Lozenge tilings of hexagons with boundary dents

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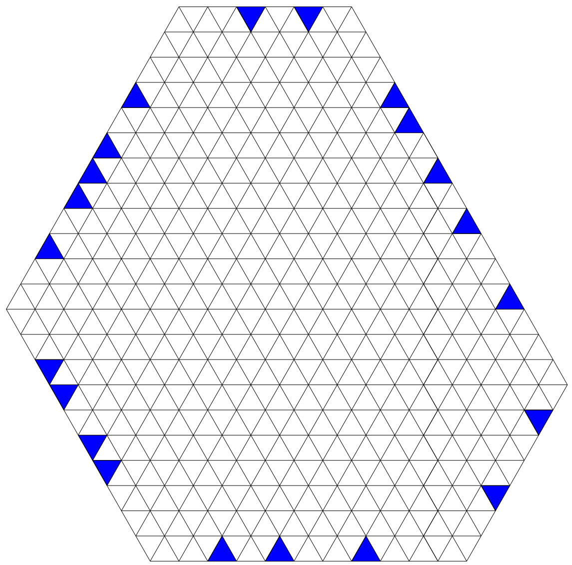
Joint work with Mihai Ciucu



lozenge: union of two adjacent unit triangles on the triangular lattice

lozenge tiling of a region R: covering of R by lozenges that has no gaps or overlaps

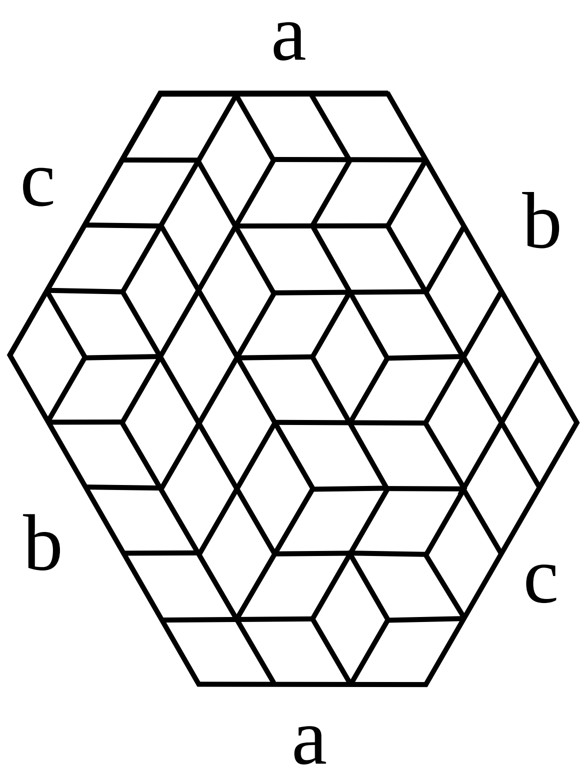
dent: unit triangle removed from along the boundary

Lozenge tilings of a hexagon with side lengths a, b, c, a, b, c

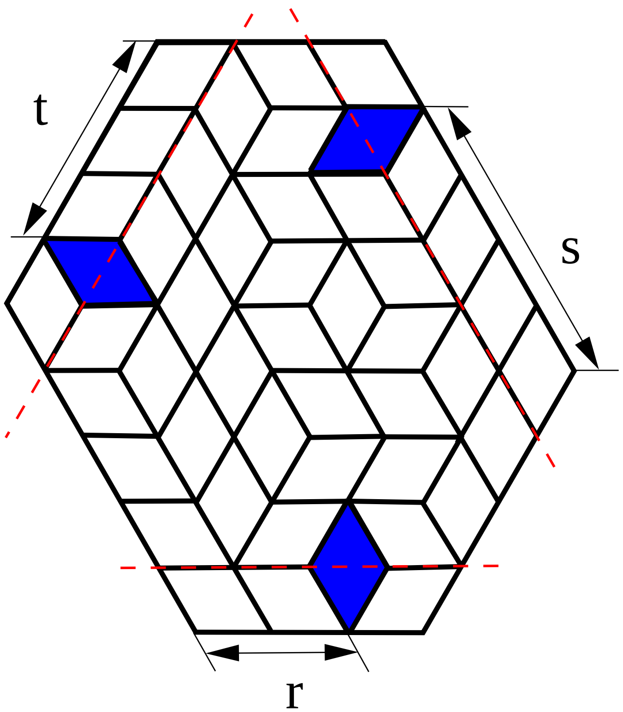
Theorem (MacMahon). The number of lozenge tilings of a hexagon with side lengths a, b, c, a, b, c is equal to

$$\prod_{i=1}^{a} \frac{(c+i)_b}{(i)_b}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$.



Lozenge tilings of a hexagon with three fixed border tiles



Theresia Eisenkölbl (1999):

$$\frac{\prod_{i=1}^{a-1} \frac{(c+i-1)_{b-1}}{(i)_{b-1}}}{(a+b-2)!(b+c-2)!(a+c-2)!}$$

