Bounded Littlewood-type identity related to alternating sign matrices

Ilse Fischer, Universität Wien

Outline

I. The classical (unbounded) Littlewood identity

II. Littlewood-type identity related to alternating sign matrices

III. Alternating sign arrays and plane partitions

IV. Arrowed Gelfand-Tsetlin patterns

V. Bounded identities

VI. Combinatorial interpretations in the bounded cases

I. The classical (unbounded) Littlewood identity

The classical (unbounded) Littlewood identity

$$\sum_{\lambda} s_\lambda(X_1, \ldots, X_n) = \prod_{i=1}^{n} \frac{1}{1 - X_i} \prod_{1 \le i < j \le n} \frac{1}{1 - X_i X_j}$$

Here $s_\lambda(X_1, \ldots, X_n)$ is the Schur polynomial of the partition λ and the sum is over all partitions λ.

Combinatorial model of Schur polynomials in terms of Gelfand-Tsetlin patterns: A Gelfand-Tsetlin pattern is a triangular array of integers of the form

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & & a_{2,2} & & \\ & \ldots & & \ldots & & \ldots & \\ a_{n,1} & & \ldots & & \ldots & & a_{n,n} \end{array}$$

with weak increase in $\nearrow$- and $\searrow$-direction.

The weight of a Gelfand-Tsetlin pattern is $\prod_{i=1}^{n} X_i^{\sum_j a_{i,j} - \sum_j a_{i-1,j}}$ and $s_\lambda(X_1, \ldots, X_n)$ is the sum of weights of all Gelfand-Tsetlin patterns with bottom row $(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$, where we allow zero parts in $\lambda = (\lambda_1, \ldots, \lambda_n)$.

Example $\lambda = (4, 2, 2)$

$$\begin{array}{ccccc} & & 2 & & \\ & 2 & & 2 & \\ 2 & & 2 & & 4 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 2 & & 3 & \\ 2 & & 2 & & 4 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 3 & \\ 2 & & 2 & & 4 \end{array}$$

$$\mathbf{X}_1^2\mathbf{X}_2^2\mathbf{X}_3^4 \qquad \mathbf{X}_1^2\mathbf{X}_2^3\mathbf{X}_3^3 \qquad \mathbf{X}_1^3\mathbf{X}_2^2\mathbf{X}_3^3$$

$$\begin{array}{ccccc} & & 2 & & \\ & 2 & & 4 & \\ 2 & & 2 & & 4 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 4 & \\ 2 & & 2 & & 4 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 2 & & 4 & \\ 2 & & 2 & & 4 \end{array}$$

$$\mathbf{X}_1^2\mathbf{X}_2^4\mathbf{X}_3^2 \qquad \mathbf{X}_1^3\mathbf{X}_2^3\mathbf{X}_3^2 \qquad \mathbf{X}_1^4\mathbf{X}_2^2\mathbf{X}_3^2$$

$$s_{(4,2,2)}(X_1, X_2, X_3) = X_1^2X_2^2X_3^2(X_1^2 + X_1X_2 + X_1X_3 + X_2^2 + X_2X_3 + X_3^2)$$

Unusual (?) combinatorial proof of the Littlewood identity

Combinatorial interpretation of the RHS:

$$\prod_{i=1}^{n} \frac{1}{1-X_i} \prod_{1 \le i < j \le n} \frac{1}{1-X_i X_j} = \prod_{i=1}^{n} \sum_{a_{i,i} \ge 0} X_i^{a_{i,i}} \prod_{1 \le i < j \le n} \sum_{a_{i,j} \ge 0} (X_i X_j)^{a_{i,j}}$$

Example:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \\ & & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ & & & a_{4,4} & a_{4,5} & a_{4,6} \\ & & & & a_{5,5} & s_{5,6} \\ & & & & & a_{6,6} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 0 & 1 & 1 \\ & 1 & 0 & 2 & 1 & 1 \\ & & 2 & 1 & 0 & 0 \\ & & & 0 & 2 & 1 \\ & & & & 0 & 2 \\ & & & & & 1 \end{pmatrix}$$

Two-line array: $a_{i,j} \to \binom{j}{i}^{a_{i,j}}$ ordered lexicographically

$$\begin{pmatrix} 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 6 \\ 1 & 1 & 1 & 2 & 3 & 3 & 2 & 2 & 3 & 1 & 2 & 4 & 4 & 1 & 2 & 4 & 5 & 5 & 6 \end{pmatrix}$$

Goal: transform this into a Gelfand-Tsetlin pattern with 6 rows.

Initial GT-pattern: 0

We insert the columns of the two-line array from left to right into the pattern.

