n=1}^\infty \omega_n$, the countable additivity of $\omega \mapsto E(\omega)y$ implies that $y = \lim_{n\to\infty} E(\omega_n)y$ for every $y \in H$. Hence y lies in the closure of $\mathcal{D}_f$ and $\mathcal{D}_f$ is dense in H.

If $x, y \in H$ and f is bounded and measurable, the Radon-Nikodym Theorem (see for instance [8]) implies that there is a measurable function g on Ω such that $|g| = 1$ on Ω and

$$gf\,dE_{x,y} = |f|\,d|E_{x,y}|.$$

Hence

$$\int_\Omega |f|\,d|E_{x,y}| = (\Psi(gf)x, y) \le \|\Psi(gf)x\|\,\|y\|. \tag{2.21}$$

As in Chapter 9.1. we get

$$\|\Psi(gf)x\|^2 = \int_\Omega |gf|^2\,dE_{x,x} = \int_\Omega |f|^2\,dE_{x,x},$$

which implies (2.19) for a bounded function f. The general case is done as in the first assertion of this proposition.

To show (2.20) we consider an arbitrary bounded measurable function h and have

$$\begin{aligned}\int_\Omega h\,dE_{x,u} &= (\Psi(h)x, u) = (\Psi(h)x, \Psi(f)v) \\ &= (\Psi(\overline{f})\Psi(h)x, v) = (\Psi(\overline{f}h)x, v) \\ &= \int_\Omega h\overline{f}\,dE_{x,v}.\end{aligned}$$

□

In the next step we carry over the results of Section 9.1 (symbolic calculus) for unbounded measurable functions.

Proposition 2.30. *Let E be a resolution of identity on Ω.*

(a) To every measurable $f : \Omega \longrightarrow \mathbb{C}$ corresponds a densely defined closed operator $\Psi(f)$ on H, with domain $dom(\Psi(f)) = \mathcal{D}_f$, which is characterized by

$$(\Psi(f)x, y) = \int_\Omega f\,dE_{x,y}\ ,\ x \in \mathcal{D}_f\ ,\ y \in H \tag{2.22}$$

and which satisfies

$$\|\Psi(f)x\|^2 = \int_\Omega |f|^2\,dE_{x,x}\ ,\ x \in \mathcal{D}_f. \tag{2.23}$$

(b) If f and g are measurable, then

$$\Psi(f)\Psi(g) \subset \Psi(fg),$$

which means that $dom(\Psi(f)\Psi(g)) \subset dom(\Psi(fg))$ and $\Psi(f)\Psi(g) = \Psi(fg)$ on $dom(\Psi(f)\Psi(g))$, and

$$dom(\Psi(f)\Psi(g)) = \mathcal{D}_g \cap \mathcal{D}_{fg}.$$

Hence $\boldsymbol{\Psi}(f)\boldsymbol{\Psi}(g) = \boldsymbol{\Psi}(fg)$ *if and only if* $\mathcal{D}_{fg} \subseteq \mathcal{D}_g$.

(c) For every measurable $f : \Omega \longrightarrow \mathbb{C}$,

$$\boldsymbol{\Psi}(f)^* = \boldsymbol{\Psi}(\overline{f}) \text{ and } \boldsymbol{\Psi}(f)\boldsymbol{\Psi}(f)^* = \boldsymbol{\Psi}(|f|^2) = \boldsymbol{\Psi}(f)^*\boldsymbol{\Psi}(f).$$

Proof. Fix $x \in \mathcal{D}_f$, then the conjugate-linear functional $y \mapsto \int_\Omega f\, dE_{x,y}$ is bounded on H (Lemma 2.29). Hence there is a unique element $\boldsymbol{\Psi}(f)x \in H$ satisfying (2.22) and

$$\|\boldsymbol{\Psi}(f)x\|^2 \leq \int_\Omega |f|^2\, dE_{x,x}\ ,\ x \in \mathcal{D}_f. \tag{2.24}$$

The linearity of $\boldsymbol{\Psi}(f)$ on $\mathcal{D}_f$ follows from (2.22) and the fact that $E_{x,y}$ is linear in x.

Now we associate with each f its truncations $f_n = f\phi_n$, where $\phi_n(p) = 1$ if $|f(p)| \leq n$, and $\phi_n(p) = 0$, if $|f(p)| > n$.

Then $\mathcal{D}_{f-f_n} = \mathcal{D}_f$, since each f_n is bounded, and therefore (2.24) shows, using the dominated convergence theorem, that

$$\|\boldsymbol{\Psi}(f)x - \boldsymbol{\Psi}(f_n)x\|^2 \leq \int_\Omega |f - f_n|^2\, dE_{x,x} \to 0\ ,\ \text{as } n \to \infty, \tag{2.25}$$

for every $x \in \mathcal{D}_f$. Since f_n is bounded, (2.23) holds for f_n. Hence (2.25) implies (2.23) for f.

This proves (a), except for the assertion that $\boldsymbol{\Psi}(f)$ is closed. This will follow from (c) (to be proved) and Lemma 2.5 with $\overline{f}$ instead of f.

(b) First assume that f is bounded. Then $\mathcal{D}_{fg} \subset \mathcal{D}_g$. If $v \in H$ and $u = \boldsymbol{\Psi}(\overline{f})v$, we get from Section 9.1 and (2.20) that

$$\begin{aligned}
(\boldsymbol{\Psi}(f)\boldsymbol{\Psi}(g), v) &= (\boldsymbol{\Psi}(g)x, \boldsymbol{\Psi}(\overline{f})v) = (\boldsymbol{\Psi}(g)x, u) \\
&= \int_\Omega g\, dE_{x,u} = \int_\Omega fg\, dE_{x,v} \\
&= (\boldsymbol{\Psi}(fg)x, v)
\end{aligned}$$

So we have shown that

$$\boldsymbol{\Psi}(f)\boldsymbol{\Psi}(g)x = \boldsymbol{\Psi}(fg)x\ ,\ x \in \mathcal{D}_g\ ,\ f \in L^\infty.$$

The last line implies that for $y = \boldsymbol{\Psi}(g)x$

$$\int_\Omega |f|^2\, dE_{y,y} = \int_\Omega |fg|^2\, dE_{x,x}\ ,\ x \in \mathcal{D}_g\ ,\ f \in L^\infty. \tag{2.26}$$

Using truncation we see that (2.26) also holds for arbitrary f (possibly unbounded). Since $\operatorname{dom}(\boldsymbol{\Psi}(f)\boldsymbol{\Psi}(g))$ consists of all $x \in \mathcal{D}_g$ such that $y = \boldsymbol{\Psi}(g)x \in \mathcal{D}_f$ and since (2.26) shows that $y \in \mathcal{D}_f$ if and only if $x \in \mathcal{D}_{fg}$, we see that

$$\operatorname{dom}(\boldsymbol{\Psi}(f)\boldsymbol{\Psi}(g)) = \mathcal{D}_g \cap \mathcal{D}_{fg}.$$

If $x \in \mathcal{D}_g \cap \mathcal{D}_{fg}$, and $y = \Psi(g)x$, and if the truncations f_n are defined as above, we obtain $f_n \to f$ in $L^2(E_{x,x})$ and $f_n g \to fg$ in $L^2(E_{y,y})$ and finally

$$\Psi(f)\Psi(g)x = \Psi(f)y = \lim_{n\to\infty} \Psi(f_n)y = \lim_{n\to\infty} \Psi(f_n g)x = \Psi(fg)x.$$

This proves (b).

(c) Suppose that $x \in \mathcal{D}_f$ and $y \in \mathcal{D}_{\overline{f}} = \mathcal{D}_f$. It follows from (2.25) and Section 9.1 that

$$(\Psi(f)x, y) = \lim_{n\to\infty} (\Psi(f_n)x, y) = \lim_{n\to\infty} (x, \Psi(\overline{f}_n)y) = (x, \Psi(\overline{f})y).$$

Hence $y \in \mathrm{dom}(\Psi(f)^*)$, and $\mathrm{dom}(\Psi(\overline{f}) \subseteq \mathrm{dom}(\Psi(f)^*)$. If we can show that each $u \in \mathrm{dom}(\Psi(f)^*)$ lies in $\mathcal{D}_f$, we obtain $\Psi(f)^* = \Psi(\overline{f})$. Fix u for this purpose and put $v = \Psi(f)^* u$. Since $f_n = f\phi_n$, the multiplication theorem yields

$$\Psi(f_n) = \Psi(f)\Psi(\phi_n).$$

Since $\Psi(\phi_n)$ is self-adjoint and bounded, we have by Lemma 2.3 and Section 9.1 that

$$\Psi(\phi_n)\Psi(f)^* = [\Psi(f)\Psi(\phi_n)]^* = \Psi(f_n)^* = \Psi(\overline{f}_n).$$

Hence

$$\Psi(\phi_n)v = \Psi(\overline{f}_n)u \ , \ n \in \mathbb{N}.$$

Since $|\phi_n| \leq 1$ we have now

$$\int_\Omega |f_n|^2 \, dE_{u,u} = \int_\Omega |\phi_n|^2 \, dE_{v,v} \leq E_{v,v}(\Omega) \ , \ n \in \mathbb{N}.$$

Hence $u \in \mathcal{D}_f$.

Finally, since $\mathcal{D}_{f\overline{f}} \subset \mathcal{D}_f$, another application of the multiplication theorem gives the last assertion of (c).

$\square$

Definition 2.31. The resolvent set of a linear operator $T : \mathrm{dom}(T) \longrightarrow H$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is an injective mapping of $\mathrm{dom}(T)$ onto H whose inverse belongs to $\mathcal{L}(H)$. The spectrum $\sigma(T)$ of T is the complement of the resolvent set of T.

First we collect some informations about the spectrum of an unbounded operator.

Lemma 2.32. *If the spectrum $\sigma(T)$ of an operator T does not coincide with the whole of the complex plane $\mathbb{C}$ then T must be a closed operator. The spectrum of a linear operator is always closed. Moreover, if $\zeta \notin \sigma(T)$ and $c := \|R_T(\zeta)\| = \|(\zeta I - T)^{-1}\|$, then the spectrum $\sigma(T)$ does not intersect the ball $\{w \in \mathbb{C} : |\zeta - w| < c^{-1}\}$. The resolvent operator R_T is a holomorphic operator valued function.*

Proof. For $\zeta \notin \sigma(T)$ let $S = (\zeta I - T)^{-1}$ which is a bounded operator. Let $x_n \in \mathrm{dom}(T)$ with $x = \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Tx_n = y$ and set $u_n = (\zeta I - T)x_n$. Then

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (\zeta x_n - Tx_n) = \zeta x - y,$$

therefore

$$S(\zeta x - y) = \lim_{n\to\infty} Su_n = \lim_{n\to\infty} x_n = x.$$

This implies $x \in \mathrm{dom}(T)$ and $(\zeta I - T)x = \zeta x - y$, or $Tx = y$. Hence T is closed.

The remainder of the proof is similar to the case when T is bounded, see Lemma 1.3. □

In the next proposition we refer to the concept of the essential range of a function with respect to a given resolution of the identity (Definition 1.24).

Proposition 2.33. *Let E be a resolution of the identity on Ω and $f : \Omega \longrightarrow \mathbb{C}$ a measurable function. For $\alpha \in \mathbb{C}$ put*

$$\omega_\alpha = \{p \in \Omega : f(p) = \alpha\}.$$

(a) If α is in the essential range of f and $E(\omega_\alpha) \neq 0$, then $\alpha I - \Psi(f)$ is not injective.

(b) If α is in the essential range of f but $E(\omega_\alpha) = 0$, then $\alpha I - \Psi(f)$ is an injective mapping of $\mathcal{D}_f$ onto a proper dense subspace of H, and there exists vectors $x_n \in H$, with $\|x_n\| = 1$, such that

$$\lim_{n\to\infty} [\alpha x_n - \Psi(f)x_n] = 0.$$

(c) $\sigma(\Psi(f))$ is the essential range of f.

One says that α lies in the point spectrum of $\Psi(f)$ in case (a) and in the continuous spectrum of $\Psi(f)$ in case (b).

Proof. Without loss of generality we can assume that $\alpha = 0$.

(a) If $E(\omega_0) \neq 0$, there exists $x_0 \in \mathrm{im}(E(\omega_0))$ with $\|x_0\| = 1$. Let ϕ_0 be the characteristic function of ω_0. Then $f\phi_0 = 0$, and $\Psi(f)\Psi(\phi_0) = 0$. Since $\Psi(\phi_0) = E(\omega_0)$, it follows that

$$\Psi(f)x_0 = \Psi(f)E(\omega_0)x_0 = \Psi(f)\Psi(\phi_0)x_0 = 0.$$

(b) Now we have $E(\omega_0) = 0$ but $E(\omega_n) \neq 0$ for $n \in \mathbb{N}$ where

$$\omega_n = \{p \in \Omega : |f(p)| < 1/n\}.$$

Let $x_n \in \operatorname{im}(E(\omega_n))$ with $\|x_n\| = 1$ and let ϕ_n be the characteristic functions of ω_n. As in (a) we obtain

$$\|\Psi(f)x_n\| = \|\Psi(f\phi_n)x_n\| \leq \|\Psi(f\phi_n)\| = \|f\phi_n\|_\infty \leq 1/n.$$

Thus $\Psi(f)x_n \to 0$ although $\|x_n\| = 1$.

If $\Psi(f)x = 0$ for some $x \in \mathcal{D}_f$, then

$$\int_\Omega |f|^2 \, dE_{x,x} = \|\Psi(f)x\|^2 = 0.$$

Since $|f| > 0$ almost everywhere $(E_{x,x})$, we must have $E_{x,x}(\Omega) = 0$. But $E_{x,x}(\Omega) = \|x\|^2$. Hence $\Psi(f)$ is injective. Similarly $\Psi(f)^* = \Psi(\overline{f})$ is injective. If $y \perp \operatorname{im}(\Psi(f))$, then $x \mapsto (\Psi(f)x, y) = 0$ is continuous in $\mathcal{D}_f$, hence $y \in \operatorname{dom}(\Psi(f)^*)$, and

$$(x, \Psi(\overline{f})y) = (\Psi(f)x, y) = 0 \ , \ x \in \mathcal{D}_f.$$

Hence $\Psi(\overline{f})y = 0$ and $y = 0$. Therefore $\operatorname{im}(\Psi(f))$ is dense in H.

