Alternating sign trapezoids and cyclically symmetric lozenge tilings with a central hole

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Partly joint work with Arvind Ayyer and Roger Behrend

I have to apologize...

No (planned) random process in this talk!

But there will be lozenge tilings...

Outline

I. ASMs, DPPs and TSSCPPs

Alternating sign matrices = ASMs

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Square matrix with entries in $\{0, \pm 1\}$ such that in each row and each column

- the non–zero entries appear with alternating signs, and
- the sum of entries is 1.

How many?

n	1	2	3	4
	(1)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$3! + \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	42

The number of $n \times n$ ASMs

Theorem (Zeilberger, Kuperberg 1996). The number of $n \times n$ alternating sign matrices is

$$\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Conjectured by Mills, Robbins and Rumsey in the 1980s.

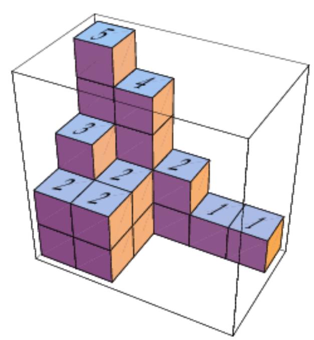
Plane partitions

A plane partition in an $a \times b \times c$ box is a subset

$$PP \subseteq \{1,2,\ldots,a\} \times \{1,2,\ldots,b\} \times \{1,2,\ldots,c\}$$

with

$$(i,j,k) \in PP \Rightarrow (i',j',k') \in PP \quad \forall (i',j',k') \leq (i,j,k).$$



$a = 5, b = 3, c = 5$

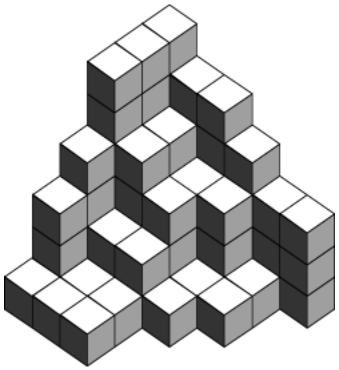
Cyclically symmetric plane partitions = CSPPs

An $n \times n \times n$ plane partition PP is cyclically symmetric if

$$(i, j, k) \in PP \Rightarrow (j, k, i) \in PP.$$

In 1979, George Andrews proved that the number of $n \times n \times n$ cyclically symmetric plane partitons is

$$\prod_{i=0}^{n-1} \frac{(3i+2)(3i)!}{(n+i)!}.$$



A determinant in Andrews' proof

In his proof, Andrews shows that the number of CSPPs of order n is given by the following determinant

$$\det_{0\le i,j\le n-1}\left(\delta_{i,j}+\binom{i+j}{i}\right)$$

and then proves that

$$\det_{0\le i,j\le n-1}\left(\delta_{i,j}+\binom{i+j}{i}\right)=\prod_{i=0}^{n-1}\frac{(3i+2)(3i)!}{(n+i)!}.$$

Then he also considered the following more general determinant:

