COCONUT Deliverable D12

Progress Report on New Techniques as Modules

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Chapter 5

Convexity and concavity detection

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5.1 Introduction

This report describes the convexity and concavity detection methods implemented in the inference engine `convex_simple` of the COCONUT solver system [2]. Not all convex or concave functions are recognized as such. It is planned to add later more sophisticated convexity tests from the literature (see, e.g., AVRIEL et al. [1].

During the solution process in a deterministic global optimization algorithm it is useful to have as much information about the structure of the objective function and the constraints.

Convexity and concavity are very important properties of constraints. First, the Karush–John first order optimality conditions simplify [3] if some constraint functions are known to be convex or concave. Second, if all constraints and the objective function are convex, the optimization problem has one local minimum only, which is the global one.

5.2 Mathematical Background

In the COCONUT system, every optimization problem is represented as a directed acyclic graph (DAG) [4]. The convexity detection works through all nodes in forward mode after constraint propagation. The ranges of the intermediate nodes are used to improve the convexity detection. We distinguish the properties *linear*, *convex*, *concave*, and *unknown convexity status*. Note that a linear function is both convex and concave.

Every node is considered as a function taking input from its child nodes, and we distinguish whether the elementary function represented by the node on its range has some convexity property, and whether its arguments have certain convexity properties. Based on this, we check the following rules to decide whether the composite function $f \circ g$ has convexity properties.

5.2.1 Rule 1:

If the function f defined by the current node depends on one argument only and is convex on the range of its argument then

1. If the child function g is linear then $f \circ g$ is convex.
2. If the child function g is convex and f is increasing in the interval of g then $f \circ g$ is convex.
3. If the child function g is concave and f is decreasing in the interval of g then $f \circ g$ is convex.

5.2.2 Rule 2:

If the function f defined by the current node depends on one argument only and is concave on the range of its argument then

1. If the child function g is linear then $f \circ g$ is concave.
2. If the child function g is convex and f is decreasing in the interval of fg then $f \circ g$ is concave.
3. If the child function g is concave and f is increasing in the interval of g then $f \circ g$ is convex.

A maximal set of nodes is recognized to be convex or concave based on repeated application of these rules. There are more known rules for deducing convexity properties (see [1]), but these do not follow simple patterns and are not yet implemented.

5.3 Convexity information for elementary functions

In the following we give a list of the elementary functions the algorithm can currently exploit:

sum:

$$f(x) = \sum_{i=1}^{n} \alpha_i x_i + c$$

The rule used is that the positive sum of convex functions is always convex. If all the child functions are convex then the function is convex. If one child function is not convex, we can say nothing about the convexity properties of the function.

min:

$$f(x) = \min\{c\} \cup \{x_i \mid i = 1, \ldots, n\}$$

If all non-linear children are concave then we check if the lower bound of any child is less then the upper bound of the node function. If this is true then we can't say anything about the node else it is concave.

max:

$$f(x) = \min\{c\} \cup \{x_i \mid i = 1, \ldots, n\}$$

If all non-linear children are convex then we check if the upper bound of any child is greater then the lower bound of the node function. If this is true then we can't say anything about the node else it is convex.

sqr:

$$f(x) = (x + p)^2$$

is a convex function. Hence, $f \circ g$ will be convex if

- g is linear.
- The lower bound of $g + p$ is greater than zero and g is convex.
- The upper bound of $g + p$ is less than zero and g is concave.

