THE MYSTERIOUS STORY OF SQUARE ICE, PILES OF CUBES, AND BIJECTIONS

ILSE FISCHER AND MATJAŽ KONVALINKA

Abstract. When combinatorialists discover two different types of objects that are counted by the same numbers, they usually want to prove this by constructing an explicit bijective correspondence. Such proofs frequently reveal many more details about the relation between the two types of objects than just equinumerosity. A famous set of problems that has resisted various attempts to find bijective proofs for almost 40 years is concerned with alternating sign matrices (which are equivalent to a well-known physics model for two-dimensional ice), and their relations to certain classes of plane partitions. In this note, we tell the story of how the first bijections were found.

1. Introduction

All mathematicians recognize $1, 1, 2, 3, 5, 8, 13, \ldots$ as *Fibonacci numbers*, and many of them, especially combinatorialists, recognize $1, 1, 2, 5, 14, 42, 132, \ldots$ as *Catalan numbers*. Both sequences appear frequently in combinatorics and other areas, they have many beautiful properties, and are extremely well studied. However, there is the mysterious sequence $1, 1, 2, 7, 42, 429, 7436, \ldots$, whose terms are sometimes called *Robbins numbers*. They can be expressed with a product formula (see (1)), and the mystery comes from the fact that they count four different families of objects that, until now, could not be translated into one another.

More precisely, there are many known objects enumerated by this sequence (13 of them are illustrated in different rows of Figure 1). There are easy bijections between some of these objects, for example between alternating sign matrices and square ice configurations, and these bijections split the families of objects into four classes (separated by a line in Figure 1): alternating sign matrices, descending plane partitions, totally symmetric self-complementary plane partitions, and alternating sign triangles. We invite readers looking for a challenge to try to guess the definitions of some of the objects in Figure 1 that are not defined in the paper, and to find bijections, for example, among the objects illustrated in the first six rows. However, until now, no bijection was known between any two of these four classes, despite the problem being open for almost four decades (for the first three classes; the fourth class was only discovered recently). In this manuscript, we describe the first such bijection, one that connects alternating sign matrices and descending plane partitions. Let us emphasize right away that the bijection is far from simple. It is, however, completely explicit. We expect that the tools used in the paper can be used to find other complicated bijections.

Let us mention that the combinatorial objects described are also commonly encountered in statistical mechanics, so this work has direct applications to questions in physics.

2. A brief history

An *alternating sign matrix* (ASM) is a square matrix with entries in $\{0, 1, -1\}$ such that in each row and each column the non-zero entries alternate and sum up to 1. See row 1 of Figure

1 for all ASMs of size 3×3. Robbins and Rumsey introduced alternating sign matrices in the 1980s [RR86] when studying their λ-*determinant* (a generalization of the classical determinant) and showing that the λ-determinant can be expressed as a sum over all alternating sign matrices of fixed size, thus generalizing Leibniz formula that expresses the ordinary determinant as a sum over all permutations.

Numerical experiments led Robbins and Rumsey to conjecture that the number of $n \times n$ alternating sign matrices is given by the surprisingly simple product formula

$$|\operatorname{ASM}_n| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{1}$$

They also conjectured a formula for the number of ASMs of size $n \times n$ with the unique 1 in the first row being in column i:

$$|\operatorname{ASM}_{n,i}| = \frac{\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}}{\binom{3n-2}{2n-1}} \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{2}$$

One of the two bijections we have discovered establishes a bijective proof of (2) which, by taking the union over all i, also leads to such a proof of (1).

Back then the surprise was even bigger when Robbins and Rumsey, now joined by Mills, learned from Stanley (see [BP99, Bre99]) that the product formula in (1) had recently also appeared in Andrews' paper [And79] on his proof of the weak Macdonald conjecture, which in turn provides a formula for the number of cyclically symmetric plane partitions (plane partitions can be visualized as piles of cubes stacked in the corner of a box). As a byproduct, Andrews had introduced *descending plane partitions* (DPPs), fillings of a shifted diagram with positive integers that decrease weakly along rows and strictly along columns, such that the first part in each row is greater than the length of its row and less than or equal to the length of the previous row; see row 8 of Figure 1. Andrews proved that the number of DPPs with parts at most n is also equal to (1), and Mills, Robbins and Rumsey [MRR82] proved that (2) is the number of such DPPs with exactly $i-1$ copies of n. Note that it is possible to interpret DPPs as certain sets of non-intersecting paths, and as cyclically symmetric lozenge tilings with a central triangular hole of size 2, see rows 9 and 10 of Figure 1. The latter objects are somewhat reminiscent of Penrose's impossible stairs.