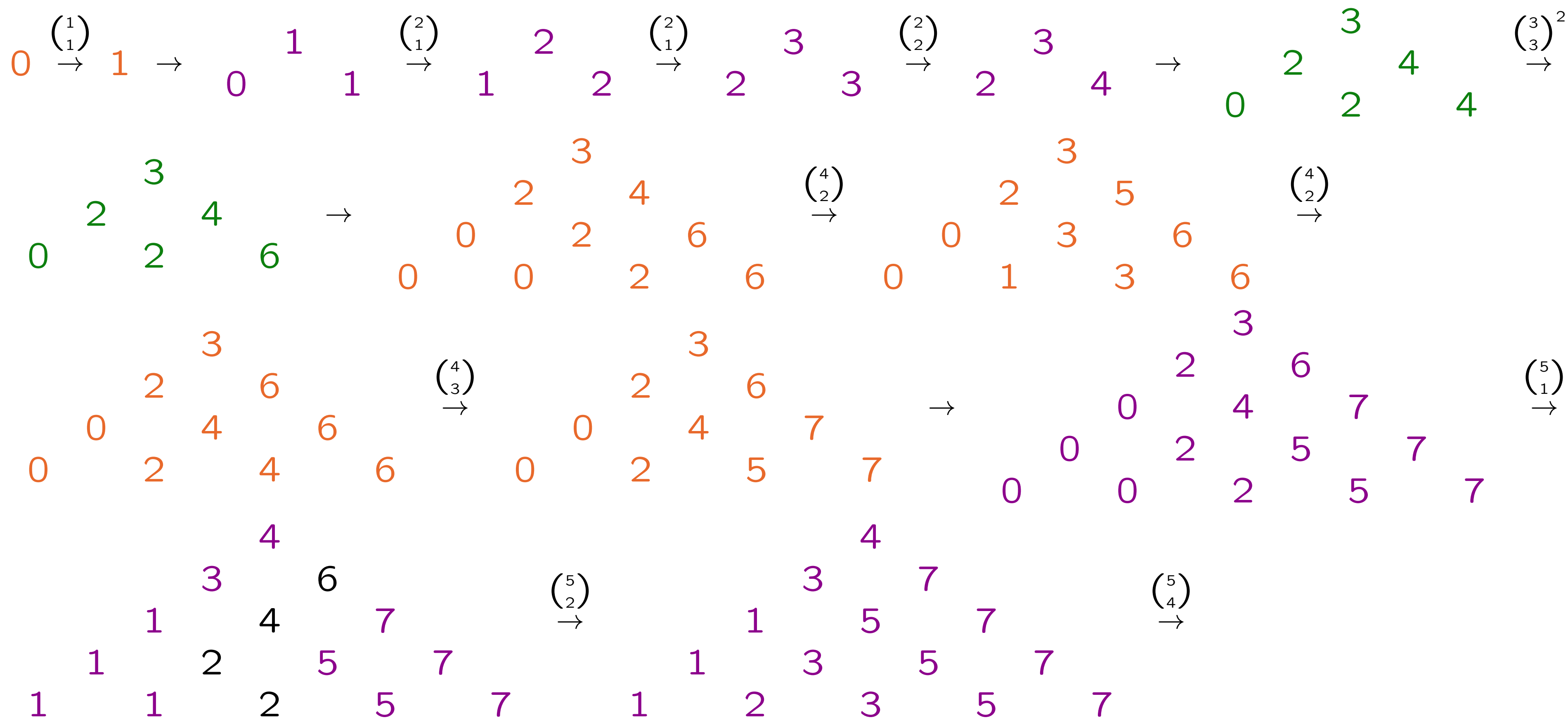
Insert column $\binom{j}{i}$**:**

• Start a path in the pattern at the end of row i with unit $\swarrow$- and $\searrow$-steps.

• Whenever the $\searrow$-neighbor of the current entry is equal to that entry, we extend our path to the next entry in $\searrow$-direction, otherwise we go to the next entry in $\swarrow$-direction. Continue until we have reached the bottom row and add 1 to all the entries in the path.

• If $i \neq j$, add 1 to the entry left of the bottom entry of the path.

When progressing from a column $\binom{j}{i}$ to a column $\binom{j+1}{k}$, copy the bottom row, prepend a 0 and add that row to the bottom of the pattern.



$$
\begin{array}{ccccccccc}
 & & & & 4 & & & & \\
 & & & 3 & & 7 & & & \\
 & & 1 & & 5 & & 7 & & \\
 & 1 & & 3 & & 5 & & 8 & \\
1 & & 2 & & 3 & & 6 & & 8
\end{array}
\quad \overset{\binom{5}{4}}{\longrightarrow} \quad
\begin{array}{ccccccccc}
 & & & & 4 & & & & \\
 & & & 3 & & 7 & & & \\
 & & 1 & & 5 & & 7 & & \\
 & 1 & & 3 & & 5 & & 9 & \\
1 & & 2 & & 3 & & 7 & & 9
\end{array}
\quad \longrightarrow
$$

$$
\begin{array}{ccccccccccc}
 & & & & & 4 & & & & & \\
 & & & & 3 & & 7 & & & & \\
 & & & 1 & & 5 & & 7 & & & \\
 & & 1 & & 3 & & 5 & & 9 & & \\
 & 1 & & 2 & & 3 & & 7 & & 9 & \\
0 & & 1 & & 2 & & 3 & & 7 & & 9
\end{array}
\quad \overset{\binom{6}{1}}{\longrightarrow} \quad
\begin{array}{ccccccccccc}
 & & & & & 5 & & & & & \\
 & & & & 4 & & 7 & & & & \\
 & & & 2 & & 5 & & 7 & & & \\
 & & 2 & & 3 & & 5 & & 9 & & \\
 & 2 & & 2 & & 3 & & 7 & & 9 & \\
1 & & 2 & & 2 & & 3 & & 7 & & 9
\end{array}
\quad \overset{\binom{6}{2}}{\longrightarrow}
$$