$$\det_{0\le i,j\le n-1}\left(\delta_{i,j}+\binom{k+i+j}{i}\right):=D_n(k)$$

$D_n(k)$ for small values of n

$$2$$

$$k+5$$

$$(k+4)(k+5)$$

$$\frac{1}{12}(k+4)^2(k+9)(k+11)$$

$$\frac{1}{72}(k+4)^2(k+6)(k+9)(k+11)^2$$

$$\frac{(k+4)^2(k+6)^2(k+11)^2(k+13)(k+15)(k+17)}{8640}$$

$$\frac{(k+4)^2(k+6)^2(k+8)(k+10)(k+11)(k+13)(k+15)^2(k+17)^2}{518400}$$

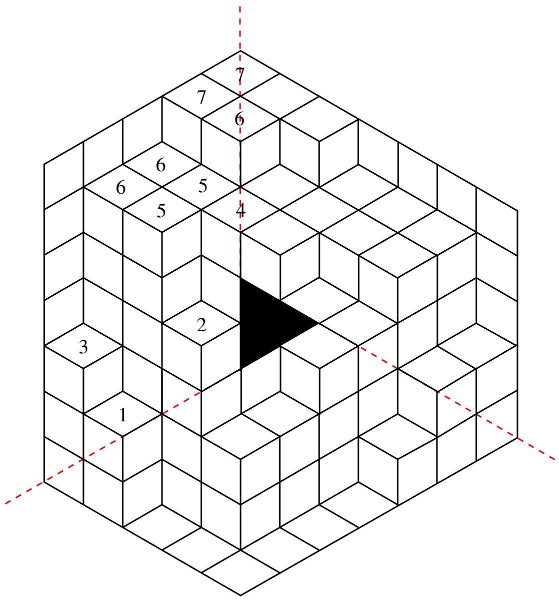
$$\frac{(k+4)^2(k+6)^2(k+8)^2(k+10)^2(k+15)^2(k+17)^3(k+19)(k+21)(k+23)}{870912000}$$

Surprise:

$$D_n(2) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

Combinatorial interpretation for $D_n(2)$

Christian Krattenthaler (2003):

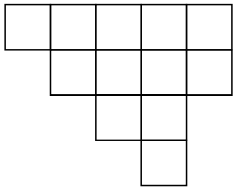


Cyclically symmetric lozenge tilings of a hexagon with side lengths $n+2, n, n+2, n, n+2, n$ with a central hole of size 2.

To obtain the combinatorial interpretation for any k, replace 2 by k!

Column strict shifted plane partitions of a fixed class aka DPPs

• With each strict partition (= partition with distinct parts), we associate a shifted Ferrers diagram. The shifted Ferrers diagram of $(5, 4, 2, 1)$ is



.

• A column strict shifted plane partition is a filling of a shifted Ferrers diagram with positive integers such that the rows are weakly decreasing and the columns are strictly decreasing.

Example.

$$\begin{array}{ccccc} 7 & 7 & 6 & 6 & 3 \\ & 6 & 5 & 5 & 1 \\ & & 4 & 2 & \end{array}$$

• A column strict shifted plane partition is of class k if the first part of each row exceeds the length of the row by precisely k. (Mills, Robbins and Rumsey 1987; Andrews 1979) The example is of class 2.

• There is a simple bijection between column strict shifted plane partitions of class k where the length of the top row does not exceed n and cyclically symmetric rhombus tilings of a hexagon with side lengths $n+k, n, n+k, n, n+k, n$ with a central triangular whole of size k.

Totally symmetric self-complementary plane partitions

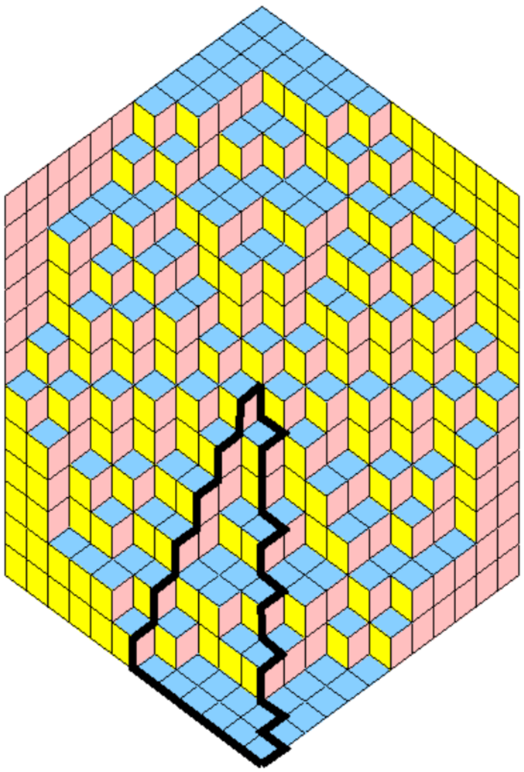


Figure by Di Francesco / Zinn-Justin

- **Totally symmetric:**
$(i,j,k) \in PP \Rightarrow \sigma(i,j,k) \in PP \ \forall \sigma \in \mathcal{S}_3$
(MacMahon 1899, 1915/16)

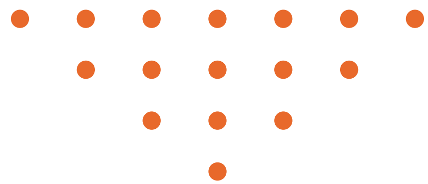
- **Self-complementary:**
Equal to its complement in the $2n \times 2n \times 2n$ box
(Mills, Robbins and Rumsey 1986)

Theorem (Andrews 1994). The number of TSSCPPs in a $2n \times 2n \times 2n$ box is (also) $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.

