First Bijective Proofs of ASM Theorems

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The talk is based on three articles:

- `arXiv:1910.04198` (Elect. J. Combin. 2020)
- `arXiv:1912.01354` (to appear in Int. Math. Res. Not.)
- `asmpnas.pdf` (PNAS 2020)

Outline

I. ASMs, DPPs, and Bijections 1 & 2

II. Signed sets and sijections

III. Some details on our constructions

I. ASMs, DPPs and Bijections 1 & 2

Alternating Sign Matrices = ASMs

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Square matrix with entries in $\{0, \pm 1\}$ such that in each row and each column

- the non–zero entries appear with alternating signs, and
- the sum of entries is 1.

How many?

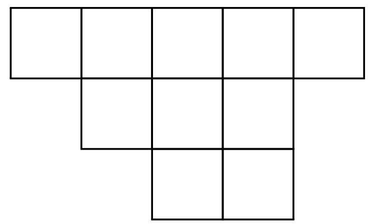
n	1	2	3	4
	(1)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$3! + \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	42

$$\# \text{ of } n \times n \text{ ASMs} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

Mills, Robbins, Rumsey, Zeilberger, Kuperberg in the 1980s and 1990s.

Descending Plane Partitions = DPPs

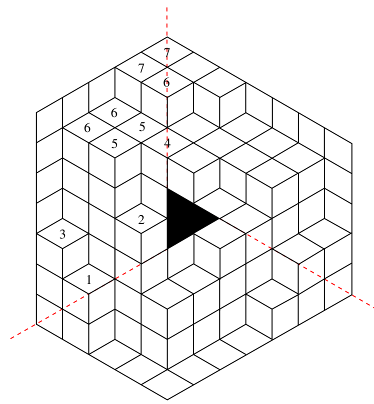
• A strict partition is a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ with distinct parts, i.e., $\lambda_1 > \lambda_2 > \dots > \lambda_l > 0$. The shifted Young diagram of shape $(5, 3, 2)$ is as follows.



• A column strict shifted plane partition is a filling of a shifted Young diagram with positive integers such that rows decrease weakly and columns decrease strictly.

6	6	5	5	2
	5	4	4	
		3	1	

• A DPP is such a column strict PP where the first part in each row is greater than the length of its row and less than or equal to the length of the previous row. Ugly condition?



• DPPs with parts no greater than 3: $\varnothing$, 2 , 3 , $3\ 1$, $3\ 2$, $3\ 3$, $\begin{matrix} 3 & 3 \\ & 2 \end{matrix}$

• The number of DPPs with parts no greater than n is also $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$ (Andrews).

Bijection 1 (Bijective Proof of the Product Formula)

ASM_n = set of $n \times n$ ASMs

$\mathsf{ASM}_{n,i}$ = set of $n \times n$ ASMs $(a_{p,q})_{1 \le p,q \le n}$ with $a_{1,i} = 1$

B_n = set of $(2n-1)$-subsets of $[3n-2] = \{1, 2, \ldots, 3n-2\}$; $|\mathsf{B}_n| = \binom{3n-2}{2n-1}$

$\mathsf{B}_{n,i}$ = set of elements of B_n whose median is $n+i-1$; $|\mathsf{B}_{n,i}| = \binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}$

DPP_n = set of DPPs with parts no greater than n

We have constructed a bijection between the following sets:

$$\mathsf{DPP}_{n-1} \times \mathsf{B}_{n,1} \times \mathsf{ASM}_{n,i} \longrightarrow \mathsf{DPP}_{n-1} \times \mathsf{ASM}_{n-1} \times \mathsf{B}_{n,i}$$

Then we also have a bijection

$$\mathsf{DPP}_{n-1} \times \mathsf{B}_{n,1} \times \mathsf{ASM}_{n} \longrightarrow \mathsf{DPP}_{n-1} \times \mathsf{ASM}_{n-1} \times \mathsf{B}_{n}\,.$$