$$\begin{array}{c} 5 \\ 4 \quad 8 \\ 2 \quad 5 \quad 8 \\ 2 \quad 3 \quad 6 \quad 9 \\ 2 \quad 2 \quad 4 \quad 7 \quad 9 \\ 1 \quad 2 \quad 3 \quad 4 \quad 7 \quad 9 \end{array} \overset{\binom{6}{4}}{\longrightarrow} \begin{array}{c} 5 \\ 4 \quad 8 \\ 2 \quad 5 \quad 8 \\ 2 \quad 3 \quad 6 \quad 10 \\ 2 \quad 2 \quad 4 \quad 7 \quad 10 \\ 1 \quad 2 \quad 3 \quad 4 \quad 8 \quad 10 \end{array}$$

$$\overset{\binom{6}{5}^2}{\longrightarrow} \begin{array}{c} 5 \\ 4 \quad 8 \\ 2 \quad 5 \quad 8 \\ 2 \quad 3 \quad 6 \quad 10 \\ 2 \quad 2 \quad 4 \quad 7 \quad 12 \\ 1 \quad 2 \quad 3 \quad 4 \quad 10 \quad 12 \end{array} \overset{\binom{6}{6}}{\longrightarrow}$$

$$\begin{array}{c} 5 \\ 4 \quad 8 \\ 2 \quad 5 \quad 8 \\ 2 \quad 3 \quad 6 \quad 10 \\ 2 \quad 2 \quad 4 \quad 7 \quad 12 \\ 1 \quad 2 \quad 3 \quad 4 \quad 10 \quad 13 \end{array}$$

Homework: Well-defined and inverse?

Rewrite the classical Littlewood identity

Bialternant formula for the Schur polynomial:

$$s_{(\lambda_1,\ldots,\lambda_n)}(X_1,\ldots,X_n) = \frac{\det_{1\le i,j\le n}\left(X_i^{\lambda_j+n-j}\right)}{\prod_{1\le i<j\le n}(X_i - X_j)} = \frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{i=1}^n X_i^{\lambda_i+n-i}\right]}{\prod_{1\le i<j\le n}(X_i - X_j)},$$

with

$$\mathbf{ASym}_{X_1,\ldots,X_n} f(X_1,\ldots,X_n) = \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}\sigma \cdot f(X_{\sigma(1)},\ldots,X_{\sigma(n)}).$$

Littlewood identity:

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\sum_{0\le k_1<k_2<\ldots<k_n} X_1^{k_1}X_2^{k_2}\cdots X_n^{k_n}\right]}{\prod_{1\le i<j\le n}(X_j - X_i)} = \prod_{i=1}^n \frac{1}{1-X_i} \prod_{1\le i<j\le n} \frac{1}{1-X_iX_j}$$

II. Littlewood-type identity related to ASMs

Littlewood-type identity related to ASMs

In two of my papers from 2019:

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i<j\le n}(1+X_j+X_iX_j)\sum_{0\le k_1<k_2<\ldots<k_n} X_1^{k_1}X_2^{k_2}\cdots X_n^{k_n}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)}$$

$$=\prod_{i=1}^{n}\frac{1}{1-X_i}\prod_{1\le i<j\le n}\frac{1+X_i+X_j}{1-X_iX_j}$$

Generalization by Hans Höngesberg:

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i<j\le n}(\mathbf{Q}+(\mathbf{Q}-1)X_i+X_j+X_iX_j)\sum_{0\le k_1<k_2<\ldots<k_n}\prod_{i=1}^{n}\left(\frac{X_i(1+X_i)}{\mathbf{Q}+X_i}\right)^{k_i}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)}$$

$$=\prod_{i=1}^{n}\frac{\mathbf{Q}+X_i}{\mathbf{Q}-X_i^2}\frac{\prod_{1\le i<j\le n}(\mathbf{Q}(1+X_i)(1+X_j)-X_iX_j)}{\prod_{1\le i<j\le n}(\mathbf{Q}-X_iX_j)}.$$

Set $\mathbf{Q}=1$ to obtain the previous identity.

Further generalizations

Different generalization:

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i<j\le n}(1+(\mathbf{1}+\mathbf{r})\mathbf{X_i}+X_j+X_iX_j)\sum_{0\le k_1<k_2<\ldots<k_n}X_1^{k_1}X_2^{k_2}\cdots X_n^{k_n}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)}$$

$$=\prod_{i=1}^{n}\frac{1}{1-X_i}\prod_{1\le i<j\le n}\frac{1+X_i+X_j+(\mathbf{1}+\mathbf{r})\mathbf{X_i}\mathbf{X_j}}{1-X_iX_j}$$

Common generalization of Hans' generalization and the above generalization:

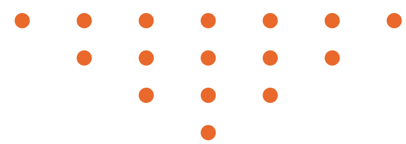
$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i<j\le n}(\mathbf{Q}+(\mathbf{Q}+\mathbf{r})\mathbf{X_i}+X_j+X_iX_j)\sum_{0\le k_1<k_2<\ldots<k_n}\prod_{i=1}^{n}\left(\frac{X_i(\mathbf{1}+\mathbf{X_i})}{\mathbf{Q}+\mathbf{X_i}}\right)^{k_i}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)}$$

$$=\prod_{i=1}^{n}\frac{\mathbf{Q}+X_i}{\mathbf{Q}-X_i^2}\frac{\prod_{1\le i<j\le n}\mathbf{Q}(1+X_i)(1+X_j)+rX_iX_j}{\prod_{1\le i<j\le n}(\mathbf{Q}-X_iX_j)}$$

III. Alternating sign arrays and plane partitions

Alternating sign triangles = ASTs

An AST of order n is a triangular array of 1's, -1's and 0's with n centered rows



such that

(1) the non-zero entries alternate in each row and each column,

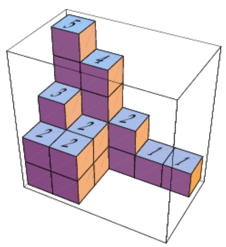
(2) all row sums are 1, and

(3) the topmost non-zero entry of each column is 1 (if such an entry exists).

