Bijective proofs of (skew) Schur polynomial factorizations

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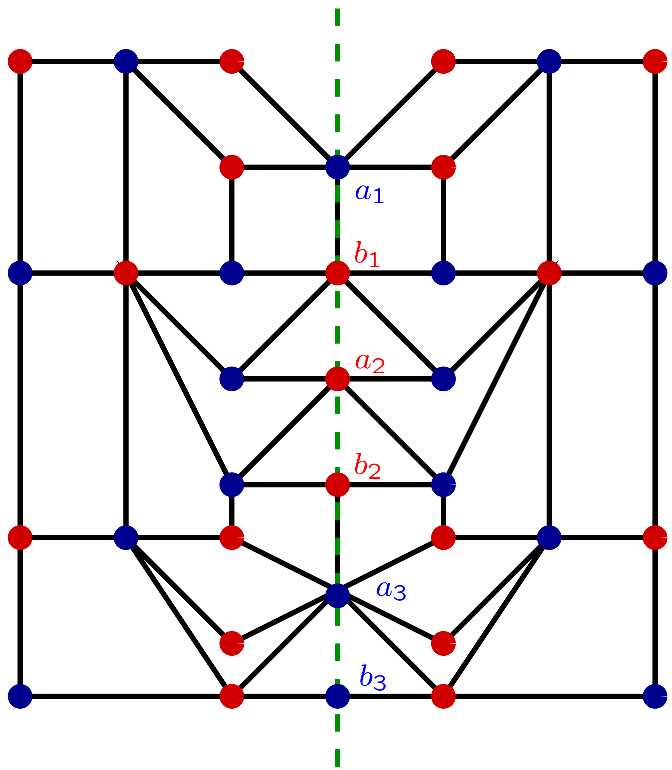
I. Ciucu's factorization theorem for perfect matching generating functions

Ciucu's factorization theorem I

Suppose G is an edge-weighted graph and we want to compute the perfect matching generating function $\mathrm{M}(G)$.

Assume G has the following properties:

- Planar, bipartite and connected.
- G symmetric w.r.t. a vertical symmetry axis ℓ, including the edge-weights!
- Removing ℓ disconnects G.
- W.l.o.g. there are an even number $2\,\mathrm{N}(G)$ of vertices on the symmetry axis. (If not, G has no perfect matching.)



Ciucu's factorization theorem II

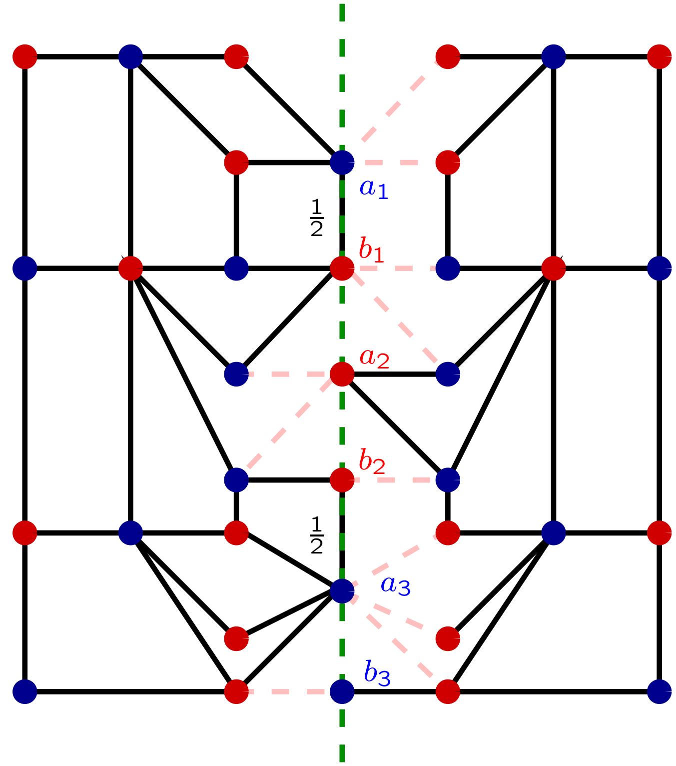
For a vertex v on ℓ, we define cutting operations:

• "Cutting left/right of v" means that we delete all incident edges left/right of ℓ.

Now we define two subgraphs of G as follows:

• **We perform the cutting operation right of all a_i's and b_i's, and left of a_i's and b_i's.**

• Reduce the weights of the edges on ℓ by half and leave all other weights unchanged. We obtain two disconnected graphs, and denote by G^+ the left graph and by G^- the right graph.