III. Alternating sign triangles

Alternating sign triangles = ASTs

An AST of order n is a triangular array of 1's, -1's and 0's with n centered rows



such that

(1) the non-zero entries alternate in each row and each column,

(2) all row sums are 1, and

(3) the topmost non-zero entry of each column is 1 (if such an entry exists).

Example:

$$\begin{array}{ccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 1 & -1 & 1 & 0 & 0 & \\ & & 1 & -1 & 1 & & \\ & & & 1 & & & \end{array}$$

ASTs of order 3

$$\begin{array}{ccccc}1&0&0&0&0\\&1&0&0&\\&&1&&\end{array}\quad\Bigg|\quad\begin{array}{ccccc}0&0&0&1&0\\&1&0&0&\\&&1&&\end{array}\quad\Bigg|\quad\begin{array}{ccccc}0&0&0&0&1\\&1&0&0&\\&&1&&\end{array}\quad\Bigg|\quad\begin{array}{ccccc}1&0&0&0&0\\&0&0&1&\\&&1&&\end{array}$$

$$\begin{array}{ccccc}0&1&0&0&0\\&0&0&1&\\&&1&&\end{array}\quad\Bigg|\quad\begin{array}{ccccc}0&0&0&0&1\\&0&0&1&\\&&1&&\end{array}\quad\Bigg|\quad\begin{array}{ccccc}0&0&1&0&0\\&1&-1&1&\\&&1&&\end{array}$$

Theorem (Ayyer, Behrend, and F., 2016). There is the same number of $n \times n$ ASMs as there is of ASTs with n rows.

Number of -1's in ASMs and ASTs

Let A be an ASM or an AST. Then we define

$$\mu(A) = \#\text{ of } -1\text{'s in } A.$$

Obviously

$$|\{A \in \mathsf{ASM}(n) \mid \mu(A) = 0\}| = n! = |\{A \in \mathsf{AST}(n) \mid \mu(A) = 0\}|.$$

Generalization of our theorem: Let m, n be non-negative integers. Then

$$|\{A \in \mathsf{ASM}(n) \mid \mu(A) = m\}| = |\{A \in \mathsf{AST}(n) \mid \mu(A) = m\}|.$$

Inversion numbers

Let $\pi = (\pi_1, \ldots, \pi_n)$ be a permutation and A be the permutation matrix of π, that is π_i is the column of the unique 1 in row i. Then

$$\mathrm{inv}(A) = \sum_{1 \le i' < i \le n, 1 \le j' \le j \le n} a_{i'j} a_{ij'}$$

is the number of inversions in π. We use this to define the inversion number of ASMs.

Let $A = (a_{i,j})_{1 \le i \le n, i \le j \le 2n-i}$ be an AST. We define

$$\mathrm{inv}(A) = \sum_{i' < i, j' \le j} a_{i'j} a_{ij'}.$$

Generalization of the generalization of our theorem: Let m, n, i be non-negative integers. Then

$$|\{A \in \mathsf{ASM}(n) \mid \mu(A) = m, \mathsf{inv}(A) = i\}| \\ = |\{A \in \mathsf{AST}(n) \mid \mu(A) = m, \mathsf{inv}(A) = i\}|.$$

The case $n = 3$

ASM	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$
μ	0	0	0	0
inv	0	1	1	2
AST	$\begin{array}{ccccc}1&0&0&0&0\\&1&0&0&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&1&0&0&0\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}1&0&0&0&0\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&0&0&1\\&1&0&0&\\&&1&&\end{array}$

ASM	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$
μ	0	0	1
inv	2	3	1
AST	$\begin{array}{ccccc}0&0&0&1&0\\&1&0&0&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&0&0&1\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&1&0&0\\&1&-1&1&\\&&1&&\end{array}$

Refined ASM-Theorem

Observation: There is a unique 1 in the top row of an ASM.