Example:

$$\begin{array}{ccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 1 & -1 & 1 & 0 & 0 & \\ & & 1 & -1 & 1 & & \\ & & & 1 & & & \end{array}$$

Totally symmetric self-complementary plane partitions = TSSCPPs



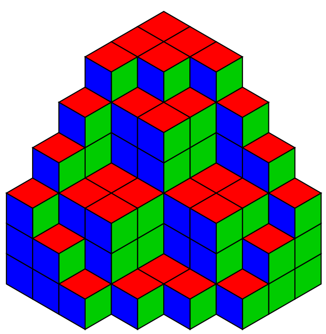
$a = 4, b = 3, c = 5$

A (boxed) plane partition in an $a \times b \times c$ box is a subset

$$PP \subseteq \{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\} \times \{1, 2, \ldots, c\}$$

with

$$(i, j, k) \in PP \Rightarrow (i', j', k') \in PP \quad \forall (i', j', k') \leq (i, j, k).$$



- **Totally symmetric:**
$(i, j, k) \in PP \Rightarrow \sigma(i, j, k) \in PP \; \forall \sigma \in \mathcal{S}_3$
(MacMahon 1899, 1915/16)

- **Self-complementary:**
Equal to its complement in the $2n \times 2n \times 2n$ box
(Mills, Robbins and Rumsey 1986)

Now: "Our" Littlewood-type identity was the crucial point in showing that there is the same number of ASTs with n rows as there is of TSSCPPs in a $2n \times 2n \times 2n$ box.

Alternating sign trapezoids

For $n \geq 1, l \geq 2^*$, an *(n, l)-alternating sign trapezoid* is an array of 1's, -1's and 0's with n centered rows and l elements in the bottom row, arranged as follows

such that the following conditions are satisfied.

(1) In each row and column, the non-zero entries alternate.

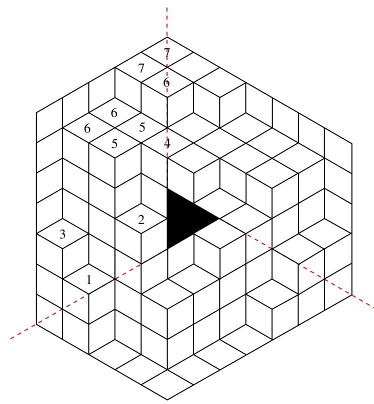
(2) All row sums are 1.

(3) The topmost non-zero entry in each column is 1.

(4) The column sums are 0 for the middle $l-2$ columns.

*Can be extended to $l = 1$.

Cyclically symmetric lozenge tilings of a hexagon with a central triangular hole



Theorem (Behrend, F. 2018). There is the same number of (n, l)-alternating sign trapezoids as there is of cyclically symmetric lozenge tilings of a hexagon with side lengths $n+l-1, n, n+l-1, n, n+l-1, n$ that has a central triangular hole of size $l-1$.

The Littlewood-type identities were also crucial in one proof of this theorem. The proof is especially useful when including several statistics.

Christian in a survey on plane partitions in 2016: "*However, the greatest, still unsolved, mystery concerns the question what plane partitions have to do with alternating sign matrices.*"

IV. Arrowed Gelfand-Tsetlin patterns

Signed intervals:

$$\underline{[a,b]} = \begin{cases} [a,b], & a \le b \\ \emptyset, & b = a-1 \\ [b+1, a-1], & b < a-1 \end{cases}$$

The interval is said to be negative in the last case.

An *arrowed Gelfand-Tsetlin pattern* is a triangular array of the following form

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & & a_{2,2} & & \\ & \cdots & & \cdots & & \cdots & \\ a_{n,1} & & \cdots & & \cdots & & a_{n,n} \end{array},$$

where each entry $a_{i,j}$ is an integer decorated with an element from $\{\nwarrow, \nearrow, \nwarrow\!\!\!\!\nearrow, \emptyset\}$ and the following is satisfied for each entry a not in the bottom row: Suppose b is the $\swarrow$-neighbor of a and c is the $\searrow$-neighbor of a, respectively, i.e.,

$$\begin{array}{ccc} & a & \\ b & & c \end{array}.$$

Depending on the decoration of b, c, denoted by $\mathrm{decor}(b)$ and $\mathrm{decor}(c)$, respectively, we need to consider four cases:

- $(\mathrm{decor}(b), \mathrm{decor}(c)) \in \{\nwarrow, \emptyset\} \times \{\nearrow, \emptyset\}$: $a \in \underline{[b, c]}$
- $(\mathrm{decor}(b), \mathrm{decor}(c)) \in \{\nwarrow, \emptyset\} \times \{\nwarrow, \nwarrow\!\!\!\!\nearrow\}$: $a \in \underline{[b, c-1]}$
- $(\mathrm{decor}(b), \mathrm{decor}(c)) \in \{\nearrow, \nwarrow\!\!\!\!\nearrow\} \times \{\nearrow, \emptyset\}$: $a \in \underline{[b+1, c]}$
- $(\mathrm{decor}(b), \mathrm{decor}(c)) \in \{\nearrow, \nwarrow\!\!\!\!\nearrow\} \times \{\nwarrow, \nwarrow\!\!\!\!\nearrow\}$: $a \in \underline{[b+1, c-1]}$