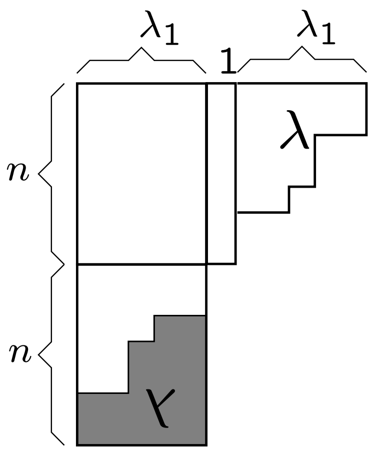


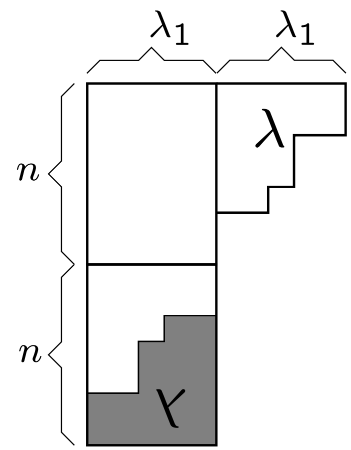
Theorem (Ciucu 1997). We have the following factorization for the perfect matching generating function $\mathrm{M}(G)$.

$$\mathrm{M}(G) = 2^{\mathrm{N}(G)}\,\mathrm{M}(G^+)\,\mathrm{M}(G^-).$$

II. Two Schur polynomial factorizations

Two families of partitions





$(\lambda_1+1,\ldots,\lambda_n+1,-\lambda_n,\ldots,-\lambda_1)+\lambda_1$ $\qquad$ $(\lambda_1,\ldots,\lambda_n,-\lambda_n,\ldots,-\lambda_1)+\lambda_1$

$(11,9,2,0)$ $\qquad$ $(10,8,2,0)$ $\qquad$ for $(\lambda_1,\lambda_2)=(5,3)$

$$
\begin{aligned}
s_{(11,9,2,0)}(x_1,\bar{x}_1,x_2,\bar{x}_2) = {} & \frac{1}{x_1^{11}x_2^{11}} \\
\times \big(& x_1^{10}x_2^{8} + x_1^{10}x_2^{6} + x_1^{10}x_2^{4} + x_1^{10}x_2^{2} + x_1^{9}x_2^{9} + 2x_1^{9}x_2^{7} + 2x_1^{9}x_2^{5} + 2x_1^{9}x_2^{3} + x_1^{9}x_2 + x_1^{8}x_2^{10} + 3x_1^{8}x_2^{8} + 4x_1^{8}x_2^{6} + 4x_1^{8}x_2^{4} + 3x_1^{8}x_2^{2} \\
& + x_1^{8} + 2x_1^{7}x_2^{9} + 4x_1^{7}x_2^{7} + 5x_1^{7}x_2^{5} + 4x_1^{7}x_2^{3} + 2x_1^{7}x_2 + x_1^{6}x_2^{10} + 4x_1^{6}x_2^{8} + 6x_1^{6}x_2^{6} + 6x_1^{6}x_2^{4} + 4x_1^{6}x_2^{2} + x_1^{6} + 2x_1^{5}x_2^{9} + 5x_1^{5}x_2^{7} \\
& + 6x_1^{5}x_2^{5} + 5x_1^{5}x_2^{3} + 2x_1^{5}x_2 + x_1^{4}x_2^{10} + 4x_1^{4}x_2^{8} + 6x_1^{4}x_2^{6} + 6x_1^{4}x_2^{4} + 4x_1^{4}x_2^{2} + x_1^{4} + 2x_1^{3}x_2^{9} + 4x_1^{3}x_2^{7} + 5x_1^{3}x_2^{5} + 4x_1^{3}x_2^{3} + 2x_1^{3}x_2 \\
& + x_1^{2}x_2^{10} + 3x_1^{2}x_2^{8} + 4x_1^{2}x_2^{6} + 4x_1^{2}x_2^{4} + 3x_1^{2}x_2^{2} + x_1^{2} + x_1x_2^{9} + 2x_1x_2^{7} + 2x_1x_2^{5} + 2x_1x_2^{3} + x_1x_2 + x_2^{8} + x_2^{6} + x_2^{4} + x_2^{2}\big) \\
\times \big(& x_1^{12}x_2^{10} + x_1^{12}x_2^{2} + x_1^{11}x_2^{11} + x_1^{11}x_2^{9} + x_1^{11}x_2^{3} + x_1^{11}x_2 + x_1^{10}x_2^{12} + x_1^{10}x_2^{10} + x_1^{10}x_2^{8} + x_1^{10}x_2^{4} + x_1^{10}x_2^{2} + x_1^{10} + x_1^{9}x_2^{11} + x_1^{9}x_2^{9} \\
& + x_1^{9}x_2^{7} + x_1^{9}x_2^{5} + x_1^{9}x_2^{3} + x_1^{9}x_2 + x_1^{8}x_2^{10} + x_1^{8}x_2^{8} + 2x_1^{8}x_2^{6} + x_1^{8}x_2^{4} + x_1^{8}x_2^{2} + x_1^{7}x_2^{9} + 2x_1^{7}x_2^{7} + 2x_1^{7}x_2^{5} + x_1^{7}x_2^{3} + 2x_1^{6}x_2^{8} + 2x_1^{6}x_2^{6} \\
& + 2x_1^{6}x_2^{4} + x_1^{5}x_2^{9} + 2x_1^{5}x_2^{7} + 2x_1^{5}x_2^{5} + x_1^{5}x_2^{3} + x_1^{4}x_2^{10} + x_1^{4}x_2^{8} + 2x_1^{4}x_2^{6} + x_1^{4}x_2^{4} + x_1^{4}x_2^{2} + x_1^{3}x_2^{11} + x_1^{3}x_2^{9} + x_1^{3}x_2^{7} + x_1^{3}x_2^{5} + x_1^{3}x_2^{3} + x_1^{3}x_2 \\
& + x_1^{2}x_2^{12} + x_1^{2}x_2^{10} + x_1^{2}x_2^{8} + x_1^{2}x_2^{4} + x_1^{2}x_2^{2} + x_1^{2} + x_1x_2^{11} + x_1x_2^{9} + x_1x_2^{3} + x_1x_2 + x_2^{10} + x_2^{2}\big)
\end{aligned}
$$

