CHARACTERISTIC CLASSES OF A–BUNDLES

Andreas Cap
Hermann Schichl

April 10, 1995

Abstract. We consider locally trivial bundles over smooth manifolds, whose fibers are finitely generated projective modules over a convenient algebra A. For such a bundle $E \to X$ and a bounded reduced cyclic cocycle c on A we construct a sequence $\mathrm{ch}_c^k(E)$ of de–Rham cohomology classes on X, which are an analog of the classical Chern character. We show that these classes depend only on the cohomology class of c and behave natural under various constructions.

1. Introduction

This paper is devoted to the problem of assigning characteristic classes to locally trivial bundles of finitely generated projective modules over a topological algebra A, so called A–bundles, over finite dimensional manifolds. Such bundles occur naturally for example in the theory of non simply connected manifolds, where one studies bundles whose fibers are finitely generated projective modules over the group C^*–algebra of the fundamental group.

In [4], M. Karoubi defined for a smooth manifold X and a Fréchet algebra A a cohomology theory $H^*_A(X)$, the de–Rham cohomology of X with values in the non–commutative de–Rham homology of A, and then constructs a Chern character of A–bundles over X in these groups.

On the other hand, in [2] A. Connes constructed a pairing $H^{even}_\lambda(A) \times K_0(A) \to \mathbb{C}$ between cyclic cohomology of an algebra A and its topological K–theory, which can be viewed as an analog of a Chern character for finitely generated projective modules over A.

In the present paper, we generalize the construction of Connes in the general setting of convenient algebras, by constructing for an A–bundle $E \to X$ over X and a reduced cyclic cohomology class $[c] \in \bar{H}^*_\lambda(A)$ a sequence of de–Rham cohomology classes $\mathrm{ch}^k_{[c]}(E) \in \bigoplus_{i\leq 2k} H^i(X, \mathbb{C})$. In that way we obtain an analog of the Chern character of Karoubi, without using the "exotic" cohomology theory $H^*_A(X)$.

We think that our approach has also some advantages from the point of view of presentation: First, almost all constructions we carry out stay in the realm of vector bundles (with infinite dimensional fibers) over X. Thus the "infinite dimensional part" of the constructions is just done pointwise, and one only has to check that

1991 *Mathematics Subject Classification.* 19L10.
Key words and phrases. Chern Character, cyclic cohomology, characteristic classes.

everything fits together smoothly, which is rather easy in the convenient setting. Moreover, our construction makes transparent how cyclic cohomology enters the construction: For traces on A (which are just cyclic zero–cocycles) one can construct the corresponding Chern character forms directly as in the classical theory. In general, there are only few traces on A, but usually there are lots of them on the universal differential forms $\Omega^*(A)$. Thus one passes from A–bundles to the associated $\Omega^*(A)$–bundles, and adapts the classical construction by taking into account that $\Omega^*(A)$ is not just an algebra but also a graded differential algebra.

2. Convenient vector spaces, algebras and modules

2.1. Convenient vector spaces. Convenient vector spaces were introduced as the appropriate spaces for differentiation theory. It turns out that they also form a very nice category of linear spaces, and this is the aspect we will mainly be interested in. We will just give a very brief outline of the theory, detailed presentations can be found in [3] or [6].

Let us start with a Hausdorff locally convex vector space V. Then a curve $c : \mathbb{R} \to V$ is called smooth if all derivatives exist and are continuous. If W is another such space then a mapping $f : V \to W$ is called smooth iff for any smooth curve $c : \mathbb{R} \to V$ the curve $f \circ c : \mathbb{R} \to W$ is smooth. It is a non trivial result that for Banach spaces this reproduces the usual notion of smoothness. For linear (and multilinear) mappings one shows that smoothness is equivalent to boundedness, so we denote by $L(V, W) \subset C^\infty(V, W)$ the space of bounded linear maps.

The vector space V is called *convenient* iff for any smooth curve c_1 there exists a smooth curve c_2 such that $c_1 = c_2'$, i.e. iff anti-derivatives of smooth curves exist. It turns out that this is not a condition on the topology of V but only on the associated bornology. In fact, this condition is equivalent to the condition that any Mackey–Cauchy sequence converges. Thus this completeness condition is weaker than sequential completeness.

It turns out that for any (even non Hausdorff) locally convex vector space V one can form a separated completion $i : V \to \tilde{V}$ where $\tilde{V}$ is convenient (and thus Hausdorff) such that any bounded linear map $\varphi : V \to W$ with W convenient can be uniquely written as $\tilde{\varphi} \circ i$ for a bounded linear map $\tilde{\varphi} : \tilde{V} \to W$.

Using the completion it is easy to show that the category of convenient vector spaces and bounded linear maps is complete and cocomplete, so all categorical limits and colimits can be formed.

2.2. For convenient vector spaces V and W one can construct a natural topology on $C^\infty(V, W)$ such that this is again a convenient vector space. Moreover, in this topology the subspace $L(V, W)$ is closed and thus again a convenient vector space.

The main feature of the convenient setting is that the category of convenient vector spaces and smooth maps is Cartesian closed, i.e. flipping coordinates induces a natural isomorphism $C^\infty(U \times V, W) \cong C^\infty(U, C^\infty(V, W))$, which is even an isomorphism of convenient vector spaces.

Similarly as for linear mappings we can topologize spaces of multilinear maps. For convenient vector spaces $V_1, \dots, V_n$ and W we denote by $L(V_1, \dots, V_n; W)$

the space of all bounded n–linear maps $V_1 \times \ldots \times V_n \to W$, with the topology induced by the inclusion into all smooth maps. One shows that flipping coordinates gives a natural isomorphism of convenient vector spaces $L(V_1, \ldots, V_n; W) \cong L(V_1, \ldots, V_i; L(V_{i+1}, \ldots, V_n; W))$.

The next important feature of convenient vector spaces is the existence of an appropriate tensor product. This is called the bornological tensor product, it is denoted by $\tilde{\otimes}$ and it has the universal property that bounded bilinear maps correspond exactly to bounded linear maps on the tensor product, so there is a natural isomorphism $L(U\tilde{\otimes}V, W) \cong L(U, L(V, W))$.

2.3. Convenient algebras. A *convenient algebra* is a convenient vector space A equipped with a bounded bilinear associative multiplication $\mu : A \times A \to A$. We will mainly be interested in complex convenient algebras, i.e. complex algebras such that the underlying real algebra is convenient. We will always assume that algebras are unital and homomorphisms preserve the units.

A standard example of a convenient algebra is the space $L(V, V)$ of bounded endomorphisms of a convenient vector space V. (Boundedness of the composition mapping follows immediately from Cartesian closedness.) Another natural example is the space $C^\infty(X, A)$ with the pointwise operations, where X is a smooth manifold and A is a convenient algebra.

For a convenient algebra A we denote by A^{op} the opposite algebra of A, which is clearly a convenient algebra, too.