Since $\Psi(f)$ is closed, so is $\Psi(f)^{-1}$. If $\operatorname{im}(\Psi(f)) = H$, the closed graph theorem would imply that $\Psi(f)^{-1} \in \mathcal{L}(H)$. This is impossible in view of the sequence $(x_n)_n$ constructed above. Hence (b) is proved.

(c) It follows from (a) and (b) that the essential range of f is a subset of $\sigma(\Psi(f))$. Now assume that 0 is not in the essential range of f. Then $g = 1/f \in L^\infty(E)$, and $fg = 1$, hence $\Psi(f)\Psi(g) = \Psi(1) = I$, which proves that $\operatorname{im}(\Psi(f)) = H$. Since $|f| > 0$, we have that $\Psi(f)$ is injective, as in the proof of (b). By the closed graph theorem, $\Psi(f)^{-1} \in \mathcal{L}(H)$. Therefore $0 \notin \sigma(\Psi(f))$ and (c) is proved.

□

In the following proposition we describe the change of measure principle.

Proposition 2.34. *Let* M *and* M' *be* σ*-algebras in the sets* $\Omega, \Omega' \subseteq \mathbb{C}$ *and let* $E : \mathrm{M} \longrightarrow \mathcal{L}(H)$ *be a resolution of identity, and suppose that* $\Phi : \Omega \longrightarrow \Omega'$ *has the property that* $\Phi^{-1}(\omega') \in \mathrm{M}$ *for every* $\omega' \in \mathrm{M}'$.

If $E'(\omega') = E(\Phi^{-1}(\omega'))$, *then* $E' : \mathrm{M}' \longrightarrow \mathcal{L}(H)$ *is a resolution of the identity, and*

$$\int_{\Omega'} f \, dE'_{x,y} = \int_\Omega (f \circ \Phi) \, dE_{x,y} \tag{2.27}$$

for every M'*-measurable* $f : \Omega' \longrightarrow \mathbb{C}$ *for which either of these integrals exists.*

Proof. A straightforward verification gives that E' is again a resolution of the identity For characteristic functions (2.27) is just the definition of E'. So (2.27) follows for simple functions and also in the general case.

□

In order to derive the general spectral theorem for unbounded self-adjoint operators we will use the Cayley transform.

The mapping
$$t \mapsto \frac{t-i}{t+i}$$
sets up a bijection between the real line and the unit circle minus the point 1. The symbolic calculus developed in Section 9.2 therefore shows that every self-adjoint operator $T \in \mathcal{L}(H)$ gives rise to a unitary operator
$$U = (T - iI)(T + iI)^{-1},$$
and that every unitary U whose spectrum does not contain the point 1 is obtained in this way. This relation will now be extended to unbounded symmetric operators.

If T is a symmetric operator, we have
$$\|Tx + ix\|^2 = (Tx + ix, Tx + ix) = \|x\|^2 + \|Tx\|^2 = \|Tx - ix\|^2,$$
for $x \in \mathrm{dom}(T)$. This implies that $(T + iI)$ is injective, and that there is an isometry U with $\mathrm{dom}(U) = \mathrm{im}(T + iI)$, and $\mathrm{im}(U) = \mathrm{im}(T - iI)$, defined by
$$U(Tx + ix) = T - ix\ ,\ x \in \mathrm{dom}(T).$$
Since $(T + iI)^{-1}$ maps $\mathrm{dom}(U)$ onto $\mathrm{dom}(T)$, we can write
$$U = (T - iI)(T + iI)^{-1}.$$
This operator U is called the Cayley transform of T.

Lemma 2.35. *Let U be an isometry, i.e. $\|Ux\| = \|x\|$ for all $x \in$ dom(U).*
(a) For $x, y \in$ dom(U), we have $(Ux, Uy) = (x, y)$.
(b) If im$(I - U)$ is dense in H, then $I - U$ is injective.
(c) If any one of the three spaces dom(U), im(U) and $\mathcal{G}(U)$ is closed, so are the other two.

Proof. (a) Follows from the polarization identity:
$$(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2) \tag{2.28}$$
(b) Let $x \in \mathrm{dom}(U)$ and $(I - U)x = 0$. Then $x = Ux$ and
$$(x, (I - U)y) = (x, y) - (x, Uy) = (Ux, Uy) - (x, Uy) = 0$$
for every $y \in \mathrm{dom}(U)$. This implies $x \perp \mathrm{im}(I - U)$, so that $x = 0$ if $\mathrm{im}(I - U)$ is dense in H.

(c) follows from

$$\|Ux - Uy\| = \|x - y\| = \frac{1}{\sqrt{2}}\|(x, Ux) - (y, Uy)\| \ , \ x, y \in \operatorname{dom}(U),$$

where in the last term $(x, Ux), (y, Uy) \in \mathcal{G}(U)$ are elements of the graph of U and

$$\|(x, Ux) - (y, Uy)\| = (\|x - y\|^2 + \|Ux - Uy\|^2)^{1/2}.$$

□

Proposition 2.36. *Let T be a symmetric operator on H (not necessarily densely defined) and let U be its Cayley transform. The following statements are true:*

(a) U is closed if and only if T is closed.

(b) $\mathrm{im}(I - U) = \mathrm{dom}(T)$, and $I - U$ is injective, and T can be reconstructed from U by

$$T = i(I + U)(I - U)^{-1}.$$

The Cayley transforms of distinct symmetric operators are distinct.

(c) U is unitary if and only if T is self-adjoint.

Conversely, if V is an operator in H which is an isometry, and if $I - V$ is injective, then V is the Cayley transform of a symmetric operator in H.

Proof. (a) The identity $\|Tx + ix\|^2 = \|x\|^2 + \|Tx\|^2$ implies that $(T + iI)x \leftrightarrow (x, Tx)$ is an isometric one-to-one correspondence between $\operatorname{im}(T + iI)$ and the graph $\mathcal{G}(T)$ of T. Hence T is closed if and only if $\operatorname{im}(T + iI)$ is closed. By Lemma 2.35, U is closed if and only if $\operatorname{dom}(U)$ is closed. But, by the definition of the Cayley transform, $\operatorname{dom}(U) = \operatorname{im}(T + iI)$, which proves (a).

(b) The one-to-one correspondence $x \leftrightarrow z$ between $\operatorname{dom}(T)$ and $\operatorname{dom}(U) = \operatorname{im}(T + iI)$, given by

$$z = Tx + ix \ , \ Uz = Tx - ix$$

can be written in the form

$$(I - U)z = 2ix \ , \ (I + U)z = 2Tx.$$

Hence $(I - U)$ is injective and $\operatorname{im}(I - U) = \operatorname{dom}(T)$, therefore $(I - U)^{-1}$ maps $\operatorname{dom}(T)$ onto $\operatorname{dom}(U)$, and

$$2Tx = (I + U)z = (I + U)(I - U)^{-1}(2ix) \ , \ x \in \operatorname{dom}(T).$$

This proves (b).

(c) Assume that T is self-adjoint. Then, by Lemma 2.13,

$$\operatorname{im}(I + T^*T) = \operatorname{im}(I + T^2) = H. \tag{2.29}$$

We have

$$(T+iI)(T-iI) = T^2 + I = (T-iI)(T+iI),$$

where all operators have domain $\mathrm{dom}(T^2)$. Hence, (2.29) implies that

$$\mathrm{dom}(U) = \mathrm{im}(T+iI) = H \tag{2.30}$$

and

$$\mathrm{im}(U) = \mathrm{im}(T-iI) = H. \tag{2.31}$$

Now $(U^*Ux, x) = (Ux, Ux) = (x, x)$ for every $x \in H$, which implies $U^*U = I$ and U is unitary.

Now assume that U is unitary. Then

$$(\mathrm{im}(I-U))^\perp = \ker(I-U)^* = \{0\},$$

and $\mathrm{dom}(T) = \mathrm{im}(I-U)$ is dense in H. Thus T^* is defined and $T \subset T^*$. Fix $y \in \mathrm{dom}(T^*)$. Since $\mathrm{im}(T+iI) = \mathrm{dom}(U) = H$, there exists $y_0 \in \mathrm{dom}(T)$ such that

$$(T^* + iI)y = (T+iI)y_0 = (T^* + iI)y_0.$$

Set $y_1 = y - y_0$. Then $y_1 \in \mathrm{dom}(T^*)$ and, for every $x \in \mathrm{dom}(T)$ we have

$$((T-iI)x, y_1) = (x, (T^* + iI)y_1) = (x, 0) = 0.$$

Thus $y_1 \perp \mathrm{im}(T-iI) = \mathrm{im}(U) = H$, so $y_1 = 0$ and $y = y_0 \in \mathrm{dom}(T)$. Hence $\mathrm{dom}(T) = \mathrm{dom}(T^*)$ and (c) is proved.

Finally, let V be as in the statement of the converse. Then there is a one-to-one correspondence $z \leftrightarrow x$ between $\mathrm{dom}(V)$ and $\mathrm{im}(I-V)$, given by

$$x = z - Vz.$$

Define S on $\mathrm{dom}(S) = \mathrm{im}(I-V)$ by

$$Sx = i(z + Vz) \text{ if } x = z - Vz. \tag{2.32}$$

If $x, y \in \mathrm{dom}(S)$, then $x = z - Vz$ and $y = u - Vu$ for some $z, u \in \mathrm{dom}(V)$. Since V is an isometry, it follows from Lemma 2.35 that

$$\begin{aligned}(Sx, y) &= i(z + Vz, u - Vu) = i(Vz, u) - i(z, Vu) \\ &= (z - Vz, iu + iVu) = (x, Sy).\end{aligned}$$

Hence S is symmetric. For $z \in \mathrm{dom}(V)$, (2.32) can be written in the form

$$2iVz = Sx - ix \ , \ 2iz = Sx + ix,$$

hence, if $x \in \mathrm{dom}(S)$, we obtain

$$V(Sx + ix) = Sx - ix$$

and that $\mathrm{dom}(V) = \mathrm{im}(S+iI)$. Therefore V is the Cayley transform of S.

□

At this point we use the methods developed above to prove a key result about the spectrum of an unbounded self-adjoint operator. It transfers properties of unbounded self-adjoint operators to the bounded resolvent operators.

Let $T \in \mathcal{L}(H)$ be a self-adjoint operator. If $\Im\lambda \neq 0$, then

$$|\Im\lambda|\,\|u\|^2 = |\Im((T-\lambda I)u,u)| \leq \|(T-\lambda I)u\|\,\|u\|, \tag{2.33}$$

for all $u \in H$, where we used that $(Tu,u) = (u,Tu)$ is real.

This implies that $T-\lambda I$ is injective and has closed range. As

$$(\operatorname{im}(T-\lambda I))^{\perp} = \ker(T-\overline{\lambda}I)$$

and this kernel reduces to $\{0\}$, we obtain that $T-\lambda I$ is bijective.

This follows also from the Lax-Milgram Theorem, once one has observed that

$$|((T-\lambda I)u,u)| \geq |\Im\lambda|\,\|u\|^2. \tag{2.34}$$

Theorem 2.37. *Let T be a bounded self-adjoint operator. Then $\sigma(T)$ is contained in $[m,M]$, where*

$$m = \inf_{u\neq 0}\frac{(Tu,u)}{(u,u)} \quad \text{and} \quad M = \sup_{u\neq 0}\frac{(Tu,u)}{(u,u)}.$$

Moreover m and M belong to the spectrum of T.

Proof. We already know that the spectrum is real. If λ Is real and $\lambda > M$, we can apply the Lax-Milgram Theorem for the sesquilinear form

$$(u,v) \mapsto \lambda(u,v) - (Tu,v),$$

to see that $\lambda \notin \sigma(T)$. To show that $M \in \sigma(T)$, we apply the Cauchy-Schwarz inequality to the scalar product

$$(u,v) \mapsto M(u,v) - (Tu,v).$$

We get

$$|(Mu-Tu,v)| \leq (Mu-Tu,u)^{1/2}(Mv-Tv,v)^{1/2}.$$

In particular

$$\|Mu-Tu\| \leq \|MI-T\|^{1/2}(Mu-Tu,u)^{1/2}. \tag{2.35}$$

Now let $(u_n)_n$ be a sequence in H such that $\|u_n\| = 1$ for each $n \in \mathbb{N}$ and $(Tu_n,u_n) \to M$ as $n \to \infty$. By (2.35)

$$\lim_{n\to\infty}(MI-T)u_n = 0,$$

and this implies $M \in \sigma(T)$, otherwise

$$u_n = (MI-T)^{-1}(MI-T)u_n$$

would tend to 0 in contradiction to $\|u_n\| = 1$. □

Corollary 2.38. *Let $T \in \mathcal{L}(H)$ be a self-adjoint operator such that $\sigma(T) = \{0\}$. Then $T = 0$.*

Proof. By Theorem 2.37 we have $m = M = 0$, hence $(Tu, u) = 0$, for each $u \in H$. As (Tu, v) can be written as a linear combination of terms of the type (Tw, w), we obtain $T = 0$.