(Notation: $\bar{x} = x^{-1}$)

$$s_{(10,8,2,0)}(x_1, \bar{x}_1, x_2, \bar{x}_2) = \frac{1}{x_1^{10} x_2^{10}}$$

$$\times \big(x_2^8x_1^{10} - x_2^7x_1^{10} + x_2^6x_1^{10} - x_2^5x_1^{10} + x_2^4x_1^{10} - x_2^3x_1^{10} + x_2^2x_1^{10} + x_2^9x_1^9 - 2x_2^8x_1^9 + 3x_2^7x_1^9 - 3x_2^6x_1^9 + 3x_2^5x_1^9 - 3x_2^4x_1^9 + 3x_2^3x_1^9$$
$$-2x_2^2x_1^9 + x_2x_1^9 + x_2^{10}x_1^8 - 2x_2^9x_1^8 + 4x_2^8x_1^8 - 5x_2^7x_1^8 + 6x_2^6x_1^8 - 6x_2^5x_1^8 + 6x_2^4x_1^8 - 5x_2^3x_1^8 + 4x_2^2x_1^8 - 2x_2x_1^8 + x_1^8 - x_2^{10}x_1^7$$
$$+3x_2^9x_1^7 - 5x_2^8x_1^7 + 7x_2^7x_1^7 - 8x_2^6x_1^7 + 9x_2^5x_1^7 - 8x_2^4x_1^7 + 7x_2^3x_1^7 - 5x_2^2x_1^7 + 3x_2x_1^7 - x_1^7 + x_2^{10}x_1^6 - 3x_2^9x_1^6 + 6x_2^8x_1^6 - 8x_2^7x_1^6$$
$$+10x_2^6x_1^6 - 10x_2^5x_1^6 + 10x_2^4x_1^6 - 8x_2^3x_1^6 + 6x_2^2x_1^6 - 3x_2x_1^6 + x_1^6 - x_2^{10}x_1^5 + 3x_2^9x_1^5 - 6x_2^8x_1^5 + 9x_2^7x_1^5 - 10x_2^6x_1^5 + 11x_2^5x_1^5$$
$$-10x_2^4x_1^5 + 9x_2^3x_1^5 - 6x_2^2x_1^5 + 3x_2x_1^5 - x_1^5 + x_2^{10}x_1^4 - 3x_2^9x_1^4 + 6x_2^8x_1^4 - 8x_2^7x_1^4 + 10x_2^6x_1^4 - 10x_2^5x_1^4 + 10x_2^4x_1^4 - 8x_2^3x_1^4$$
$$+6x_2^2x_1^4 - 3x_2x_1^4 + x_1^4 - x_2^{10}x_1^3 + 3x_2^9x_1^3 - 5x_2^8x_1^3 + 7x_2^7x_1^3 - 8x_2^6x_1^3 + 9x_2^5x_1^3 - 8x_2^4x_1^3 + 7x_2^3x_1^3 - 5x_2^2x_1^3 + 3x_2x_1^3 - x_1^3$$
$$+x_2^{10}x_1^2 - 2x_2^9x_1^2 + 4x_2^8x_1^2 - 5x_2^7x_1^2 + 6x_2^6x_1^2 - 6x_2^5x_1^2 + 6x_2^4x_1^2 - 5x_2^3x_1^2 + 4x_2^2x_1^2 - 2x_2x_1^2 + x_1^2 + x_2^9x_1 - 2x_2^8x_1 + 3x_2^7x_1$$
$$-3x_2^6x_1 + 3x_2^5x_1 - 3x_2^4x_1 + 3x_2^3x_1 - 2x_2^2x_1 + x_2x_1 + x_2^8 - x_2^7 + x_2^6 - x_2^5 + x_2^4 - x_2^3 + x_2^2 \big)$$