Since then the problem of finding an explicit bijection between alternating sign matrices and descending plane partitions has attracted considerable attention from combinatorialists, and to many of them it is a miracle that such a bijection has not been found so far. All the more so because Mills, Robbins and Rumsey also introduced several "statistics" on alternating sign matrices and on descending plane partitions for which they had strong numerical evidence that the joint distributions coincide as well, see [MRR83]. On the other hand, some believe that a natural bijection is in some sense impossible, but it is unclear how to interpret such a statement mathematically and even more unclear how to prove it. The second bijection we have discovered explains for the first time the relation between ASMs and DPPs bijectively.

There were a few further surprises yet to come. Robbins introduced a new operation on plane partitions, *complementation*, and had strong numerical evidence that totally symmetric self-complementary plane partitions (TSSCPPs) in a box of dimensions $2n \times 2n \times 2n$ are also counted by (1), see row 11 of Figure 1. Note that a plane partition is totally symmetric if it is invariant under every permutation of the coordinate axes, and that there is a simple bijection between TSSCPPs and certain triangular shifted plane partitions, see row 12 of Figure 1. Again this was further supported by statistics that have the same joint distribution as well as certain refinements, see

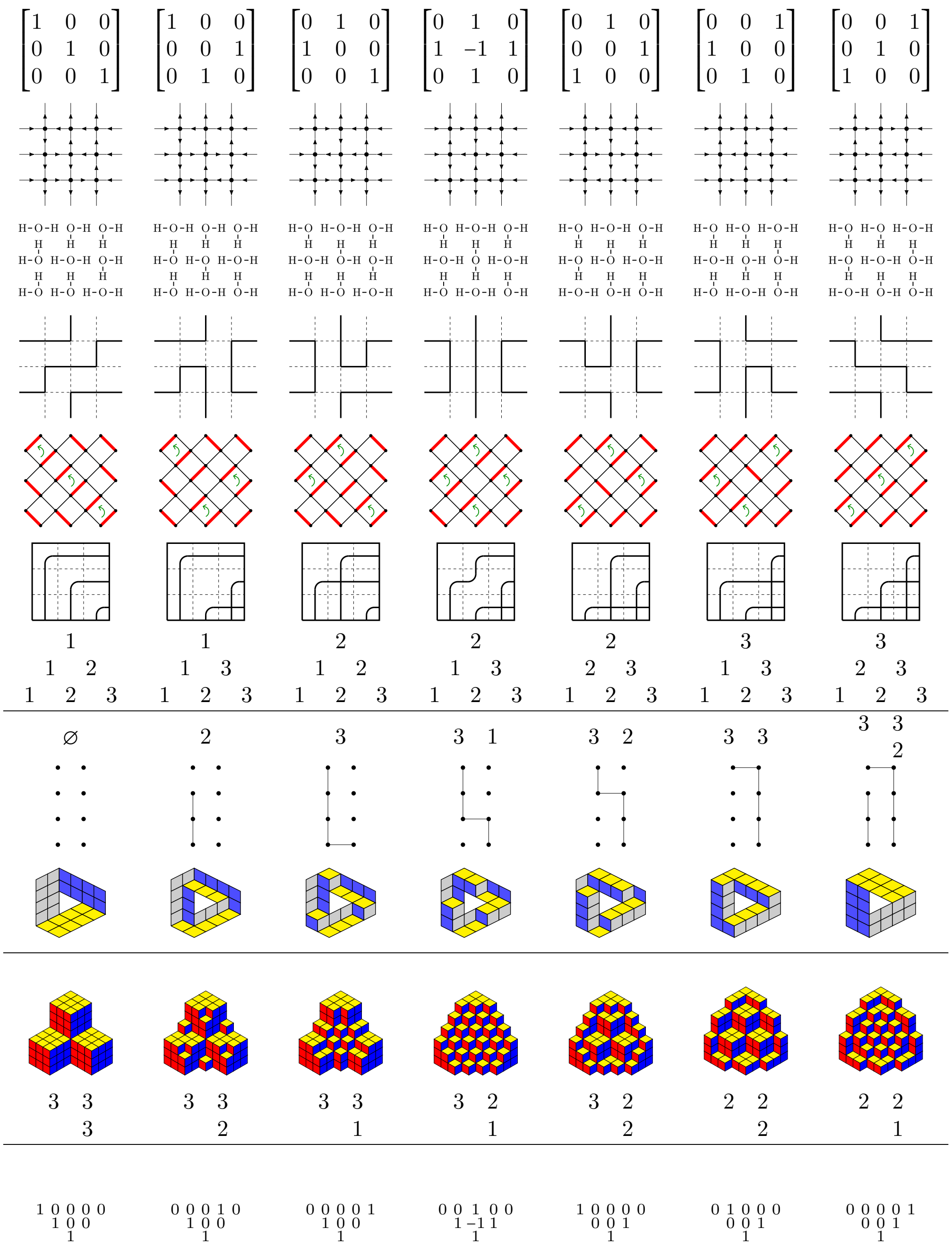

FIGURE 1. Families of objects counted by the same enumeration formula, for $n = 3$: ASMs; the six vertex model with domain wall boundary conditions; square ice; fully packed loop configurations; classes of perfect matchings of the Aztec diamond graph; (not necessarily reduced) bumpless pipe dreams; monotone triangles with bottom row $12\ldots n$; DPPs; certain non-intersecting paths; cyclically symmetric lozenge tilings with a central hole; TSSCPPs; certain triangular shifted plane partitions; ASTs.

[MRR86, Kra96, Kra16, BC16]. We still lack an explicit bijection between TSSCPPs and ASMs, as well as between TSSCPPs and DPPs, but we are optimistic that the methods we sketch in this note will also lead to such bijections.

In his collection of bijective proof problems (which is available from his web page) Stanley says the following about the problem of finding all these bijections: "*This is one of the most intriguing open problems in the area of bijective proofs.*" In Krattenthaler's survey on plane partitions [Kra16] he expresses his opinion by saying: "*The greatest, still unsolved, mystery concerns the question of what plane partitions have to do with alternating sign matrices.*"