□

Proposition 2.39. *The spectrum $\sigma(T)$ of any self-adjoint operator T is real and non-empty. If $\zeta \notin \mathbb{R}$ then*

$$\|(\zeta I - T)^{-1}\| \leq |\Im\zeta|^{-1}. \tag{2.36}$$

Moreover,

$$(\overline{\zeta} I - T)^{-1} = ((\zeta I - T)^{-1})^*. \tag{2.37}$$

Proof. Let $\zeta = \xi + i\eta$ and $\eta \neq 0$ and set $K = \frac{1}{\eta}(T - \xi I)$. Using Lemma 2.4, it follows that $K^* = K$. Let $f \in \mathrm{dom}(K)$ such that $Kf = K^*f = if$, then $i(f, f) = (Kf, f) = (f, Kf) = -i(f, f)$, which implies $f = 0$ and that $K - iI$ is injective. In a similar way one shows that $K + iI$ is injective. The proof of Proposition 2.36 part (a) implies that $\mathrm{im}(K \pm iI)$ is closed. Now we obtain from Lemma 2.8 that $\mathrm{im}(K \pm iI)^\perp = \ker(K \pm iI) = \{0\}$. Therefore $(K \pm iI)^{-1}$ is defined on the whole of H. Since we have

$$\|Kx \pm ix\|^2 = \|Kx\|^2 + \|x\|^2 \ , \ x \in \mathrm{dom}(K),$$

we get

$$\|(K \pm iI)^{-1}y\| = \|(K \pm iI)^{-1}(K \pm iI)x\| = \|x\| \leq \|(K \pm iI)x\| = \|y\|,$$

for each $y \in H$, which implies that

$$\|(K \pm iI)^{-1}\| \leq 1. \tag{2.38}$$

Thus $\pm i \notin \sigma(K)$ and hence $\zeta \notin \sigma(T)$. In addition (2.38) implies (2.36).

Now let $x_1, x_2 \in \mathrm{dom}(T)$. Then

$$((T - \zeta I)x_1, x_2) = (x_1, (T - \overline{\zeta} I)x_2).$$

Putting $y_1 = (T - \zeta I)x_1$ and $y_2 = (T - \overline{\zeta} I)x_2$ and rewriting the last equation in terms of y_1 and y_2 yields (2.37).

Suppose T has empty spectrum. Then T^{-1} is a bounded self-adjoint operator. We claim that $\sigma(T^{-1}) = \{0\}$. For $\lambda \neq 0$ we can write the inverse of $T^{-1} - \lambda I$ in the form

$$(T^{-1} - \lambda I)^{-1} = \lambda^{-1} T(\lambda^{-1} I - T)^{-1} = -\lambda^{-1} I + \lambda^{-2}(\lambda^{-1} I - T)^{-1},$$

which is a bounded operator. Now Corollary 2.38 gives that $T^{-1} = 0$, which contradicts that $T \circ T^{-1} = I$. □

The Cayley transform is now used to reduce the construction of the spectral decomposition of an unbounded self-adjoint operator to the spectral decomposition of a unitary operator.

Proposition 2.40. *Let T be an unbounded self-adjoint operator ($T = T^*$ and $dom(T) = dom(T^*)$). Then there exists a uniquely determined resolution of the identity E on the Borel subsets of $\mathbb{R}$, such that*

$$(Tx, y) = \int_{-\infty}^{\infty} t\, dE_{x,y}(t) \ , \ x \in dom(T) \ , \ y \in H. \tag{2.39}$$

Moreover, E is concentrated on the spectrum $\sigma(T) \subset \mathbb{R}$ of T, in the sense that $E(\sigma(T)) = I$.

Proof. Let U be the Cayley transform of T and let $\boldsymbol{\Omega}$ be the unit circle with the point 1 removed. Let E' be the spectral decomposition of U (Proposition 1.25). By Proposition 2.36 $I - U$ is injective and, by Proposition 1.27 $E'(\{1\}) = 0$. Hence

$$(Ux, y) = \int_{\boldsymbol{\Omega}} \lambda\, dE'_{x,y}(\lambda) \ , x, y \in H. \tag{2.40}$$

Define

$$f(\lambda) = \frac{i(1+\lambda)}{(1-\lambda)} \ , \ \lambda \in \boldsymbol{\Omega}.$$

We define $\boldsymbol{\Psi}(f)$ as in Proposition 2.30 with E' in place of E.

$$(\boldsymbol{\Psi}(f)x, y) = \int_{\boldsymbol{\Omega}} f\, dE'_{x,y} \ , \ x \in \mathcal{D}_f \ , \ y \in H. \tag{2.41}$$

Since f is real-valued, $\boldsymbol{\Psi}(f)$ is self-adjoint (Proposition 2.30). Since

$$f(\lambda)(1-\lambda) = i(1+\lambda),$$

the symbolic calculus gives

$$\boldsymbol{\Psi}(f)(I - U) = i(I + U). \tag{2.42}$$

This implies in particular

$$\mathrm{im}(I - U) \subset \mathrm{dom}(\boldsymbol{\Psi}(f)). \tag{2.43}$$

By Proposition 2.36

$$T(I - U) = i(I + U) \tag{2.44}$$

and $\mathrm{dom}(T) = \mathrm{im}(I - U) \subset \mathrm{dom}(\boldsymbol{\Psi}(f))$. (2.43) and (2.44) imply that $\boldsymbol{\Psi}(f)$ is a self-adjoint extension of the self-adjoint operator T. Thus we have $T \subset \boldsymbol{\Psi}(f)$ and $\boldsymbol{\Psi}(f) = \boldsymbol{\Psi}(f)^* \subset T^* = T$, hence $\boldsymbol{\Psi}(f) = T$. In addition we get

$$(Tx, y) = \int_{\boldsymbol{\Omega}} f\, dE'_{x,y} \ , x \in \mathrm{dom}(T) \ , y \in H. \tag{2.45}$$

By Proposition 2.33 (c) , $\sigma(T)$ is the essential range of f. Thus $\sigma(T) \subset \mathbb{R}$. Since f is injective in $\mathbf{\Omega}$, we can define $E(f(\omega)) = E'(\omega)$ for every Borel set $\omega \subset \mathbf{\Omega}$, to obtain the desired resolution E which converts (2.45) into (2.39).The uniqueness of E follows from the uniqueness of the representation (2.40).

□

The symbolic calculus is now used to prove the following assertions.

Proposition 2.41. *Let T be a self-adjoint operator on H.*

(a) $(Tx, x) \geq 0$ for every $x \in dom(T)$ (briefly $T \geq 0$) if and only if $\sigma(T) \subset [0, \infty)$.

(b) If $T \geq 0$, there exists a unique self-adjoint operator $S \geq 0$ such that $dom(S) \supseteq dom(T)$ and $S^2 = T$ on $dom(T)$, The symbolic calculus implies that $x \in dom(T)$ if and only if $x \in dom(S)$ and also $Sx \in dom(S)$. In addition $dom(T)$ is a core of S.

Proof. (a) see Proposition 1.29 (a).

(b) Assume $T \geq 0$, so that $\sigma(T) \subset [0, \infty)$, and

$$(Tx, y) = \int_0^\infty t \, dE_{x,y}(t) \ , x \in \mathrm{dom}(T) \ , y \in H, \tag{2.46}$$

where $\mathrm{dom}(T) = \{x \in H : \int_0^\infty t^2 \, dE_{x,x}(t) < \infty\}$. Let $s(t)$ be the nonnegative square root of $t \geq 0$ and put $\Psi(s) = S$, explicitly

$$(Sx, y) = \int_0^\infty s(t) \, dE_{x,y}(t) \ , x \in \mathcal{D}_s \ , y \in H. \tag{2.47}$$

By Proposition 2.30 we obtain that $S^2 = T$ on $\mathrm{dom}(T)$ and $S \geq 0$.

To prove uniqueness, suppose R is self-adjoint, $R \geq 0$ and $R^2 = T$, and E^R is its spectral decomposition:

$$(Rx, y) = \int_0^\infty t \, dE^R_{x,y}(t) \ , x \in \mathrm{dom}(R) \ , y \in H. \tag{2.48}$$

We apply Proposition 2.34 with $\mathbf{\Omega} = [0, \infty)$, $\phi(t) = t^2$, $f(t) = t$, and

$$E'(\phi(\omega)) = E^R(\omega) \text{ for } \omega \subset [0, \infty), \tag{2.49}$$

to obtain

$$(Tx, y) = (R^2 x, y) = \int_0^\infty t^2 \, dE^R_{x,y(t)} = \int_0^\infty t \, dE'_{x,y}(t). \tag{2.50}$$

(2.46) and (2.50) and the uniqueness statement in Proposition 2.40 show that $E' = E$. By (2.49), E determines E^R, and hence R.

The statement about the domains of the operators S and T follows from the symbolic calculus and the definition of the domains there (see Proposition 2.30). As $S^* = S$, Lemma 2.16 immediately implies that $\operatorname{dom}(T)$ is a core of S. □

As applications of these results we prove a useful characterization of self-adjoint and essentially self-adjoint operators.

Proposition 2.42. *Let T be a closed symmetric operator. Then the following statements are equivalent:*
(i) T is self-adjoint;
(ii) $\ker(T^* + iI) = \{0\}$ and $\ker(T^* - iI) = \{0\}$;
(iii) $\operatorname{im}(T + iI) = H$ and $\operatorname{im}(T - iI) = H$.

Proof. (i) implies (ii): by Proposition 2.39 $\pm i \notin \sigma(T)$.
(ii) implies (iii): Notice that $\ker(T^* \pm iI) = \{0\}$ if and only if $\operatorname{im}(T \mp iI)$ is dense in H. This follows easily from

$$(Tu \pm iu, v) = (u, T^* \mp iv),$$

for $u, v \in \operatorname{dom}(T)$. So it remains to show that $\operatorname{im}(T \mp iI)$ is closed. The symmetry of T implies that

$$\|(T \mp iI)u\|^2 = \|Tu\|^2 + \|u\|^2, \tag{2.51}$$

for $u \in \operatorname{dom}(T)$. Now, since T is closed, we easily obtain that $\operatorname{im}(T \mp iI)$ is closed.
(iii) implies (i): Let $u \in \operatorname{dom}(T^*)$. By (iii) there exists $v \in \operatorname{dom}(T)$ such that

$$(T - iI)v = (T^* - iI)u.$$

Since T is symmetric, we have also $(T^* - iI)(v - u) = 0$. But, if $(T + iI)$ is surjective, then $(T^* - iI)$ is injective (Lemma 2.8) and we obtain $u = v$. This proves that $u \in \operatorname{dom}(T)$ and that T is self-adjoint. □

We proved during the assertion (ii) implies (iii) that

Lemma 2.43. *If T is closed and symmetric, then $\operatorname{im}(T \pm iI)$ is closed.*

In a similar way we obtain a characterization for essentially self-adjoint operators.

Proposition 2.44. *Let A be a symmetric operator. Then the following statements are equivalent:*
(i) A is essentially self-adjoint;
(ii) $\ker(A^* + iI) = \{0\}$ and $\ker(A^* - iI) = \{0\}$;
(iii) $\operatorname{im}(A + iI)$ and $\operatorname{im}(A - iI)$ are dense in H.

Proof. We apply Proposition 2.42 to $\overline{A}$ and notice that $\overline{A}$ is symmetric and that Lemma 2.5 implies that $A^* = (\overline{A})^*$. In addition we use Lemma 2.43. □

If A is also a positive operator, we get

Proposition 2.45. *Let A be a positive, symmetric operator. Then the following statements are equivalent:*
(i) A is essentially self-adjoint;
(ii) ker$(A^* + bI) = \{0\}$ for some $b > 0$;
(iii) im$(A + bI)$ is dense in H.

Proof. We proceed in a similar way as before and notice that for a positive, symmetric operator A we have

$$((A + bI)u, u) \geq b\|u\|^2, \tag{2.52}$$

for $u \in \mathrm{dom}(A)$, which is a good substitute for (2.51).

By Lemma 2.8, (ii) and (iii) are equivalent. Since the closure of a positive, symmetric operator is again positive and symmetric, it remains to show that a closed, positive symmetric operator T is self-adjoint if and only if $\ker(T^*+bI) = \{0\}$ for some $b > 0$.

We can suppose that $b = 1$. If T is self-adjoint, then the spectrum $\sigma(T) \subseteq \mathbb{R}^+$, hence $\ker(T + I) = \ker(T^* + I) = \{0\}$.

For the converse, we first show that $\mathrm{im}(T+I)$ is closed: let $(y_k)_k \subset \mathrm{im}(T+I)$ be a convergent sequence. There exists a sequence $(x_k)_k \subset \mathrm{dom}(T)$ such that $y_k = (T + I)x_k$. Then

$$(x_k, y_k) = (x_k, Tx_k) + \|x_k\|^2 \geq \|x_k\|^2,$$

and, by Cauchy-Schwarz,

$$\|x_k\| \leq \|y_k\|. \tag{2.53}$$

Since $(y_k)_k$ is convergent, $\sup_k \|y_k\| < \infty$, and, by (2.53), $\sup_k \|x_k\| < \infty$. Now, positivity implies

$$\begin{aligned} \|x_k - x_\ell\|^2 &\leq ((x_k - x_\ell, (T + I)(x_k - x_\ell)) \\ &\leq (\|x_k\| + \|x_\ell\|)\|y_k - y_\ell\| \\ &\leq C\|y_k - y_\ell\|. \end{aligned}$$

Hence $(x_k)_k$ is a Cauchy sequence. Since we supposed that T is closed, there exists $x \in \mathrm{dom}(T)$ such that $x = \lim_{k\to\infty} x_k$ and $(T + I)x = y = \lim_{k\to\infty} y_k$. Hence $\mathrm{im}(T + I)$ is closed.