$$\times \big(x_2^8x_1^{10} + x_2^7x_1^{10} + x_2^6x_1^{10} + x_2^5x_1^{10} + x_2^4x_1^{10} + x_2^3x_1^{10} + x_2^2x_1^{10} + x_2^9x_1^9 + 2x_2^8x_1^9 + 3x_2^7x_1^9 + 3x_2^6x_1^9 + 3x_2^5x_1^9 + 3x_2^4x_1^9 + 3x_2^3x_1^9$$
$$+2x_2^2x_1^9 + x_2x_1^9 + x_2^{10}x_1^8 + 2x_2^9x_1^8 + 4x_2^8x_1^8 + 5x_2^7x_1^8 + 6x_2^6x_1^8 + 6x_2^5x_1^8 + 6x_2^4x_1^8 + 5x_2^3x_1^8 + 4x_2^2x_1^8 + 2x_2x_1^8 + x_1^8 + x_2^{10}x_1^7$$
$$+3x_2^9x_1^7 + 5x_2^8x_1^7 + 7x_2^7x_1^7 + 8x_2^6x_1^7 + 9x_2^5x_1^7 + 8x_2^4x_1^7 + 7x_2^3x_1^7 + 5x_2^2x_1^7 + 3x_2x_1^7 + x_1^7 + x_2^{10}x_1^6 + 3x_2^9x_1^6 + 6x_2^8x_1^6 + 8x_2^7x_1^6$$
$$+10x_2^6x_1^6 + 10x_2^5x_1^6 + 10x_2^4x_1^6 + 8x_2^3x_1^6 + 6x_2^2x_1^6 + 3x_2x_1^6 + x_1^6 + x_2^{10}x_1^5 + 3x_2^9x_1^5 + 6x_2^8x_1^5 + 9x_2^7x_1^5 + 10x_2^6x_1^5 + 11x_2^5x_1^5$$
$$+10x_2^4x_1^5 + 9x_2^3x_1^5 + 6x_2^2x_1^5 + 3x_2x_1^5 + x_1^5 + x_2^{10}x_1^4 + 3x_2^9x_1^4 + 6x_2^8x_1^4 + 8x_2^7x_1^4 + 10x_2^6x_1^4 + 10x_2^5x_1^4 + 10x_2^4x_1^4 + 8x_2^3x_1^4$$
$$+6x_2^2x_1^4 + 3x_2x_1^4 + x_1^4 + x_2^{10}x_1^3 + 3x_2^9x_1^3 + 5x_2^8x_1^3 + 7x_2^7x_1^3 + 8x_2^6x_1^3 + 9x_2^5x_1^3 + 8x_2^4x_1^3 + 7x_2^3x_1^3 + 5x_2^2x_1^3 + 3x_2x_1^3 + x_1^3$$
$$+x_2^{10}x_1^2 + 2x_2^9x_1^2 + 4x_2^8x_1^2 + 5x_2^7x_1^2 + 6x_2^6x_1^2 + 6x_2^5x_1^2 + 6x_2^4x_1^2 + 5x_2^3x_1^2 + 4x_2^2x_1^2 + 2x_2x_1^2 + x_1^2 + x_2^9x_1 + 2x_2^8x_1 + 3x_2^7x_1$$
$$+3x_2^6x_1 + 3x_2^5x_1 + 3x_2^4x_1 + 3x_2^3x_1 + 2x_2^2x_1 + x_2x_1 + x_2^8 + x_2^7 + x_2^6 + x_2^5 + x_2^4 + x_2^3 + x_2^2 \big)$$

Characters of classical groups

• Symplectic characters ($Sp_{2n}(\mathbb{C})$):

$$sp_\lambda(x_1,\ldots,x_n) = \frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j+1} - \bar{x}_i^{\lambda_j+n-j+1}\right)}{\det_{1\le i,j\le n}\left(x_i^{n-j+1} - \bar{x}_i^{n-j+1}\right)}.$$

• Even orthogonal characters ($O_{2n}(\mathbb{C})$):

$$o_\lambda^{\text{even}}(x_1,\ldots,x_n) = (1+[\lambda_n \ne 0])\frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j} + \bar{x}_i^{\lambda_j+n-j}\right)}{\det_{1\le i,j\le n}\left(x_i^{n-j} + \bar{x}_i^{n-j}\right)}.$$

• Odd orthogonal characters ($SO_{2n+1}(\mathbb{C})$):

$$so_\lambda^{\text{odd}}(x_1,\ldots,x_n) = \frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j+1/2} - \bar{x}_i^{\lambda_j+n-j+1/2}\right)}{\det_{1\le i,j\le n}\left(x_i^{n-j+1/2} - \bar{x}_i^{n-j+1/2}\right)}.$$

• A half-integer is an odd integer divided by 2, and a half-integer partition is a finite weakly decreasing sequence of positive half-integers. In the case of the even orthogonal group, the character formula has a representation theoretic meaning as characters of spin covering groups when λ is a half-integer partition.

• General linear characters ($GL_n(\mathbb{C})$):

$$s_\lambda(x_1,\ldots,x_n) = \frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j}\right)}{\det_{1\le i,j\le n}\left(x_i^{n-j}\right)}$$

The factorization theorem

Theorem (Ayyer, Behrend, 2019). Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition.