Example:

$$
\begin{array}{ccccccccccc}
 & & & & & \nwarrow 2 & & & & & \\
 & & & & 2 & & \nwarrow 3 \nearrow & & & & \\
 & & & \nwarrow 2 & & 2 \nearrow & & 3 \nearrow & & & \\
 & & 3 & & \nwarrow 2 & & \nwarrow 3 \nearrow & & \nwarrow 3 \nearrow & & \\
 & 2 \nearrow & & 4 & & \nwarrow 2 \nearrow & & 3 \nearrow & & 2 & \\
\nwarrow 6 & & \nwarrow 2 \nearrow & & 5 & & 1 \nearrow & & \nwarrow 4 & & \nwarrow 2 \nearrow
\end{array}
$$

Sign: Each negative interval $\underline{[a_{i+1,j}(+1), a_{i+1,j+1}(-1)]}$ with $i \geq 1$ and $j \leq i$ contributes a multiplicative -1.

In our example, there are no negative intervals in rows $1, 2, 3$, two in rows $4, 5$ and three in row 6, so that the sign of the pattern is -1.

We associate the following weight to a given arrowed Gelfand-Tsetlin pattern $A = (a_{i,j})_{1 \leq j \leq i \leq n}$:

$$\mathrm{W}(A) = \mathrm{sgn}(A) \cdot t^{\#\emptyset} u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\nearrow} \prod_{i=1}^{n} X_i^{\sum_{j=1}^{i} a_{i,j} - \sum_{j=1}^{i-1} a_{i-1,j} + \#\nearrow \text{in row } i - \#\nwarrow \text{in row } i}$$

The weight of our example is

$$-t^5 u^5 v^5 w^6 X_1 X_2^3 X_3^3 X_4^3 X_5^4 X_6^6.$$

Compare to the Schur function weight for Gelfand-Tsetlin patterns!

Generating function of arrowed Gelfand-Tsetlin patterns with prescribed bottom row

Theorem (F., Schreier-Aigner). The generating function of arrowed Gelfand-Tsetlin patterns with bottom row $k_1, \ldots, k_n$ is

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1 \le i \le j \le n} (v + wX_i + tX_j + uX_iX_j) \prod_{i=1}^n X_i^{k_i-1}\right]}{\prod_{1 \le i < j \le n}(X_j - X_i)}.$$

Our Littlewood-type identity, slightly rewritten:

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1 \le i \le j \le n}(1 + wX_i + X_j + X_iX_j) \sum_{0 \le k_1 < k_2 < \ldots < k_n} X_1^{k_1-1} X_2^{k_2-1} \cdots X_n^{k_n-1}\right]}{\prod_{1 \le i < j \le n}(X_j - X_i)}$$
$$= \prod_{i=1}^n \frac{X_i^{-1} + (1+w) + X_i}{1 - X_i} \prod_{1 \le i < j \le n} \frac{1 + X_i + X_j + wX_iX_j}{1 - X_iX_j}$$

The left-hand side is the generating function of all arrowed Gelfand-Tsetlin patterns with strictly increasing bottom row of non-negative integers, setting $t = u = v = 1$.

Combinatorial interpretation of the RHS

$$\prod_{i=1}^{n} \frac{X_i^{-1} + (1+w) + X_i}{1 - X_i} \prod_{1 \le i < j \le n} \frac{1 + X_i + X_j + w X_i X_j}{1 - X_i X_j}$$

Generating function of two-line arrays with entries in $\{1, 2, \ldots, n\}$ that are

- ordered lexicographically,
- the top element of each column is greater than or equal to the bottom element,
- for each i, j, there is a distinguished column $\binom{j}{i}$ that is either overlined, underlined, both or neither.

Weight:

- columns $\binom{j}{i}$ different from the distinguished columns contribute $X_i X_j$ if $i \neq j$ and X_i if $i = j$,
- an overline of a column $\binom{j}{i}$ contributes X_j,
- an underline of a column $\binom{j}{i}$ contributes X_i if $i \neq j$ and X_i^{-1} if $i = j$,
- a column that is overlined and underlined contributes w.

Open problem: Find a combinatorial proof!

From these over- and underlined two-line arrays, we need to construct arrowed monotone triangles.

$\rightarrow$ **Flo, Hans, Moritz, Seamus**

V. Bounded identities

Bounded classical Littlewood identity

Bounded? $\sum_{0 \le k_1 < k_2 < \ldots < k_n} \to \sum_{0 \le k_1 < k_2 < \ldots < k_n \le m}$

$$\sum_{\lambda \subseteq (M^n)} s_\lambda(X_1, \ldots, X_n) = \sum_{0 \le k_1 \le k_2 \le \ldots \le k_n \le m} s_{(k_n, k_{n-1}, \ldots, k_1)}(X_1, \ldots, X_n)$$
$$= \frac{\det_{1 \le i,j \le n}\left(X_i^{j-1} - X_i^{M+2n-j}\right)}{\prod_{i=1}^n (1 - X_i) \prod_{1 \le i < j \le n} (X_j - X_i)(1 - X_i X_j)}$$

Macdonald in his book. (Note that $m = M + n - 1$.)