Iterating this, we obtain a bijection

$$\mathsf{DPP}_0 \times \cdots \times \mathsf{DPP}_{n-1} \times \mathsf{B}_{1,1} \times \cdots \times \mathsf{B}_{n,1} \times \mathsf{ASM}_n \longrightarrow \mathsf{DPP}_0 \times \cdots \times \mathsf{DPP}_{n-1} \times \mathsf{B}_1 \times \cdots \times \mathsf{B}_n\,.$$

Example: $\mathrm{DPP}_2 \times \mathrm{B}_{3,1} \times \mathrm{ASM}_{3,2} \longrightarrow \mathrm{DPP}_2 \times \mathrm{ASM}_2 \times \mathrm{B}_{3,2}$

for $x = 0$

$$
\begin{aligned}
\left(\varnothing, 12345, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) &\leftrightarrow (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23457)\\
\left(\varnothing, 12346, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) &\leftrightarrow (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13457)\\
\left(\varnothing, 12347, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) &\leftrightarrow (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12457)\\
\left(\varnothing, 12356, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) &\leftrightarrow (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13456)\\
\left(\varnothing, 12357, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) &\leftrightarrow (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13457)\\
\left(\varnothing, 12367, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) &\leftrightarrow (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13467)\\
\left(2, 12345, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) &\leftrightarrow (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23467)\\
\left(2, 12346, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) &\leftrightarrow (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13467)\\
\left(2, 12347, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) &\leftrightarrow (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12467)\\
\left(2, 12356, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) &\leftrightarrow (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23456)\\
\left(2, 12357, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) &\leftrightarrow (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23457)\\
\left(2, 12367, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) &\leftrightarrow (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23467)
\end{aligned}
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\begin{aligned}
\left(\varnothing, 12345, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) &\leftrightarrow (\varnothing, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23456)\\
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\left(2, 12357, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) &\leftrightarrow (2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23457)\\
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\left(\varnothing, 12346, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) &\leftrightarrow (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13456)\\
\left(\varnothing, 12347, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) &\leftrightarrow (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12456)\\
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\left(\varnothing, 12367, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) &\leftrightarrow (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12467)\\
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\left(2, 12346, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) &\leftrightarrow (\varnothing, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13457)\\
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\left(2, 12367, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) &\leftrightarrow (2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13467)
\end{aligned}
$$

The python code is available at `https://www.fmf.uni-lj.si/~konvalinka/asmcode.html`.

Bijection 2 (ASMs and DPPs)

$\mathrm{DPP}_{n,i}$ = subset of DPP_n with DPPs that have $i-1$ occurrences of n.

We have constructed a bijection between the following sets:

$$\mathrm{DPP}_{n-1} \times \mathrm{ASM}_{n,i} \longrightarrow \mathrm{ASM}_{n-1} \times \mathrm{DPP}_{n,i}$$

- Once such a bijection is constructed, it follows that

$$|\mathrm{DPP}_{n-1}| \cdot |\mathrm{ASM}_{n,i}| = |\mathrm{ASM}_{n-1}| \cdot |\mathrm{DPP}_{n,i}|.$$

- By induction, we can assume $|\mathrm{DPP}_{n-1}| = |\mathrm{ASM}_{n-1}|$ and so $|\mathrm{ASM}_{n,i}| = |\mathrm{DPP}_{n,i}|$.
- Summing this over all i implies $|\mathrm{DPP}_n| = |\mathrm{ASM}_n|$.