The assumption $\ker(T^* + I) = \{0\}$ now gives $\operatorname{im}(T + I) = H$. In order to show that T is self-adjoint. it suffices to show that $\operatorname{dom}(T^*) \subseteq \operatorname{dom}(T)$. Let $x \in \operatorname{dom}(T^*)$. There exists $y \in \operatorname{dom}(T)$ such that

$$(T + I)y = (T^* + I)y = (T^* + I)x,$$

since $\operatorname{dom}(T) \subseteq \operatorname{dom}(T^*)$. This implies $(T^* + I)(x - y) = 0$, and hence $x = y \in \operatorname{dom}(T)$. □

Finally we mention that every self-adjoint operator is unitarily equivalent to a multiplication operator, which is important for applications to the solution of other spectral problems. We omit the proof, as we will not use this version of the spectral theorem in the sequel. Using the Riesz representation theorem in measure theory (see for instance [8]), this version follows easily from Proposition 2.40.

Theorem 2.46. *Let T be a self-adjoint operator on H with spectrum $\sigma(T)$. Then there exists a finite measure μ on $\sigma(T) \times \mathbb{N}$ and a unitary operator*

$$U : H \longrightarrow L^2(\sigma(T) \times \mathbb{N}, d\mu)$$

with the following properties: if

$$g : \sigma(T) \times \mathbb{N} \longrightarrow \mathbb{R}$$

is the function $g(t, n) = t$, then $x \in H$ lies in $dom(T)$ if and only if $g \cdot Ux \in L^2(\sigma(T) \times \mathbb{N}, d\mu)$. In addition we have

$$UTU^{-1}h = gh$$

for all $h \in U(dom(T))$, and

$$Uf(T)U^{-1}h = f(g)h$$

for all bounded Borel functions f on $\sigma(T)$.

In particular, $f(T)$ is a bounded operator and

$$\|f(T)\| = \|f\|_\infty. \tag{2.54}$$

In the proof of this version of the spectral theorem one starts with a function $f \in \mathcal{C}_0(\sigma(T))$, a continuous function which vanishes at infinity. Then one considers the linear functional

$$\Lambda(f) := (f(T)x, x),$$

where $x \in H$ is an appropriate vector (a cyclic vector). The Riesz representation theorem (see [8]) implies that there exists a finite countably additive measure μ on $\mathbb{R}$ such that

$$(f(T)x, x) = \int_{\sigma(T)} f(t)\, d\mu(t).$$

This is the main step. The rest of the proof consists of an application of the symbolic calculus (see [3] for all details).

2.5 Determination of the spectrum

In this part we prove some results of the spectral theory of unbounded self-adjoint operators, which are used later on for the applications to the $\Box$-operator, to Schrödinger operators with magnetic field, and to Pauli and Dirac operators. These results enable us to determine the spectrum of the $\Box$-operator in some special cases and they yield methods to decide whether the corresponding differential operators are with compact resolvent.

First we prove some general results about the spectrum of an unbounded self-adjoint operator.

Lemma 2.47. *Let T be an unbounded self-adjoint operator. Then $\lambda \in \sigma(T)$ if and only if there exists a sequence $(x_k)_k$ in dom(T) such that $\|x_k\| = 1$ for each $k \in \mathbb{N}$ and*

$$\lim_{k\to\infty} \|(T - \lambda I)x_k\| = 0.$$

If T and S are self-adjoint operators such that $T = U^{-1}SU$ for some unitary operator U, where dom$(T) = U^{-1}($dom$(S))$, then $\sigma(T) = \sigma(S)$.

Proof. If $\lambda \in \sigma(T)$, then $T - \lambda I$ is not injective and one can find $x \in \mathrm{dom}(T)$ such that $\|x\| = 1$ and $(T - \lambda I)x = 0$. So the constant sequence $x_k = x$ has the desired property.

Conversely, suppose that $\lambda \notin \sigma(T)$. Then $(T - \lambda I)^{-1}$ is a bounded operator. If there exists a sequence $(x_k)_k$ in $\mathrm{dom}(T)$ such that $\|x_k\| = 1$ for each $k \in \mathbb{N}$ and $\lim_{k\to\infty} \|(T - \lambda I)x_k\| = 0$, then

$$\lim_{k\to\infty} (T - \lambda I)^{-1}(T - \lambda I)x_k = \lim_{k\to\infty} x_k = 0$$

yields a contradiction to $\|x_k\| = 1$ for each $k \in \mathbb{N}$.

To prove the second assertion take $\lambda \in \sigma(S)$ and a sequence $(y_k)_k$ in $\mathrm{dom}(S)$ such that $\|y_k\| = 1$ for each $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \|(S - \lambda I)y_k\| = 0.$$

Then for $U^{-1}y_k = z_k \in \mathrm{dom}(T)$ we have $\|z_k\| = 1$, since U is unitary and

$$\|(T-\lambda I)z_k\| = \|U^{-1}(Sy_k - \lambda y_k)\| = \|(S-\lambda I)y_k\|.$$

This shows that $\lambda \in \sigma(T)$. $\square$

Lemma 2.48. *Let T be an unbounded self-adjoint operator and E the uniquely determined resolution of the identity E on the Borel subsets of $\mathbb{R}$, such that*

$$(Tx, y) = \int_{-\infty}^{\infty} t\, dE_{x,y}(t)\ ,\ x \in dom(T)\ ,\ y \in H$$

(see Proposition 2.40).

Then

$$\sigma(T) = \{\lambda \in \mathbb{R} : E((\lambda-\epsilon, \lambda+\epsilon)) \neq 0, \forall \epsilon > 0\}. \tag{2.55}$$

Proof. Let $\lambda \in \sigma(T)$. Suppose that there exists $\epsilon_0 > 0$ such that $E((\lambda-\epsilon_0, \lambda+\epsilon_0)) = 0$. Then there exists a continuous function f on $\mathbb{R}$ such that $f(t) = (t-\lambda)^{-1}$ on the support of the measure $dE(t)$. By the symbolic calculus, we obtain a bounded operator $(T-\lambda I)^{-1}$ and hence a contradiction.

Conversely, let λ belong to the right hand side of (2.55). For each $k \in \mathbb{N}$ we can now find $x_k \in \mathrm{dom}(T)$ such that $\|x_k\| = 1$ and

$$E((\lambda - 1/k, \lambda + 1/k))x_k = x_k.$$

Consider the bounded Borel functions

$$f_k(t) = (t-\lambda)\,\chi_{(\lambda-1/k,\lambda+1/k)}(t),\ t \in \mathbb{R},\ k \in \mathbb{N},$$

where $\chi_{(\lambda-1/k,\lambda+1/k)}$ is the characteristic function of the intervall $(\lambda - 1/k, \lambda + 1/k)$. Then, by (2.54),

$$\|(T-\lambda I)x_k\| = \|(T-\lambda I)\,E((\lambda-1/k,\lambda+1/k))x_k\| \leq 1/k\,\|x_k\| = 1/k.$$

Using Lemma 2.47, we get $\lambda \in \sigma(T)$. $\square$

Lemma 2.49. *Let T be a symmetric operator on H with domain $dom(T)$, and suppose that $(x_k)_k$ is a complete orthonormal system in H. If each x_k lies in $dom(T)$, and there exist $\lambda_k \in \mathbb{R}$ such that*

$$Tx_k = \lambda_k x_k$$

for every $k \in \mathbb{N}$, then T is essentially self-adjoint. Moreover the spectrum of $\overline{T}$ is the closure in $\mathbb{R}$ of the set of all λ_k.

Proof. If $x = \sum_{k=1}^{\infty} \alpha_k x_k$ belongs to $\operatorname{dom}(T)$, and

$$y = Tx = \sum_{k=1}^{\infty} \beta_k x_k,$$

then

$$\beta_k = (y, x_k) = (Tx, x_k) = (x, Tx_k) = \lambda_k (x, x_k) = \lambda_k \alpha_k.$$

Since $x, y \in H$ we have

$$\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty \ , \ \sum_{k=1}^{\infty} |\beta_k|^2 < \infty$$

and hence

$$\sum_{k=1}^{\infty} (1 + \lambda_k^2) |\alpha_k|^2 < \infty.$$

We define an operator $\tilde{T}$ as follows: let

$$\operatorname{dom}(\tilde{T}) = \{x \in H : x = \sum_{k=1}^{\infty} \alpha_k x_k \text{ with } \sum_{k=1}^{\infty} (1 + \lambda_k^2) |\alpha_k|^2 < \infty\}$$

and define

$$\tilde{T}x = \sum_{k=1}^{\infty} \alpha_k \lambda_k x_k$$

for $x \in \operatorname{dom}(\tilde{T})$. It follows that $\tilde{T}$ is an extension of T.

Let Σ be the closure of the set $\{\lambda_k : k \in \mathbb{N}\}$. Each λ_k is an eigenvalue of $\tilde{T}$ and, by Lemma 2.32, $\sigma(\tilde{T})$ is closed, so $\Sigma \subseteq \sigma(\tilde{T})$. For $z \notin \Sigma$ and $x = \sum_{k=1}^{\infty} \alpha_k x_k$ we define the operator S on H by

$$Sx = S\left(\sum_{k=1}^{\infty} \alpha_k x_k\right) := \sum_{k=1}^{\infty} \alpha_k (z - \lambda_k)^{-1} x_k.$$

It follows that S is injective and that

$$\|S(\sum_{k=1}^{\infty} \alpha_k x_k)\| \leq \sup_k |z - \lambda_k|^{-1} \left(\sum_{k=1}^{\infty} |\alpha_k|^2\right)^{1/2} = C\,\|x\|,$$

which implies that the operator S is bounded. Its range is precisely $\operatorname{dom}(\tilde{T})$ and $(zI - \tilde{T})Sx = x$ for all $x \in H$. Thus $z \notin \sigma(\tilde{T})$ and $S = (zI - \tilde{T})^{-1}$. This implies $\Sigma = \sigma(\tilde{T})$.

Now we claim that $\tilde{T}$ is the closure of T. Since $\sigma(\tilde{T})$ is not equal to $\mathbb{C}$, Lemma 2.32 implies that $\tilde{T}$ is a closed operator. Let

$$u = \sum_{k=1}^{\infty} \alpha_k x_k \in \operatorname{dom}(\tilde{T})$$

and put $u_m = \sum_{k=1}^{m} \alpha_k x_k$. Then $\lim_{m\to\infty} u_m = u$ and

$$\lim_{m\to\infty} T u_m = \lim_{m\to\infty} \sum_{k=1}^{m} \alpha_k \lambda_k x_k = \sum_{k=1}^{\infty} \alpha_k \lambda_k x_k = \tilde{T} u.$$

Hence $\tilde{T} = \overline{T}$.

Finally we prove that $\tilde{T}$ is self-adjoint. For $x \in \operatorname{dom}(\tilde{T}^*)$ and $\tilde{T}^* x = y$ we have

$$(y, x_k) = (\tilde{T}^* x, x_k) = (x, \tilde{T} x_k) = \lambda_k (x, x_k).$$

If $x = \sum_{k=1}^{\infty} \alpha_k x_k$ the above implies that $y = \sum_{k=1}^{\infty} \alpha_k \lambda_k x_k$. Hence $x \in \operatorname{dom}(\tilde{T})$, thus $\operatorname{dom}(\tilde{T}^*) = \operatorname{dom}(\tilde{T})$, and $\tilde{T} = \tilde{T}^*$. □

Definition 2.50. Let T be a self-adjoint operator on H. The discrete spectrum $\sigma_d(T)$ of T is the set of all eigenvalues λ of finite multiplicity which are isolated in the sense that the intervals $(\lambda - \epsilon, \lambda)$ and $(\lambda, \lambda + \epsilon)$ are disjoint from the spectrum for some $\epsilon > 0$. The non-discrete part of the spectrum of T is called the essential spectrum of T, and is denoted by $\sigma_e(T)$.

A closed linear subspace L of H is called invariant if

$$(\zeta I - T)^{-1}(L) \subseteq L,$$

for all $\zeta \notin \mathbb{R}$.

Lemma 2.51. *Let T be a self-adjoint operator. Then*

$$\sigma_d(T) = \{\lambda \in \sigma(T) : \exists \epsilon > 0,\ \mathit{dim}\,\mathit{im}(E((\lambda - \epsilon, \lambda + \epsilon))) < \infty\}. \tag{2.56}$$

Proof. Let λ belong to the right hand side of (2.56). Then there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ the projection $E((\lambda - \epsilon, \lambda + \epsilon))$ becomes a projection with finite range independent of ϵ. This is actually the projection $E(\{\lambda\})$ and we observe that $E((\lambda - \epsilon_0, \lambda)) = 0$ and $E((\lambda, \lambda + \epsilon_0)) = 0$. This shows that $\lambda \in \sigma_d(T)$.

If $\lambda \in \sigma_d(T)$, then λ is an eigenvalue of finite multiplicity and there exists $\epsilon > 0$ such that the intervals $(\lambda - \epsilon, \lambda)$ and $(\lambda, \lambda + \epsilon)$ are disjoint from the spectrum, which means that $\dim \operatorname{im}(E((\lambda - \epsilon, \lambda + \epsilon))) < \infty$. □

As the whole spectrum of a self-adjoint operator is closed, it now follows that the essential spectrum is closed. The following characterization of the essential spectrum of a self-adjoint operator is similar to the characterization of the whole spectrum, compare with Lemma 2.47.

Lemma 2.52. *Let T be a self-adjoint operator. λ belongs to the essential spectrum $\sigma_e(T)$ if and only if there exists a sequence $(x_k)_k$ in dom(T) such that $\|x_k\| = 1$ for each $k \in \mathbb{N}$, such that x_k converges weakly to 0 and*

$$\lim_{k\to\infty} \|(T - \lambda I)x_k\| = 0.$$

Proof. The sequence appearing in the lemma is called a Weyl sequence. We say that λ belongs to the Weyl spectrum $W(T)$ if there exists an associated Weyl sequence. By Lemma 2.47, we already know that

$$W(T) \subseteq \sigma(T).$$

Let $\lambda \in W(T)$ and suppose that $\lambda \in \sigma_d(T)$. Then the spectral projection $E(\{\lambda\})$ has finite dimensional range and hence is compact. So, by Proposition 1.11,

$$\lim_{k\to\infty} E(\{\lambda\})x_k = 0$$

in H. Now let $y_k := (I - E(\{\lambda\}))x_k$. Then, as $E(\{\lambda\})T = TE(\{\lambda\})$ (Proposition 2.40), we get $\lim_{k\to\infty} \|y_k\| = 1$ and

$$\lim_{k\to\infty} (T - \lambda I)y_k = \lim_{k\to\infty} (I - E(\{\lambda\}))(T - \lambda I)x_k = 0.$$

But $(T - \lambda I)$ is invertible on $\mathrm{im}(I - E(\{\lambda\}))$, so we obtain $\lim_{k\to\infty} y_k = 0$, which is a contradiction. Hence $\lambda \in \sigma_e(T)$.