- For $\widehat{\lambda} = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_n + 1, -\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1) + \lambda_1$, we have

$$s_{\widehat{\lambda}}(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n) = sp_\lambda(x_1, \ldots, x_n)\, o^{\text{even}}_{\lambda+1}(x_1, \ldots, x_n).$$

- For $\widehat{\lambda} = (\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1) + \lambda_1$, we have

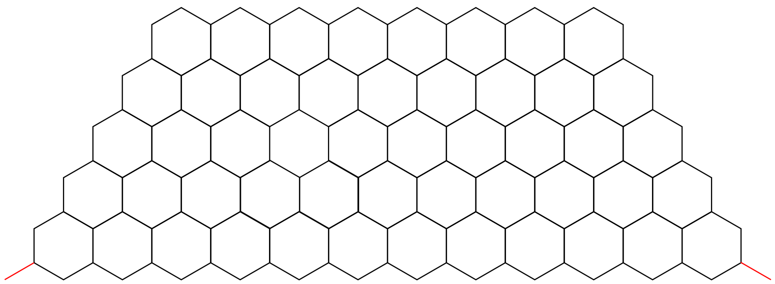
$$s_{\widehat{\lambda}}(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n) = \prod_{i=1}^{n} (x_i^{1/2} + \bar{x}_i^{1/2})^{-1} so^{\text{odd}}_\lambda(x_1, \ldots, x_n)\, o^{\text{even}}_{\lambda+\frac{1}{2}}(x_1, \ldots, x_n).$$

Extends work of Ciucu and Krattenthaler.

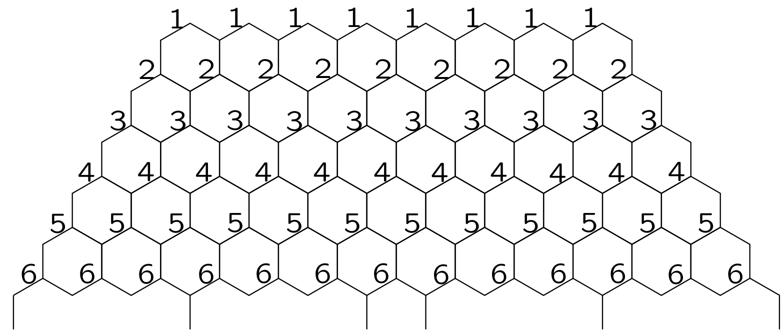
Goal of our work: Bijective proof

III. Graphical models of characters

Trapezoidal honeycomb graph



$T_{6,8}$



$T_{6,8}^{1,4,7,8,11,14}(x_1,\dots,x_6)$

(Edge weight x_i is abbreviated by i.)

Theorem. For a partition $\lambda = (\lambda_1,\dots,\lambda_n)$, we have

$$s_\lambda(x_1,\dots,x_n) = \mathrm{M}(\mathsf{T}_{n,\lambda_1}^{\lambda_n+1,\lambda_{n-1}+2,\dots,\lambda_1+n}(x_1,\dots,x_n)),$$

where $\mathrm{M}(G)$ denotes the perfect matching generating function of the edge weighted graph G.

From a tableau of shape λ to a matching of $T_{n,\lambda_1}^{\lambda_n+1,\lambda_{n-1}+2,\ldots,\lambda_1+n}$

Semistandard tableau of shape $(8, 6, 4, 4, 2, 0)$:

1	1	1	2	2	5	5	6
2	2	3	3	5	6		
3	4	4	4				
5	5	6	6				
6	6						

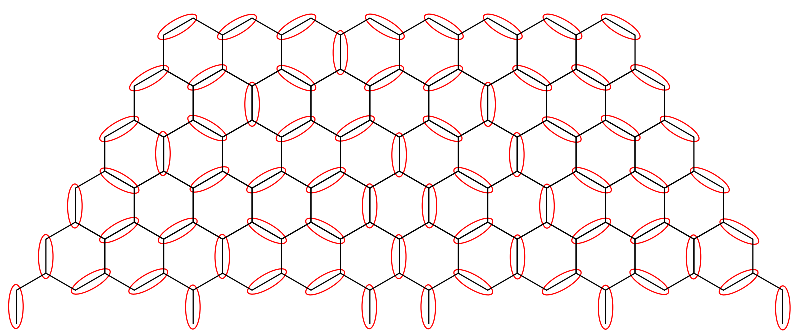
Corresponding Gelfand-Tsetlin pattern:

$$\begin{array}{ccccccccccc}
 & & & & & 3 & & & & & \\
 & & & & 2 & & 5 & & & & \\
 & & & 1 & & 4 & & 5 & & & \\
 & & 0 & & 4 & & 4 & & 5 & & \\
 & 0 & & 2 & & 4 & & 5 & & 7 & \\
0 & & 2 & & 4 & & 4 & & 6 & & 8
\end{array}$$

We add i to the i-th $\nearrow$-diagonal:

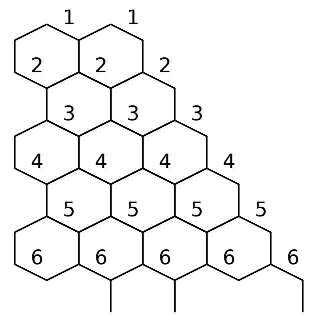
$$\begin{array}{ccccccccccc} & & & & & 4 & & & & & \\ & & & & 3 & & 7 & & & & \\ & & & 2 & & 6 & & 8 & & & \\ & & 1 & & 6 & & 7 & & 9 & & \\ & 1 & & 4 & & 7 & & 9 & & 12 & \\ 1 & & 4 & & 7 & & 8 & & 11 & & 14 \end{array}$$