Many of the above-mentioned conjectures have since been proved by non-bijective means. Zeilberger [Zei96a] was the first who proved that $n \times n$ ASMs are counted by (1). Kuperberg gave a shorter proof [Kup96] based on the remarkable observation that the *six vertex model* (which had been introduced by physicists several decades earlier) with domain wall boundary conditions (see row 2 of Figure 1) is equivalent to ASMs, and he used the techniques that had been developed by physicists to study this model. Note that other equivalent ways to think about ASMs are the square ice model, fully packed loop configurations, classes of perfect matchings of the Aztec diamond graph and (not necessarily reduced) bumpless pipe dreams, see rows 3–6 of Figure 1. Andrews enumerated TSSCPPs in [And94]. The equivalence of certain statistics for ASMs and of certain statistics for DPPs was proved in [BDFZJ13], while for ASMs and TSSCPPs see [Zei96b, FZJ08], and note in particular that already in Zeilberger's first ASM paper [Zei96a] he could deal with an important refinement. Further work including the study of *symmetry classes* has been accomplished; for a more detailed description of this we defer to [BFK17]. Then, in very recent work, *alternating sign triangles* (ASTs) were introduced in [ABF16], which establishes a fourth class of objects that are equinumerous with ASMs, see row 13 of Figure 1, and also in this case nobody has so far been able to construct a bijection. Also in this case, we expect that the approach presented in this note will be used to construct such a bijection. In fact, the planned bijection between ASMs and TSSCPPs will be most likely via ASTs.

Starting in around 2005, the first author published a series of papers in which *monotone triangles* feature very prominently [Fis05, Fis06, Fis07, Fis10, Fis11, Fis12a, Fis12b, Fis16]. Alluding to Krattenthaler's citation above, one could argue that among the objects that are in easy bijective correspondence with ASMs, monotone triangles are the closest to plane partitions. In order to define them note that a *Gelfand–Tsetlin pattern* (or GT pattern) is a triangular array of integers that are weakly increasing along ↗- and ↘-diagonals. A *monotone triangle* is a GT pattern with strictly increasing rows. There are 8 GT patterns with bottom row 123, and all but one of them (the one with two 2s in row 2) are monotone triangles; see row 7 of Figure 1. There is an easy bijection between ASMs of size $n \times n$ and monotone triangles with bottom row $1, 2, \ldots, n$: add to each entry the entries that are above in the same column, and record the positions of the 1s in the rows of the new matrix. There is a simple product formula for the number of GT patterns with fixed bottom row, and the first author found an *operator formula* for the number of monotone triangles with fixed bottom row. This formula will also be crucial in our construction (see (8)), which will give a bijective proof of the fact that the number of ASMs of size $n \times n$ equals the number of DPPs with entries at most n.