Example $\mathrm{DPP}_3 \times \mathrm{ASM}_{4,2} \longrightarrow \mathrm{ASM}_3 \times \mathrm{DPP}_{4,2}$ **for** $x = 0$

$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, 4\ 2\ 1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, 4\ 1\ 1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, 4\ 1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\ 2\right)$
$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\ 1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, 4\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, 4\ 1\ 1\right)$
$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\ 1\ 1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, 4\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, 4\ 2\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, 4\right)$
$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, 4\ 1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, 4\ 1\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\ 3\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, 4\ 3\right)$
$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, 4\ 2\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, 4\ 3\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, 4\ 3\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, 4\ 3\right)$
$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, 4\ 2\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, 4\ 2\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, 4\ 3\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, 4\ 3\right)$
$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, 4\ 2\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, 4\ 2\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, 4\ 1\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, 4\ 1\right)$
$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\ 2\ 1\right)$	$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\ 2\ 2\right)$
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II. Signed sets and sijections

A short introduction to signed sets

A signed set is a pair of disjoint finite sets: $\underline{S} = (S^+, S^-)$ with $S^+ \cap S^- = \varnothing$.

- The size of a signed set $\underline{S}$ is $|\underline{S}| = |S^+| - |S^-|$.
- The opposite signed set of $\underline{S}$ is $-\underline{S} = (S^-, S^+)$.
- The Cartesian product of signed sets $\underline{S}$ and $\underline{T}$ is

$$\underline{S} \times \underline{T} = (S^+ \times T^+ \cup S^- \times T^-, S^+ \times T^- \cup S^- \times T^+).$$

- The disjoint union of signed sets $\underline{S}$ and $\underline{T}$ is

$$\underline{S} \sqcup \underline{T} = (\underline{S} \times (\{0\}, \varnothing)) \cup (\underline{T} \times (\{1\}, \varnothing)).$$

- The disjoint union of a family of signed sets $\underline{S}_t$ indexed with a signed set $\underline{T}$ is

$$\bigsqcup_{t \in \underline{T}} \underline{S}_t = \bigcup_{t \in \underline{T}} (\underline{S}_t \times \underline{\{t\}}).$$

Our approach

- We translate some of my non-bijective proofs into combinatorics!

- Note that $|\underline{S} \sqcup \underline{T}| = |\underline{S}| + |\underline{T}|$, $|-\underline{S}| = -|\underline{S}|$, and $|\underline{S} \times \underline{T}| = |\underline{S}| \cdot |\underline{T}|$, and so we can deal with all arithmetic operations accept for division. (The latter explains the "redundant" factors in our bijections.)

- In the original proofs, signs are unavoidable and this makes it necessary to work with signed sets.

- Is there a non-bijective proof that avoids signs? Is there a bijective proof that avoids signed sets (and can this proof be translated into a computation)?

Crucial example: Signed intervals

For $a, b \in \mathbb{Z}$, we set

$$\underline{[a,b]} = \begin{cases} ([a,b], \varnothing) & \text{if } a \le b \\ (\varnothing, [b+1, a-1]) & \text{if } a > b \end{cases},$$

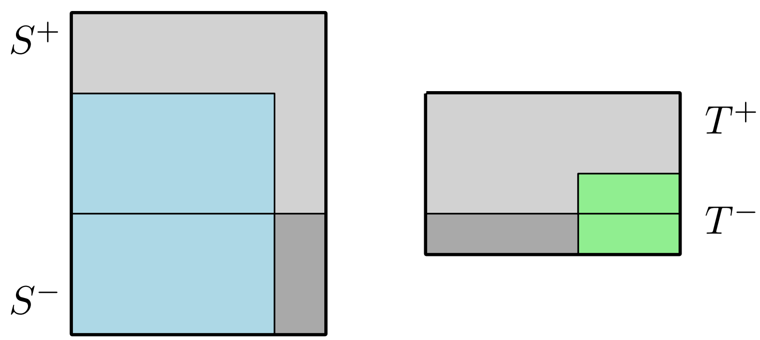
where $[a,b]$ stands for an interval in $\mathbb{Z}$ in the usual sense.

The signed sets in our constructions are typically signed boxes (= Cartesian products of signed intervals) and disjoint unions of signed boxes.