Corresponding perfect matching of $T^{1,4,7,8,11,14}_{6,8}(x_1,\ldots,x_6)$:



Symplectic characters

The half-trapezoidal honeycomb graph $\text{HT-}_{6,2}^{2,3,5}(x_1,\ldots,x_6)$:



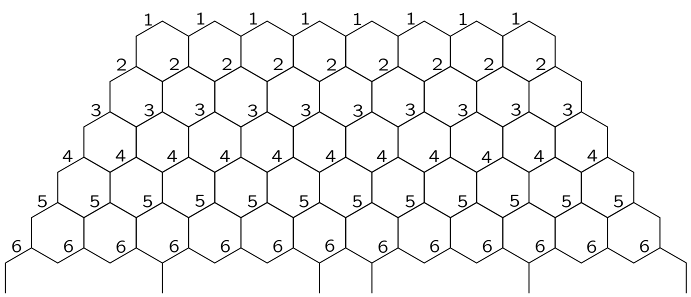
Theorem. For a partition $\lambda = (\lambda_1,\ldots,\lambda_n)$, we have

$$sp_\lambda(x_1,\ldots,x_n) = M(\text{HT-}_{2n,\lambda_1}^{\lambda_n+1,\lambda_{n-1}+2,\ldots,\lambda_1+n}(x_1,\bar{x}_1,\ldots,x_n,\bar{x}_n)).$$

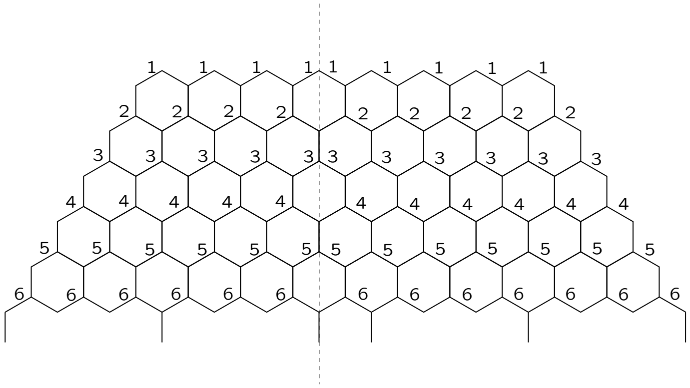
Similar results exist also for even orthogonal and odd orthogonal characters (also for half-integer partitions when needed). The graphs are always approximately **half** of the original trapezoidal graph with possibly some edge weights close to the "cut".

IV. Symmetrizing

Trapezoidal honeycomb graph with different edge weights



$\Rightarrow$



$T_{6,8}^{1,4,7,8,11,14}(x_1,\dots,x_6)$ $\quad\quad$ $\mathrm{ST}_{6,8,4}^{1,4,7,8,11,14}(x_1\dots,x_6)$

Important: Vertical line contains "peak vertex" in top row!

Lemma. Let $1 \le j \le k$. Then the matching generating function

$$\mathrm{M}(\mathrm{T}^{\mathrm{p}}_{2n,k}(x_1,\bar{x}_1,\dots,x_n,\bar{x}_n))$$

is equal to

$$\frac{1}{2^n}\sum_{\sigma\in\mathcal{I}_n}\mathrm{M}(\sigma\,\mathrm{ST}^{\mathrm{p}}_{2n,k,j}(x_1,\bar{x}_1,\dots,x_n,\bar{x}_n)),$$

where $\mathcal{I}_n$ denotes the permutations of the edge weights $\{x_1,\bar{x}_1,x_2,\bar{x}_2,\dots,x_n,\bar{x}_n\}$ that are generated by the transpositions $(x_1,\bar{x}_1),(x_2,\bar{x}_2),\dots,(x_n,\bar{x}_n)$.

“Randomized” bijection

• Suppose A, B are finite sets, then a randomized map from A to B is a randomized algorithm that assigns to each element $a \in A$ an element $b \in B$ with some probability $p_{a,b}$ such that

$$\sum_{b \in B} p_{a,b} = 1$$

for all $a \in A$.

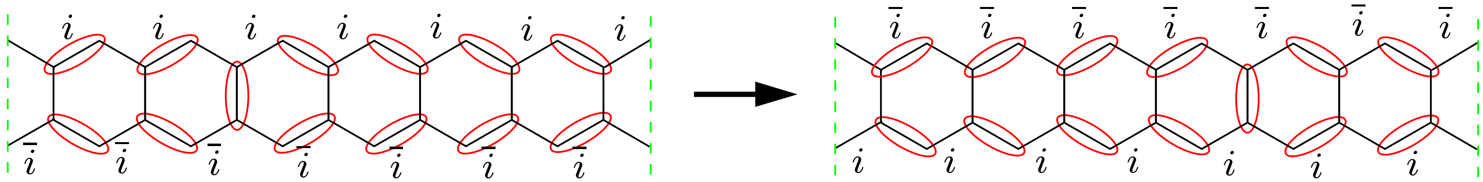
• A randomized map is a randomized bijection, if there exists a randomized map from B to A such that the corresponding randomized algorithm sends $b \in B$ to $a \in A$ with probability $p_{a,b}$ (which implies then also $\sum_{a \in A} p_{a,b} = 1$ for all $b \in B$).