2.4. Convenient modules. Let A be a convenient algebra. A *convenient left (right) module* over A is a convenient vector space M together with a bounded algebra homomorphism $\lambda : A \to L(M, M)$ ($\rho : A \to L(M, M)^{op}$). By Cartesian closedness this is equivalent to having bounded bilinear maps $A \times M \to M$ respectively $M \times A \to M$, which satisfy the usual properties.

Let V be a convenient vector space. Then we can make $V\tilde{\otimes}A$ into a right A–module using the multiplication on A. One easily verifies that this is in fact a convenient module, called the free module corresponding to V. (In fact, forming the free module defines a functor from convenient vector spaces to convenient right A–modules which is left adjoint to the forgetful functor.) In particular, if we take $V = \mathbb{R}^n$ for some n we obtain the finitely generated free modules A^n.

An A–module M is called *projective* if there is a free module F and bounded A–module homomorphisms $i : M \to F$ and $p : F \to M$ such that $p \circ i = id_M$. M is called *finitely generated projective* (or f.g.p.) if this F can be chosen to be A^n for some n.

2.5. In [1] it was shown that in the convenient setting there is a nice tensor product of modules, so if M is a right convenient A–module and N is a left convenient A–module then one can form a convenient vector space $M\tilde{\otimes}_A N$ which has the universal property that bounded bilinear maps $f : M \times N \to V$ into any convenient vector space V such that $f(m \cdot a, n) = f(m, a \cdot n)$ for any $a \in A$, correspond bijectively to bounded linear maps $M\tilde{\otimes}_A N \to V$. This tensor product is well behaved with respect to additional module structures on M and N. In particular, if N is also a convenient right B module such that the actions of A and B on N commute, then $M\tilde{\otimes}_A N$ is a right B–module in a canonical way.

In particular, if $\varphi : A \to B$ is a bounded homomorphism between two convenient algebras and M is a right A module, then via φ we can view B as a convenient left A–module, so $\varphi_* M := M \tilde{\otimes}_A B$ is canonically a right B–module. It is easy to verify that if M is projective respectively f.g.p. then the same is true for $\varphi_* M$.

2.6. Traces. Let A be a convenient algebra, V a convenient vector space. A *V–valued trace* on A is a bounded linear map $t : A \to V$ such that $t(ab) = t(ba)$ for all $a, b \in A$.

Let M be a f.g.p. right A–module and consider the space $M^* := \operatorname{Hom}^A(M, A)$ of bounded right module homomorphisms (in fact any such homomorphism is bounded). It is easy to show that multiplication from the left on the values makes this space into a convenient left A–module. There is an obvious map $M \times M^* \to \operatorname{Hom}^A(M, M)$ defined by mapping (m, φ) to the map $m' \mapsto m \cdot \varphi(m')$ and clearly $(m \cdot a, \varphi)$ and $(m, a \cdot \varphi)$ have the same image, so it gives rise to a unique bounded linear map $M \tilde{\otimes}_A M^* \to \operatorname{Hom}^A(M, M)$.

Next let $i : M \to A^n$ and $p : A^n \to M$ be bounded module homomorphisms such that $p \circ i = id_M$, let e_j be the j–th unit vector in A^n, $\pi_j : A^n \to A$ the j–th projection and put $\varphi_j := \pi_j \circ i \in M^*$. It is easy to verify that mapping a module homomorphism $\Phi : M \to M$ to $\sum_{j=1}^n \Phi(p(e_j)) \otimes \varphi_j$ defines a bounded linear map $\operatorname{Hom}^A(M, M) \to M \tilde{\otimes}_A M^*$ which is inverse to the map constructed above. Thus for a f.g.p. module M the space $\operatorname{Hom}^A(M, M)$ is canonically isomorphic to $M \tilde{\otimes}_A M^*$.

Next let $t : A \to V$ be a trace, and consider the map $M \times M^* \to V$ defined by $(m, \varphi) \mapsto t(\varphi(m))$. By Cartesian closedness this map is bounded and since t is a trace it induces a unique bounded linear map $\tilde{t} : M \tilde{\otimes}_A M^* \to V$. Composing this with the canonical isomorphism from above we get a bounded linear map $\operatorname{Tr}_t : \operatorname{Hom}^A(M, M) \to V$. Now it is easy to verify that under the canonical isomorphism from above the composition of homomorphisms corresponds to the map $(m \otimes \varphi, m' \otimes \varphi') \mapsto m \otimes \varphi(m') \varphi'$, and using this one immediately verifies that Tr_t is again a trace.

2.7. Finally, we will need graded algebras and graded differential algebras and modules over them. By a graded convenient algebra we just mean a convenient algebra $\mathcal{A}$ such that the underlying vector space is a graded convenient vector space, i.e. can be written as a direct sum $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n$ of convenient vector spaces. Similar as above, the bounded endomorphisms of any graded convenient vector space form a graded convenient algebra, so the notion of a graded module makes no problem. (We always assume that the corresponding homomorphism $\mathcal{A} \to L(\mathcal{M}, \mathcal{M})$ is homogeneous of degree zero.) Also, the notion of free and projective modules makes no problem in this setting.

By a convenient graded differential algebra we mean just a convenient graded algebra $\mathcal{A}$ together with a bounded linear differential $d = d_{\mathcal{A}}$ which is homogeneous of degree one and a graded derivation.

3. Bounded Cyclic Cohomology

Following closely [2] we develop in this section bounded Hochschild and cyclic cohomology for convenient algebras and discuss the relations of these to traces on the algebra of universal differential forms.

3.1. Definition. For a convenient algebra A and $n \geq 0$ we denote by $C^n(A)$ the convenient vector space $L^{n+1}(A, \mathbb{C})$ of bounded $n+1$–linear maps from A^{n+1} to $\mathbb{C}$. By $C^n_\lambda(A)$ we denote the closed subspace of those maps $\varphi \in C^n(A)$ which are cyclically invariant, i.e. satisfy $\varphi(a_n, a_0, \dots, a_{n-1}) = (-1)^n \varphi(a_0, \dots, a_n)$. Next by $\bar{C}^n(A)$ we denote the closed subspace of those maps $\varphi \in C^n(A)$ which satisfy $\varphi(a_0, \dots, a_n) = 0$ if for some $i \geq 1$ we have $a_i = 1$. Finally, we put $\bar{C}^n_\lambda(A) := \bar{C}^n(A) \cap C^n_\lambda(A)$.

Next we denote by $b : C^{n-1}(A) \to C^n(A)$ the Hochschild differential, defined as usually by

$$(b\varphi)(a_0, \dots, a_n) = \sum_{i=0}^{n-1} (-1)^i \varphi(a_0, \dots, a_i a_{i+1}, \dots, a_n) + (-1)^n \varphi(a_n a_0, a_1, \dots, a_{n-1})$$

Clearly, this is a bounded linear mapping. Obviously $b(\bar{C}^{n-1}(A)) \subset \bar{C}^n(A)$. Moreover, it is well known, see [2] or [7], that $b(C^{n-1}_\lambda(A)) \subset C^n_\lambda(A)$, and thus also $b(\bar{C}^{n-1}_\lambda(A)) \subset \bar{C}^n_\lambda(A)$. It is a classical result that $b^2 = b \circ b = 0$, so we have the corresponding cohomologies which we denote by $H^*(A)$, $\bar{H}^*(A)$, $H^*_\lambda(A)$ and $\bar{H}^*_\lambda(A)$ and call Hochschild cohomology, reduced Hochschild cohomology, cyclic cohomology and reduced cyclic cohomology, respectively. Note that in general these cohomology spaces are not Hausdorff.