Theorem (Zeilberger, 1996): The number of $n \times n$ ASMs with a 1 in the top row and column r is

$$\binom{n+r-2}{n-1}\frac{(2n-r-1)!}{(n-r)!}\prod_{j=0}^{n-2}\frac{(3j+1)!}{(n+j)!} = A_{n,r}.$$

Find a statistic on ASTs that has the same distribution as the column of the 1 in the top row of an ASM!

Equivalent statistic on ASTs

In an AST, the elements of a column add up to 0 or 1. We say that a column is a 1-column if they add up to 1.

Let T be an AST with n rows. Define

$$\rho(T) = (\#\text{1-columns in the left half of } T \text{ that have a 0 at the bottom}) + (\#\text{1-columns in the right half of } T \text{ that have a 1 at the bottom}) + 1.$$

Theorem (F. 2019). The number of ASTs T with n rows and $\rho(T) = r$ is equal to $A_{n,r}$.

The case $n = 3$

ASM	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$
μ	0	0	0	0
inv	0	1	1	2
Top 1	1	1	2	2
ρ	1	3	2	2
AST	1 0 0 0 0 1 0 0 1	0 1 0 0 0 0 0 1 1	1 0 0 0 0 0 0 1 1	0 0 0 0 1 1 0 0 1

ASM	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$
μ	0	0	1
inv	2	3	1
Top 1	3	3	2
ρ	1	3	2
AST	0 0 0 1 0 1 0 0 1	0 0 0 0 1 0 0 1 1	0 0 1 0 0 1 -1 1 1

III. Alternating sign trapezoids

Back to Andrews' determinant

$$D_n(k) = \mathrm{det}_{0\leq i,j\leq n-1}\left(\delta_{i,j} + \binom{k+i+j}{i}\right)$$

Recall:

• $D_n(2)$ is the number of $n \times n$ ASMs as well as the number of ASTs with n rows.

• $D_n(k)$ is the number of cyclically symmetric lozenge tilings of a hexagon with central triangular hole of size k.

Is there a combinatorial realization of $D_n(k)$ in terms of ASM-like objects ?

Alternating sign trapezoids

For $n \geq 1, l \geq 2^*$, an *(n,l)-alternating sign trapezoid* is an array of 1's, -1's and 0's with n centered rows and l elements in the bottom row, arranged as follows

such that the following conditions are satisfied.

(1) In each row and column, the non-zero entries alternate.

(2) All row sums are 1.

(3) The topmost non-zero entry in each column is 1.

(4) The column sums are 0 for the middle $l-2$ columns.

*Can be extended to $l = 1$.

Example

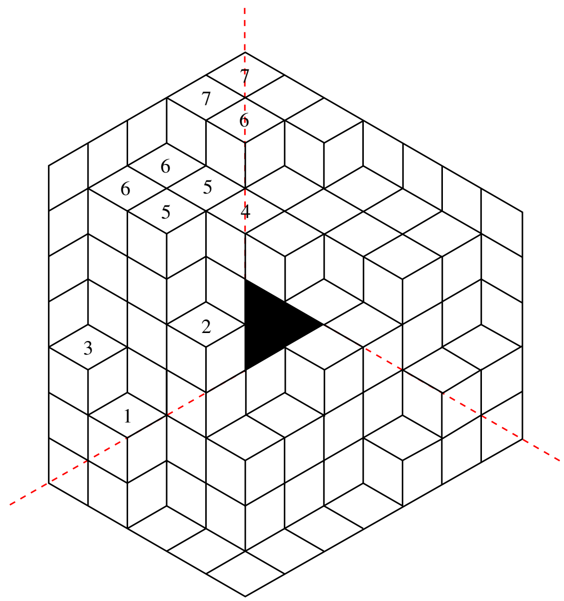
A $(5,4)$-alternating sign trapezoid.

$$
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & \\
 & & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 1 & & \\
 & & & 0 & 0 & 0 & 1 & -1 & 1 & & & \\
 & & & & 1 & 0 & -1 & 1 & & & &
\end{array}
$$

ASTs with n rows are equivalent to $(n-1,3)$-alternating sign trapezoids. (Delete the bottom row of the AST.)