Conversely, let $\lambda \in \sigma_e(T)$. Then, by Lemma 2.51,

$$\dim \mathrm{im}(E((\lambda - \epsilon, \lambda + \epsilon))) = \infty$$

for any $\epsilon > 0$. Let ϵ_k be a decreasing sequence of positive numbers such that $\lim_{k\to\infty} \epsilon_k = 0$. We can now choose an orthonormal system $(x_k)_k$ such that

$$x_k \in \mathrm{im}(E((\lambda - \epsilon_k, \lambda + \epsilon_k))).$$

Then, by Bessel's inequality, x_k converges weakly to 0 and the same reasoning as in Lemma 2.48 yields

$$\lim_{k\to\infty} \|(T - \lambda I)x_k\| = 0.$$

□

2.6 The Laplacian

Consider the Laplace operator

$$-\triangle = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

on $\mathbb{R}^n$. We extend its domain as

$$\mathrm{dom}(-\triangle) = \{f \in L^2(\mathbb{R}^n) : D^\alpha f \in L^2(\mathbb{R}^n)\,, |\alpha| \le 2\} = W^2(\mathbb{R}^n),$$

and obtain, by a similar reasoning as before, a closed operator from $\mathrm{dom}(-\triangle)$ to $L^2(\mathbb{R}^n)$, which is in addition symmetric and positive, since we have

$$(-\triangle u, u) = \sum_{j=1}^{n} (D_j u, D_j u),$$

for $u \in \mathrm{dom}(-\triangle)$. We will show that this operator is essentially self-adjoint and hence has a unique self-adjoint extension.

Using the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x.\xi}\, d\lambda(x),$$

one can show that $W^2(\mathbb{R}^n)$ coincides with the space of all functions $f \in L^2(\mathbb{R}^n)$ such that $(1 + |\xi|^2)\hat{f}$ belongs to $L^2(\mathbb{R}^n)$.

The operator $-\triangle + I$ has a bounded inverse

$$B : L^2(\mathbb{R}^n) \longrightarrow W^2(\mathbb{R}^n)$$

given by

$$\widehat{Bf}(\xi) = (1 + 4\pi^2|\xi|^2)^{-1}\hat{f}(\xi).$$

It is easily seen that $\mathrm{im}(B) = W^2(\mathbb{R}^n)$ and $\mathrm{im}(-\triangle + I) = L^2(\mathbb{R}^n)$.

We will now determine the spectrum of $-\triangle$ on $\mathbb{R}^n$ and show that $\sigma_e(-\triangle) = [0, \infty)$.

Let $\mu \in [0, \infty)$ and choose $w \in \mathbb{R}^n$ such that $4\pi|w|^2 = \mu$. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a cut-off function such that $\chi = 1$ on the unit ball $\mathbb{B}(0, 1)$ of $\mathbb{R}^n$ and $\chi = 0$ on $\mathbb{R}^n \setminus \mathbb{B}(0, 2)$. Let

$$\nu_w(x) = e^{2\pi i x.w},\ x \in \mathbb{R}^n,$$

and

$$f_k(x) = c_k \chi(x/k)\nu_w(x),\ k \in \mathbb{N},$$

where the normalizing constant is

$$c_k = (k^{n/2}\|\chi\|_{L^2})^{-1}.$$

We compute

$$(\mu + \triangle) f_k(x) = \\ c_k(\mu\nu_w + \triangle\nu_w)\chi(x/k) + (2c_k/k)\nabla\nu_w(x)\cdot\nabla\chi(x/k) + (c_k/k^2)\nu_w(x)\triangle\chi(x/k).$$

The first term vanishes pointwise because $\triangle\nu_w = -4\pi|w|^2\nu_w$. The function ν_w is bounded, and since, by a change of variable

$$\frac{c_k}{k^2}\,\|\triangle\chi(\cdot/k)\|_{L^2} = \frac{1}{k^2}\,\frac{\|\triangle\chi\|_{L^2}}{\|\chi\|_{L^2}},$$

the L^2-norm of the third term tends to 0 as $k \to \infty$. In a similar way one shows that the L^2-norm of the second term vanishes as $k \to \infty$. Hence

$$\|(\mu I + \triangle) f_k\|_{L^2} \to 0, \text{ as } k \to \infty.$$

Since $|\nu_w(x)| = 1$ for all x, the definition of c_k gives

$$\|f_k\|_{L^2} = 1,\ k \in \mathbb{N}.$$

Now we show that $(f_k)_k$ converges weakly to 0. For this purpose we decompose an arbitrary function $f \in L^2(\mathbb{R}^n)$ in the form $f = g+h$, where $g = f$ on $\mathbb{B}(0,R)$ and $g = 0$ elsewhere, and $h = f$ on $\mathbb{R}^n \setminus \mathbb{B}(0,R)$ and $h = 0$ elsewhere, $R > 0$. Then we have

$$(f, f_k) = (g, f_k) + (h, f_k),$$

and

$$|(g, f_k)| \leq c_k\|\chi\|_{L^\infty}\|g\|_{L^1} \to 0, \text{ as } k \to \infty.$$

By Cauchy-Schwarz we see

$$\limsup_{k\to\infty} |(h, f_k)| \leq \|h\|_{L^2}.$$

The last term can be made arbitrarily small by letting $R \to \infty$, and so

$$\lim_{k\to\infty} (f, f_k) = 0.$$

Now, by Lemma 2.52, we have $\mu \in \sigma_e(-\triangle)$.

Assume that $\mu \in \mathbb{C}\setminus[0,\infty)$ and let $\delta = \operatorname{dist}(\mu, [0,\infty))$ so that $\delta > 0$. Let $(f_k)_k$ be a corresponding Weyl sequence, which means that $\lim_k \|(-\triangle-\mu I)f_k\|_{L^2} = 0$ and $\|f_k\|_{L^2} = 1, k \in \mathbb{N}$. We set $g_k = (-\triangle - \mu I)f_k$. Taking Fourier transforms we observe that

$$\hat{g}_k(\xi) = (4\pi^2|\xi|^2 - \mu)\hat{f}_k(\xi),$$

so that

$$|\hat{f}_k(\xi)| \leq \delta^{-1}\,|\hat{g}_k(\xi)|.$$

This implies

$$\|f_k\|_{L^2} = \|\hat{f}_k\|_{L^2} \leq \delta^{-1} \|\hat{g}_k\|_{L^2} = \delta^{-1} \|g_k\|_{L^2} \to 0,$$

which contradicts $\|f_k\|_{L^2} = 1, k \in \mathbb{N}$. Therefore $\mu \notin \sigma_e(-\triangle)$. We have now shown that $\sigma_e(-\triangle) = [0, \infty)$. The calculation from above shows that $\mu \in \rho(-\triangle)$, when $\mu \notin [0, \infty)$, so

$$\sigma(-\triangle) = \sigma_e(-\triangle) = [0, \infty).$$

If we consider $-\triangle$ as an unbounded operator on $L^2(\mathbf{\Omega})$, where $\mathbf{\Omega}$ is a bounded domain with $\mathcal{C}^2$-boundary, we first remark that $-\triangle$ fails to be essentially self-adjoint. The Dirichlet realization $-\triangle_D$ corresponds to the zero boundary condition and its domain is

$$\mathrm{dom}(-\triangle_D) = H^2(\mathbf{\Omega}) \cap H_0^1(\mathbf{\Omega}).$$

In this case the spectrum $\sigma(-\triangle_D) \subset (0, \infty)$ is discrete and consists of positive eigenvalues tending to ∞. (see [6])

2.7 Compact resolvents

Next, we will characterize the situation when $\sigma_e(T) = \emptyset$. For this purpose we need some preparations.

Lemma 2.53. *Let T be a self-adjoint operator on H and let $\lambda \in \mathbb{R}$ be an eigenvalue of T. Then*

$$L_\lambda = \{x \in dom(T) : Tx = \lambda x\}$$

is a closed invariant subspace of H.

If L is an invariant subspace of T, then $L^\perp$ is also invariant.

Proof. $L_\lambda = \ker(T - \lambda I)$, and $T - \lambda I$ is a closed operator, hence, by Lemma 2.4, L_λ is a closed subspace. Now let $x \in L_\lambda$ and $\zeta \notin \mathbb{R}$. Then

$$\begin{aligned}
(\zeta I - T)^{-1}x &= (\lambda - \zeta)^{-1}(\zeta I - T)^{-1}(\lambda - \zeta)x \\
&= (\lambda - \zeta)^{-1}(\zeta I - T)^{-1}(T - \zeta I)x \\
&= -(\lambda - \zeta)^{-1}x,
\end{aligned}$$

hence $(\zeta I - T)^{-1}(L_\lambda) \subseteq L_\lambda$.

Let L be an invariant subspace and $y \in L^\perp$. Then, by (2.37)

$$((\zeta I - T)^{-1}y, x) = (y, (\overline{\zeta} I - T)^{-1}x) = 0$$

for all $x \in L$ and $\zeta \notin \mathbb{R}$. Therefore $(\zeta I - T)^{-1}y \in L^\perp$ and $L^\perp$ is invariant. $\square$

For the following results we always suppose that the underlying Hilbert space H is separable and infinite-dimensional, i.e. each complete orthonormal system is countably infinite.

Proposition 2.54. *Let T be a self-adjoint operator on H. The essential spectrum $\sigma_e(T)$ of T is empty if and only if there exists a complete orthonormal system of eigenvectors $\{x_n\}_n$ of T such that the corresponding eigenvalues λ_n converge in absolute value to ∞ as $n \to \infty$.*

Proof. If $\sigma_e(T) = \emptyset$, then the spectrum $\sigma(T)$ consists of a set $\{r_n\}_n$ of isolated eigenvalues of finite multiplicity, which can only converge to $\pm\infty$. We enumerate the eigenvalues in order of increasing absolute values and repeat each eigenvalue according to its multiplicity. In this way we get an associated orthonormal system of eigenvectors $\{x_n\}_n$ of T. Suppose that this system is not complete. Then, by Lemma 2.53, the subspace

$$L = \{x \in H : (x, x_n) = 0\,,\ n \in \mathbb{N}\}$$

is invariant with respect to T in the sense of Definition 2.49. If $x \in \mathrm{dom}(T) \cap L$, then

$$(Tx, x_n) = (x, Tx_n) = r_n(x, x_n) = 0,$$

hence $T(\mathrm{dom}(T) \cap L) \subseteq L$. The essential spectrum of the restriction of T to L is also empty. But the spectrum of this restriction is non-empty (Proposition 2.39), therefore T has further eigenvalues and eigenvectors not accounted for in the above list.

Conversely suppose that a sequence of eigenvalues and eigenvectors with the stated properties exists, and let $\{s_n\}_n$ be the set of distinct eigenvalues. By the assumption that λ_n converge in absolute value to ∞ as $n \to \infty$, we deduce that s_n are isolated eigenvalues of finite multiplicity. It follows from Lemma 2.49 that

$$\sigma(T) = \bigcup_{n=1}^{\infty} \{s_n\}.$$

Thus $\sigma_e(T) = \emptyset$. □

Proposition 2.55. *Let T be an unbounded self-adjoint operator on H which is non-negative in the sense that $\sigma(T) \subseteq [0, \infty)$. Then the following conditions are equivalent:*

(i) The resolvent operator $(I + T)^{-1}$ is compact.

(ii) $\sigma_e(T) = \emptyset$.

(iii) There exists a complete orthonormal system of eigenvectors $\{x_n\}_n$ of T with corresponding eigenvalues $\mu_n \geq 0$ which converge to $+\infty$ as $n \to \infty$.

Proof. (i) $\Rightarrow$ (iii): The operator $(I+T)^{-1} : H \longrightarrow \mathrm{dom}(T)$ is compact and self-adjoint and has dense image. By Proposition 1.12 there exists a complete orthonormal system of eigenvectors x_n of $(I+T)^{-1}$ and eigenvalues λ_n (of finite multiplicity) tending to 0 such that

$$(I+T)^{-1}x = \sum_{n=1}^{\infty} \lambda_n (x, x_n) x_n.$$

Since all $x_n \in \mathrm{dom}(T)$, we have $T(I+T)^{-1}x_n = \lambda_n T x_n$. If we add $(I+T)^{-1}x_n$ to this equality we get

$$\lambda_n T x_n + (I+T)^{-1}x_n = T(I+T)^{-1}x_n + (I+T)^{-1}x_n = (I+T)(I+T)^{-1}x_n = x_n,$$

which implies that

$$\lambda_n T x_n + \lambda_n x_n = x_n,$$

and therefore

$$T x_n = \frac{1-\lambda_n}{\lambda_n} x_n.$$

Setting $\mu_n = \frac{1-\lambda_n}{\lambda_n}$ we get (iii).