$$\times (r+1)_{b-2}(s+1)_{c-2}(t+1)_{a-2}$$

$$\times (a+1-r)_{c-2}(b+1-s)_{a-2}(c+1-t)_{b-2}$$

$$\times ((a-1)(b-1)(c-1)(a-r)(b-s)(c-t)$$

$$+ (a-1)(b-1)(c-1)\, r\, s\, t$$

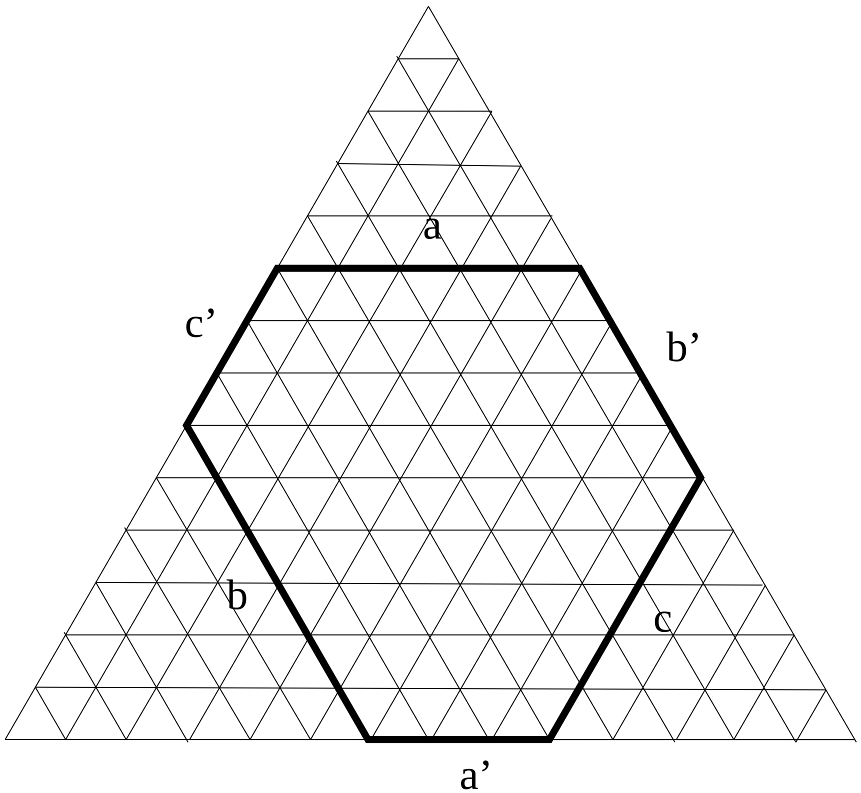
$$- (a-r)(b-s)(c-t)\, r\, s\, t$$

$$+ (a-1)(c-1)(b-s)(c-t)\, r\, s$$

$$+ (a-1)(b-1)(a-r)(c-t)\, s\, t$$

$$+ (b-1)(c-1)(a-r)(b-s)\, r\, t)$$

General hexagon on a triangular grid



$n =$ side length of big $\triangle$

$k = n - a - b - c$

$a' = n - b - c = a + k$

$b' = b + k, c' = c + k$

$H^k_{a,b,c}$: Hexagon with side lengths

$$a, b + k, c, a + k, b, c + k.$$

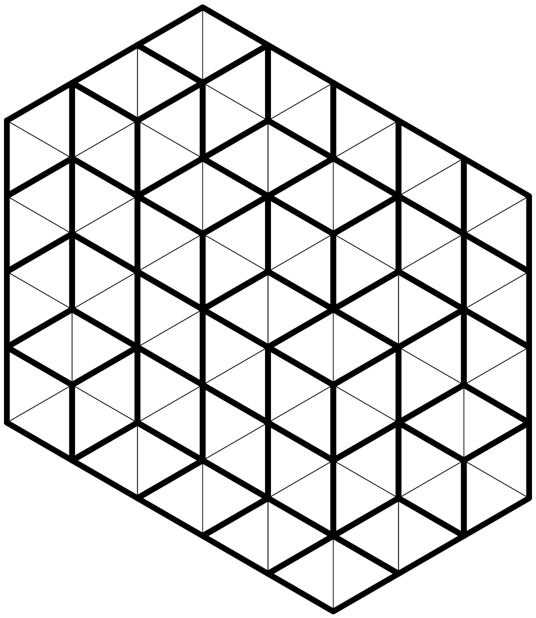
W.l.o.g. $k \geq 0$.

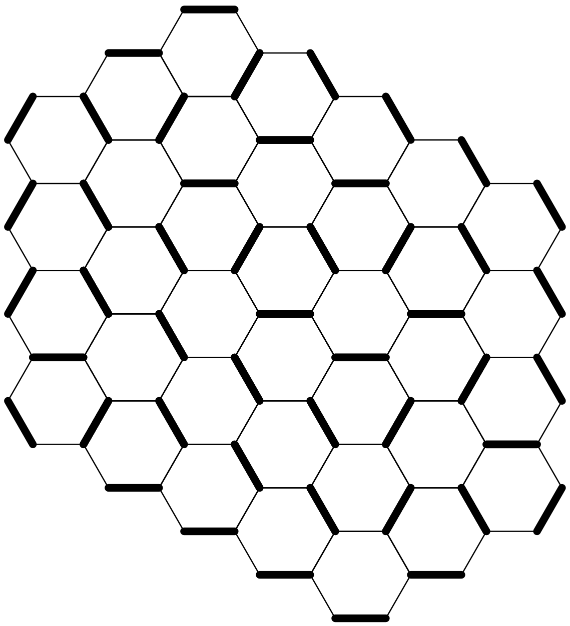
- If $k > 0$, then $H^k_{a,b,c}$ has no lozenge tiling.
- $\# \triangle = \# \triangledown + k \Rightarrow$ We need to have k more $\triangle$-dents than $\triangledown$-dents.

Tool: Ciucu's extension of Kuo's graphical condensation

Kuo's graphical condensation is useful to count perfect matchings in planar graphs.

Lozenge tilings are perfect matchings of a hexagonal grid!





Kuo's graphical condensation

For a graph G, $\mathrm{M}(G)$ denotes the number of perfect matchings.

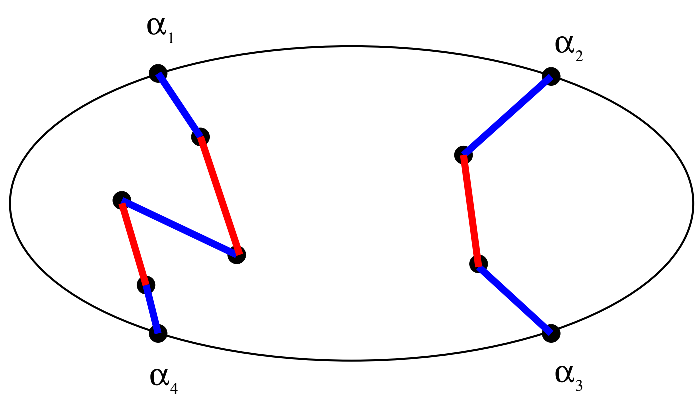
Theorem. Let G be a planar graph with four vertices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ that appear in cyclic order on a face of G. Then

$$\mathrm{M}(G)\,\mathrm{M}(G - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}) + M(G - \{\alpha_1, \alpha_3\})M(G - \{\alpha_2, \alpha_4\})$$
$$= \mathrm{M}(G-\{\alpha_1, \alpha_2\})\,\mathrm{M}(G-\{\alpha_3, \alpha_4\}) + M(G-\{\alpha_1, \alpha_4\})M(G-\{\alpha_2, \alpha_3\}).$$

G bipartite: $V = V_1 \dot\cup V_2$
$\alpha_1, \alpha_3 \in V_1, \alpha_2, \alpha_4 \in V_2, |V_1| = |V_2|$: second term vanishes
$\alpha_1, \alpha_2 \in V_1, \alpha_3, \alpha_4 \in V_2, |V_1| = |V_2|$: third term vanishes
$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in V_1, |V_1| = |V_2| + 2$: first term vanishes

Idea of the proof

• Superimpose a perfect matching of G (blue) and a perfect matching of $G - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ (red).
• There is a blue-red-alternating path from α_1 to α_i for an $i \in \{2, 3, 4\}$.
• Two blue-red-alternating paths cannot cross and thus $i \neq 3$.
• Switch the edges in the path of α_1 and obtain a pair of matchings of $M(G - \{\alpha_1, \alpha_2\})$ and $M(G - \{\alpha_3, \alpha_4\})$ or of $M(G - \{\alpha_1, \alpha_4\})$ and $M(G - \{\alpha_2, \alpha_3\})$.
• When thinking about the reverse mapping, one observes that this is not a bijection and that the term $M(G - \{\alpha_1, \alpha_3\})M(G - \{\alpha_2, \alpha_4\})$ is also necessary.