Alternating sign trapezoids and cyclically symmetric rhombus tilings of a holey hexagon

Theorem (Behrend, F. 2018). There is the same number of (n, l)-alternating sign trapezoids as there is of cyclically symmetric rhombus tilings of a hexagon with side lengths $n+l-1, n, n+l-1, n, n+l-1, n$ that has a central triangular hole of size $l-1$.



Product formula

Corollary. The number of (n,l)-alternating sign trapezoids is

$$2^n \prod_{i=0}^{n-1} q_i(l)^{n-i-1},$$

where

$$q_i(l) = \begin{cases} \frac{(l+3i)(2+l+3i)(4+l+3i)}{(l+2i)(2+l+2i)(4+4i)}, & i \text{ even}, \\ \frac{2\left(-\frac{1}{2}+l+\frac{3}{2}i\right)\left(\frac{1}{2}+l+\frac{3}{2}i\right)\left(\frac{3}{2}+l+\frac{3}{2}i\right)}{(l+2i)(2+l+2i)(l+i)}, & i \text{ odd}. \end{cases}$$

Three statistics on alternating sign trapezoids

- A 1-column is a column with sum 1.
- A 10-column is a 1-column whose bottom element is 0.

Simple fact: An (n,l)-alternating sign trapezoid has n 1-columns

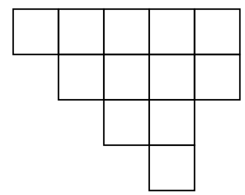
The statistics on (n,l)-alternating sign trapezoids T:

$$\mathsf{p}(T) = \# \text{ of 10-columns among the } n \text{ leftmost columns,}$$
$$\mathsf{q}(T) = \# \text{ of 10-columns among the } n \text{ rightmost columns,}$$
$$\mathsf{r}(T) = \# \text{ of 1-columns among the } n \text{ leftmost columns.}$$

In the example above, we have $\mathsf{p}(T) = 1, \mathsf{q}(T) = 0, \mathsf{r}(T) = 2$.

Column strict shifted plane partitions of a fixed class aka DPPs

• With each strict partition (= partition with distinct parts), we associate a shifted Ferrers diagram. The shifted Ferrers diagram of $(5, 4, 2, 1)$ is



.

• A column strict shifted plane partition is a filling of a shifted Ferrers diagram with positive integers such that the rows are weakly decreasing and the columns are strictly decreasing.

Example.

$$\begin{array}{ccccc} 7 & 7 & 6 & 6 & 3 \\ & 6 & 5 & 5 & 1 \\ & & 4 & 2 & \end{array}$$

• A column strict shifted plane partition is of class k if the first part of each row exceeds the length of the row by precisely k. (Mills, Robbins and Rumsey 1987; Andrews 1979) The example is of class 2.

• There is a simple bijection between column strict shifted plane partitions of class k where the length of the top row does not exceed n and cyclically symmetric rhombus tilings of a hexagon with side lengths $n+k, n, n+k, n, n+k, n$ with a central triangular whole of size k.

Three statistics on column strict shifted plane partitions

For $d \in \{1, \ldots, k\}$ and a column strict shifted plane partition C of class k, we define

$$\mathsf{p}_d(C) = \# \text{ of parts } j - i + d \text{ where } i \text{ is the row and } j \text{ is the column,}$$
$$\mathsf{q}(C) = \# \text{ of 1's,}$$
$$\mathsf{r}(C) = \# \text{ of rows.}$$

In the example above, we have $\mathsf{p}_1(C) = 1, \mathsf{q}(C) = 1, \mathsf{r}(C) = 3$.

Theorem (F. 2019). The number of (n, l)-alternating sign trapezoids T with $\mathsf{p}(T) = p, \mathsf{q}(T) = q, \mathsf{r}(T) = r$ is equal to the number of column strict shifted plane partitions of class $l - 1$ with $\mathsf{p}_d(C) = p, \mathsf{q}(C) = q, \mathsf{r}(C) = r$, where the length of the first row does not exceed n.

Recently, Höngesberg could add another statistic (number of -1's on the alternating sign trapezoid side).