Sijections

The role of bijections for signed sets is played by "signed bijections", which we call sijections.

A sijection φ from $\underline{S}$ to $\underline{T}$, $\varphi\colon \underline{S} \Rightarrow \underline{T}$, is an involution on the set $(S^+ \cup S^-) \sqcup (T^+ \cup T^-)$ with $\varphi(S^+ \sqcup T^-) = S^- \sqcup T^+$.



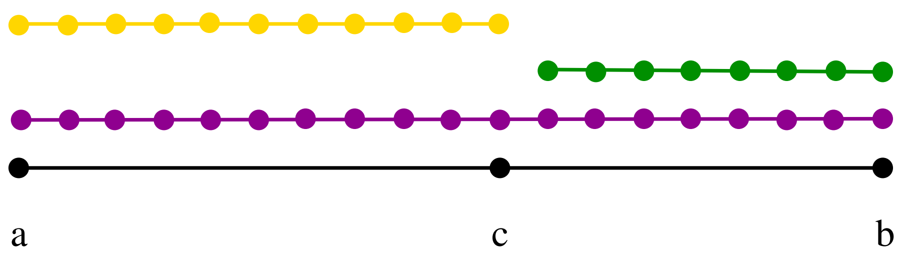
This implies: $|\underline{S}| = |S^+| - |S^-| = |T^+| - |T^-| = |\underline{T}|$

The fundamental sijection

Given $a, b, c \in \mathbb{Z}$, construct a sijection

$$\alpha = \alpha_{a,b,c} \colon \underline{[a, c]} \Rightarrow \underline{[a, b]} \sqcup \underline{[b + 1, c]}.$$

Construction: For $a \le b \le c$ and $c < b < a$, there is nothing to prove. For, say, $a \le c < b$, we have that $\underline{[b + 1, c]} = -\underline{[c + 1, b]}$ is "contained" in $\underline{[a, b]}$, but due to its opposite sign this subset "cancels" and what remains is $\underline{[a, c]}$.



The cases $b < a \le c$, $b \le c < a$, and $c < a \le b$ are analogous.

Cartesian product and disjoint union of sijections

• $\underline{S}_1 \times \cdots \times \underline{S}_k \Rightarrow \underline{T}_1 \times \cdots \times \underline{T}_k$: Suppose we have sijections $\varphi_i: \underline{S}_i \Rightarrow \underline{T}_i$, $i = 1, \ldots, k$. Then define $\varphi = \varphi_1 \times \cdots \times \varphi_k$ by

$$\varphi(s_1, \ldots, s_k) = \begin{cases} (\varphi_1(s_1), \ldots, \varphi_k(s_k)) & \text{if } \varphi_i(s_i) \in \underline{T}_i \text{ for } i = 1, \ldots, k \\ (s_1, \ldots, s_{j-1}, \varphi_j(s_j), s_{j+1}, \ldots, s_k) & \text{if } \varphi_j(s_j) \in \underline{S}_j, \varphi_i(s_i) \in \underline{T}_i \text{ for } i < j \end{cases}$$

if $(s_1, \ldots, s_k) \in \underline{S}_1 \times \cdots \times \underline{S}_k$ and

$$\varphi(t_1, \ldots, t_k) = \begin{cases} (\varphi_1(t_1), \ldots, \varphi_k(t_k)) & \text{if } \varphi_i(t_i) \in \underline{S}_i \text{ for } i = 1, \ldots, k \\ (t_1, \ldots, t_{j-1}, \varphi_j(t_j), t_{j+1}, \ldots, t_k) & \text{if } \varphi_j(t_j) \in \underline{T}_j, \varphi_i(t_i) \in \underline{S}_i \text{ for } i < j \end{cases}$$