Pfaffian

Let $A = (a_{i,j})$ be a $2n \times 2n$ antisymmetric matrix and Π_n the set of all perfect matchings of K_{2n}. Then

$$\mathrm{Pf}(A) = \sum_{\pi=\{(i_1,j_1),\ldots,(i_n,j_n)\}\in\Pi_n} \mathrm{sgn}\,\pi \prod_{k=1}^{n} a_{i_k,j_k}$$

where $\mathrm{sgn}\,\pi = \mathrm{sgn}\, i_1 j_1 i_2 j_2 \ldots i_n j_n$. (There are several ways to write π as $\{(i_1, j_1), \ldots, (i_n, j_n)\}$. To see that $\mathrm{Pf}(A)$ is still well-defined, we can assume $i_k < j_k$ and $i_1 < i_2 < \ldots < i_n$ or show that it does not matter which representative we choose.)

Recall that

$$\mathrm{Pf}(A)^2 = \det(A).$$

$n = 2$: $\mathrm{Pf}(A) = a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}$

Ciucu's extension of Kuo's graphical condensation

Theorem (Ciucu, 2014). Let G be a planar graph with the vertices $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ appearing in cyclic order on a face of G. Consider the $2n \times 2n$ skew symmetric matrix $A = (a_{i,j})$ with

$$a_{i,j} = \mathrm{M}(G - \{\alpha_i, \alpha_j\}) \qquad \text{if } i < j.$$

Then we have that

$$\mathrm{M}(G - \{\alpha_1, \ldots, \alpha_{2n}\}) = \frac{\mathrm{Pf}(A)}{\mathrm{M}(G)^{n-1}}.$$

Problem in our case: $\mathrm{M}(G) = 0$ if $k > 0$!

We restrict to the case $k = 0$ for the moment.

Four cases – two are trivial

Our application: G is the hexagonal grid/triangulated hexagon $H^0_{a,b,c}$, α_i are vertices of degree 2/dents along the boundary.

We need to compute $\mathrm{M}(G - \{\alpha_i, \alpha_j\})$.

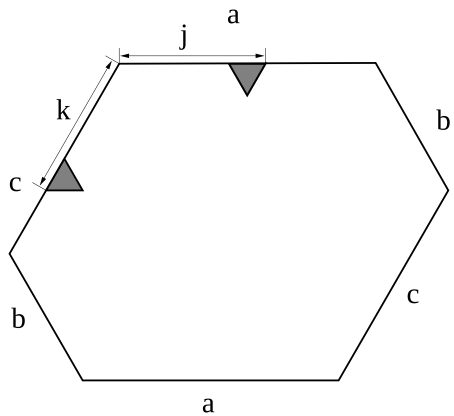
(1) α_i, α_j are on the same side. The dents are of the same type: $\mathrm{M}(G - \{\alpha_i, \alpha_j\}) = 0$

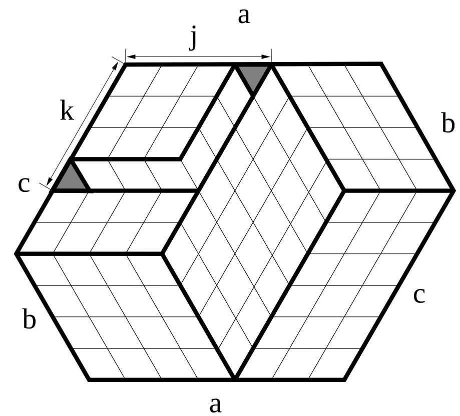
(2) α_i, α_j are on adjacent sides.

(3) α_i, α_j are on different sides that share an adjacent side: $\mathrm{M}(G - \{\alpha_i, \alpha_j\}) = 0$

(4) α_i, α_j are on opposite sides.

Dents on adjacent sides





Proposition 1. Let a, b, c, j, k be non-negative integers with $1 \le j \le a$ and $1 \le k \le c$. The number of lozenge tilings of the hexagon $H^0_{a,b,c}$ with two dents on adjacent sides of length a and c in positions j and k, respectively, as counted from the common vertex of the two sides is

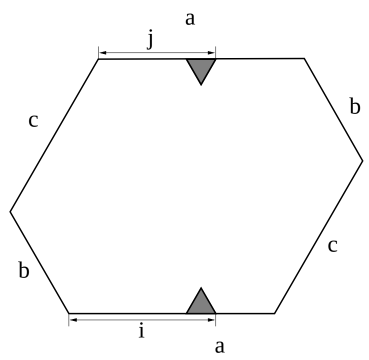
$$\prod_{i=0}^{a-1} \frac{(c+i)_b}{(1+i)_b} {}_3F_2\left[\begin{matrix} -a+j,\ b,\ -c+k \\ 1-a-c,\ 1+b \end{matrix}; 1\right] \frac{(1+b)_{a-j}(j)_{k-1}(1+c-k)_{k-1}}{(1)_{a-j}(1)_{k-1}(1+b+c-k)_{k-1}}.$$

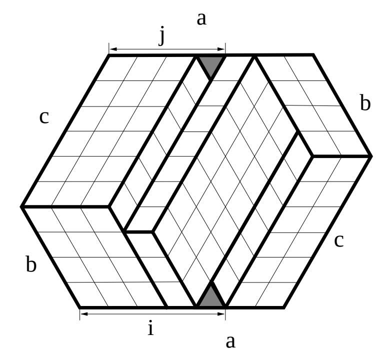
A reminder: Hypergeometric notation

The hypergeometric series of parameters $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ is defined as

$$ {}_rF_s\left[\begin{matrix} a_1, & \ldots, & a_r \\ b_1, & \ldots, & b_s \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}. $$

Dents on opposite sides





Proposition 2. Let a, b, c, i, j be positive integers with $1 \le i, j \le a$. The number of lozenge tilings of the hexagon $H^0_{a,b,c}$ with two dents in positions i and j along opposite sides of length a is

$$\prod_{k=0}^{a-2} \frac{(1+c+k)_b}{(1+k)_b} {}_4F_3\left[\begin{matrix} 1-i,\ 1-j,\ 1-c-j,\ 1+a+b-j \\ 2-c-j,\ 1+b-j,\ 2+a-i-j \end{matrix}; 1\right] \times \frac{(c)_{j-1}(1+b-j)_{i-1}(2+a-i-j)_{i+j-2}}{(1)_{i-1}(1)_{j-1}(1+a+c-i)_{i-1}(1+a+b-j)_{j-1}}.$$