The case $n = 2, l = 4$

Alternating sign trapezoids:

$$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & \\ & & \multicolumn{2}{c}{(0,0,2)} & & \end{array} \quad \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & \\ & & \multicolumn{2}{c}{(0,0,1)} & & \end{array} \quad \begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & \\ & & \multicolumn{2}{c}{(1,0,1)} & & \end{array} \quad \begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ & 1 & -1 & 0 & 1 & \\ & & \multicolumn{2}{c}{(0,0,1)} & & \end{array}$$

$$\begin{array}{cccccc} 0 & 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & -1 & 1 & \\ & & \multicolumn{2}{c}{(0,0,1)} & & \end{array} \quad \begin{array}{cccccc} 0 & 0 & 0 & 0 & 1 & 0 \\ & 1 & 0 & 0 & 0 & \\ & & \multicolumn{2}{c}{(0,1,1)} & & \end{array} \quad \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 0 & \\ & & \multicolumn{2}{c}{(0,0,1)} & & \end{array} \quad \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 1 & \\ & & \multicolumn{2}{c}{(0,0,0)} & & \end{array}$$

Column strict shifted plane partitions:

	$\emptyset$	4	5 1	5 2	5 3	5 4	5 5	5 5 4
$d = 1$	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(1,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,2)$
$d = 2$	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(0,0,1)$	$(1,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,2)$
$d = 3$	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(0,0,1)$	$(0,0,1)$	$(1,0,1)$	$(0,0,1)$	$(0,0,2)$

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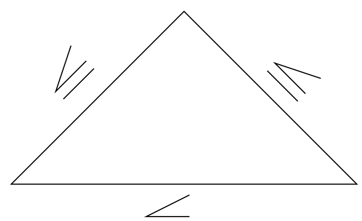
IV. Proofs

Monotone triangles

Triangular arrays of integers with monotonicity requirements:



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 2 & & 5 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Monotone triangles with bottom row $1, 2, \ldots, n \Leftrightarrow n \times n$ ASMs

Formula for the number of monotone triangles with prescribed bottom row

Antisymmetrizer:

$$\mathbf{ASym}_{Y_1,\ldots,Y_n} F(Y_1,\ldots,Y_n) = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma F(Y_{\sigma(1)},\ldots,Y_{\sigma(n)})$$

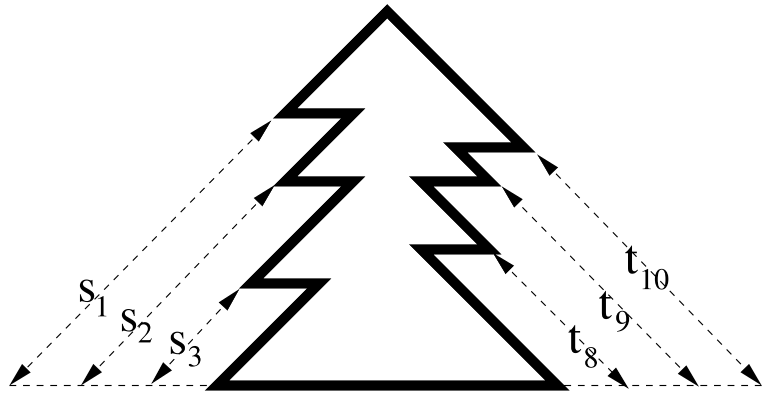
Define

$$M_n(\mathbf{x}) = \operatorname{CT}_{Y_1,\ldots,Y_n} \frac{\mathbf{ASym}_{Y_1,\ldots,Y_n}\left[\prod_{i=1}^{n}(1+Y_i)^{x_i}\prod_{1\le i<j\le n}(1+Y_j+Y_iY_j)\right]}{\prod_{1\le i<j\le n}(Y_j-Y_i)},$$

where $\operatorname{CT}_{Y_1,\ldots,Y_n}$ denotes the constant term in $Y_1,\ldots,Y_n$.

Theorem (F., 2006). The number of monotone triangles with bottom row $b_1,\ldots,b_n$ is $M_n(b_1,\ldots,b_n)$.