3.2. Next, we need the B–operator of Connes. First we define $B_0 : C^{n+1}(A) \to C^n(A)$ by

$$(B_0\varphi)(a_0, \dots, a_n) := \varphi(1, a_0, \dots, a_n) - (-1)^{n+1} \varphi(a_0, \dots, a_n, 1).$$

Let $N : C^n(A) \to C^n_\lambda(A)$ be the cyclization, which is given by

$$(N\varphi)(a_0, \dots, a_n) := \sum_{i=0}^{n} (-1)^{in} \varphi(a_i, \dots, a_n, a_0, \dots, a_{i-1}).$$

Clearly, for $\varphi \in C^n_\lambda(A)$ we have $N\varphi = (n+1)\varphi$. Now we put $B := N \circ B_0 : C^{n+1}(A) \to C^n_\lambda(A)$. One immediately verifies that, although N does not map $\bar{C}^n(A)$ to $\bar{C}^n_\lambda(A)$, we have $B(\bar{C}^{n+1}(A)) \subset \bar{C}^n_\lambda(A)$. It is well known, see [2] or [7], that $B^2 = B \circ B = 0$ and $B \circ b + b \circ B = 0$. In particular, this shows that if $\varphi \in \bar{Z}^{n+1}(A)$ is a reduced Hochschild cocycle then $B\varphi \in \bar{C}^n_\lambda(A)$ is in fact a cyclic cocycle and that B induces a well defined map from reduced Hochschild to reduced cyclic cohomology.

3.3. Lemma. *The space $\bar{B}^n_\lambda(A)$ of reduced cyclic n–coboundaries is contained in the image of $\bar{Z}^{n+1}(A)$ under B.*

Proof. (see also [2]). Take $\psi \in \bar{C}^{n-1}_\lambda(A)$ be a reduced cochain. Let $f : A \to \mathbb{C}$ be any linear functional such that $f(1) = 1$, and define $\omega \in \bar{C}^n(A)$ by $\omega(a_0, \dots, a_n) := f(a_0)\psi(a_1, \dots, a_n)$. (Note that by cyclicity ψ vanishes if any of its arguments is equal to one.) Obviously $\psi = B_0\omega$, so $\psi = \frac{1}{n} B\omega$. But then $b\psi = \frac{1}{n} bB\omega = -\frac{1}{n} Bb\omega$. $\square$

3.4. Cyclic Cohomology and universal differential forms. Recall the construction of the convenient graded differential algebra $\Omega^*(A)$ of universal differential forms over a convenient algebra A from [1, Section 2]. Here we use the analogous construction for complex convenient algebras, but this means just that we replace $\mathbb{R}$ by $\mathbb{C}$ in all constructions. Note that in the proof of theorem 2.9 of [1] it was shown that $\Omega^k(A) \cong A\tilde{\otimes}\bar{A}\tilde{\otimes}\dots\tilde{\otimes}\bar{A}$, where there are k factors $\bar{A} = A/\mathbb{C}\cdot 1$, and $\tilde{\otimes}$ denotes the bornological tensor product. This isomorphism is given by mapping $a_0\tilde{\otimes}\bar{a}_1\tilde{\otimes}\dots\tilde{\otimes}\bar{a}_n$ to $a_0 da_1 \dots da_n$. By the universal property of the bornological tensor product we see immediately that $\bar{C}^n(A) \cong L(\Omega^n(A), \mathbb{C})$, the space of bounded linear maps. From now on we will always use the same letter for reduced cochains and the corresponding functionals on the universal forms.

3.5. Proposition. *Let $\varphi \in \bar{C}^n(A)$ be a reduced cochain which we view as a map $\Omega^n(A) \to \mathbb{C}$. Then*

1. *If φ is a graded trace, i.e. $\varphi(\beta\alpha) = (-1)^{|\alpha||\beta|}\varphi(\alpha\beta)$, then $b\varphi = 0$.*
2. *φ is a closed graded trace, i.e. in addition $\varphi(d\alpha) = 0$ if and only if $\varphi \in \bar{Z}^n_\lambda(A)$.*
3. *If $\varphi \in \bar{Z}^n_\lambda(A)$ then the trace defined by φ can be written as the composition of a graded trace with d if and only if φ lies in $B(\bar{Z}^{n+1}(A))$.*

Proof. (1) If φ defines a graded trace, then applying the trace property to the 0–form a_{n+1} and the n–form $a_0 da_1 \dots da_n$ we get

$$\varphi(a_{n+1}a_0, a_1, \dots, a_n) = \varphi(a_{n+1}(a_0 da_1 \dots da_n)) = \varphi((a_0 da_1 \dots da_n)a_{n+1}).$$

Using the formula $d(ab) = (da)b + adb$ we can write $(a_0 da_1 \dots da_n)a_{n+1}$ as a sum of terms of the form $adb \dots dc$, and doing this we arrive at a formula for $\varphi(a_{n+1}a_0, a_1, \dots, a_n)$ which is immediately seen to be equivalent to $b\varphi = 0$.

(2) If φ defines a closed graded trace then by (1) it is a Hochschild cocycle. Moreover, expanding $(da_1 \dots da_n)a_0$ as above we get $(-1)^n a_1 da_2 \dots da_n da_0$ plus a sum of closed forms. Thus applying the trace property to the zero form a_0 and the n–form $da_1 \dots da_n$ we see that φ is cyclic.

Conversely, if φ is a reduced cyclic cocycle, then $\varphi(1, a_1, \dots, a_n) = 0$, so φ vanishes on closed forms. Moreover, the computation of (1) shows that φ has the trace property if one of the forms has degree zero. Clearly, it suffices to show that φ also has the trace property if one of the forms is an exact one form, i.e. that $\varphi(a_0 da_1 \dots da_n) = (-1)^{n-1}\varphi(da_n(a_0 da_1 \dots da_{n-1}))$. But

$$da_n(a_0 da_1 \dots da_{n-1}) = d(a_n a_0)da_1 \dots da_{n-1} - a_n da_0 \dots da_{n-1},$$

so the result immediately follows from cyclicity of φ.

(3) If $\varphi(\omega) = \tau(d\omega)$ for a graded trace τ, then clearly $\varphi = B_0\tau$ and by (1) τ is a Hochschild cocycle, and since φ is cyclic we have $\varphi = \frac{1}{n+1}B\tau$.