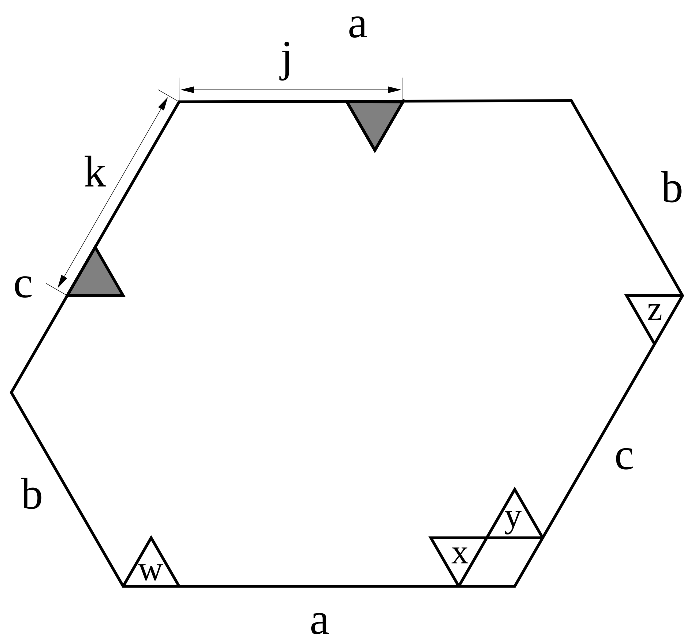
Proof of Proposition 1

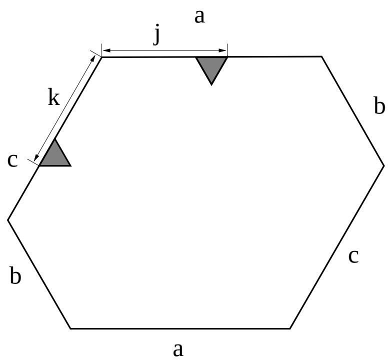
Yet another application of Kuo's condensation.

Theorem. Let $G = (V_1, V_2, E)$ be a bipartite planar graph and w, x, y, z vertices of G that appear in cyclic order on a face of G. If $w, y \in V_1$ and $x, z \in V_2$ then

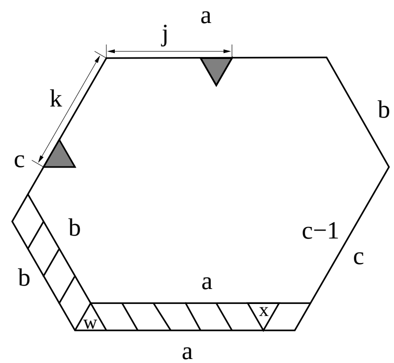
$$\mathrm{M}(G)\,\mathrm{M}(G-\{w,x,y,z\}) = \mathrm{M}(G-\{w,x\})\,\mathrm{M}(G-\{y,z\}) + \mathrm{M}(G-\{w,z\})\,\mathrm{M}(G-\{x,y\}).$$

Choice of w, x, y, z in our application:

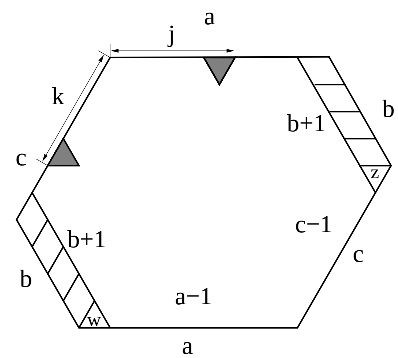




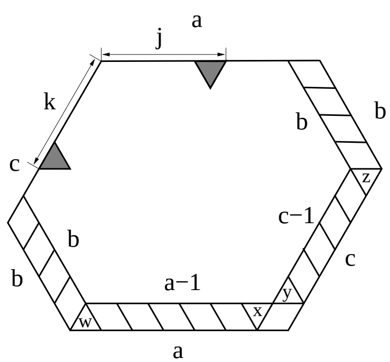
G



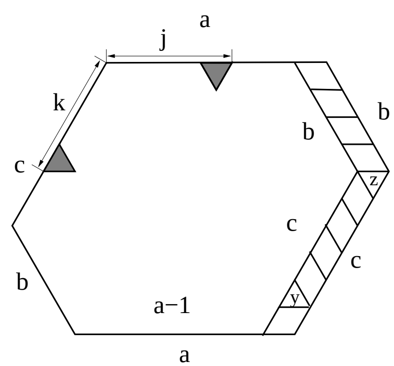
$G - \{w, x\}$



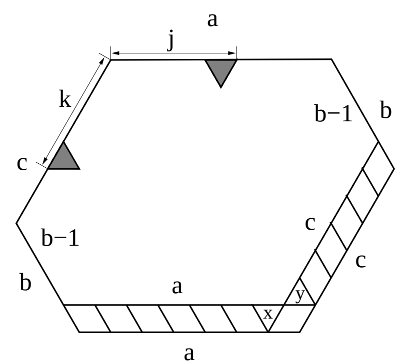
$G - \{w, z\}$



$G - \{w, x, y, z\}$



$G - \{y, z\}$



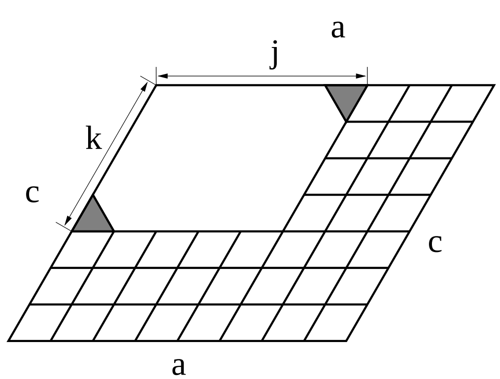
$G - \{x, y\}$

$$\mathsf{ADJ}(a,b,c)_{j,k}\,\mathsf{ADJ}(a-1,b,c-1)_{j,k}$$
$$= \mathsf{ADJ}(a,b,c-1)_{j,k}\,\mathsf{ADJ}(a-1,b,c)_{j,k} + \mathsf{ADJ}(a-1,b+1,c-1)_{j,k}\,\mathsf{ADJ}(a,b-1,c)_{j,k}$$

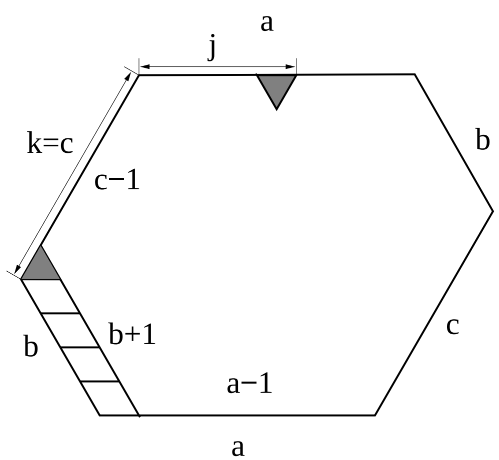
Induction w.r.t. $a+b+c$

Base case: It suffices to show the formula for the cases $a=1$, $b=0$ and $c=1$. For our argument, we also need to check the cases $a=j$ and $c=k$.