Our method of proof involves signed sets and sijections (signed bijections). We are able to build complicated sijections out of simple building blocks by extending classical notions such as the Cartesian product, disjoint union, and composition to signed sets. In some sense, this framework is implicit in, say, the groundbreaking works of Garsia and Milne, see [And86, GM81a, GM81b], but we needed to make it more explicit and to extend it to be able to deal with the more complicated situation. For example, the *Garsia–Milne involution principle* is equivalent to the special case of the composition of two sijections when only the "intermediate" set has a non-empty negative part. The "naturalness" of the composition might let us argue that the involution principle is not as bad as the reputation it sometimes has. After all, enumeration results often have natural extensions to certain signed sets, and then sijections and compositions thereof are unavoidable. We also believe that the tools employed in our constructions could prove very useful in the search for bijective proofs of other identities that are of interest to combinatorialists. Roughly speaking, the framework should be applicable for translating "computational" proofs of identities that only involve additions, subtractions and multiplications (but not divisions), as detailed below. Note that it is probably more complicated to transfer ASM proofs using the six vertex model approach (see [Kup96]), as such proofs typically employ an interpolation argument.

3. Signed sets and sijections

It is widely accepted in combinatorics that bijective proofs of identities are "the best" in most circumstances: they typically bring the most clarity to a statement, they yield interesting generalizations, and they are usually esthetically pleasing. For example, the statement $\sum_{k=0}^{n}\binom{n}{k}=2^n$ can be proved in a variety of ways, e.g. by induction, or by finding the expansion of $(1+x)^n$ and plugging in $x=1$. On the other hand, a bijective proof of this statement is the simple observation that the right-hand side counts all subsets of an n-element set, while the left-hand side splits them according to size.

Things are a little different when the identity involves signs. For example, consider the identity $\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$ for $n\in\mathbb{Z}_{>0}$. In this case, a "bijective" proof means that we find the right cancellations: we have to cancel a set of even size with a set of odd size. For example, we could map a set A to $A\setminus\{n\}$ or $A\cup\{n\}$, depending on whether or not $n\in A$. This map has the added benefit of proving $\sum_{k=0}^{m}(-1)^k\binom{n}{k}=(-1)^m\binom{n-1}{m}$ at no extra cost.

Since the identity (2) has no signs, a proof that would avoid signs and cancellations would be preferable. Our proof, however, uses them quite substantially. This stems from the fact that this proof has been developed from the non-bijective proof by the first author which contains calculations that involve signs. This also raises the question of whether a possible bijective proof that avoids signs can in turn be translated into a "computational" proof that avoids signs. No such non-bijective proof is currently known.

In the remainder of this section, we briefly introduce the concepts of *signed sets* and *sijections*, signed bijections between signed sets. We present the basic concepts here, and refer the reader to [FK19a, §2] for all the details and more examples.

A *signed set* is a pair of disjoint finite sets: $\underline{S}=(S^+,S^-)$ with $S^+\cap S^-=\varnothing$. Equivalently, a signed set is a finite set S together with a sign function $\operatorname{sign}\colon S\to\{1,-1\}$. Signed sets are usually underlined throughout the manuscript with the following exception: an ordinary set S always induces a signed set $\underline{S}=(S,\varnothing)$, and in this case we identify $\underline{S}$ with S.

We summarize related notions. The *size* of a signed set $\underline{S}$ is $|\underline{S}|=|S^+|-|S^-|$. The *opposite* signed set of $\underline{S}$ is

$$-\underline{S}=(S^-,S^+). \tag{3}$$

The *Cartesian product* of signed sets $\underline{S}$ and $\underline{T}$ is $\underline{S}\times\underline{T}=(S^+\times T^+\cup S^-\times T^-,S^+\times T^-\cup S^-\times T^+)$. The *disjoint union* of signed sets $\underline{S}$ and $\underline{T}$ is $\underline{S}\sqcup\underline{T}=(\underline{S}\times(\{0\},\varnothing))\cup(\underline{T}\times(\{1\},\varnothing))$. These constructions correspond as usual to arithmetic operations on the sizes, i.e.,

$$|\underline{S}\times\underline{T}|=|\underline{S}|\cdot|\underline{T}|\quad\text{and}\quad|\underline{S}\sqcup\underline{T}|=|\underline{S}|+|\underline{T}|. \tag{4}$$