Bounded Littlewood identity related to ASMs

$$\frac{1}{\prod_{1\le i<j\le n}(X_j-X_i)}\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i<j\le n}(Q+(Q+r)X_i+X_j+X_iX_j)\right.$$
$$\left.\times\sum_{0\le k_1<k_2<\ldots<k_n\le m}\left(\frac{X_1(1+X_1)}{Q+X_1}\right)^{k_1}\left(\frac{X_2(1+X_2)}{Q+X_2}\right)^{k_2}\cdots\left(\frac{X_n(1+X_n)}{Q+X_n}\right)^{k_n}\right]$$
$$=\frac{\det_{1\le i,j\le n}\left(a_{j,m,n}(Q,r;X_i)\right)}{\prod_{1\le i\le j\le n}(Q-X_iX_j)\prod_{1\le i<j\le n}(X_j-X_i)}$$

with

$$a_{j,m,n}(Q,r;X)=(1+QX^{-1})X^j(1+X)^{j-1}(Q+rX+QX)^{n-j}$$
$$-X^{2n}Q^{-n}\left(\frac{(1+X)X}{Q+X}\right)^m(1+X)\left(QX^{-1}\right)^j(1+QX^{-1})^{j-1}(Q+rQX^{-1}+Q^2X^{-1})^{n-j}.$$

The proof has more than 7 pages, but it is elementary.

The case $Q = 1$

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i\le j\le n}(1+wX_i+X_j+X_iX_j)\sum_{0\le k_1<k_2<\ldots<k_n\le m} X_1^{k_1-1}X_2^{k_2-1}\cdots X_n^{k_n-1}\right]}{\prod\limits_{1\le i<j\le n}(X_j-X_i)}$$

$$= \prod_{i=1}^{n}(X_i^{-1}+1+w+X_i)$$

$$\times \frac{\det_{1\le i,j\le n}\left(X_i^{j-1}(1+X_i)^{j-1}(1+wX_i)^{n-j} - X_i^{m+2n-j}(1+X_i^{-1})^{j-1}(1+wX_i^{-1})^{n-j}\right)}{\prod\limits_{i=1}^{n}(1-X_i)\prod\limits_{1\le i<j\le n}(1-X_iX_j)(X_j-X_i)}.$$

We are interested in finding a combinatorial proof.

V. Combinatorial interpretations in the bounded cases

The classical case

The classical bounded Littlewood identity:

$$\sum_{\lambda\subseteq(m^n)} s_\lambda(X_1,\ldots,X_n) = \frac{\det_{1\le i,j\le n}\left(X_i^{j-1} - X_i^{m+2n-j}\right)}{\prod_{i=1}^n (1-X_i)\prod_{1\le i<j\le n}(X_j - X_i)(1-X_iX_j)}$$

This identity is equivalent to

$$\sum_{\lambda\subseteq(m^n)} s_\lambda(X_1,\ldots,X_n) = \prod_{i=1}^n X_i^{m/2}\, so^{\text{odd}}_{(m/2,m/2,\ldots,m/2)}(X_1,\ldots,X_n),$$

where $so^{\text{odd}}_\lambda(X_1,\ldots,X_n)$ is the irreducible character of the special orthogonal group $SO_{2n+1}(\mathbb{C})$ associated with the partition $\lambda = (\lambda_1,\ldots,\lambda_n)$.

Combinatorial interpretation of the irreducible characters of $SO_{2n+1}(\mathbb{C})$

A $2n$-split orthogonal pattern is an array of non-negative integers or non-negative half-integers with $2n$ rows of lengths $1, 1, 2, 2, \ldots, n, n$, which are aligned as follows for $n = 3$

$$\begin{array}{cccccc} a_{1,1} & & & & & \\ & a_{2,1} & & & & \\ a_{3,1} & & a_{3,2} & & & \\ & a_{4,1} & & a_{4,2} & & \\ a_{5,1} & & a_{5,2} & & a_{5,3} & \\ & a_{6,1} & & a_{6,2} & & a_{6,3} \end{array},$$

such that

• the entries are weakly increasing along $\nearrow$-diagonals and $\searrow$-diagonals,

• the entries, except for the first entries in the odd rows (called odd starters), are either all non-negative integers or all non-negative half-integers, and

• each starter is independently either a non-negative integer or a non-negative half-integer.

The weight of a $2n$-split orthogonal pattern is

$$\prod_{i=1}^{n} X_i^{r_{2i} - 2r_{2i-1} + r_{2i-2}}$$

where r_i is the sum of entries in row i and $r_0 = 0$.

Formula for $so^{\mathbf{odd}}_\lambda(X_1,\dots,X_n)$

Theorem (Proctor 1994). Let $\lambda=(\lambda_1,\dots,\lambda_n)$ be a partition (allowing zero parts) or a half-integer partition. Then the generating function of $2n$-split orthogonal patterns with respect to the above weight that have λ as bottom row, written in increasing order, is

$$\prod_{i=1}^{n} X_i^{n-1/2}\frac{\det_{1\le i,j\le n}\left(X_i^{-\lambda_j-n+j-1/2}-X_i^{\lambda_j+n-j+1/2}\right)}{(1+[\lambda_n=0])\prod_{i=1}^{n}(1-X_i)\prod_{1\le i<j\le n}(X_j-X_i)(1-X_iX_j)}.$$

This gives a combinatorial interpretation of $\prod_{i=1}^{n} X_i^{m/2} so^{\text{odd}}_{(m/2,m/2,\dots,m/2)}(X_1,\dots,X_n)$, which is the RHS of the classical bounded Littlewood identity.