Truncated monotone triangles: (s, t)-trees



• $\mathrm{s} = (s_1, s_2, \ldots, s_l)$ weakly decreasing sequence: prescribes the number of entries deleted at the bottom of the $\nearrow$-diagonals.

• $\mathrm{t} = (t_{n-r+1}, \ldots, t_n)$ weakly increasing sequence: prescribes the number of entries deleted at the bottom of the $\searrow$-diagonals.

The number of (s,t)-trees

Forward difference operator: $\overline{\Delta}_x\, p(x) = p(x+1) - p(x)$
Backward difference operator: $\underline{\Delta}_x\, p(x) = p(x) - p(x-1)$

The evaluation of

$$(-\overline{\Delta}_{x_1})^{s_1} \cdots (-\overline{\Delta}_{x_l})^{s_l} \underline{\Delta}_{x_{n-r+1}}^{t_{n-r+1}} \cdots \underline{\Delta}_{x_n}^{t_n} M_n(\mathbf{x})$$

at $\mathbf{x} = (b_1, \ldots, b_n)$ is the number of $(\mathbf{s},\mathbf{t})$-trees of order n with the following properties:

- The bottom entry of the i-th $\nearrow$-diagonal is b_i for $1 \le i \le n-r$.
- The bottom entry of the i-th $\searrow$-diagonal is b_i for $n-r+1 \le i \le n$.

From alternating sign triangles to truncated monotone triangles

$$
\begin{array}{c}
0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
1 \quad 0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 1 \quad 0 \quad -1 \quad 1 \\
0 \quad 1 \quad -1 \quad 0 \quad 1 \\
0 \quad 0 \quad 1 \\
1
\end{array}
$$

1-column = a column with sum 1.

- An AST with n rows has precisely n 1-columns.

- First goal: Constant term formula for the number of ASTs with prescribed positions of the 1-columns.

$$\begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$$\Downarrow$$

$$\begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{matrix}$$

−5	−4	−3	−2	−1	0	1	2	3	4	5
0	0	0	1	0	0	0	0	0	0	0
0	1	0	0	0	0	0	1	0	0	0
0	1	0	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	1	0	0
0	1	0	0	1	0	1	1	1	0	0
0	1	0	0	1	1	1	1	1	0	0

$$\Downarrow$$

−2

−4 2

−4 0 3

−4 −1 2 3

−4 −1 1 2 3

−4 −1 0 1 2 3

• Orange entries are redundant. Delete them in order to obtain an $(\mathbf{s},\mathbf{t})$-tree.

• Number of deleted entries in a fixed diagonal equals the absolute value of the bottom entry in the truncated diagonal.

The number of ASTs with prescribed positions of the 1-columns

Using the formula for the number of $(\mathbf{s},\mathbf{t})$-tree, we can deduce the following (after a few pages of calculations).

Theorem (F. 2019). The number of ASTs with n rows that have the 1-columns in positions $j_1, j_2, \ldots, j_{n-1}$, where we count from the left starting with 0 and disregard the central column, is the coefficient of $X_1^{j_1} X_2^{j_2} \ldots X_{n-1}^{j_{n-1}}$ in

$$\prod_{i=1}^{n-1} (1 + X_i) \prod_{1 \le i < j \le n-1} (1 + X_i + X_i X_j)(X_j - X_i).$$

Total number of ASTs

The number of ASTs with n rows is the constant term of

$$\sum_{0\le j_1<j_2<\ldots<j_{n-1}} \prod_{i=1}^{n-1}(1+X_i^{-1})X_i^{j_i} \prod_{1\le i<j\le n-1}(1+X_i^{-1}+X_i^{-1}X_j^{-1})(X_j^{-1}-X_i^{-1})$$

$$= \frac{\prod_{i=1}^{n-1}(1+X_i)X_i^{i-2n+2} \prod_{1\le i<j\le n-1}(1+X_j+X_iX_j)(X_i-X_j)}{\prod_{i=1}^{n-1}\left(1-\prod_{j=i}^{n-1}X_j\right)}.$$

"Trick:" Apply the symmetrizer in $X_1,\ldots,X_n$. The constant term is then multiplied by $n!$.