The *disjoint union of a family of signed sets* $\underline{S}_t$ indexed with a signed set $\underline{T}$ is

$$\bigsqcup_{t\in\underline{T}} \underline{S}_t = \bigcup_{t\in\underline{T}} (\underline{S}_t \times \underline{\{t\}}).$$

Here $\underline{\{t\}}$ is $(\{t\},\varnothing)$ if $t \in T^+$ and $(\varnothing,\{t\})$ if $t \in T^-$. Most of the usual properties of Cartesian products and disjoint unions (commutativity, distributivity etc.) of ordinary sets extend to signed sets.

An important type of signed sets are *signed intervals*: for $a,b \in \mathbb{Z}$, define

$$\underline{[a,b]} = \begin{cases} ([a,b],\varnothing) & \text{if } a \le b \\ (\varnothing,[b+1,a-1]) & \text{if } a > b \end{cases}.$$

Here $[a,b]$ stands for the usual interval in $\mathbb{Z}$, defined when $a \le b$. Note that we always have $|\underline{[a,b]}| = b-a+1$. The signed sets that are of relevance in this manuscript are usually constructed from signed intervals using Cartesian products and disjoint unions.

The role of bijections for signed sets is played by "signed bijections", which we call *sijections*, and they are manifestations of the fact that two signed sets have the same size. A sijection φ from $\underline{S}$ to $\underline{T}$,

$$\varphi\colon \underline{S} \Longrightarrow \underline{T},$$

is an involution on the set $(S^+ \cup S^-) \sqcup (T^+ \cup T^-)$ with the property $\varphi(S^+ \sqcup T^-) = S^- \sqcup T^+$. It follows that also $\varphi(S^- \sqcup T^+) = S^+ \sqcup T^-$. A sijection can also be thought of as a collection of a sign-reversing involution on a subset of $\underline{S}$, a sign-reversing involution on a subset of $\underline{T}$, and a sign-preserving matching between the remaining elements of $\underline{S}$ with the remaining elements of $\underline{T}$. The existence of a sijection $\varphi\colon \underline{S} \Rightarrow \underline{T}$ clearly implies $|\underline{S}| = |S^+| - |S^-| = |T^+| - |T^-| = |\underline{T}|$. See Figure 2, left drawing; the sijection is a bijection between the blue (resp. green) parts of S^+ and S^- (resp. T^+ and T^-), and between the light gray (resp. dark gray) parts of S^+ and T^+ (resp. S^- and T^-).

A sijection between two signed sets with no negative elements is clearly a bijection. Our two bijections are constructed from two chains of sijections with several intermediate sets connecting the two pairs of sets for which we want to show equinumerosity. However, in order to be able to use these sijections to construct the two bijections, we need a notion of *composing* sijections. While composing bijections is of course trivial, this turns to be slightly more complicated for general sijections. There seems to be only one natural choice for how to do this; indeed, the construction is a generalization of the *Garsia-Milne involution principle*. For an illustration of this, see Figure 2, right drawing. There we have a sijection φ between $\underline{S}$ and $\underline{T}$ (solid lines), and a sijection ψ between $\underline{T}$ and $\underline{U}$ (dashed lines); through φ (resp. ψ), we have a bijection between the blue (resp. green) parts of $\underline{S}$ (resp. $\underline{U}$), and all other elements of S^+ or U^- are mapped to a unique element of S^- or U^+ via an alternating sequence of applications of φ and ψ. For the formal definition of composition as well as of the Cartesian product and the disjoint union of sijections see Proposition 2 of [FK19a].

Just as jeu de taquin is a building block for several constructions in algebraic combinatorics, the fundamental sijection that is underlying many of our constructions is the following.