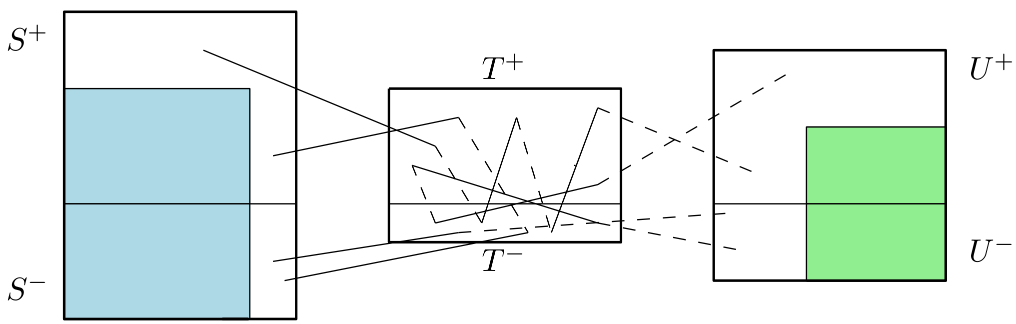
if $(t_1, \ldots, t_k) \in \underline{T}_1 \times \cdots \times \underline{T}_k$.

• $\bigsqcup_{t \in \underline{T}} \underline{S}_t \Rightarrow \bigsqcup_{t \in \underline{\widetilde{T}}} \underline{S}_t$: Suppose we have signed sets $\underline{T}, \underline{\widetilde{T}}$ and a sijection $\psi: \underline{T} \Rightarrow \underline{\widetilde{T}}$. Furthermore, suppose that for every $t \in \underline{T} \sqcup \underline{\widetilde{T}}$, we have a signed set $\underline{S}_t$ and a sijection $\varphi_t: \underline{S}_t \Rightarrow \underline{S}_{\psi(t)}$ satisfying $\varphi_{\psi(t)} = \varphi_t^{-1}$. Then define $\varphi = \bigsqcup_{t \in \underline{T} \sqcup \underline{\widetilde{T}}} \varphi_t$ by

$$\varphi(s_t, t) = \begin{cases} (\varphi_t(s_t), t) & \text{if } s_t \in \underline{S}_t, \varphi_t(s_t) \in \underline{S}_t \\ (\varphi_t(s_t), \psi(t)) & \text{if } s_t \in \underline{S}_t, \varphi_t(s_t) \in \underline{S}_{\psi(t)} \end{cases}.$$

Composition of sijections

Suppose $\underline{S}, \underline{T}, \underline{U}$ are signed sets and $\varphi : \underline{S} \Rightarrow \underline{T}$, $\psi : \underline{T} \Rightarrow \underline{U}$, then we can construct a sijection $\psi \circ \varphi : \underline{S} \to \underline{U}$ by alternating applications of φ (solid lines) and ψ (dashed lines) as sketched next.



The special case $S^- = U^- = \varnothing$ is the **Garsia-Milne involution principle.**

III. Some details on our constructions

ASMs $\longrightarrow$ Monotone Triangles

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 2 & & 5 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

- A monotone triangle is a triangular array that increases weakly in $\nearrow$-direction and in $\searrow$-direction, and strictly along rows. The set of monotone triangles with bottom row $k_1, \ldots, k_n$ is denoted by $\underline{MT}(k_1, \ldots, k_n)$.

- If we drop the condition that rows are strictly increasing, then we obtain the well-known Gelfand-Tsetlin patterns.

Gelfand-Tsetlin patterns with arbitrary bottom row in $\mathbb{Z}^n$

A Gelfand-Tsetlin pattern with bottom row $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ is a triangular array of integers

$$\begin{array}{ccccccccccccc} & & & & & & a_{1,1} & & & & & & \\ & & & & & a_{2,1} & & a_{2,2} & & & & & \\ & & & & a_{3,1} & & a_{3,2} & & a_{3,3} & & & & \\ & & & \ldots & & \ldots & & a_{i,j} & & \ldots & & & \\ & & \ldots & & \ldots & & a_{i+1,j} & & a_{i+1,j+1} & & \ldots & & \\ & a_{n,1} = k_1 & & a_{n,2} = k_2 & & \ldots & & \ldots & & a_{n,n-1} = k_{n-1} & & a_{n,n} = k_n, & \end{array}$$

such that $a_{i,j} \in \underline{[a_{i+1,j}, a_{i+1,j+1}]}$ for $1 \le j \le i < n$.