ASM-related identity

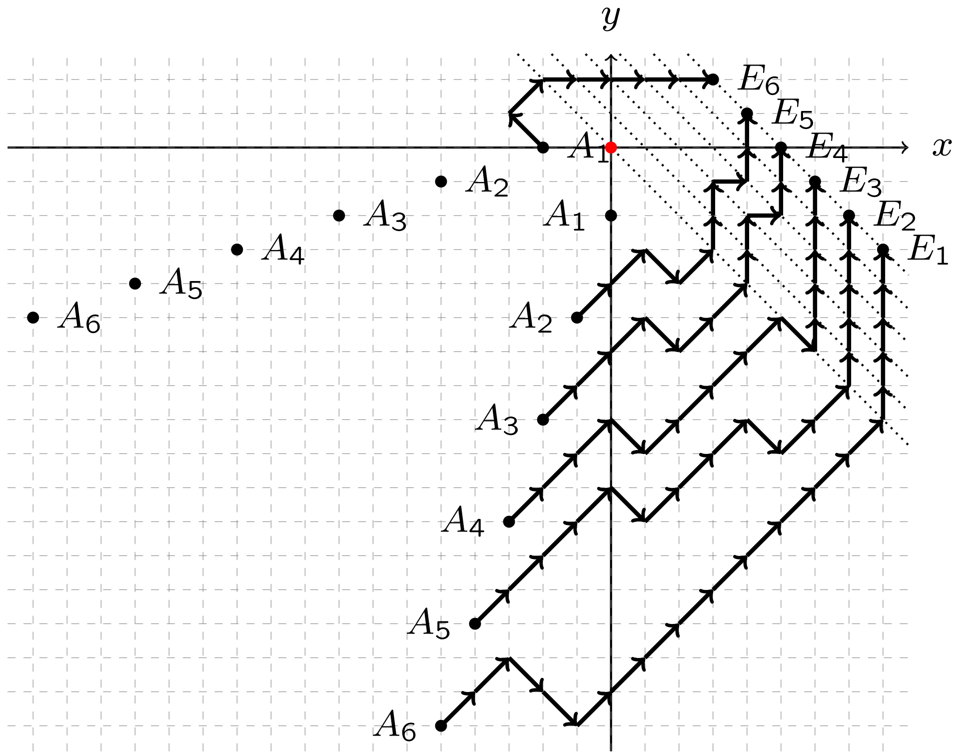
$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i\le j\le n}(1+wX_i+X_j+X_iX_j)\sum_{0\le k_1<k_2<\ldots<k_n\le m}X_1^{k_1-1}X_2^{k_2-1}\cdots X_n^{k_n-1}\right]}{\prod\limits_{1\le i<j\le n}(X_j-X_i)}$$

$$=\prod_{i=1}^{n}(X_i^{-1}+1+w+X_i)$$

$$\times\frac{\det_{1\le i,j\le n}\left(X_i^{j-1}(1+X_i)^{j-1}(1+wX_i)^{n-j}-X_i^{m+2n-j}(1+X_i^{-1})^{j-1}(1+wX_i^{-1})^{n-j}\right)}{\prod\limits_{i=1}^{n}(1-X_i)\prod\limits_{1\le i<j\le n}(1-X_iX_j)(X_j-X_i)}.$$

The LHS has a combinatorial interpretation in terms of arrowed Gelfand-Tsetlin patterns with bounded bottom row.

Combinatorial interpretation of the RHS in terms of non-intersecting lattice paths



Case $m = 2l + 1$

The RHS is the weighted count of families of n lattice paths.

• The i-th lattice path starts in $A_i = \{(-3i+1, -i+1), (-i+1, -3i+1)\}$ and the end points are $E_j = (n - j + l + 1, j - l - 2)$.

• Below and on the line $x+y=0$, the step set is $\{(1,1), (-1,1)\}$ for steps that start in $(-3i+1, -i+1)$ and it is $\{(1,1), (1,-1)\}$ for steps that start in $(-i+1, -3i+1)$.

• Steps of type $(-1,1)$ and $(1,-1)$ with distance $0, 2, 4, \ldots$ from $x+y=0$ are equipped with the weights $X_1, X_2, X_3, \ldots$, while such steps with distance $1, 3, 5, \ldots$ are equipped with the weights $X_1^{-1}, X_2^{-1}, X_3^{-1}, \ldots$, respectively.