• A randomized bijection can only exist if A and B have the same cardinality, as

$$|A| = \sum_{a \in A} 1 = \sum_{a \in A} \sum_{b \in B} p_{a,b} = \sum_{b \in B} \sum_{a \in A} p_{a,b} = \sum_{b \in B} 1 = |B|.$$

Typical "move"

Reflecting the matching edges vertically and the edge weights horizontally leaves the total weight of $\bar{x}_i^2$ unchanged...

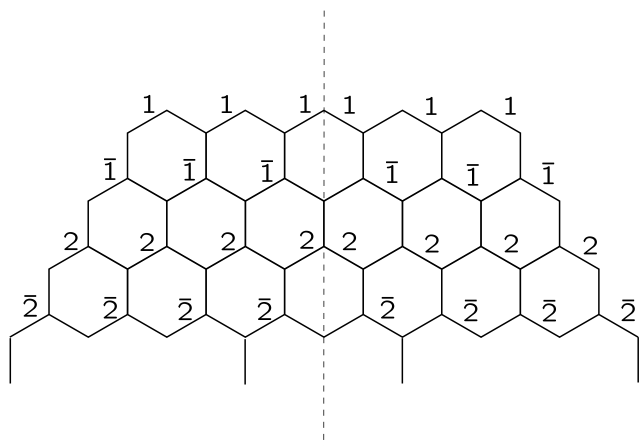


... after all it is just a rotation by 180° of the whole picture.

"Randomized": For each i, we randomly choose either the left or the right side of the fixed vertical line (if possible) to which we apply this move.

First identity: $s_{\widehat{\lambda}}(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n) = sp_\lambda(x_1, \ldots, x_n)\, o^{\text{even}}_{\lambda+1}(x_1, \ldots, x_n)$

For $n = 2$ and $\lambda = (2, 0)$ we obtain the graph $\mathrm{ST}^{1,4,6,9}_{4,5,3}(x_1, \bar{x}_1, x_2, \bar{x}_2)$:



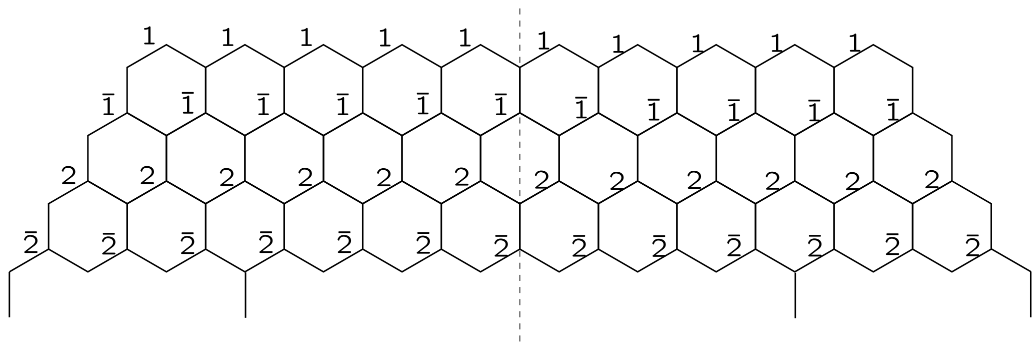
The weighted graph is symmetric with respect to the vertical symmetry axis!

V. Doubling for the second case

Second identity:

$$s_{\widehat{\lambda}}(x_1,\bar{x}_1,\dots,x_n,\bar{x}_n)=\prod_{i=1}^n(x_i^{1/2}+\bar{x}_i^{1/2})^{-1}so_\lambda^{\text{odd}}(x_1,\dots,x_n)\,o_{\lambda+\frac{1}{2}}^{\text{even}}(x_1,\dots,x_n)$$

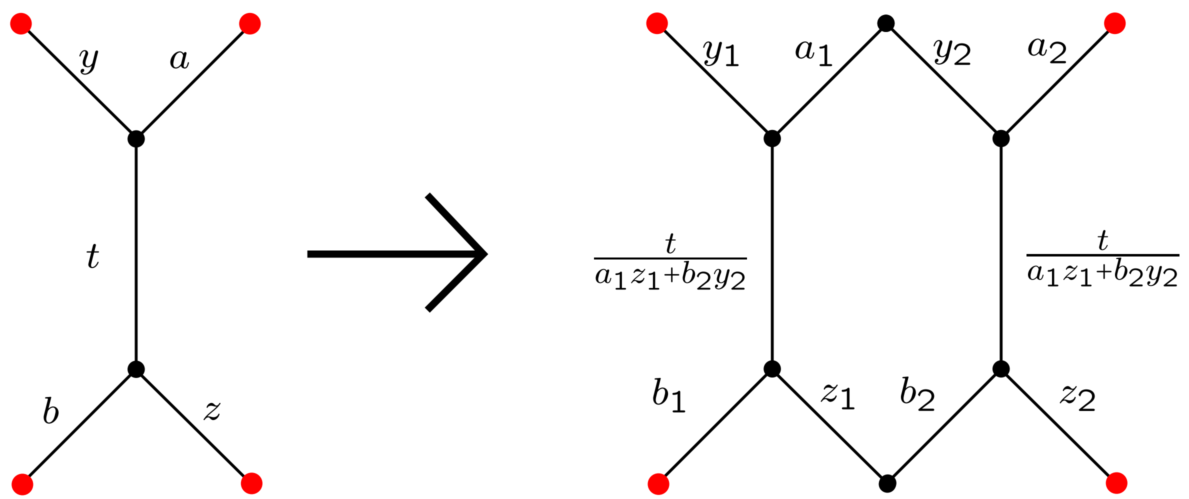
Example $\lambda=(5,3)$, $\widehat{\lambda}=(10,8,2,0)$:



Wrong type of symmetry axis: In this case, the lemma cannot be applied to achieve symmetric edge weights!