Now suppose that (iii) holds. We rearrange the eigenvectors x_n so that the sequence $\{\mu_n\}_n$ is non-decreasing. Each $x \in H$ can be written in the form $x = \sum_{n=1}^{\infty}(x, x_n)x_n$ and we obtain

$$(I+T)^{-1}x = \sum_{n=1}^{\infty}(x, x_n)(I+T)^{-1}x_n = \sum_{n=1}^{\infty} \frac{1}{1+\mu_n}(x, x_n)x_n.$$

Let

$$A_N = \sum_{n=1}^{N} \frac{1}{1+\mu_n}(x, x_n)x_n.$$

Then the operators A_N are of finite rank and, by Bessel's inequality, we obtain

$$\|((I+T)^{-1} - A_N)x\| = \| \sum_{n=N+1}^{\infty} \frac{1}{1+\mu_n}(x, x_n)x_n\| \leq \frac{1}{1+\mu_N}\|x\|,$$

from which we see that A_N converges in operator norm to $(I+T)^{-1}$ as $N \to \infty$. Now, by Proposition 1.13, $(I+T)^{-1}$ is a compact operator.

The equivalence of (ii) and (iii) is a consequence of Proposition 2.54.

$\square$

The following general result explains the approach to the $\overline{\partial}$-Neumann operator (2.11) by means of the embedding

$$j : \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*) \hookrightarrow L^2_{(0,q)}(\Omega),$$

where $\text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*)$ is endowed with the graph-norm

$$u \mapsto (\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}.$$

It will also be crucial for the question of compactness of the $\overline{\partial}$-Neumann operator, which will be discussed in the following chapters.

Proposition 2.56. *Let A be a non-negative self-adjoint operator (i.e. $\sigma(A)$ is contained in $[0, \infty)$). There exists a unique self-adjoint square root $A^{1/2}$ of A and dom$(A^{1/2}) \supseteq$ dom(A). In addition dom(A) endowed with the norm*

$$\|f\|_{\mathcal{D}} := (\|A^{1/2}f\|^2 + \|f\|^2)^{1/2}$$

becomes a Hilbert space, the norm $\|.\|_{\mathcal{D}}$ stems from the inner product

$$(f, g)_{\mathcal{D}} = (A^{1/2}f, A^{1/2}g) + (f, g).$$

Let dom(A) be endowed with the norm $\|.\|_{\mathcal{D}}$. Then, A has compact resolvent if and only if the canonical imbedding

$$j : dom(A) \hookrightarrow H$$

is a compact linear operator.

Furthermore, A has compact resolvent if and only if $A^{1/2}$ has compact resolvent.

Proof. In Proposition 2.41 we proved existence and uniqueness of the square root of A.

For $n \in \mathbb{N}$ we define the functions

$$q_n(t) = \frac{nt}{n+t} \ , \ t \in [0, \infty).$$

These are continuous functions with $q_n(t) \leq q_{n+1}(t)$ and $\lim_{n\to\infty} q_n(t) = t$. Moreover $q_n(t) \leq n$ for each $t \in [0, \infty)$. By Theorem 2.46 the operator $nA(nI + A)^{-1}$ is bounded on H and the function

$$Q_n(x) := (nA(nI + A)^{-1}x, x) \ , \ x \in H$$

is bounded on the unit ball of H and continuous. The functional calculus implies that $Q_n(x)$ increases monotonically to $Q(x)$, where

$$Q(x) = \begin{cases} (A^{1/2}x, A^{1/2}x) & \text{for } x \in \text{dom}(A), \\ +\infty & \text{otherwise.} \end{cases}$$

A function $\Theta : H \longrightarrow (-\infty, +\infty]$ is said to be lower semicontinuous if for every convergent sequence $x_n \to x$ in H we have

$$\Theta(x) \leq \liminf_{n\to\infty} \Theta(x_n).$$

It is easily seen that a function Θ is lower semicontinuous if and only if

$$\{x : \Theta(x) > \alpha\}$$

is open for every real α, and that the pointwise limit of an increasing sequence of continuous functions is a lower semicontinuous function. Therefore the function Q is lower semicontinuous.

Now let $\{x_n\}_n$ be a Cauchy sequence with respect to $\|.\|_{\mathcal{D}}$. Then $\{x_n\}_n$ is also a Cauchy sequence with respect to $\|.\|$ and therefore converges to $x \in H$. Given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$Q(x_m - x_n) + \|x_m - x_n\|^2 < \epsilon^2$$

for all $m, n \geq N$. Letting $m \to \infty$ and using the lower semicontinuity of Q, we deduce that $x \in \mathrm{dom}(A)$ and

$$Q(x - x_n) + \|x - x_n\|^2 \leq \epsilon^2$$

for all $n > N$. Hence $\|x - x_n\|_{\mathcal{D}} \leq \epsilon$ and $\mathrm{dom}(A)$ endowed with the norm $\|.\|_{\mathcal{D}}$ is complete.

Since $-1 \notin \sigma(A)$, we know that $(I + A)^{-1}$ is a bounded operator on H. From (1.1) we get that $R_A(-1) = (I + A)^{-1}$ is compact if and only if $R_A(z)$ is compact for any $z \notin \sigma(A)$.

Let $u \in H$ and $v \in \mathrm{dom}(A)$. Then

$$\begin{aligned}
(j^*u, v)_{\mathcal{D}} &= (u, jv) = (u, v) = ((I + A)(I + A)^{-1}u, v) \\
&= ((I + A)^{-1}u, (I + A)v) \\
&= ((I + A)^{-1}u, Av) + ((I + A)^{-1}u, v) \\
&= (A^{1/2}(I + A)^{-1}u, A^{1/2}v) + ((I + A)^{-1}u, v) \\
&= ((I + A)^{-1}u, v)_{\mathcal{D}},
\end{aligned}$$

This implies that $j^* = (I + A)^{-1}$ as operator on $\mathrm{dom}(A)$ and $j \circ j^* = (I + A)^{-1}$ as operator on H. So we deduce the desired conclusion by the fact that j is compact if and only if $j \circ j^*$ is compact (Theorem 1.8).

The last statement follows from $(iI + A^{1/2})^* = -iI + A^{1/2}$ and

$$(I + A) = (iI + A^{1/2})(-iI + A^{1/2}).$$

□

Our next aim is to compare two self-adjoint, strictly positive operators and to prove Ruelle's Lemma.

Remark 2.57. Let S be a self-adjoint operator. Suppose that $(Sx, x) \geq 0$ for each $x \in \operatorname{dom}(S)$. We say that S is strictly positive, if $(Sx, x) > 0$ for each $x \in \operatorname{dom}(S) \setminus \{0\}$. Using the fact that $\operatorname{dom}(S)$ is a core of $S^{1/2}$ (Proposition 2.41), one can easily show that S is strictly positive if and only if $\|S^{1/2}x\| > 0$ for each $x \in \operatorname{dom}(S^{1/2}) \setminus \{0\}$. We also indicate that a strictly positive self-adjoint operator S is injective and $\operatorname{im}(S)$ is dense in H and S^{-1} is self-adjoint (Lemma 2.12).

Lemma 2.58. *Let S and T be strictly positive self-adjoint operators. Then the following assertions are equivalent.*

(a) $\operatorname{dom}(T^{1/2}) \subset \operatorname{dom}(S^{1/2})$ and $\|S^{1/2}x\| \leq \|T^{1/2}x\|$ for $x \in \operatorname{dom}(T^{1/2})$;
(b) $\operatorname{dom}(T^{1/2}) \subset \operatorname{dom}(S^{1/2})$ and $\|S^{1/2}T^{-1/2}x\| \leq \|x\|$ for $x \in \operatorname{dom}(T^{-1/2})$;
(c) $\operatorname{dom}(S^{-1/2}) \subset \operatorname{dom}(T^{-1/2})$ and $\|T^{-1/2}S^{1/2}x\| \leq \|x\|$ for $x \in \operatorname{dom}(S^{1/2})$;
(d) $\operatorname{dom}(S^{-1/2}) \subset \operatorname{dom}(T^{-1/2})$ and $\|T^{-1/2}x\| \leq \|S^{-1/2}x\|$ for $x \in \operatorname{dom}(S^{-1/2})$.

Proof. We show that (a) and (b) are equivalent and that (b) implies (c). The rest of the implications follows by symmetry and by replacing S and T by S^{-1} and T^{-1}.

Suppose that (a) holds. If $x \in \operatorname{dom}(T^{-1/2})$, then $T^{-1/2}x \in \operatorname{dom}(T^{1/2})$, so we get $\|S^{1/2}T^{-1/2}x\| \leq \|x\|$.

If (b) holds and $x \in \operatorname{dom}(T^{1/2})$, we have $T^{1/2}x \in \operatorname{dom}(T^{-1/2})$ and hence

$$\|S^{1/2}x\| = \|S^{1/2}T^{-1/2}T^{1/2}x\| \leq \|T^{1/2}x\|.$$

Suppose that (b) holds. We observe that $\operatorname{dom}(S^{-1/2}) = \operatorname{im}(S^{1/2})$ and that we have to show

$$\operatorname{im}(S^{1/2}) \subset \operatorname{dom}(T^{-1/2}) \text{ and } \|T^{-1/2}S^{1/2}x\| \leq \|x\| \text{ for } x \in \operatorname{dom}(S^{1/2}).$$

If $x \in \operatorname{dom}(S^{1/2})$, then $S^{1/2}x \in \operatorname{im}(S^{1/2})$. We consider the linear functional

$$\psi_x(y) := (S^{1/2}x, T^{-1/2}y) \text{ for } y \in \operatorname{dom}(T^{-1/2}).$$

Then

$$|\psi_x(y)| = |(S^{1/2}x, T^{-1/2}y)| = |(x, S^{1/2}T^{-1/2}y)| \leq \|x\| \, \|y\|.$$

This implies that $S^{1/2}x \in \operatorname{dom}((T^{-1/2})^*) = \operatorname{dom}(T^{-1/2})$. Finally we get

$$|(T^{-1/2}S^{1/2}x, y)| \leq \|x\| \, \|y\|, \text{ for } y \in \operatorname{dom}((T^{-1/2}).$$

And as $\operatorname{dom}(T^{-1/2})$ is dense in H, we obtain $\|T^{-1/2}S^{1/2}x\| \leq \|x\|$. □

For Ruelle's Lemma we consider strictly positive, self-adjoint operators S and T and we write $S \leq T$, if and only if $\operatorname{dom}(T) \subseteq \operatorname{dom}(S)$ and $(Sx, x) \leq (Tx, x)$ for each $x \in \operatorname{dom}(T)$. By Proposition 2.41 the square roots of S and T exist and are themselves positive, self-adjoint operators.

Lemma 2.59 (Ruelle's Lemma). *Let S and T be strictly positive self-adjoint operators. Suppose that $S \leq T$ and that $0 \in \rho(S)$. Then $T^{-1} \leq S^{-1}$.*

Proof. We have $\operatorname{dom}(T) \subseteq \operatorname{dom}(S)$ and

$$\|S^{1/2}x\|^2 = (Sx, x) \leq (Tx, x) = \|T^{1/2}x\|^2, \tag{2.57}$$

for all $x \in \operatorname{dom}(T)$.

Next we show that $\operatorname{dom}(T^{1/2}) \subset \operatorname{dom}(S^{1/2})$. Let $x \in \operatorname{dom}(T^{1/2})$. By Proposition 2.41, $\operatorname{dom}(T)$ is a core of $\operatorname{dom}(T^{1/2})$. Hence there exists a sequence $(x_k)_k$ in $\operatorname{dom}(T)$ such that $x_k \to x$ and $T^{1/2}x_k \to T^{1/2}x$. By (2.57) we have

$$\|S^{1/2}(x_m - x_k)\| \leq \|T^{1/2}(x_m - x_k)\|,$$

which implies that $(S^{1/2}x_k)_k$ is a Cauchy sequence. But $S^{1/2}$ is also a closed operator, and so $x \in \operatorname{dom}(S^{1/2})$ and $S^{1/2}x_k \to S^{1/2}x$. In addition, by (2.57), we have

$$\|S^{1/2}x\| = \lim_{k\to\infty} \|S^{1/2}x_k\| \leq \lim_{k\to\infty} \|T^{1/2}x_k\| = \|T^{1/2}x\|,$$

for each $x \in \operatorname{dom}(T^{1/2})$.

Now we can apply Lemma 2.58 (d) and get $\operatorname{dom}(S^{-1/2}) \subset \operatorname{dom}(T^{-1/2})$ and $\|T^{-1/2}x\| \leq \|S^{-1/2}x\|$ for $x \in \operatorname{dom}(S^{-1/2})$. Our assumption $0 \in \rho(S)$ implies that $H = \operatorname{dom}(S^{-1}) = \operatorname{dom}(S^{-1/2})$, which proves the lemma. □

Example 2.60. Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}^+$ be a plurisubharmonic $\mathcal{C}^2$-weight function and define the space

$$L^2(\mathbb{C}^n, e^{-\varphi}) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \ : \ \int_{\mathbb{C}^n} |f|^2 \, e^{-\varphi} \, d\lambda < \infty\},$$

where λ denotes the Lebesgue measure, the space $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ of $(0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, e^{-\varphi})$ and the space $L^2_{(0,2)}(\mathbb{C}^n, e^{-\varphi})$ of $(0,2)$-forms with coefficients in $L^2(\mathbb{C}^n, e^{-\varphi})$. Let

$$(f, g)_\varphi = \int_{\mathbb{C}^n} f \, \overline{g} e^{-\varphi} \, d\lambda$$

denote the inner product and

$$\|f\|^2_\varphi = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} \, d\lambda$$

the norm in $L^2(\mathbb{C}^n, e^{-\varphi})$.