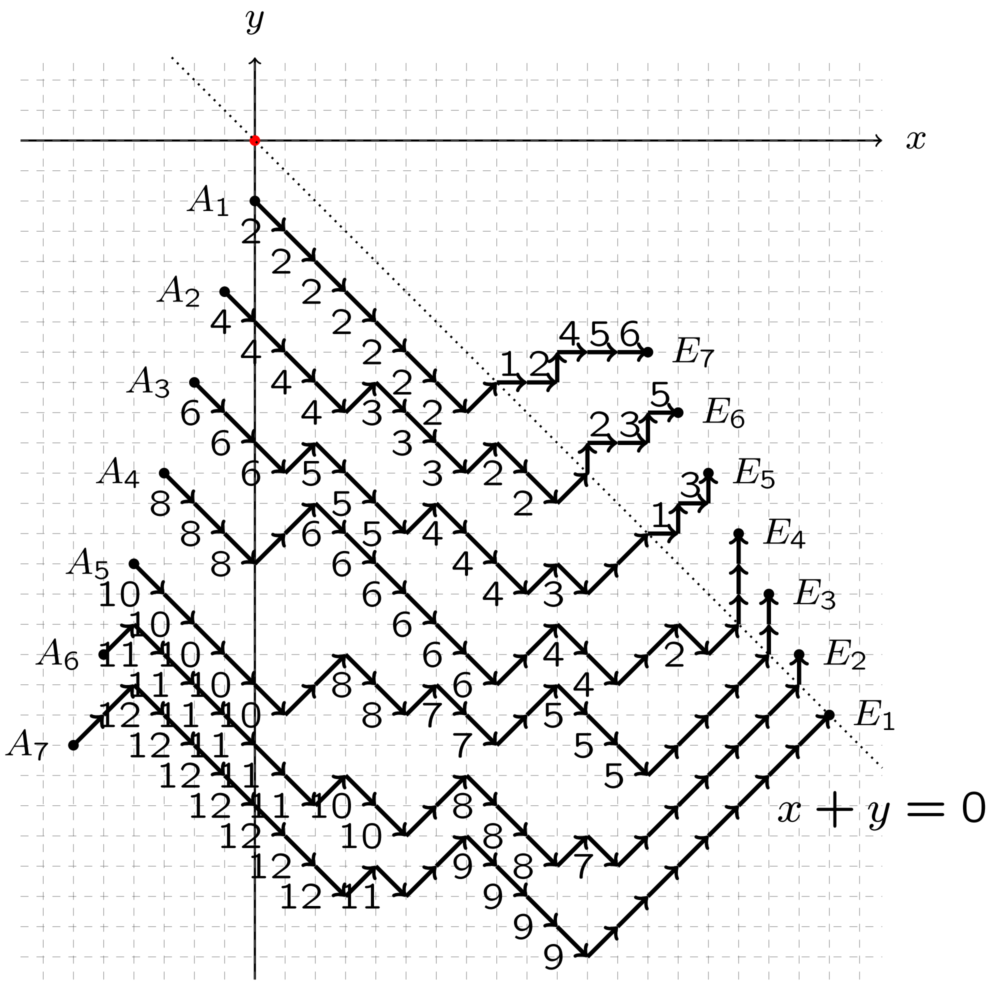
• Above the line $x+y=0$, the step set is $\{(1,0), (0,1)\}$. Above the line $x+y=j-1$, horizontal steps of the path that ends in E_j are equipped with the weight w.

• The paths can be assumed to be non-intersecting below the line $x+y=0$. In case $w=1$, we can also assume them to be non-intersecting above the line $x+y=0$. If $w=0$, E_j can be replaced by $E'_j = (n-j+l+1, 2j-n-l-2)$, $j = 1, 2, \ldots, n$, and then we can also assume the paths to be non-intersecting above the line $x+y=0$.

• The sign of family of paths is the sign of the permutation σ with the property that the i-th path connects A_i to $E_{\sigma(i)}$ with an extra contribution of -1 if we choose $(-i+1, -3i+1)$ from A_i. Moreover, we have an overall factor of $(-1)^{\binom{n+1}{2}} \prod_{i=1}^{n} X_i^l (X_i^{-1} + 1 + w + X_i)(1 + X_i)$.

• In case $w = 0, 1$, when restricting to non-intersecting paths, let $1 \le i_1 < i_2, \ldots < i_m < n$ be the indices for which we chose $(-3i+1, -i+1)$ from A_i. Then the sign can assumed to be $(-1)^{i_1 + \ldots + i_m}$ and the overall factor is $\prod_{i=1}^{n} X_i^l (X_i^{-1} + 1 + w + X_i)(1 + X_i)$.

The interpretation is signless if $w = 0$ and $l \geq n - 2$



12	12	12	12	12	12	12	11	9	9	9	9
11	11	11	11	11	11	10	10	8	8	8	7
10	10	10	10	10	8	8	7	7	5	5	5
8	8	8	6	6	6	6	6	6	4	4	2
6	6	6	5	5	5	4	4	4	3		
4	4	4	4	3	3	3	2	2			
2	2	2	2	2	2	2					

,

6	5	4	2	1
5	3	2		
3	1			

Theorem (F., 2022). Assume that $w = 0$ and $m = 2l + 1$. In case $l \geq n - 2$, the RHS is the generating function of pairs of plane partitions (P, Q) of shape λ and μ, respectively, where

- μ is the complement of λ in the $n \times l$-rectangle,
- P is a column-strict plane partition such that the entries in the i-th row are bounded by $2n + 2 - 2i$, and
- Q is a row-strict plane partition such that the entries in the i-th row are bounded by $n - i$.

The weight is

$$\prod_{i=1}^{n-1} X_i^l (X_i^{-1} + 1 + X_i)(1 + X_i) X_i^{\#\text{of } 2i-1 \text{ in P}} X_i^{-\#\text{of } 2i \text{ in P}}.$$

Remark. The Q's are in easy bijection with $2n \times 2n \times 2n$ TSSCPPs. The P's are in easy bijection with symplectic tableaux.

Thanks!