The sign of such an array is 1 if the number of (i,j) with $a_{i+1,j} > a_{i+1,j+1}$ is even, otherwise -1.

The signed set of Gelfand-Tsetlin patterns with bottom row $\mathbf{k} = (k_1, \ldots, k_n)$ is denoted by $\underline{GT}(\mathbf{k}) = \underline{GT}(k_1, \ldots, k_n)$.

We have

$$|\underline{GT}(k_1, \ldots, k_n)| = \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}.$$

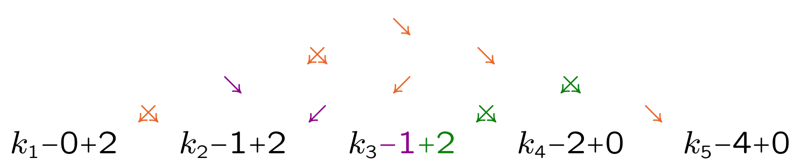
Arrow patterns ...

... are triangular arrays

$$T = \begin{array}{ccccccccc} & & & & t_{1,n} & & & & \\ & & & t_{1,n-1} & & t_{2,n} & & & \\ & & t_{1,n-2} & & t_{2,n-1} & & t_{3,n} & & \\ & \cdot^{\cdot^{\cdot}} & \vdots & \ddots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \ddots & \\ t_{1,2} & & t_{2,3} & & \ldots & & \ldots & & t_{n-1,n} \end{array},$$

with $t_{p,q} \in \{\swarrow, \searrow, \swarrow\!\!\!\!\searrow\}$. The sign of an arrow pattern is 1 if the number of $\swarrow\!\!\!\!\searrow$'s is even and -1 otherwise, and the signed set of arrow patterns of order n is denoted by $\underline{AP}_n$.

The role of an arrow pattern of order n is that it induces a deformation of $(k_1, \ldots, k_n)$ as indicated on the following example:



We let $d(\mathbf{k}, T)$ denote this deformation for $\mathbf{k} = (k_1, \ldots, k_n)$ and $T \in \underline{AP}_n$.

Shifted Gelfand-Tsetlin patterns

For $\mathbf{k} = (k_1, \ldots, k_n)$, a shifted Gelfand-Tsetlin pattern is the disjoint union of deformed Gelfand-Tsetlin patterns over arrow patterns of order n:

$$\underline{SGT}(\mathbf{k}) = \bigsqcup_{T \in \underline{AP}_n} \underline{GT}(d(\mathbf{k}, T)).$$

Given $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $x \in \mathbb{Z}$, we have constructed a sijection

$$\Gamma = \Gamma_{\mathbf{k},x} \colon \underline{MT}(\mathbf{k}) \Rightarrow \underline{SGT}(\mathbf{k}).$$

Remarks.

- $\underline{SGT}(\mathbf{k})$ and $\underline{MT}(\mathbf{k})$ can be extended naturally to all integers sequences $k_1, \ldots, k_n$ (not necessarily increasing). Then the negative part of $\underline{MT}(\mathbf{k})$ is non-empty, and sijections and compositions thereof cannot be avoided (including the involution principle).

- The merit of this sijection is that from now on we can replace $\underline{MT}$ with $\underline{SGT}$.