Conversely, assume that $\varphi = B\psi$ for some $\psi \in \bar{Z}^{n+1}(A)$. Then consider

$$\theta := (n+1)B_0\psi - \varphi \in \bar{C}^n(A).$$

This is by construction in the kernel of N, so it can be written as $\tau - t\tau$ for some $\tau \in \bar{C}^n(A)$, where $(t\tau)(a_0, \dots, a_n) = (-1)^n\tau(a_n, a_0, \dots, a_{n-1})$. (Explicitly, we can write $\tau = \frac{1}{n+1}(n\theta + (n-1)t\theta + \cdots + t^{n-1}\theta)$, and from this formula we see that τ vanishes if any of its entries equals 1.) Thus, if we consider $b\tau \in \bar{C}^{n+1}(A)$, we immediately get $B_0 b\tau = (1-t)\tau$, and hence $\varphi = B_0((n+1)\psi - b\tau)$.

So $\varphi = B_0\psi$ for some $\psi \in \bar{Z}^{n+1}(A)$. To finish the proof, we have to show that this ψ in fact defines a graded trace on $\Omega(A)$, and as above it suffices to prove the trace property if one of the forms is an exact one form. Now we have

$$\psi(da_n(a_0da_1\dots da_{n-1})) = B_0\psi(a_na_0, a_1, \dots, a_{n-1}) - \psi(a_n, a_0, \dots, a_{n-1}).$$

Since $B_0\psi$ is a cocycle we can expand the right hand side as

$$\begin{aligned} -(-1)^n \sum_{i=0}^{n-1}(-1)^i B_0\psi(a_0, \dots, a_ia_{i+1}, \dots, a_n) - \psi(a_n, a_0, \dots, a_{n-1}) = \\ = (-1)^n(b\psi)(1, a_0, \dots, a_n) - (-1)^n\psi(a_0, \dots, a_n), \end{aligned}$$

and since ψ is a cocycle the result follows. □

3.6. Corollary. *The quotient of the space of all closed graded traces on $\Omega^*(A)$ modulo those which may be written as the composition of a graded trace with d is isomorphic to $\bar{Z}^*_\lambda(A)/B(\bar{Z}^{*+1}(A)) \cong \bar{H}^*_\lambda(A)/B(\bar{H}^{*+1}(A))$.*

Proof. The first isomorphism is clear by 3.5, (2) and (3), while the last isomorphism follows using 3.3. □

4. Bundles of modules

4.1. The basic definitions and notions of the theory of vector bundles can be generalized to vector bundles with fiber a convenient vector space without any problems. So if V is a convenient vector space and X is a (finite dimensional, second countable, Hausdorff) smooth manifold, then it is clear, how to define locally trivial vector bundles with standard fiber V, and one gets transition functions as in the usual case. Moreover, it is simple to prove that the sections of such a bundle form a convenient vector space (cf. [3, Section 4.6] for a more general approach).

4.2. Definition. Let A be a convenient algebra, X a smooth manifold as above. An *A–bundle over X* is a locally trivial vector bundle with standard fiber a right f.g.p. A–module and bounded A–module homomorphisms as transition functions.

A homomorphism of A–bundles is a smooth vector bundle homomorphism covering the identity such that the restriction to each fiber is an A–module homomorphism.

If $E \to X$ is such an A–bundle with fiber M then one shows that the pointwise operations make the space $\Gamma(E)$ of sections into a convenient right module over the convenient algebra $C^\infty(X, A)$ of A–valued smooth functions on X. In fact, one can show that this module is f.g.p., but we will not use this fact. Clearly, any homomorphism of A–bundles induces a $C^\infty(X, A)$–module homomorphism between the spaces of sections. In fact, any such module homomorphism is of this form:

4.3. Lemma. *Let $E \to X$ be an A–bundle and $F \to X$ a bundle of A–modules (not necessarily f.g.p.), and let $\Phi : \Gamma(E) \to \Gamma(F)$ be a bounded linear $C^\infty(X, A)$–module homomorphism. Then Φ is induced by a homomorphism of A–bundles.*

Proof. Clearly, any such map is a local operator, i.e. if s is a section which is identically zero locally around a point $x \in X$ then also $\Phi(s)$ is identically zero locally around x. Thus we may assume that $E = X \times M$ for a f.g.p. A–module M. What we have to show is just that $\Phi(s)(x)$ depends on $s(x)$ only. Thus let us assume that $s(x) = 0$. We can choose bounded A–module homomorphisms $i : M \to A^m$, $p : A^m \to M$ such that $p \circ i = id_M$. Then $i \circ s$ can be written uniquely as $i(s(x)) = \sum e_j \cdot s_j(x)$ for the standard unit vectors e_j and smooth functions $s_j : X \to A$. Thus $s(x) = \sum p(e_j) \cdot s_j(x)$ and thus $\Phi(s)(x) = \sum \Phi(p(e_j)) \cdot s_j(x)$ and all s_j vanish in x. □

4.4. Constructions with A–bundles. In the classical theory of vector bundles many constructions, like Whitney sums, tensor products, exterior powers and so on can be carried out using smooth functors, see for example [5, Section 6.7]. A similar approach is possible in the setting of convenient vector bundles and A–bundles. For example, consider the map $M \mapsto \varphi_*(M)$ induced by an algebra homomorphism $\varphi : A \to B$ as in 2.5. This in fact defines a functor between the categories of f.g.p. modules. In these categories any space of morphisms is canonically a convenient vector space (see 2.2), and one easily verifies that this functor induces bounded linear maps on the spaces of morphisms. Thus for an A–bundle $E \to X$ one gets a B–bundle $\varphi_* E \to X$ by applying the functor φ_* to each fiber.

Another construction which can be immediately generalized to vector bundles with convenient fibers is the pullback. If $f : X \to Y$ is a smooth map between smooth manifolds and $E \to X$ is such a vector bundle, then one defines f^*E as usual as a subset of $E \times Y$ and shows that it is a convenient vector bundle as in the finite dimensional case. Clearly, if E is an A–bundle, then so is f^*E.

5. Characteristic classes for A–bundles

5.1. The double complex $\Omega^*(X, \Omega^*(A))$. Let X be a finite dimensional smooth manifold, and let $E \to X$ be a convenient vector bundle over X. Then we define the space $\Omega^*(X, E)$ of *E–valued smooth differential forms* on X to be the space of smooth sections of the convenient vector bundle $L(\Lambda^* TX, E)$, whose fiber over a point $x \in X$ is the space $L(\Lambda^* T_x X, E_x)$ of bounded linear maps. (One easily verifies that $L(\ ,\ )$ is a smooth bifunctor and uses 4.4.)

In particular, if V is a convenient vector space we define $\Omega^*(X, V)$ using the trivial vector bundle $X \times V \to X$. In this case we have the exterior derivative $d_X : \Omega^*(X, V) \to \Omega^{*+1}(X, V)$ defined by the usual formula

$$d_X \omega(\xi_0, \dots, \xi_k) := \sum_{i=0}^{k} (-1)^i \xi_i \cdot \omega(\xi_0, \dots, \hat{\xi_i}, \dots, \xi_k) + \\ + \sum_{i<j} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \dots, \hat{\xi_i}, \dots, \hat{\xi_j}, \dots, \xi_k)$$

for smooth vector fields ξ_i on X. As in the case of finite dimensional V one verifies that this is well defined and that $d_X^2 = 0$. Moreover d_X is easily seen to be bounded.