Since $a=1$ implies $a=j$ and by the symmetry of a and c, it suffices to consider the cases $b=0$ and $c=k$.



$b=0$ MacMahon



$c=k$ “easy”

Induction step

We need to verify that the formula in the proposition fulfills the recursion.

$$-{}_3F_2\left[\begin{matrix}-a+j,\ -1+b,\ -c+k\\ 1-a-c,\ b\end{matrix};1\right]{}_3F_2\left[\begin{matrix}1-a+j,\ 1+b,\ 1-c+k\\ 3-a-c,\ 2+b\end{matrix};1\right]\times \mathrm{QPP}_1$$
$$-{}_3F_2\left[\begin{matrix}-a+j,\ b,\ 1-c+k\\ 2-a-c,\ 1+b\end{matrix};1\right]{}_3F_2\left[\begin{matrix}1-a+j,\ b,\ -c+k\\ 2-a-c,\ 1+b\end{matrix};1\right]\times \mathrm{QPP}_2$$
$$+{}_3F_2\left[\begin{matrix}-a+j,\ b,\ -c+k\\ 1-a-c,\ 1+b\end{matrix};1\right]{}_3F_2\left[\begin{matrix}1-a+j,\ b,\ 1-c+k\\ 3-a-c,\ 1+b\end{matrix};1\right]\times \mathrm{QPP}_3 = 0$$

QPP_i = quotient of products of Pochhammer functions

Arguments differ by small integer values!

Another tool: Contiguous relations for hypergeometric series

$$ {}_rF_s\left[\begin{matrix} x,\ (A) \\ y,\ (B) \end{matrix}; z\right] = {}_rF_s\left[\begin{matrix} x-1,\ (A) \\ y-1,\ (B) \end{matrix}; z\right] + \frac{(y-x)z}{(y-1)y}\frac{\prod_{i=1}^{r-1} A_i}{\prod_{i=1}^{s-1} B_i}\, {}_rF_s\left[\begin{matrix} x,\ (A+1) \\ y+1,\ (B+1) \end{matrix}; z\right] \qquad \texttt{C40}[x,y] $$

$$ {}_rF_s\left[\begin{matrix} x,\ (A) \\ y,\ (B) \end{matrix}; z\right] = \frac{(y-2)(y-1)}{(y-x-1)z}\frac{\prod_{i=1}^{s-1}(B_i-1)}{\prod_{i=1}^{r-1}(A_i-1)}\, {}_rF_s\left[\begin{matrix} x,\ (A-1) \\ y-1,\ (B-1) \end{matrix}; z\right] $$

$$ - \frac{(y-2)(y-1)}{(y-x-1)z}\frac{\prod_{i=1}^{s-1}(B_i-1)}{\prod_{i=1}^{r-1}(A_i-1)}\, {}_rF_s\left[\begin{matrix} x-1,\ (A-1) \\ y-2,\ (B-1) \end{matrix}; z\right] \qquad \texttt{C42}[x,y] $$

$$ {}_rF_s\left[\begin{matrix} w,\ x,\ (A) \\ y,\ (B) \end{matrix}; z\right] = \frac{x(y-w)}{(x-w)y}\, {}_rF_s\left[\begin{matrix} w,\ x+1,\ (A) \\ y+1,\ (B) \end{matrix}; z\right] + \frac{w(y-x)}{(w-x)y}\, {}_rF_s\left[\begin{matrix} w+1,\ x,\ (A) \\ y+1,\ (B) \end{matrix}; z\right] \qquad \texttt{C54}[w,x,y] $$

$$ {}_rF_s\left[\begin{matrix} w,\ x,\ (A) \\ y,\ (B) \end{matrix}; z\right] = \frac{(1-w+x)(y-1)}{(w-1)(1+x-y)}\, {}_rF_s\left[\begin{matrix} w-1,\ x,\ (A) \\ y-1,\ (B) \end{matrix}; z\right] $$

$$ + \frac{x(y-w)}{(w-1)(y-x-1)}\, {}_rF_s\left[\begin{matrix} w-1,\ x+1,\ (A) \\ y,\ (B) \end{matrix}; z\right] \qquad \texttt{C55}[w,x,y] $$

(A) and (B) stand for lists $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$ of the appropriate lengths.

C40[x,y]

$$\frac{(x)_n \prod_{i=1}^{r-1}(A_i)_n}{(y)_n \prod_{i=1}^{s-1}(B_i)_n}\frac{z^n}{n!} = \frac{(x-1)_n \prod_{i=1}^{r-1}(A_i)_n}{(y-1)_n \prod_{i=1}^{s-1}(B_i)_n}\frac{z^n}{n!}$$

$$+ \frac{(y-x)z}{(y-1)y}\frac{\prod_{i=1}^{r-1} A_i}{\prod_{i=1}^{s-1} B_i}\frac{(x)_{n-1} \prod_{i=1}^{r-1}(A_i+1)_{n-1}}{(y+1)_{n-1} \prod_{i=1}^{s-1}(B_i+1)_{n-1}}\frac{z^{n-1}}{(n-1)!}$$

Simple calculation:

$$\frac{x+n-1}{y \cdot n} = \frac{(x-1)(y+n-1)}{(y-1)y \cdot n} + \frac{y-x}{(y-1)y}$$

Application

C55[1-c+k,b,3-a-c]:

$$
{}_3F_2\left[{1-a+j,\ b,\ 1-c+k \atop 3-a-c,\ 1+b};1\right] = \frac{(2-a-c)(b+c-k)}{(-2+a+b+c)(-c+k)}\,{}_3F_2\left[{1-a+j,\ b,\ -c+k \atop 2-a-c,\ 1+b};1\right] + \frac{b(2-a-k)}{(2-a-b-c)(-c+k)}\,{}_3F_2\left[{1-a+j,\ b+1,\ -c+k \atop 3-a-c,\ b+1};1\right]
$$

The second ${}_3F_2$ on the right-hand side can be computed using Chu-Vandermonde summation:

$$
{}_3F_2\left[{1-a+j,\ b+1,\ -c+k \atop 3-a-c,\ b+1};1\right] = {}_2F_1\left[{1-a+j,\ -c+k \atop 3-a-c};1\right] = \frac{(3-a-k)_{a-j+1}}{(3-a-c)_{a-j+1}}
$$