A graph operation

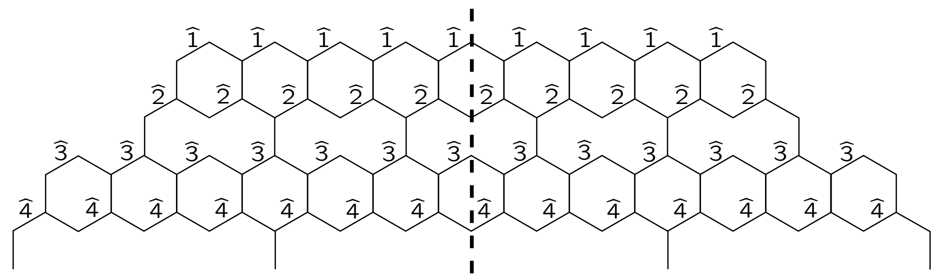
Lemma. Let $a_1 \cdot a_2 = a$, $b_1 \cdot b_2 = b$, $y_1 \cdot y_2 = y$ and $z_1 \cdot z_2 = z$. The following replacement rule in a weighted graph leaves the matching generating function invariant, where in the replacement the degree of the black vertices does not change and the red vertices are the connecting points.



Application to $\mathrm{T}^{1,3,6,8}_{4,4}(x_1,x_2,x_3,x_4)$

By $\mathrm{DT}^{\mathbf{p}}_{n,k}(x_1,\dots,x_n)$ we denote the graph that is obtained from $\mathrm{T}^{\mathbf{p}}_{n,k}(x_1,\dots,x_n)$ by applying the rule to the vertical edges in all odd rows $1,3,5,\dots$ and choose the weights cleverly.

The graph $\mathrm{DT}^{1,3,6,8}_{4,4}(x_1,x_2,x_3,x_4)$:

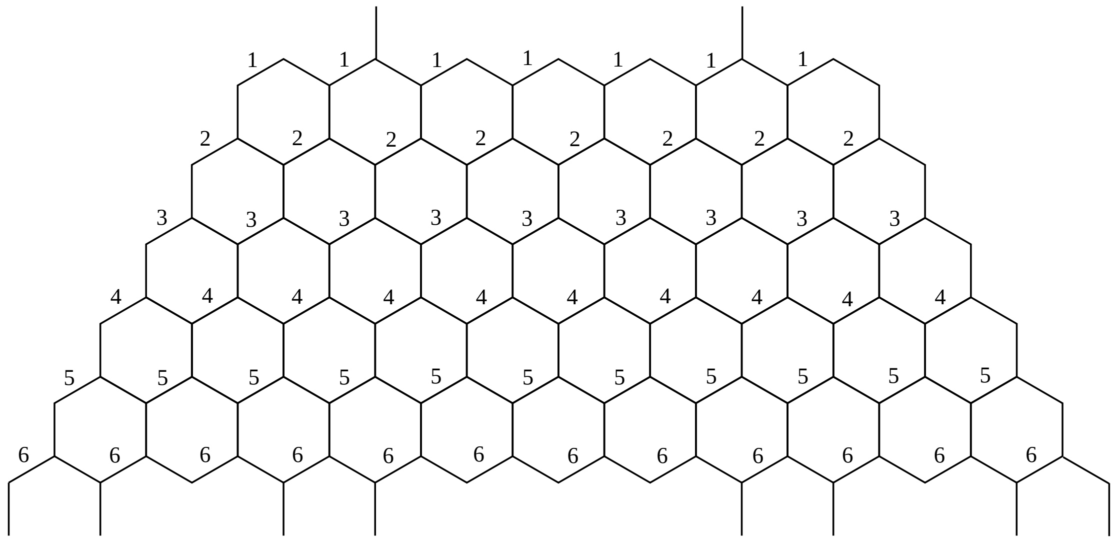


The vertical edges in row 1 carry the weight $\left(x_1^{1/2}+x_2^{1/2}\right)^{-1}$, while the vertical edges in row 3 carry the weight $\left(x_3^{1/2}+x_4^{1/2}\right)^{-1}$, and $\widehat{i}$ stands for $x_i^{1/2}$.

Final steps:

- Symmetrize weights.
- Apply Ciucu's factorization theorem.
- Reverse "doubling" for the two halves.

The skew case



- Edges sticking out at the top encode the inner shape.

- The "power" of bijective proof: Everything goes through without any changes (except for the edges at the top of course).

Thank you !