Next let A be a convenient algebra and let $\Omega^*(A)$ be the convenient graded differential algebra of universal differential forms on A. Then on $\Omega^*(X, \Omega^*(A))$ there is a second differential $d_A : \Omega^*(X, \Omega^*(A)) \to \Omega^*(X, \Omega^{*+1}(A))$ induced by the differential on $\Omega^*(A)$. Since the latter differential is a bounded linear mapping we may differentiate through it and thus from the formula for d_X above we see that $d_X d_A = d_A d_X$. Thus $(\Omega^*(X, \Omega^*(A)), d_X, d_A)$ is a double complex with bounded differentials. On $\Omega^*(X, \Omega^*(A))$ we define a multiplication as follows: For $\omega \in \Omega^k(X, \Omega^\ell(A))$ and $\omega' \in \Omega^{k'}(X, \Omega^{\ell'}(A))$ and tangent vectors $\xi_1, \dots, \xi_{k+k'} \in T_xX$ we define

$$(\omega\omega')(\xi_1, \dots, \xi_{k+k'}) = \tfrac{(-1)^{k'\ell}}{k!k'!} \sum_{\sigma \in \mathfrak{S}_{k+k'}} sgn(\sigma)\omega(\xi_{\sigma 1}, \dots, \xi_{\sigma k})\omega'(\xi_{\sigma(k+1)}, \dots, \xi_{\sigma(k+k')}),$$

where the sum is over all permutations of $k + k'$ elements. The sign $(-1)^{k'\ell}$ is motivated by the usual definition of the graded tensor product of graded algebras. Next, we define the differential d on $\Omega^*(X, \Omega^*(A))$ by putting $d = d_X + (-1)^k d_A$ on $\Omega^k(X, \Omega^*(A))$. One easily verifies that d is in fact a graded derivation (with respect to the total degree) for the multiplication defined above.

5.2. Let $E \to X$ be an A–bundle. Then by 4.4 we can form the associated $\Omega^*(A)$–bundle $\tilde{E} := i_*E = E\tilde{\otimes}_A\Omega^*(A)$, where $i : A \to \Omega^*(A)$ is the inclusion, and from 5.1 we get the space $\Omega^*(X, \tilde{E})$ of differential forms on X with values in this bundle. These forms are a convenient right module over $\Omega^*(X, \Omega^*(A))$ with action defined by

$$(\varphi \cdot \omega)(\xi_1, \dots, \xi_{k+k'}) = \tfrac{(-1)^{k'\ell}}{k!k'!} \sum_{\sigma \in \mathfrak{S}_{k+k'}} sgn(\sigma)\varphi(\xi_{\sigma 1}, \dots, \xi_{\sigma k})\omega(\xi_{\sigma(k+1)}, \dots, \xi_{\sigma(k+k')}),$$

where $\varphi \in \Omega^k(X, \tilde{E}^\ell)$, $\omega \in \Omega^{k'}(X, \Omega^*(A))$, the ξ_i are tangent vectors and $\tilde{E}^\ell$ denotes the homogeneous component of degree ℓ. In fact, one can prove that $\Omega^*(X, \tilde{E})$ is a finitely generated projective module over $\Omega^*(X, \Omega^*(A))$, but we will not use this fact.

5.3. Definition. A *covariant derivative* on E is a bounded linear map

$$D : \Gamma(E) \to \Omega^1(X, E) \oplus C^\infty(X, \tilde{E}^1),$$

which satisfies the Leibniz rule $D(sf) = D(s)f + s \cdot df$ for any section $s \in \Gamma(E)$ and any $f \in C^\infty(X, A)$.

5.4. Lemma. *On any A–bundle $E \to X$ as above, there exists a covariant derivative, and the space of all covariant derivatives is an affine space modeled on the convenient vector space of bounded right $C^\infty(X, A)$–module homomorphisms from $\Gamma(E)$ to the degree one component of $\Omega^*(X, \tilde{E})$, which by 4.3 is isomorphic to the space of A–bundle homomorphisms from E to the degree one component of the bundle $L(\Lambda^*TX, \tilde{E})$.*

Moreover, any covariant derivative D extends uniquely to a bounded linear map $D : \Omega^*(X, \tilde{E}) \to \Omega^*(X, \tilde{E})$, which is homogeneous of degree 1 and satisfies the Leibniz rule $D(\varphi\omega) = D(\varphi)\omega + (-1)^{|\varphi|}\varphi(d\omega)$ for any $\varphi \in \Omega^*(X, \tilde{E})$ which is homogeneous of total degree $|\varphi|$ and $\omega \in \Omega^*(X, \Omega^*(A))$.

Proof. For a trivial bundle with fiber A^n one can simply take the map d from 5.1 in each component to get a covariant derivative. Next, for a trivial bundle with fiber any f.g.p. module M one chooses homomorphisms $i : M \to A^n$ and $p : A^n \to M$ such that $p \circ i = id_M$. Then p induces an $\Omega^*(X, \Omega^*(A))$–module homomorphism $\tilde{p} : \Omega^*(X, A^n \tilde{\otimes}_A \Omega^*(A)) \to \Omega^*(X, M \tilde{\otimes}_A \Omega^*(A))$, and for any covariant derivative D on the trivial A^n–bundle we get a covariant derivative $\hat{D}$ on the trivial M–bundle by defining $\hat{D}s = \tilde{p}(D(i \circ s))$.

On a general A–bundle, take an atlas corresponding to a covering $\{U_\alpha\}$ of X. As above one can then define operators D_α which act as covariant derivatives on sections having support in U_α. Then let $\{f_\alpha\}$ be a partition of unity subordinate to the covering $\{U_\alpha\}$ and define $D(s) := \sum_\alpha D_\alpha(sf_\alpha)$. One immediately verifies that this is in fact a covariant derivative.

The Leibniz rule immediately implies that the difference of two covariant derivatives is a $C^\infty(X, A)$–module homomorphism. Conversely, a sum of a covariant derivative and a $C^\infty(X, A)$–module homomorphism clearly satisfies the Leibniz rule, and thus is again a covariant derivative, so the structure of an affine space follows.