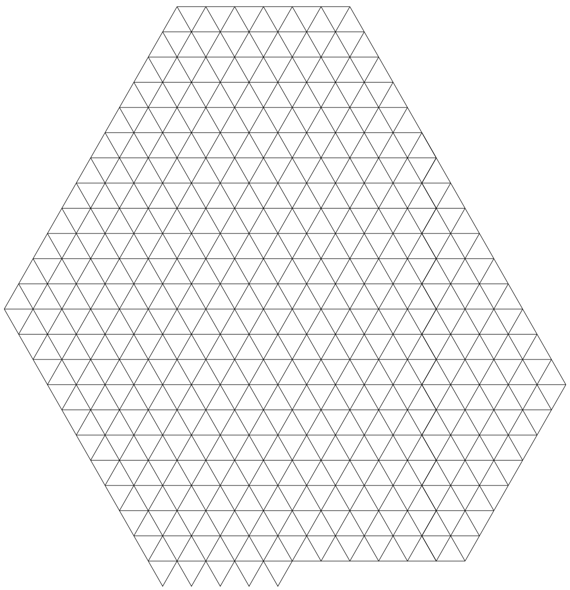
The first ${}_3F_2$ on the right-hand side is another ${}_3F_2$ that appears in our identity. We have reduced the number of ${}_3F_2$ to 5. We apply four more transformations of this type such that in the end we obtain a polynomial of degree at most 2 in

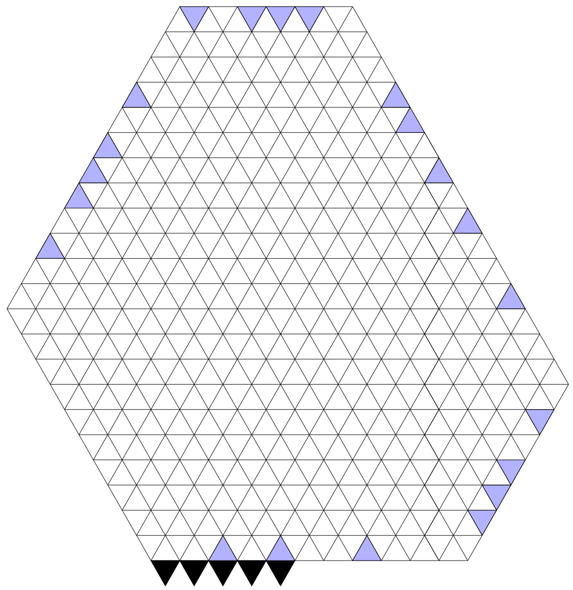
$$
{}_3F_2\left[{1-a+j,\ b,\ -c+k \atop 2-a-c,\ 1+b};1\right].
$$

The coefficients of the polynomial are sums of quotients of products of Pochhammer functions. It is routine to check that the coefficients are in fact zero.

General case $k \geq 0$

$\overline{H}^k_{a,b,c}$ denotes the region obtained from $H^k_{a,b,c}$ by augmenting it by a string of k contiguous $\bigtriangledown$ along its bottom row.



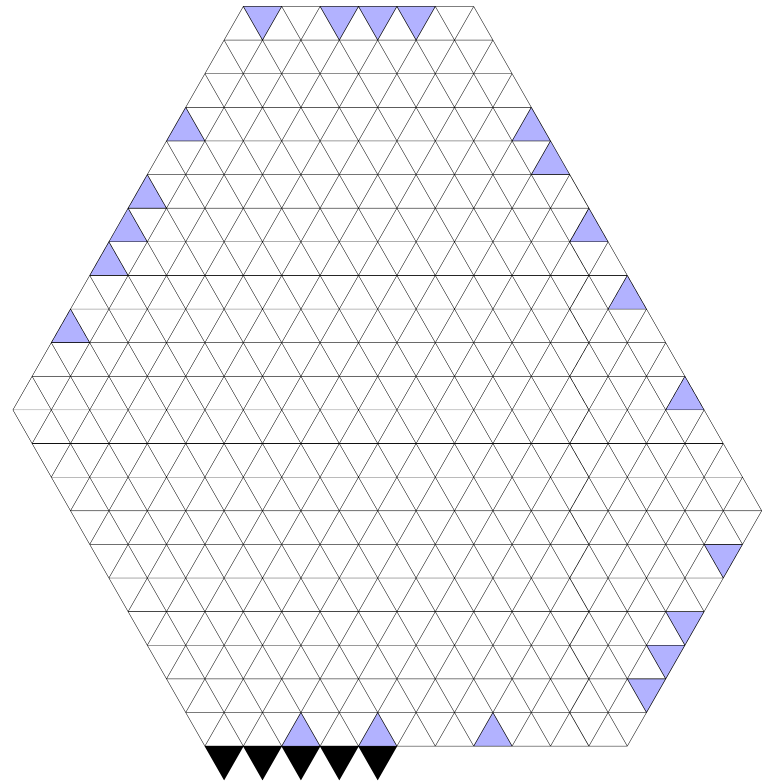


M($\overline{H}^k_{a,b,c}$) $\neq 0$

Notation:

- $\alpha_1, \alpha_2, \ldots, \alpha_{n+k}$: $\triangle$-dents
- $\beta_1, \beta_2, \ldots, \beta_n$: $\bigtriangledown$-dents
- $\gamma_1, \gamma_2, \ldots, \gamma_k$: $\bigtriangledown$-triangles below bottom row

Let $\delta_1, \delta_2, \ldots, \delta_{2n+2k}$ be a cyclic order of these elements



$\gamma_1, \gamma_2, \alpha_1, \gamma_3, \gamma_4, \alpha_2, \gamma_5, \alpha_3, \beta_1, \beta_2,$
$\beta_3, \beta_4, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \beta_5, \beta_6, \beta_7,$
$\beta_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}$

Ciucu's extension of Kuo's graphical condensation implies

$$\mathrm{M}(H^k_{a,b,c} - \{\alpha_1, \dots, \alpha_{n+k}, \beta_1, \dots, \beta_n\}) = \frac{\mathrm{Pf}[\mathrm{M}(\overline{H}^k_{a,b,c} - \{\delta_i, \delta_j\})_{1 \le i < j \le 2n+2k}]}{\mathrm{M}(\overline{H}^k_{a,b,c})^{n+k-1}}.$$

We need to show: Each quantity on the right-hand side can be computed.

Assumption: one of the three sides on which $\bigtriangledown$-dents can occur does not actually have any dents. W.l.o.g. let this be the southwestern side.

Easy cases

Denominator: $\mathrm{M}(\overline{H}^k_{a,b,c}) = \mathrm{M}(H^0_{a,b+k,c})$

Numerator:

$$\mathrm{M}(\overline{H}^k_{a,b,c} - \{\alpha_i, \alpha_j\}) = 0$$
$$\mathrm{M}(\overline{H}^k_{a,b,c} - \{\beta_i, \beta_j\}) = 0$$
$$\mathrm{M}(\overline{H}^k_{a,b,c} - \{\beta_i, \gamma_j\}) = 0$$
$$\mathrm{M}(\overline{H}^k_{a,b,c} - \{\gamma_i, \gamma_j\}) = 0$$

$$\mathrm{M}(\overline{H}^{k}_{a,b,c} - \{\alpha_i, \beta_j\}))$$

The number is zero if either

- α_i shares an edge with one of the γ_l, or

- α_i is on the northwestern side, at distance at most $k-1$ from the western corner.