We define $\mathrm{dom}(\overline{\partial})$ to be the space of all functions $f \in L^2(\mathbb{C}^n, e^{-\varphi})$ such that $\overline{\partial} f$, in the sense of distributions, belongs to $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$, and consider the weighted $\overline{\partial}$-complex

$$L^2(\mathbb{C}^n, e^{-\varphi}) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,2)}(\mathbb{C}^n, e^{-\varphi}), \tag{2.58}$$

where $\overline{\partial}^*_\varphi$ is the adjoint operator to $\overline{\partial}$ with respect to the weighted inner product. For a smooth $(0,1)$-form $u = \sum_{j=1}^n u_j d\overline{z}_j \in \mathrm{dom}(\overline{\partial}^*_\varphi)$ one has

$$\overline{\partial}^*_\varphi u = -\sum_{j=1}^n \left(\frac{\partial}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} \right) u_j. \tag{2.59}$$

The complex Laplacian on $(0,1)$-forms is defined as

$$\Box_\varphi := \overline{\partial}\, \overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi \overline{\partial},$$

and $\mathrm{dom}(\Box_\varphi)$ is the space of all $f \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ such that

$$f \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*_\varphi)$$

and $\overline{\partial} f \in \mathrm{dom}(\overline{\partial}^*_\varphi)$ and $\overline{\partial}^*_\varphi f \in \mathrm{dom}(\overline{\partial})$.

Let

$$M_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)_{jk}$$

denote the Levi - matrix of φ. The Kohn-Morrey formula follows from integartion by parts

$$(M_\varphi u, u)_\varphi \le (\Box_\varphi u, u)_\varphi$$

for a $(0,1)$-form $u \in dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_\varphi)$. Using Ruelle's Lemma 2.59 we see that

$$(N_\varphi u, u)_\varphi \le (M_\varphi^{-1} u, u)_\varphi.$$

Setting $\overline{\partial} v = u$ we get

$$\|v\|^2_\varphi = (v,v)_\varphi = (v, \overline{\partial}^*_\varphi N_\varphi u)_\varphi = (\overline{\partial} v, N_\varphi u)_\varphi = (u, N_\varphi u)_\varphi \le (M_\varphi^{-1} \overline{\partial} v, \overline{\partial} v)_\varphi$$

for each $v \in dom\,(\overline{\partial})$ orthogonal to $ker\,(\overline{\partial})$.

This gives a different proof of Hörmander's L^2-estimates similar to the Brascamp-Lieb inequality (see [5] and [7]).

2.8 Variational characterization of the discrete spectrum

Here we explain the max-min principle to describe the lowest part of the spectrum of a self-adjoint operator when it is discrete. This is done in the setting of semibounded operators.

Definition 2.61. Let T be a symmetric unbounded operator with $\mathrm{dom}(T)$. We say that T is semibounded (from below) if there exists a constant $C > 0$ such that

$$(Tu, u) \geq -C\|u\|^2, \ \forall u \in \mathrm{dom}(T).$$

See the next chapter for examples of semibounded operators.

One can show that a symmetric semibounded operator T admits a self-adjoint extension (Friedrichs' extension).

Proposition 2.62. *Let A be a self-adjoint semibounded operator. Let*

$$\Sigma = \inf \sigma_e(A).$$

The set $\sigma(A) \cap (-\infty, \Sigma)$ can be described as a sequence (finite or infinite) of eigenvalues λ_j ordered increasingly. Then one has

$$\lambda_1 = \inf\{(A\phi, \phi)\, \|\phi\|^{-2} : \phi \in dom(A),\, \phi \neq 0\}, \tag{2.60}$$

and for $k \geq 2$

$$\lambda_k = \inf\{(A\phi, \phi)\, \|\phi\|^{-2} : \phi \in dom(A) \cap \mathcal{K}_{k-1}^{\perp},\, \phi \neq 0\}, \tag{2.61}$$

where

$$\mathcal{K}_j = \bigoplus_{m \leq j} ker(A - \lambda_m I).$$

Proof. Let μ_1 denote the right hand side of (2.60). If ϕ_1 is an eigenfunction for the eigenvalue λ_1, we get $\mu_1 \leq \lambda_1$. On the other side we have $\sigma(A - \lambda_1 I) \subseteq [0, \infty)$, hence, by Proposition 2.41,

$$(A\phi - \lambda_1\phi, \phi) \geq 0,$$

for each $\phi \in \mathrm{dom}(A)$, which implies $\lambda_1 \leq \mu_1$.

Actually, we have shown that if $\mu_1 < \Sigma$, then the spectrum below Σ is not empty. In particular the bottom of the spectrum is an eigenvalue equal to μ_1.

For $k = 2, 3, \ldots$, we apply the first step of the proof to $A\,|_{\mathrm{dom}(A)\cap\mathcal{K}_{k-1}^{\perp}}$ and use the spectral theorem.

□

Our next aim is to generalize the following minmax-Lemma for positive, compact operators:

Lemma 2.63. *Let $A : H \longrightarrow H$ be a positive, compact operator (i.e. $(Ax, x) \geq 0$, $\forall x \in H$) with spectral decomposition*

$$Ax = \sum_{n=0}^{\infty} \lambda_n (x, x_n) x_n,$$

where $\lambda_0 \geq \lambda_1 \geq \dots$. Then

(i)

$$\lambda_0 = \max_{x \in H} \frac{(Ax, x)}{(x, x)},$$

where the maximum is attained by an eigenvector with eigenvalue λ_0.

(ii)

$$\lambda_j = \min_{L \in N_j} \max_{x \in L^\perp} \frac{(Ax, x)}{(x, x)}, \ j \geq 1,$$

where N_j denotes the set of all j-dimensional subspaces of H. The minimum is attained by the subspace $L = L_j = \langle x_0, \dots, x_{j-1} \rangle$, i.e.

$$\lambda_j = \max_{x \in L_j^\perp} \frac{(Ax, x)}{(x, x)}.$$

Proof. (i) follows directly from the proof of the spectral theorem 1.12.

For $j \geq 1$ we have $\lambda_j = (Ax_j, x_j)/(x_j, x_j)$. The assertion follows, if we can show that for each j-dimensional subspace L there exists $z_0 \perp L$ with $z_0 \neq 0$ and $z_0 = \sum_{k=0}^{j} (z_0, x_k) x_k$. Because then we have

$$\frac{(Az_0, z_0)}{(z_0, z_0)} = \frac{\sum_{k=0}^{j} \lambda_k |(z_0, x_k)|^2}{\sum_{k=0}^{j} |(z_0, x_k)|^2} \geq \lambda_j,$$

as $\lambda_j \leq \lambda_i$ for $0 \leq i \leq j$ and

$$\max_{x \in L^\perp} \frac{(Ax, x)}{(x, x)} \geq \lambda_j.$$

The existence of z_0 follows from the fact that for a basis $\{y_k : k = 0, \dots, j-1\}$ of L the system of linear equations

$$\sum_{i=0}^{j} a_i (x_i, y_k) = 0 \ , \ k = 0, \dots j-1,$$

has a non-trivial solution. Set $z_0 = \sum_{i=0}^{j} a_i x_i$, then $z_0 \perp L$.

□

So if one has positive compact operators A and B such that $A \leq B$, which means $(Ax, x) \leq (Bx, x)$, then the eigenvalues of A and B satisfy

$$\lambda_j(A) \leq \lambda_j(B),\ j = 0, 1, 2, \ldots$$

The corresponding result for unbounded operator yields information about the bottom of the spectrum and of the essential spectrum.

Proposition 2.64. *Let H be a Hilbert space of infinite dimension. Let A be a self-adjoint semibounded operator with domain $dom(A)$. Let*

$$\mu_1(A) = \inf\{(A\phi, \phi) : \phi \in dom(A), \|\phi\| = 1\}$$

and for $n \geq 2$

$$\mu_n(A) = \sup_{L \in N_{n-1}} \inf\{(A\phi, \phi) : \phi \in L^\perp \cap dom(A), \|\phi\| = 1\} \tag{2.62}$$

where N_{n-1} denotes the set of all subspaces of H of dimension $\leq n-1$. Then either

(a) $\mu_n(A)$ is the n-th eigenvalue when the eigenvalues are increasingly ordered (counting the multiplicities) and A has a discrete spectrum in $(-\infty, \mu_n(A)]$, or

(b) $\mu_n(A)$ corresponds to the bottom of the essential spectrum. In this case we have $\mu_j(A) = \mu_n(A)$ for all $j \geq n$.

Proof. Let E be the uniquely determined resolution of identity on the Borel subsets of $\mathbb{R}$. First we show that

$$\dim \operatorname{im} E((-\infty, a)) < n, \text{ if } a < \mu_n(A), \tag{2.63}$$

$$\dim \operatorname{im} E((-\infty, a)) \geq n, \text{ if } a > \mu_n(A). \tag{2.64}$$

To prove (2.63) let $a < \mu_n(A)$ and suppose that

$$\dim \operatorname{im} E((-\infty, a)) \geq n.$$

If $y \in \operatorname{im} E((-\infty, a))$, we get $y = E((-\infty, a))x$, for some $x \in H$ and therefore

$$E_{y,y}(\omega) = (E(\omega)y, y) = (E(\omega)E((-\infty, a))x, E((-\infty, a))x) = 0, \tag{2.65}$$

for each Borel subset ω of $\mathbb{R}$ such that $(-\infty, a) \cap \omega = \emptyset$. Since A is bounded from below we have $(Ax, x) \geq -C\|x\|^2$ for all $x \in \operatorname{dom}(A)$ and Proposition 2.41 and Lemma 2.29 imply that

$$\operatorname{dom}(A) = \{u \in H : \int_{-C}^{\infty} t^2 \, dE_{u,u}(t) < \infty\},$$

so we obtain for $y \in \mathrm{im}E((-\infty, a))$ by (2.65) that

$$\int_{-C}^{\infty} t^2 \, dE_{y,y}(t) \le C' \max(C^2, a^2) < \infty,$$

which implies that $\mathrm{im}E((-\infty, a)) \subseteq \mathrm{dom}(A)$. So we can find an n-dimensional subspace $L \subset \mathrm{dom}(A)$, such that

$$(Au, u) = \int_{-C}^{a} t \, dE_{u,u} \le a(u, u), \ \forall u \in L. \tag{2.66}$$

But then, given any $\psi_1, \ldots, \psi_{n-1} \in H$, we can find

$$\phi \in L \cap \langle \psi_1, \ldots, \psi_{n-1} \rangle^{\perp}$$

such that $\|\phi\| = 1$ and $(A\phi, \phi) \le a$. Returning to the definition of $\mu_n(A)$ we would have $\mu_n(A) \le a$, which is a contradiction. Hence we have shown (2.63).

To prove (2.64), let $a > \mu_n(A)$ and suppose that

$$\dim \mathrm{im}E((-\infty, a)) \le n - 1.$$

Then we can find $(n-1)$ generators $\psi_1, \ldots, \psi_{n-1}$ of this space and any

$$\phi \in \mathrm{dom}(A) \cap \langle \psi_1, \ldots, \psi_{n-1} \rangle^{\perp}$$

is in $\mathrm{im}E([a, +\infty))$, so

$$(A\phi, \phi) \ge a\|\phi\|^2,$$

which is again a contradiction, and we get (2.64).

In the next step we will show that $\mu_n(A) < +\infty$ for each $n \in \mathbb{N}$. Since A is semibounded from below $\mu_n(A)$ has a uniform lower bound. Suppose that $\mu_n(A) = +\infty$. By (2.63), this means that

$$\dim \mathrm{im}E((-\infty, a)) < n,$$

for all $a \in \mathbb{R}$. Hence H must be of finite dimension and we arrive at a contradiction.(If H is finite dimensional, we have $\mu_n(A) \ge \|A\|$.)

For the rest of the poof we distinguish between the following two cases:

$$\dim \mathrm{im}E((-\infty, \mu_n(A) + \epsilon)) = \infty, \ \forall \epsilon > 0; \tag{2.67}$$

and

$$\dim \mathrm{im}E((-\infty, \mu_n(A) + \epsilon_0)) < \infty, \text{ for some } \epsilon_0 > 0. \tag{2.68}$$

Assuming (2.67) we claim that the assertion (b) of the proposition holds: using (2.63) in this case we obtain

$$\dim \mathrm{im}E((\mu_n(A) - \epsilon, \mu_n(A) + \epsilon)) = \infty \, , \forall \epsilon > 0.$$

By Lemma 2.51, this shows that $\mu_n(A) \in \sigma_e(A)$.

Using (2.63) once more, we see that the intervall $(-\infty, \mu_n(A))$ does not contain any point of the essential spectrum. Hence

$$\mu_n(A) = \inf\{\lambda : \lambda \in \sigma_e(A)\}.$$

From (2.62) we obtain that $\mu_{n+1}(A) \geq \mu_n(A)$. But if $\mu_{n+1}(A) > \mu_n(A)$, (2.63) would also be satisfied for $\mu_{n+1}(A)$. This is a contradiction to (2.67). Hence assertion (b) is proved.

Finally suppose that (2.68) holds. By Lemma 2.51 it is clear that the spectrum of A is discrete in $(-\infty, \mu_n(A) + \epsilon_0)$. Then, for $\epsilon_1 > 0$ small enough

$$\mathrm{im} E((-\infty, \mu_n(A)]) = \mathrm{im} E((-\infty, \mu_n(A) + \epsilon_1)),$$

and by (2.64)

$$\dim\, \mathrm{im} E((-\infty, \mu_n(A)]) \geq n.$$

So, there are at least n eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \ldots \lambda_n \leq \mu_n(A)$$

for A. If λ_n were strictly less than $\mu_n(A)$, then

$$\dim\, \mathrm{im} E((-\infty, \lambda_n]) = n,$$

which yields a contradiction to (2.63). So $\lambda_n = \mu_n(A)$, and $\mu_n(A)$ is an eigenvalue. This proves assertion (a). □

We note that the proof of (2.63) gives

Proposition 2.65. *Suppose that there exists an a and an n-dimensional subspace $L \subset dom(A)$ such that* (2.66) *is satisfied. Then $\mu_n(A) \leq a$.*

Using Proposition 2.64 we now get

Corollary 2.66. *Under the same assumptions as in Proposition 2.65, if a is below the bottom of the essential spectrum of A, then A has at least n eigenvalues (counted with multiplicities).*

The last results permit to compare the spectra of two operators. If $(Au, u) \leq (Bu, u)$ for all $u \in \mathrm{dom}(B) \subseteq \mathrm{dom}(A)$, then $\lambda_n(A) \leq \lambda_n(B)$.