To construct the extensions of covariant derivatives we start with the trivial bundle with fiber A^n. In this case $\tilde{E} \cong X \times \Omega^*(A)^n$, so any form $\varphi \in \Omega^*(X, \tilde{E})$ can be uniquely written as $\varphi = \sum e_i \cdot \varphi_i$ for the constant sections $e_i \in \Gamma(E)$ and forms $\varphi_i \in \Omega^*(X, \Omega^*(A))$. But then clearly $D(\varphi) = \sum(D(e_i) \cdot \varphi_i + e_i \cdot d\varphi_i)$ is the unique extension which has the required properties. For the trivial bundle with fiber any f.g.p. module M we can first extend the covariant derivative to one which is defined on the trivial bundle with fiber some A^n, then extend this and project it down to the original bundle as above. Finally, for a non trivial bundle we can construct from a given covariant derivative locally defined covariant derivatives on trivial bundles over charts, extend these, and by uniqueness they fit together to define a global extension. □

5.5. Curvature. Let D be a covariant derivative on an A–bundle $E \to X$ and denote by D also the extension to all differential forms as constructed in 5.4 above. Then we define the *curvature* of D to be the bounded linear map $R = R(D) : \Omega^*(X, \tilde{E}) \to \Omega^*(X, \tilde{E})$ given by $R = D^2 = D \circ D = \frac{1}{2}[D, D]$.

From the general Leibniz rule it follows immediately that $R(D)$ is a bounded $\Omega^*(X, \Omega^*(A))$–module homomorphism, which is homogeneous of degree 2.

5.6. Consider a bounded $\Omega^*(X, \Omega^*(A))$–module homomorphism $\Phi : \Omega^*(X, \tilde{E}) \to \Omega^*(X, \tilde{E})$. From 4.3 we know that this is induced by an $\Omega^*(A)$–bundle endomorphism on $L(\Lambda^* TX, \tilde{E})$. (Note that since X is finite dimensional this is in fact an $\Omega^*(A)$–bundle.) In particular, this means that for each $x \in X$ it induces a bounded linear map Φ_x from $L(\Lambda^* T_x X, \tilde{E}_x)$ to itself. Since $T_x X$ is finite dimensional, the latter space is isomorphic to $\Lambda^* T_x^* X \otimes \tilde{E}_x$. From the fact that Φ is an $\Omega^*(X, \Omega^*(A))$–module endomorphism one easily sees that Φ_x is actually determined by its restriction to $\mathbb{R} \otimes \tilde{E}_x \subset \Lambda^* T_x^* X \otimes \tilde{E}_x$, and this restriction is an element of

$\mathrm{Hom}^{\Omega^*(A)}(\tilde{E}_x, \Lambda^* T_x^* X \otimes \tilde{E}_x)$. A slight generalization of the argument used in 2.6 to describe the endomorphisms of a f.g.p. module shows that this space is actually isomorphic to $\Lambda^* T_x^* X \otimes \mathrm{Hom}^{\Omega^*(A)}(\tilde{E}_x, \tilde{E}_x)$. Thus we may view Φ as a section of the corresponding bundle, i.e. an element of $\Omega^*(X, \mathrm{Hom}^{\Omega^*(A)}(\tilde{E}, \tilde{E}))$. In fact, one can verify that the above construction actually describes a bounded linear isomorphism

$$\mathrm{Hom}^{\Omega^*(X,\Omega^*(A))}(\Omega^*(X, \tilde{E}), \Omega^*(X, \tilde{E})) \to \Omega^*(X, \mathrm{Hom}^{\Omega^*(A)}(\tilde{E}, \tilde{E})).$$

Next let c be a bounded graded $\mathbb{C}$–valued trace on $\Omega^*(A)$. From 2.6 we see that this gives rise to a graded trace Tr_c on the space of endomorphisms of any f.g.p. module over $\Omega^*(A)$. Now if we take the bundle $\tilde{E}$ of f.g.p. $\Omega^*(A)$–modules then the transition functions of the convenient vector bundle $\mathrm{Hom}^{\Omega^*(A)}(\tilde{E}, \tilde{E})$ are just the conjugations with the transition functions of $\tilde{E}$ which are homogeneous of degree zero. Thus the trace property implies that Tr_c gives rise to a vector bundle homomorphism from $\mathrm{Hom}^{\Omega^*(A)}(\tilde{E}, \tilde{E})$ to the trivial complex line bundle over X, which is a graded trace in each point. This in turn induces a vector bundle homomorphism $L(\Lambda^* TX, \mathrm{Hom}^{\Omega^*(A)}(\tilde{E}, \tilde{E})) \to L(\Lambda^* TX, \mathbb{C})$ and thus a bounded linear map $\mathrm{Tr}_c : \Omega^*(X, \mathrm{Hom}^{\Omega^*(A)}(\tilde{E}, \tilde{E})) \to \Omega^*(X, \mathbb{C})$. Using the above isomorphism we can thus assign to the homomorphism Φ from above a differential form $\mathrm{Tr}_c(\Phi) \in \Omega^*(X, \mathbb{C})$.

5.7. Lemma.

(1) *The map* $\mathrm{Tr}_c : \mathrm{Hom}^{\Omega^*(X,\Omega^*(A))}(\Omega^*(X, \tilde{E}), \Omega^*(X, \tilde{E})) \to \Omega^*(X, \mathbb{C})$ *defined above is a bounded graded trace.*

(2) *Let D be a covariant derivative on the A–bundle $E \to X$. Then for a $\Omega^*(X, \Omega^*(A))$–module endomorphism Φ of $\Omega^*(X, \tilde{E})$ which is homogeneous of total degree $|\Phi|$, the map $\tilde{D}(\Phi)$ defined by*

$$\tilde{D}(\Phi)(\omega) = D(\Phi(\omega)) - (-1)^{|\Phi|}\Phi(D(\omega))$$

is a module endomorphism, too.

(3) *The curvature R satisfies the Bianchi–identity $\tilde{D}(R) = 0$, even $\tilde{D}(R^k) = 0$.*

Proof. (1) From 5.6 we know that a homomorphism Φ in the left hand space is induced by a homomorphism of $\Omega^*(A)$–bundles, and by construction $\mathrm{Tr}_c(\Phi)(x)$ depends only on Φ_x. Thus we can verify the trace property in one fiber, which is of the form $\Lambda^* T_x^* X \otimes \tilde{E}$. Choosing a basis for $T_x^* X$ one verifies the trace property by a direct computation.
(2) follows directly from the Leibniz rule.
(3) This is just the fact, that D^{2k} commutes with D. $\square$

5.8 Chern–character forms. From 3.5(2) and 5.7 we now see that given a covariant derivative D on an A–bundle, any bounded reduced cyclic cocycle c on A gives rise to a sequence $\{\mathrm{ch}_c^k(D)\}$ of differential forms on X, defined by $\mathrm{ch}_c^k(D) := \frac{1}{k!}\mathrm{Tr}_c(R(D)^k) \in \bigoplus_{i \leq 2k} \Omega^i(X, \mathbb{C})$. The form $\mathrm{ch}_c^k(D)$ is called the *k–th Chern–character form* of D corresponding to c.