Problem 1. [FK19a, Problem 1] *Given $a,b,c \in \mathbb{Z}$, construct a sijection*

$$\alpha = \alpha_{a,b,c}\colon \underline{[a,c]} \Longrightarrow \underline{[a,b]} \sqcup \underline{[b+1,c]} = \underline{[a,b]} \sqcup -\underline{[c+1,b]}.$$

Construction. The sijection is very simple, but we do have to split it into cases. If $a \le b \le c$, we take the natural bijection $[a,c] \to [a,b] \sqcup [b+1,c]$. If $a \le c < b$, then $[a,b] = [a,c] \cup [c+1,b]$ and $\underline{[b+1,c]} = (\varnothing,[c+1,b])$, so we simply cancel the two copies of elements in $[c+1,b]$. Other cases are similar. □

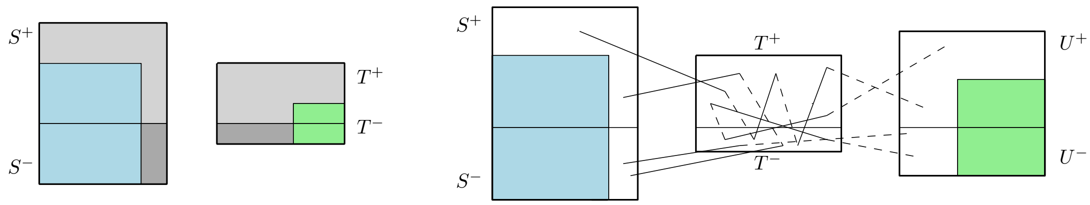

FIGURE 2. Illustration of a sijection, and of composition of sijections.

Note that to emphasize that we are not merely interested in the fact that two signed sets have the same size, but that we want to use the constructed signed bijection later on, we will be using a slightly unorthodox convention. Instead of listing our results as lemmas and theorems with their corresponding proofs, we will be using the Problem–Construction terminology. See for instance [Voe17] and [Bau19].

Finally, let us mention two crucial combinatorial objects: *generalized GT patterns* and *generalized monotone triangles*. The difference is that now the rows are not necessarily increasing. For $k \in \mathbb{Z}$, define $\underline{\mathrm{GT}}(k)$ to have a single positive element, and for $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$, define recursively

$$\underline{\mathrm{GT}}(\mathbf{k}) = \underline{\mathrm{GT}}(k_1, \dots, k_n) = \bigsqcup_{\mathbf{l} \in \underline{[k_1,k_2]} \times \cdots \times \underline{[k_{n-1},k_n]}} \underline{\mathrm{GT}}(l_1, \dots, l_{n-1}).$$

One can think of an element of $\underline{\mathrm{GT}}(\mathbf{k})$ as a triangular array $A = (A_{i,j})_{1 \le j \le i \le n}$ of $\binom{n+1}{2}$ numbers, where we have $A_{i,j} \le A_{i-1,j} \le A_{i,j+1}$ or $A_{i,j} > A_{i-1,j} > A_{i,j+1}$ whenever all three terms are defined, and the sign of a GT pattern is 1 if and only if the number of descents $A_{i,j} > A_{i,j+1}$ is even.

We skip the full definition of a generalized monotone triangle, and just reiterate that if the bottom row of a monotone triangle is strictly increasing, then a monotone triangle is simply a GT pattern with strictly increasing rows, and its sign is 1.

4. Main steps of the construction

Recall that our goal is to find a bijective proof of the fact that the number of ASM of size $n \times n$ equals the number of DPP with elements $\le n$, and also that this number equals $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$. There are four main steps of the proof. The first step is described in detail in [FK19a], and the other three in [FK19b].

1. First, we construct a sijection between generalized monotone triangles and a certain disjoint union of GT patterns (called *shifted GT patterns*). That alone is a sijective proof of the above-mentioned operator formula for the number of monotone triangles with a fixed bottom row, but it allows us also to replace the complicated generalized monotone triangles with the more convenient shifted GT patterns in what follows. However, this is the first place in our proof where we produce signs; signs are necessary for shifted GT patterns even for strictly increasing bottom rows.
2. Then we show that the shifted GT patterns enjoy a certain rotational invariance (and, thanks to the sijection from the previous item, the same is true also for generalized monotone triangles). More precisely, performing a cyclic rotation of the prescribed bottom entries together with a certain shift leads, up to sign, to a signed set of the same size. This is proved by means of a sijection. For this step, it is necessary to allow also bottom rows that

are not necessarily increasing (simply because the "rotation" of an increasing bottom row is not increasing) and this makes the use of signs again unavoidable.

(3) In the next step, we use this sijection to construct "linear equations" for refined enumerations of ASMs. Such equations make sense for signed sets using the constructions we have introduced to mimic basic arithmetic operations, see (4).