In the remaining cases we can't say anything about the node.

sqrt:

$$f(x) = \sqrt{x + p}, \quad x \geq -p$$

This function is always concave because only positive root are considered. The function $f o g$ is concave if g is concave. In the remaining cases we can't say anything about the node.

exp:

$$f(x) = e^{x+p}$$

This function is convex everywhere. The function $f \circ g$ is convex if g is convex. In the remaining cases we can't say anything about the node.

log:

$$f(x) = \log(x + p), \quad x \geq -p$$

This function is concave everywhere. Therefore, $f \circ g$ is concave if g is concave. In the remaining cases we can't say anything about the node.

even power: The function f is convex everywhere, and $f \circ g$ will be convex if x is nonnegative and g is concave; it will be convex if x is nonpositive and g is convex. In the remaining cases we can't say anything about the node.

odd positive power: If the lower bound of the argument is nonnegative the function is convex, and if the upper bound is nonpositive the function is concave. If 0 is in the interior of the range of g, we can't say anything about the node because there occurs a saddle point. The function $f \circ g$ is convex if the lower bound of g is nonnegative and g is convex. The composite $f \circ g$ is concave if the upper bound of g is nonpositive and g is concave. In the remaining cases we can't say anything about the node.

odd negative power: This function f is convex if x is greater than 0 and concave if x is less than 0. The composite function $f \circ g$ will be convex if x greater than 0 and g is concave. The function $f \circ g$ will be concave if x less than 0 and g is convex. In the remaining cases we can't say anything about the node.

sin:

$$f(x) = \sin(x + p)$$

In the following cases the function is neither convex nor concave:

- the difference between upper bound of g and lower bound of g is greater than π.
- The sines of the lower bound and of the upper bound have different signs.

If the sine of the lower bound and of the upper bound are positive then $f \circ g$ is concave if any of the following is true:

- g is linear.
- g is convex and the cosine of the lower and upper bounds are negative.
- g is concave and the cosine of lower and upper bounds is positive.

If the sine of the lower and upper bounds are negative then $f \circ g$ is convex if any of the following is true:

- g is linear.
- g is concave and the cosine of the lower and upper bounds is negative.
- g is convex and the cosine of lower and upper bounds is positive.

In the remaining cases we can't say anything about the node.

cos: In the following case the function is neither convex or concave

- The width of the range of the argument function g is greater than π.
- The cosine of lower bound and the cosine of upper bound have different signs.

If the cosine of lower and upper bounds are positive, $f \circ g$ is concave if any of the following is true:

- g is linear.
- g is convex and the sine of lower and upper bounds are negative.
- g is concave and the sine of lower and upper bounds are positive.

If the cosine of the lower and upper bounds are negative, then $f \circ g$ is convex if any of the following is true:

- g is linear.
- g is concave and the sine of lower and upper bounds are negative.
- g is convex and the sine of lower and upper bound are positive.

In the remaining cases we can't say anything about the node.

gauss:

$$f(x) = e^{-(x-m)^2/s^2}$$

The Gauss function may be convex or concave in its domain. It is concave if $(x - m)/|s|$ is in the interval $[-1/\sqrt{2}, 1/\sqrt{2}]$ and convex in the intervals $]-\infty, -1/\sqrt{2}]$ and $[1/\sqrt{2}, \infty[$.

The function $f \circ g$ is convex in the following cases:

- The lower bound of $(g - m)/|s|$ is greater than $1/\sqrt{2}$, and g is concave.
- The upper bound of $(g - m)/|s|$ is less than $-1/\sqrt{2}$ and g is convex.

The function $f \circ g$ is concave in the following cases:

- The lower bound of $(g - m)/|s|$ is greater than $-1/\sqrt{2}$ and the upper bound of $(g - m)/|s|$ is less than $-1/\sqrt{2}$ and any of the following is true
  1. g is linear.
  2. The lower bound of $g - m$ is greater than zero and g is convex.
  3. The upper bound of $g - m$ is less than zero and g is concave.

In the remaining cases we can't say anything about the node.

others: The remaining elementary operations are not analyzed, so we can't say anything about the node.

Bibliography

[1] M. Avriel, W.E. Diewert, S. Schaible and I. Zang, Generalized Concavity, Plenum, New York 1988.

[2] The COCONUT project home page.
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[3] Arnold Neumaier and Hermann Schichl. Sharpening the Karush-John optimality conditions. preprint (2003).

[4] Hermann Schichl and Arnold Neumaier. Interval analysis on directed acyclic graphs for global optimization. preprint (2003).