Theorem. *Let $c \in \bar{Z}^n_\lambda(A)$ be a bounded reduced cyclic cocycle, $E \to X$ an A–bundle, and D a covariant derivative on E. Then*

1. $d\,\mathrm{ch}^k_c(D) = 0$.
2. *If $c \in \bar{B}^n_\lambda(A)$ a bounded reduced cyclic coboundary. Then the differential form $\mathrm{ch}^k_c(D)$ is exact.*
3. *The cohomology class of $\mathrm{ch}^k_c(D)$ is independent of the choice of D.*

Proof. Since the questions above are local questions, we may assume that the bundle is trivial, so we assume $E = X \times M$ for a f.g.p. A–module M. Choose A–module homomorphisms $i : M \to A^m$ and $p : A^m \to M$ such that $p \circ i = id_M$. Using these homomorphisms we define $\hat{d} : \Omega^*(X, \tilde{E}) \to \Omega^*(X, \tilde{E})$ as the covariant derivative induced by the canonical covariant derivative on the trivial A^m–bundle as in the proof of 5.4. By $\hat{d}_X$ and $\hat{d}_A$ we denote the two "components" of this derivative, where we include the sign in the second one.

Now let $\Phi \in \mathrm{Hom}^{\Omega^*(X,\Omega^*(A))}(\Omega^*(X, \tilde{E}), \Omega^*(X, \tilde{E}))$ be any homomorphism and let $c : \Omega^*(A) \to \mathbb{C}$ be a bounded graded trace. Then we get the differential form $\mathrm{Tr}_c(\Phi)$ as above.

Claim:
$$d\,\mathrm{Tr}_c(\Phi) = \mathrm{Tr}_c(\tilde{D}(\Phi)) - \mathrm{Tr}_c(\hat{d}_A \circ \Phi)$$

Proof of the Claim. For $j = 1, \dots, m$ let e_j be the image under $\tilde{p}$ of the constant unit section of the trivial bundle $X \times \Omega^*(A)^m$. Using Lemma 5.4 we see that $\Psi = D - \hat{d}$, the difference of the extensions of the covariant derivatives D and $\hat{d}$, is a bounded $\Omega^*(X, \Omega^*(A))$–module homomorphism of total degree one. Then a simple direct computation using that $\hat{d}(e_j) = 0$ shows that $\tilde{D}(\Phi)(e_j) = \hat{d}(\Phi(e_j)) + [\Psi, \Phi](e_j)$.

Next let $\pi_j : \Omega^*(X, \tilde{E}) \to \Omega^*(X, \Omega^*(A))$ be the $\Omega^*(X, \Omega^*(A))$–module homomorphism induced by the composition of the j–th projection $A^m \to A$ with the inclusion $i : M \to A^m$, and let $c_* : \Omega^*(X, \Omega^*(A)) \to \Omega^*(X, \mathbb{C})$ be the map induced by $c : \Omega^*(A) \to \mathbb{C}$. Specializing the definition of $\mathrm{Tr}_c(\Phi)$ to the case of a trivial bundle we see that $\mathrm{Tr}_c(\Phi) = \sum_{j=1}^m c_*(\pi_j(\Phi(e_j)))$.

Now we compute

$$\begin{aligned}\mathrm{Tr}_c(\tilde{D}(\Phi)) &= \sum_j c_*(\pi_j(\hat{d}(\Phi(e_j)) + [\Phi, \Psi](e_j))) = \\ &= \sum_j c_*(\pi_j(\hat{d}_X(\Phi(e_j)))) + \sum_j c_*(\pi_j(\hat{d}_A(\Phi(e_j)))),\end{aligned}$$

since Tr_c is a graded trace. Since $c_* \circ \pi_j$ is the map $\Omega^*(X, \tilde{E}) \to \Omega^*(X, \mathbb{C})$ induced by a bounded linear map $\tilde{E} \to X \times \mathbb{C}$ it is clear that $c_* \circ \pi_j \circ \hat{d}_X = d \circ c_* \circ \pi_j$, so the first term just gives $d\,\mathrm{Tr}_c(\Phi)$, while the second one obviously gives $\mathrm{Tr}_c(\hat{d}_A \circ \Phi)$. □

Note that if $c \in \bar{Z}^n_\lambda(A)$, then c defines a closed graded trace, and therefore $\mathrm{Tr}_c(\hat{d}_A \circ \Phi) = 0$ in this case.

(1) $d\,\mathrm{ch}^k_c(D) = d\,\mathrm{Tr}_c(R^k) = \mathrm{Tr}_c(\tilde{D}(R^k)) = 0$ by the claim above and Lemma 5.7(3).
(2) If $c \in \bar{B}^n_\lambda(A)$, then by Lemma 3.3 $c \in B(\bar{Z}^{n+1}(A))$. Therefore, by Proposition 3.5(3) we can find a graded trace τ on $\Omega^*(A)$ with $c(\omega) = \tau(d\omega)$. Then

$d\,\mathrm{Tr}_\tau(R^k) = \mathrm{Tr}_\tau(\tilde{D}(R^k)) - \mathrm{Tr}_\tau(\hat{d}_A \circ R^k) = (-1)^{n+1}\,\mathrm{Tr}_c(R^k)$ by the claim and the fact that $\mathrm{Tr}_\tau(\hat{d}_A \circ \Phi) = (-1)^{|\Phi|-n}\,\mathrm{Tr}_c(\Phi)$, which can be easily verified directly.
(3) Let D_t be a smooth curve of covariant derivatives. Let R_t be the corresponding curve of curvatures. Then $\frac{d}{dt}R_t = \tilde{D}_t(\frac{d}{dt}D_t)$. Using the trace property of Tr_c, Lemma 5.7(3), and the claim we get

$$\begin{aligned}\tfrac{d}{dt}\tfrac{1}{k}\,\mathrm{Tr}_c(R_t^k) &= \mathrm{Tr}_c((\tfrac{d}{dt}R_t)R_t^{k-1}) = \mathrm{Tr}_c([D_t,\tfrac{d}{dt}D_t]R_t^{k-1}) = \\ &= \mathrm{Tr}_c([D_t,(\tfrac{d}{dt}D_t)R_t^{k-1}]) = d\,\mathrm{Tr}_c((\tfrac{d}{dt}D_t)R_t^{k-1}).\end{aligned}$$

Now let D_0 and D_1 be two covariant derivatives. Define $D_t := (1-t)D_0 + tD_1$, $\Psi := \frac{d}{dt}D_t = D_1 - D_0$. Then, using the calculation above, we get

$$ch_c^k(D_1) - ch_c^k(D_0) = \tfrac{1}{(k-1)!}d\int_0^1 \mathrm{Tr}_c(\Psi R_t^{k-1})\,dt. \qquad \square$$

5.9. From theorem 5.8 above it is now clear that if $E \to X$ is an A–bundle, D is a covariant derivative on E, and $[c] \in \bar{H}_\lambda^*(A)$ is a reduced cyclic cohomology class, then the cohomology class of $\mathrm{ch}_c^k(D)$ in the de–Rham cohomology of X is independent of the choices of D and c. We denote this class by $\mathrm{ch}_{[c]}^k(E)$, and call it the *k–th Chern character class* of E corresponding to $[c]$.