Otherwise it is given by Proposition 1 if α_i and β_j are along adjacent sides, and by Proposition 2 if α_i and β_j are along opposite sides.

$$\mathsf{M}(\overline{H}^k_{a,b,c} - \{\alpha_i, \gamma_j\})$$

The number is zero if either

- α_i shares an edge with one of the γ_l with $l \neq j$, or

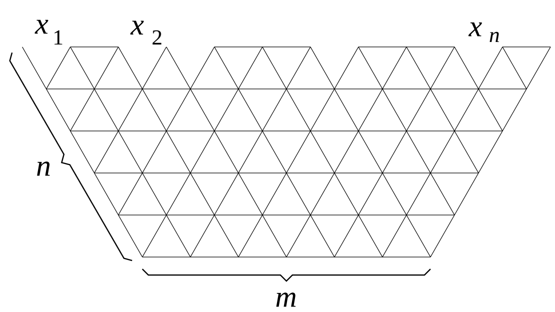
- α_i is on the northwestern side, at distance at most $j - 2$ from the western corner.

If α_i and γ_j are along the same side, the result follows from Proposition 3 and if α_i and γ_j are along different sides, the result follows from Proposition 4.

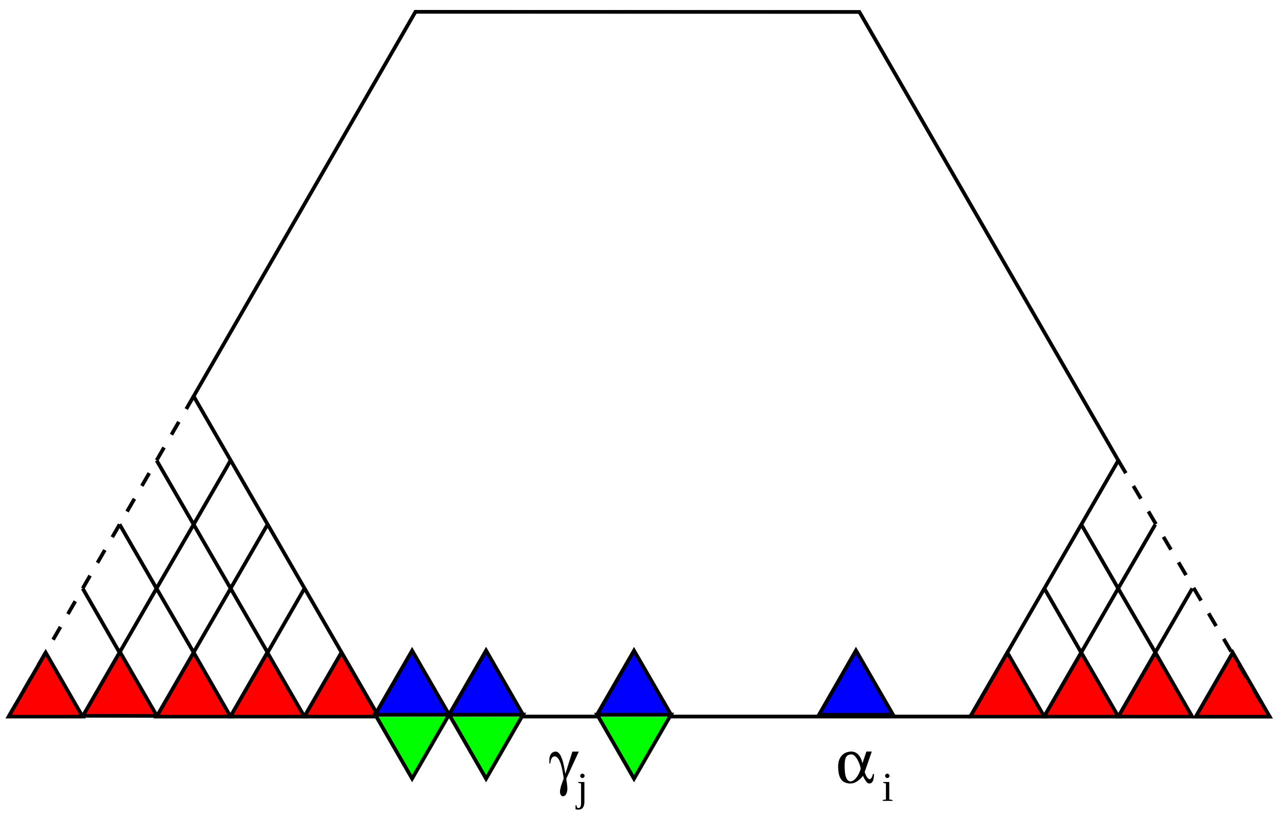
α_i and γ_j are on the same side

Proposition 3. Let $T_{m,n}(x_1,\ldots,x_n)$ be the region obtained from the trapezoid of side lengths m, n, $m+n$, n (clockwise from bottom) by removing the down-pointing unit triangles from along its top that are in positions $x_1, x_2, \ldots, x_n$ as counted from left to right. Then