"Magic:" The symmetrizer can actually be computed!

End of the proof

Lemma. Let $n \geq 1$. Then

$$\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i<j\le n}(1+X_j+X_iX_j)\prod_{i=1}^{n}X_i^{i-1}\left(1-\prod_{j=i}^{n}X_j\right)^{-1}\right]$$
$$=\prod_{i=1}^{n}(1-X_i)^{-1}\prod_{1\le i<j\le n}\frac{(1+X_i+X_j)(X_j-X_i)}{(1-X_iX_j)}.$$

After some further steps, one can see that the number is the constant term of

$$(-1)^{\binom{n-1}{2}}\sum_{0\le b_1<b_2<\ldots<b_{n-1}}\det_{1\le i,j\le n-1}\left((1+X_j)^i X_j^{i-2n+2+b_j}\right),$$

and this leads directly to an expression that gives the number of totally symmetric self-complementary plane partitions in a $2n\times 2n\times 2n$ box (Lindström-Gessel-Viennot).

The number of alternating sign trapezoids with prescribed positions of the 1-columns

Theorem (Aigner). The number of (n,l)-alternating sign trapezoids with the 1-columns in positions $0 \leq j_1 < j_2 < \ldots < j_n \leq 2n-1$ where we index the columns from left to right starting with 0 and disregard the $l-2$ central columns is the coefficient of $X_1^{j_1} X_2^{j_2} \ldots X_n^{j_n}$ in

$$\prod_{i=1}^{m} (1+X_i) \prod_{i=m+1}^{n} X_i^{-l+3} (1+X_i)^{l-2} \prod_{1 \leq i < j \leq n} (X_j - X_i)(1 + X_i + X_i X_j),$$

where m is maximal such that $j_m \leq n-1$.

Only for $l=3$ (ASTs!), there is no dependency on m.

Crucial step in the enumeration of AS-trapezoids

Definition.

$$\textbf{Subsets}_m\, F(Y_1,\dots,Y_n) = \sum_{\substack{\sigma\in\mathcal{S}_n \\ \sigma(1)<\sigma(2)<\ldots<\sigma(m),\sigma(m+1)<\sigma(m+2)<\ldots<\sigma(n)}} F(Y_{\sigma(1)},\dots,Y_{\sigma(n)})$$

The number is the constant term of

$$\sum_{m=0}^{n} \textbf{Subsets}_m \prod_{i=m+1}^{n} (1+Y_i)^{l-1} \prod_{i,j=1}^{m} \frac{1}{1+Y_i+Y_j} \prod_{i,j=m+1}^{n} \frac{1}{1-Y_iY_j} \times \prod_{i=1}^{m} \prod_{j=m+1}^{n} \frac{1+Y_j+Y_iY_j}{(Y_j-Y_i)(1+Y_i+Y_j)(1-Y_iY_j)}.$$

Using the Cauchy determinant, it follows that

$$\prod_{i=m+1}^{n}(1+Y_i)^{l-1}\prod_{i,j=1}^{m}\frac{1}{1+Y_i+Y_j}\prod_{i,j=m+1}^{n}\frac{1}{1-Y_iY_j}$$
$$\times\prod_{i=1}^{m}\prod_{j=m+1}^{n}\frac{1+Y_j+Y_iY_j}{(Y_j-Y_i)(1+Y_i+Y_j)(1-Y_iY_j)}$$
$$=\frac{\det_{1\le i,j\le n}\left(\begin{cases}\frac{1}{1+Y_i+Y_j}, & i\le m\\ \frac{(1+Y_i)^{l-1}}{1-Y_iY_j}, & i>m\end{cases}\right)}{\prod_{1\le i<j\le n}(Y_j-Y_i)^2}.$$

Applying $\mathbf{Subsets}_m$ and summing over all m then gives

$$\frac{\det_{1\le i,j\le n}\left(\frac{1}{1+Y_i+Y_j}+\frac{(1+Y_i)^{l-1}}{1-Y_iY_j}\right)}{\prod_{1\le i<j\le n}(Y_j-Y_i)^2}.$$

After some further manipulations we obtain Andrews generalization of the determinant for the number of cyclically symmetric plane partitions.

Thank you!