Chapter 3

Schrödinger operators

3.1 Magnetic field

Let $z \in \mathbb{C}$. Define

$$\mathrm{sgn} z = \begin{cases} \overline{z}/|z| & z \neq 0 \\ 0 & z = 0. \end{cases}$$

Proposition 3.1. *Suppose that $f \in L^1_{loc}(\mathbb{R}^n)$ with $\nabla f \in L^1_{loc}(\mathbb{R}^n)$. Then*

$$\nabla |f| \in L^1_{loc}(\mathbb{R}^n)$$

and

$$\nabla |f|(x) = \Re[sgn(f(x))\, \nabla f(x)] \tag{3.1}$$

almost everywhere. In particular, we have

$$|\nabla |f|| \leq |\nabla f|, \tag{3.2}$$

almost everywhere.

Proof. Let $z \in \mathbb{C}$ and $\epsilon > 0$. We define

$$|z|_\epsilon := \sqrt{|z|^2 + \epsilon^2} - \epsilon$$

and observe that

$$0 \leq |z|_\epsilon \leq |z| \text{ and } \lim_{\epsilon \to 0} |z|_\epsilon = |z|.$$

If $u \in \mathcal{C}^\infty(\mathbb{R}^n)$, then $|u|_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^n)$ and as $|u|^2 = u\,\overline{u}$ we get

$$\nabla |u|_\epsilon = \frac{\Re(u\, \nabla u)}{\sqrt{|u|^2 + \epsilon^2}}. \tag{3.3}$$

Now let f be as assumed, take an approximation to the identity $(\chi_\delta)_\delta$ and define

$$f_\delta = f * \chi_\delta.$$

Then $f_\delta \to f$, $|f_\delta| \to |f|$, and $\nabla f_\delta \to \nabla f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $\delta \to 0$.

Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a test function. There exists a subsequence $\delta_k \to 0$ such that $f_{\delta_k}(x) \to f(x)$ for almost every $x \in \operatorname{supp}\phi$. For simplicity we omit the index k now. Using the dominated convergence theorem and (3.3) we get

$$\begin{aligned}\int (\nabla\phi)\,|f|\,d\lambda &= \lim_{\epsilon\to 0}\int (\nabla\phi)\,|f|_\epsilon\,d\lambda \\ &= \lim_{\epsilon\to 0}\lim_{\delta\to 0}\int (\nabla\phi)\,|f_\delta|_\epsilon\,d\lambda \\ &= -\lim_{\epsilon\to 0}\lim_{\delta\to 0}\int \phi\,\frac{\Re(\overline{f}_\delta\nabla f_\delta)}{\sqrt{|f_\delta|^2+\epsilon^2}}\,d\lambda.\end{aligned}$$

Since $\nabla f_\delta \to \nabla f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we get taking the limit $\delta \to 0$ that

$$\int (\nabla\phi)\,|f|\,d\lambda = -\lim_{\epsilon\to 0}\int \phi\,\frac{\Re(\overline{f}\nabla f)}{\sqrt{|f|^2+\epsilon^2}}\,d\lambda,$$

and since $\phi\nabla f \in L^1(\mathbb{R}^n$ and $\overline{f}/\sqrt{|f|^2+\epsilon^2} \to \operatorname{sgn} f$ as $\epsilon \to 0$ we get the desired result by applying once more dominated convergence. □

We consider differential operators $H(A,V)$ of the form

$$H(A,V) = -\Delta_A + V, \tag{3.4}$$

where $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the electric potential and

$$A = \sum_{j=1}^n A_j dx_j$$

is a 1-form, and

$$\Delta_A = -\sum_{j=1}^n \left(\frac{\partial}{\partial x_j} - iA_j\right)^2.$$

The 2-form

$$B = dA = \sum_{j<k}\left(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}\right) dx_j \wedge dx_k$$

is the magnetic field, which is responsible for specific spectral properties of the operator $H(A,V)$, as will be seen later.

Under appropriate assumptions on A and V the operator $H(A,V)$ acts as an unbounded self-adjoint operator on $L^2(\mathbb{R}^n)$. In many aspects of the spectral theory of the Schrödinger operator with magnetic field $H(A,V)$, it is convenient to compare this operator with the ordinary Schrödinger operator

$$H(0,V) = -\Delta + V,$$

and then to employ well-known properties of $H(0, V)$.

Let $X_j = (-i\frac{\partial}{\partial x_j} - A_j)$ for $j = 1, \ldots, n$. Then

$$-\triangle_A = \sum_{j=1}^{n} X_j^2, \tag{3.5}$$

and for $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we have

$$(-\triangle_A u, u) = \sum_{j=1}^{n} \|X_j u\|^2. \tag{3.6}$$

Proposition 3.2. *Let $A \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ and V be a continuous real-valued function on $\mathbb{R}^m$, such that*

$$V(x) \geq -C, \, \forall x \in \mathbb{R}^m,$$

where $C > 0$ is a positive constant. Let $dom(H(A, V)) = \mathcal{C}_0^\infty(\mathbb{R}^m)$. Then $H(A, V)$ is a symmetric, semibounded operator on $L^2(\mathbb{R}^n)$.

Proof. For $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned}
(H(A, V)u, u) &= \int_{\mathbb{R}^n} (-\triangle_A u + Vu) \, \overline{u} \, d\lambda \\
&= \int_{\mathbb{R}^n} \sum_{j=1}^{n} |X_j u|^2 \, d\lambda + \int_{\mathbb{R}^n} V|u|^2 \, d\lambda \\
&\geq -C \, \|u\|^2.
\end{aligned}$$

□

Recall that a function $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is the distributional derivative of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with respect to x_j (formally $g = \partial f / \partial x_j$), if

$$(g, \phi) = -\left(f, \frac{\partial \phi}{\partial x_j}\right),$$

for each $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

Let $f_k, f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We say that f_k converges to f in the distributional sense, if

$$(f_k, \phi) \to (f, \phi)$$

for each $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

Let $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We say that $f \geq g$ in the distributional sense, if

$$(f, \phi) \geq (g, \phi),$$

for all positive $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

A useful tool for spectral analysis of Schrödinger operators is Kato's inequality sometimes also called the diamagnetic inequality:

Proposition 3.3. *Let $A \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$. Then, for all $f \in L^1_{loc}(\mathbb{R}^n)$ with $(-i\nabla + A)^2 f \in L^2_{loc}(\mathbb{R}^n)$, we have*

$$\triangle|f| \geq -\Re(sgn(f)(-i\nabla + A)^2 f) = \Re(sgn(f)\triangle_A f), \tag{3.7}$$

in the distributional sense, where sgn is defined in Chapter 5.

Proof. Let $A_1, \ldots, A_n$ be the components of A. Notice that

$$-\triangle_A f = (-i\nabla + A)^2 f = \sum_{j=1}^{n} (-i\frac{\partial}{\partial x_j} + A_j)^2 f.$$

The assumption $(-i\nabla + A)^2 f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, and the standard regularity property of second-order elliptic operators (see [4]) imply that $f \in W^2_{\mathrm{loc}}(\mathbb{R}^n)$, in particular $\triangle f, \nabla f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

First suppose that u is smooth.Then, with $|u|_\epsilon = \sqrt{|u|^2 + \epsilon^2} - \epsilon$, we get

$$\nabla|u|_\epsilon = \frac{\Re(\overline{u}\nabla u)}{\sqrt{|u|^2 + \epsilon^2}} = \frac{\Re(\overline{u}(\nabla + iA)u)}{\sqrt{|u|^2 + \epsilon^2}}. \tag{3.8}$$

A straightforward calculation shows that for a smooth function g we have

$$g\triangle g = \operatorname{div}(g\nabla g) - |\nabla g|^2.$$

Hence we obtain

$$\begin{aligned}
\sqrt{|u|^2 + \epsilon^2}\,\triangle|u|_\epsilon &= \operatorname{div}(\sqrt{|u|^2 + \epsilon^2}\;\nabla|u|_\epsilon) - |\nabla|u|_\epsilon|^2 \\
&= \Re\left[\overline{\nabla u} \cdot (\nabla + iA)u + \overline{u}\operatorname{div}((\nabla + iA)u)\right] - |\nabla|u|_\epsilon|^2 \\
&= \Re[\overline{(\nabla u + iAu)} \cdot (\nabla + iA)u \\
&\quad + iA\overline{u} \cdot (\nabla + iA)u + \overline{u}\operatorname{div}((\nabla + iA)u)] - |\nabla|u|_\epsilon|^2 \\
&= |(\nabla + iA)u|^2 - |\nabla|u|_\epsilon|^2 \\
&\quad + \Re\left[iA\overline{u} \cdot (\nabla + iA)u + \overline{u}\operatorname{div}((\nabla + iA)u)\right].
\end{aligned}$$

An easy calculation shows that

$$iA\overline{u} \cdot (\nabla + iA)u + \overline{u}\operatorname{div}((\nabla + iA)u) = \overline{u}\,(\nabla + iA)^2.$$

Using the Cauchy-Schwarz inequality for (3.8) we get

$$|(\nabla + iA)u|^2 \geq |\nabla|u|_\epsilon|^2. \tag{3.9}$$

So we finally see that

$$\triangle|u|_\epsilon \geq \Re \frac{\overline{u}\,(\nabla + iA)^2 u}{\sqrt{|u|^2+\epsilon^2}}. \tag{3.10}$$

The rest of the proof uses approximative units and follows the same lines as the proof of the Proposition 3.1. □

Using Kato's inequality and a criterion for essential self-adjointness we obtain

Proposition 3.4. *Let $A \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ and $V \in L^2_{loc}(\mathbb{R}^n)$ and $V \geq 0$. Then the Schrödinger operator $H(A,V) = -\triangle_A + V$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^n)$. In this case the Friedrichs extension is the uniquely determined self-adjoint extension (see [6]).*

Proof. By Proposition 2.42, it is sufficient to show that

$$\ker(H(A,V)^* + I) = \{0\}.$$

Since $\mathrm{dom}(H(A,V)^*) \subseteq L^2(\mathbb{R}^n)$, the triviality of the kernel follows from the statement: if

$$-\triangle_A u + Vu + u = 0, \tag{3.11}$$

for $u \in L^2(\mathbb{R}^n)$, then $u = 0$.

If $u \in L^2(\mathbb{R}^n)$ and $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, one has $uV \in\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. In addition we have the inclusion

$$L^2(\mathbb{R}^n) \subset L^2_{\mathrm{loc}}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

which follows from the estimate

$$\int_K |u|\, d\lambda \leq |K|\, (\int_K |u|^2\, d\lambda)^{1/2}.$$

Hence we have $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and, by (3.11), that $\triangle u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where the derivative is taken in the sense of distributions.

From (3.7) and (3.11) we obtain

$$\begin{aligned} \triangle|u| &\geq \Re(\mathrm{sgn}(u)\, \triangle_A u) \\ &= \Re(\mathrm{sgn}(u)\, (V+1)u) \\ &= |u|\,(V+1) \geq 0. \end{aligned}$$

If $(\chi_\epsilon)_\epsilon$ is an approximate unit, we get

$$\triangle(\chi_\epsilon * |u|) = \chi_\epsilon * \triangle|u| \geq 0. \tag{3.12}$$

Since $\chi_\epsilon * |u| \in \mathrm{dom}(\triangle)$, we have

$$(\triangle(\chi_\epsilon * |u|), \chi_\epsilon * |u|) = -\|\nabla(\chi_\epsilon * |u|)\|^2 \leq 0. \tag{3.13}$$

By (3.12), the left side of (3.13) is nonnegative, so $\nabla(\chi_\epsilon * |u|) = 0$ and hence $\chi_\epsilon * |u| = c \geq 0$. But $|u| \in L^2(\mathbb{R}^n)$ and $\chi_\epsilon * |u| \to |u|$ in $L^2(\mathbb{R}^n)$, and so $c = 0$. Hence $\chi_\epsilon * |u| = 0$, so $|u| = 0$ and $u = 0$. □

3.2 Properties of the spectrum

Proposition 3.5. *Let $A \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ and $V \in L^2_{loc}(\mathbb{R}^n)$ and $V \geq 0$. Then*

$$\inf \sigma(H(A, V)) \geq \inf \sigma(H(0, V)). \tag{3.14}$$

Proof. By Kato's inequality (3.7), we have

$$\triangle|f| \leq \Re(\operatorname{sgn}(f)(-\triangle_A f)),$$

so we get

$$\begin{aligned}
(|f|, H(0, V)|f|) &\leq \int_{\mathbb{R}^n} |f|\, \Re(\operatorname{sgn}(f)(-\triangle_A f))\, d\lambda \\
&= \Re \int_{\mathbb{R}^n} \overline{f} H(A, V) f \, d\lambda \\
&= (H(A, V)f, f).
\end{aligned}$$

Now we can apply Proposition 2.64 to obtain the desired result. □

Finally we still mention the gauge invariance of the spectrum of $H(A, V)$:

Proposition 3.6. *Let $A, A' \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ and $V \in L^2_{loc}(\mathbb{R}^n)$ and $V \geq 0$ be such that $dA = dA'$. Then $\sigma(H(A, V)) = \sigma(H(A', V))$.*

Proof. By the Poincaré lemma, we have $A' = A + dg$, where $g \in \mathcal{C}^1(\mathbb{R}^n)$.

Let $X_j = (-i\frac{\partial}{\partial x_j} - A_j)$ and $X'_j = (-i\frac{\partial}{\partial x_j} - A'_j)$ for $j = 1, \ldots, n$. Then

$$X'_j = e^{-ig}\, X_j\, e^{ig}.$$

Hence

$$H(A', V) = e^{-ig}\, H(A, V)\, e^{ig}.$$

Therefore the operators $H(A', V)$ and $H(A, V)$ are unitarily equivalent, hence, by Lemma 2.47 have the same spectrum.

□

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