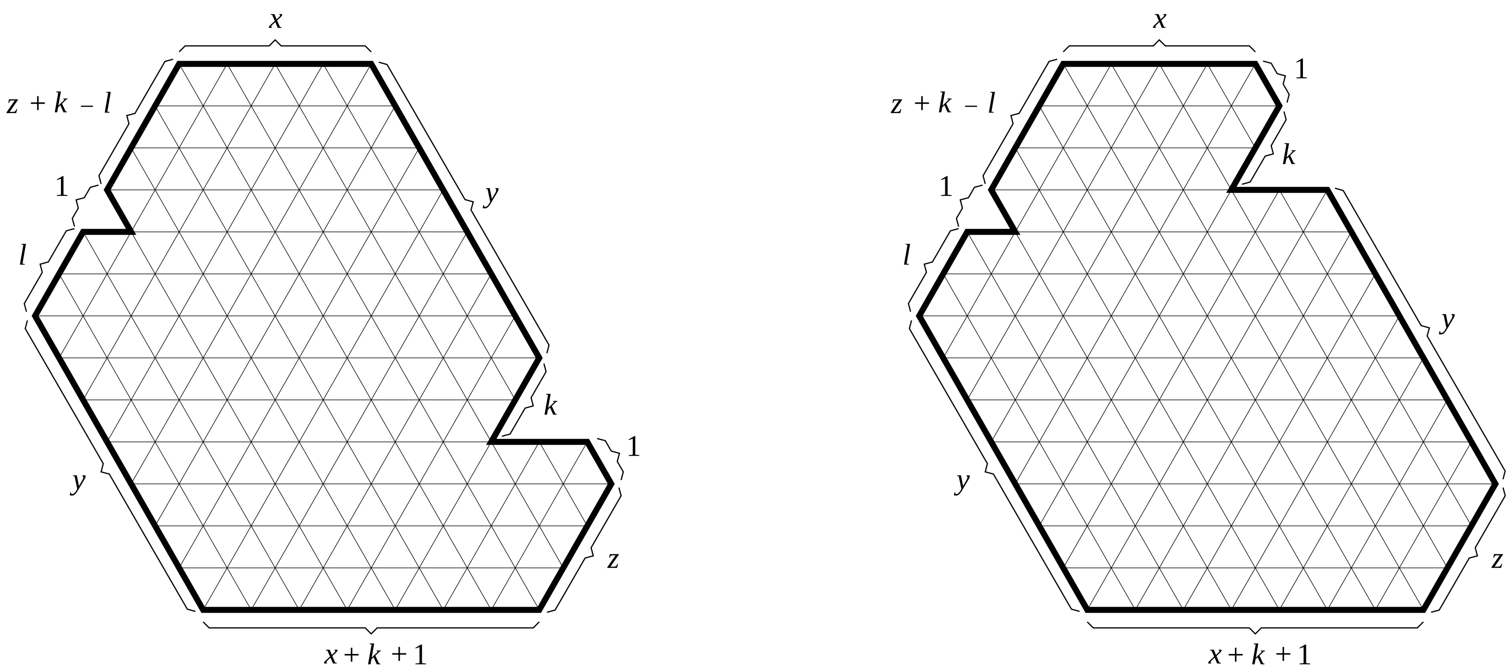
$$\mathrm{M}(T_{m,n}(x_1,\ldots,x_n)) = \prod_{1\le i<j\le n} \frac{x_j - x_i}{j-i}.$$



Special case



α_i and γ_j are on different sides



Proposition 4, part 1

Let $H_{a,b,c}(k,l)$ be the region obtained from the hexagon of side lengths a, $b+k+1$, c, $a+k+1$, b, $c+k+1$ (clockwise from top) by removing an up-pointing unit triangle from its northwestern side, l units above the western corner, and an up-pointing triangle of side k from its northeastern side, one unit above the eastern corner. Let $m = \min(a,b)$ and $M = \max(a,b)$. Then we have

$$\mathrm{M}(H_{a,b,c}(k,l)) = \mathrm{M}(H_{a,b,c})\frac{p(c,l)}{p(0,0)},$$

where the polynomial $p(c,l)$ is defined to be

$$\begin{aligned}
&(l+1)_b(c+k-l+1)_a \\
&\quad\times (c+k+2)(c+k+3)^2\cdots(c+k+m+1)^m(c+k+m+2)^m\cdots(c+k+M+1)^m \\
&\quad\quad\times (c+k+M+2)^{m-1}(c+k+M+3)^{m-2}\cdots(c+k+M+m) \\
&\quad\quad\times \sum_{i=1}^{k+1}\frac{(-1)^{i-1}}{(i-1)!(k-i+1)!}(l-k+i)_{k-i+1}(l+b+1)_{i-1}(c+1)_{i-1}(c+i+1)_{k-i+1}.
\end{aligned}$$

Proposition 4, part 2

Let $H'_{a,b,c}(k,l)$ be the region defined precisely as $H_{a,b,c}(k,l)$, with the one exception that the up-pointing triangle of side k is one unit below the northeastern corner, rather than one unit above the eastern corner.

Let $\nu = \min(b-1,k)$, and define $r(c)$ by

$$r(c) := \begin{cases} (c+2)^1 \cdots (c+\nu+1)^\nu \cdots (c+b+k-\nu)^\nu \cdots (c+b+k-1)^1, & \nu \geq 1 \\ 1, & \nu = 0 \\ \dfrac{1}{(c+1)_k}, & \nu = -1 \end{cases}$$

(in the first branch the bases are incremented by 1 from each factor to the next; the exponents are incremented by one until they reach ν, stay equal to ν across the middle portion, and then they decrease by one unit from each factor to the next).

Then we have

$$\mathrm{M}(H'_{a,b,c}(k,l)) = \binom{a+k}{k}\frac{q(c,l)}{q(0,0)},$$

where the polynomial $q(c,l)$ is defined to be

$$\begin{aligned}
&r(c)\,(l+1)_b(z+k-l+1)_a \\
&\qquad \times (c+k+2)(c+k+3)^2\cdots(c+k+m+1)^m(c+k+m+2)^m\cdots(c+k+M+1)^m \\
&\qquad\qquad \times (c+k+M+2)^{m-1}(c+k+M+3)^{m-2}\cdots(c+k+M+m) \\
&\qquad \times \sum_{i=1}^{k+1}\frac{(-1)^{i-1}}{(i-1)!(k-i+1)!}(l-k+i)_{k-i+1}(l+b+1)_{i-1}(l-k-c)_{i-1}(l-k-c+i)_{k-i+1}
\end{aligned}$$

(as in part 1, $m = \min(a,b)$ and $M = \max(a,b)$).

Allowing dents also on the southwestern side: nested Pfaffian

Theorem. Let $\alpha_1, \ldots, \alpha_{n+k}$ be arbitrary $\triangle$-dents and $\beta_1, \ldots, \beta_n$ arbitrary $\bigtriangledown$-dents along the boundary of $H^k_{a,b,c}$. Then

$$\mathrm{M}(H^k_{a,b,c} - \{\alpha_1, \ldots, \alpha_{n+k}, \beta_1, \ldots, \beta_k\})$$

is equal to a Pfaffian of a $2n \times 2n$ matrix whose entries are Pfaffians of $(2k+2) \times (2k+2)$ matrices of the type as in the previous "theorem" (that was never stated).

Proof

$D =$ region obtained from $H^k_{a,b,c}$ by removing the dents $\alpha_{n+1}, \ldots, \alpha_{n+k}$.

Apply Ciucu's extension of Kuo's condensation to the "dual" of D, with the removed vertices to be $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$.

We obtain a $2n \times 2n$ Pfaffian with entries of the form $\mathrm{M}(D - \{\alpha_i, \beta_j\})$, $i, j \in \{1, 2, \ldots, n\}$.

Now $\mathrm{M}(D - \{\alpha_i, \beta_j\})$ is a dented hexagon with all dents confined to four of its sides (dents of type $\triangle$ can only occur along the northwestern, northeastern and southern side, and there is a single dent of type $\bigtriangledown$).

By our previous theorem, $\mathrm{M}(D - \{\alpha_i, \beta_j\})$ can be expressed as the Pfaffian of a $(2k + 2) \times (2k + 2)$ matrix.