5.10 Proposition (Properties of Chern character classes). *Let X and Y be smooth manifolds, A and B convenient algebras, $f : Y \to X$ a smooth map, $\varphi : A \to B$ a bounded homomorphism, $E \to X$ and $F \to X$ A–bundles, $c, c' \in \bar{Z}_\lambda^*(A)$, and $c'' \in \bar{Z}_\lambda^*(B)$. Then*

1. $\mathrm{ch}_{[c]}^k(f^*E) = f^*(\mathrm{ch}_{[c]}^k(E))$
2. $\mathrm{ch}_{[\varphi^*c'']}^k(E) = \mathrm{ch}_{[c'']}^k(\varphi_*E)$
3. $\mathrm{ch}_{[c]+[c']}^k(E) = \mathrm{ch}_{[c]}^k(E) + \mathrm{ch}_{[c']}^k(E)$
4. $\mathrm{ch}_{[c]}^k(E \oplus F) = \mathrm{ch}_{[c]}^k(E) + \mathrm{ch}_{[c]}^k(F)$

Proof. (1) First note that $\widetilde{f^*E} = f^*\tilde{E}$, which follows immediately from the fact that the passage from E to $\tilde{E}$ is just a fiberwise construction. Using this it is easy to verify that $\mathrm{Hom}^{\Omega^*(A)}(f^*\tilde{E}, f^*\tilde{E}) \cong f^*\,\mathrm{Hom}^{\Omega^*(A)}(\tilde{E},\tilde{E})$ as a vector bundle.

As for finite dimensional vector bundles one then shows that if D is a covariant derivative on E, then there is a unique covariant derivative f^*D on f^*E characterized by $f^*D(f^*s) = f^*(D(s))$.

Next, since the curvature $R(f^*D)$ is a module homomorphism it immediately follows that it is determined by its value on sections of the form f^*s. Using this one easily sees that, viewed as an element of $\Omega^*(Y, f^*\,\mathrm{Hom}^{\Omega^*(A)}(\tilde{E},\tilde{E}))$ we have $R(f^*D) = f^*(R(D))$. Then the result is obvious from the construction.
(2) This follows easily from the construction, taking into account that, viewed as a trace $\Omega^*(A) \to \mathbb{C}$ we have $\varphi^*(c'') = c'' \circ \Omega^*(\varphi)$, where $\Omega^*(\varphi) : \Omega^*(A) \to \Omega^*(B)$ is the homomorphism induced by φ.
(3) By definition of Tr_c.
(4) As in the finite dimensional case one shows that two covariant derivatives D_E and D_F on E and F induce a covariant derivative $D = D_E \oplus D_F$ on $E \oplus F$ with curvature $R = R(D_E) \oplus R(D_F)$. As in the finite dimensional case this implies that $\mathrm{Tr}_c(R^k) = \mathrm{Tr}_c(R(D_E)^k) + \mathrm{Tr}_c(R(D_F)^k)$. $\square$

5.11. Remarks. (1) The (inhomogeneous) total Chern character $\mathrm{ch}_{[c]}(E) = \sum_k \mathrm{ch}^k_{[c]}(E)$ is not well defined in general, since it could be necessary to sum up infinitely many non–trivial terms in each component. However, if it is possible to form e^R as a homomorphism defined by the power series, then $\mathrm{ch}_{[c]}(E)$ can be formed for any c. This is for example automatically the case if A is a Banach algebra.
(2) By 5.10(4) it is clear that each $\mathrm{ch}^k_{[c]}$ induces a map from the Grothendieck group $K_A(X)$ of the monoid of isomorphism classes of A–bundles over X to $H^*(X, \mathbb{C})$. Thus our construction (for each k) can be interpreted as a pairing $\bar{H}^*_\lambda(A) \times K_A(X) \to H^*(X, \mathbb{C})$.
(3) Going through our constructions one easily verifies that for a fixed covariant derivative D on an A–bundle $E \to X$ the mapping $\bar{Z}^n_\lambda(A) \to H^*(X, \mathbb{C})$ given by $c \mapsto \mathrm{ch}^k_c(D)$ is bounded, and thus continuous for the bornological topology on $\bar{Z}^n_\lambda(A)$. Now suppose that the de–Rham cohomology of X is finite dimensional. Since the projection from differential forms to de–Rham cohomology is continuous it is clear that the cohomology class of $\mathrm{ch}^k_c(D)$ depends only on the class of c in the quotient of the kernel of b by the closure (in the bornological topology) of the image of b. Thus in this case we may pass to the separated reduced cyclic cohomology of A.

5.12. Examples. (1) Let us consider these mappings for $A = \mathbb{C}$. A $\mathbb{C}$–bundle is just a normal complex vector bundle E and $\Omega^*(\mathbb{C}) = \mathbb{C}$. $\bar{H}^0_\lambda(\mathbb{C}) = \mathbb{C}$ and $\bar{H}^n_\lambda(\mathbb{C}) = 0$ for $n > 0$. A covariant derivative on the $\mathbb{C}$–bundle E is just a usual covariant derivative and R the usual curvature. Thus, if we take $[c] = [1]$, we get $\mathrm{ch}^k_{[c]}(E) = \mathrm{ch}^k(E)$ the classical k-th Chern character of E up to a scalar multiple.
(2) Consider the case where $X = pt$ is a single point. Then $H^*(X, \mathbb{C}) = \mathbb{C}$, and going through our constructions it is obvious that only classes in $\bar{H}^{even}_\lambda(A)$ can give rise to nontrivial results. Passing to K–theory, we clearly have $K_A(pt) \cong K_0(A)$, and our construction gives the pairing $\bar{H}^{even}_\lambda(A) \times K_0(A) \to \mathbb{C}$ constructed in [2].

References

1. Cap, A.; Kriegl, A.; Michor, P.W.; Vanžura, J., *The Frölicher–Nijenhuis bracket in non commutative differential geometry*, Acta Math. Univ. Commenianae **LXII, 1** (1993), 17–49.
2. Connes, A., *Non commutative differential geometry*, Publ. Math. IHES **52** (1985), 257–360.
3. Frölicher, A.; Kriegl, A., *Linear Spaces and Differentiation Theory*, Pure and Applied Mathematics, J. Wiley, Chichester, 1988.
4. Karoubi, M., *Homologie cyclique et K–théorie*, Asterisque 149, Société Mathématique de France, 1987.
5. Kolář, I.; Michor, P.W.; Slovák, J., *Natural Operations in Differential Geometry*, Springer, 1993.
6. Kriegl, A.; Michor, P.W., *Foundations of Global Analysis*, in preparation.
7. Loday, J.L.; Quillen, D., *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv. **59** (1984), 565–591.

Institute of Mathematics, University of Vienna, Strudlhofg. 4, A–1090 Wien, Austria and Erwin Schrödinger Institute for Mathematical Physics, Pasteurg. 6/7, A–1090 Wien, Austria

E-mail address: Andreas.Cap@esi.ac.at, Hermann.Schichl@esi.ac.at