Example $\mathbf{k} = (1,2,3)$ and $x = 0$

$$\begin{array}{c}1\\1\;2\\1\;2\;3\end{array} \leftrightarrow \left(\begin{array}{c}1\\1\;1\\1\;1\;1\end{array}, \begin{array}{c}\searrow\\\searrow\;\searrow\end{array}\right) \qquad \begin{array}{c}2\\1\;2\\1\;2\;3\end{array} \leftrightarrow \left(\begin{array}{c}2\\1\;2\\1\;2\;2\end{array}, \begin{array}{c}\searrow\\\searrow\;\swarrow\end{array}\right) \qquad \begin{array}{c}1\\1\;3\\1\;2\;3\end{array} \leftrightarrow \left(\begin{array}{c}1\\1\;2\\1\;2\;2\end{array}, \begin{array}{c}\searrow\\\searrow\;\swarrow\end{array}\right)$$

$$\begin{array}{c}2\\1\;3\\1\;2\;3\end{array} \leftrightarrow \left(\begin{array}{c}2\\2\;3\\2\;2\;3\end{array}, \begin{array}{c}\swarrow\\\searrow\;\swarrow\end{array}\right) \qquad \begin{array}{c}3\\1\;3\\1\;2\;3\end{array} \leftrightarrow \left(\begin{array}{c}3\\2\;3\\2\;2\;3\end{array}, \begin{array}{c}\swarrow\\\searrow\;\swarrow\end{array}\right) \qquad \begin{array}{c}2\\2\;3\\1\;2\;3\end{array} \leftrightarrow \left(\begin{array}{c}2\\2\;2\\3\;1\;2\end{array}, \begin{array}{c}\swarrow\\\searrow\!\!\!\!\!\swarrow\;\searrow\end{array}\right)$$

$$\begin{array}{c}3\\2\;3\\1\;2\;3\end{array} \leftrightarrow \left(\begin{array}{c}3\\3\;3\\3\;3\;3\end{array}, \begin{array}{c}\swarrow\\\swarrow\;\swarrow\end{array}\right) \qquad \left(\begin{array}{c}2\\2\;2\\2\;2\;3\end{array}, \begin{array}{c}\swarrow\\\searrow\;\swarrow\end{array}\right) \leftrightarrow \left(\begin{array}{c}2\\2\;2\\2\;2\;2\end{array}, \begin{array}{c}\searrow\!\!\!\!\!\swarrow\\\searrow\;\swarrow\end{array}\right)$$

$$\left(\begin{array}{c}2\\2\;2\\2\;3\;1\end{array}, \begin{array}{c}\searrow\\\swarrow\;\searrow\!\!\!\!\!\swarrow\end{array}\right) \leftrightarrow \left(\begin{array}{c}2\\2\;2\\2\;2\;2\end{array}, \begin{array}{c}\swarrow\\\searrow\;\searrow\!\!\!\!\!\swarrow\end{array}\right) \qquad \left(\begin{array}{c}2\\2\;2\\1\;2\;2\end{array}, \begin{array}{c}\searrow\\\searrow\;\swarrow\end{array}\right) \leftrightarrow \left(\begin{array}{c}2\\2\;2\\2\;2\;2\end{array}, \begin{array}{c}\searrow\\\searrow\!\!\!\!\!\swarrow\;\swarrow\end{array}\right)$$

Rotation of monotone triangles

Given $(k_1, \ldots, k_n)$**, we have constructed a sijection**

$$\underline{MT}(k_1, \ldots, k_n) \Longrightarrow (\text{-}1)^{n-1}\, \underline{MT}(k_2, \ldots, k_n, k_1 - n).$$

Using $\Gamma : \underline{MT} \Longrightarrow \underline{SGT}$, it suffices to construct a sijection

$$\underline{SGT}(k_1, \ldots, k_n) \Longrightarrow (\text{-}1)^{n-1}\, \underline{SGT}(k_2, \ldots, k_n, k_1 - n).$$

After many more steps with obtain Bijections 1 & 2.

Thank you for your attention!