(4) Finally, we use some "bijective linear algebra" (namely, we define the determinant of a family $[\underline{P}_{ij}]_{i,j=1}^m$ of signed sets as a signed set in a natural way, and then use a sijective version of Cramer's rule) to "solve" the system of linear equations, i.e., to construct bijections

$$\mathrm{DPP}_{n-1} \times \mathrm{ASM}_{n,i} \longrightarrow \mathrm{ASM}_{n-1} \times \mathrm{DPP}_{n,i}, \tag{5}$$

where ASM_n is the set of all ASMs of size $n \times n$, $\mathrm{ASM}_{n,i}$ is the subset of those with 1 in position $(1,i)$, DPP_n is the set of all DPPs with elements $\leq n$, and $\mathrm{DPP}_{n,i}$ is the subset of those with exactly $i-1$ occurrences of n. The system of linear equations can also be used to construct bijections

$$\mathrm{DPP}_{n-1} \times \mathrm{B}_{n,1} \times \mathrm{ASM}_{n,i} \longrightarrow \mathrm{DPP}_{n-1} \times \mathrm{ASM}_{n,1} \times \mathrm{B}_{n,i}, \tag{6}$$

where $\mathrm{B}_{n,i}$ is the set of $(2n-1)$-subsets of $\{1,2,\ldots,3n-2\}$ with median $n+i-1$. It can readily be checked that this serves as a bijective proof of (2).

Note that (5) and (6) involve seemingly unnecessary factors, which cancel when taking cardinalities. On the level of bijections it is somewhat more natural to keep these factors because division cannot be mimicked as naturally as the three other basic arithmetic operations (see (3) and (4)) by a construction for signed sets.

A method we use several times in Step 1 is to use disjoint unions and Cartesian products of the sijection α to construct some sijections for disjoint unions of signed boxes (Cartesian products of signed intervals), then to use disjoint unions of those to construct sijections for disjoint unions of GT patterns, and then to use those to construct sijections for monotone triangles. As an example, let us sketch one such construction.

Problem 2. [FK19a, Problem 2] *Given* $\mathbf{a} = (a_1,\ldots,a_{n-1}) \in \mathbb{Z}^{n-1}$, $\mathbf{b} = (b_1,\ldots,b_{n-1}) \in \mathbb{Z}^{n-1}$, $x \in \mathbb{Z}$, *write* $\underline{S}_i = (\{a_i\},\varnothing) \sqcup (\varnothing,\{b_i+1\})$, *and construct a sijection*

$$\beta = \beta_{\mathbf{a},\mathbf{b},x} \colon \underline{[a_1,b_1]} \times \cdots \times \underline{[a_{n-1},b_{n-1}]} \Longrightarrow \bigsqcup_{(l_1,\ldots,l_{n-1}) \in \underline{S}_1 \times \cdots \times \underline{S}_{n-1}} \underline{[l_1,l_2]} \times \underline{[l_2,l_3]} \times \cdots \times \underline{[l_{n-2},l_{n-1}]} \times \underline{[l_{n-1},x]}.$$

Construction. The case $n=2$ is constructed in Problem 1. For $n \geq 3$, we get, by induction, a sijection to

$$\left(\underline{[a_1,b_1]} \times \bigsqcup_{(l_3,\ldots,l_{n-1}) \in \underline{S}_3 \times \cdots \times \underline{S}_{n-1}} \underline{[a_2,l_3]} \times \cdots \times \underline{[l_{n-1},x]}\right) \sqcup \left(\underline{[a_1,b_1]} \times \bigsqcup_{(l_3,\ldots,l_{n-1}) \in \underline{S}_3 \times \cdots \times \underline{S}_{n-1}} (-\underline{[b_2+1,l_3]}) \times \cdots \times \underline{[l_{n-1},x]}\right),$$

and we use sijections α from $\underline{[a_1,b_1]}$ to $\underline{[a_1,a_2]} \sqcup (-\underline{[b_1+1,a_2]})$ and $\underline{[a_1,b_2+1]} \sqcup (-\underline{[b_1+1,b_2+1]})$, respectively. □

Problem 3. [FK19a, Problem 4] *Given* $\mathbf{a} = (a_1,\ldots,a_{n-1}) \in \mathbb{Z}^{n-1}$, $\mathbf{b} = (b_1,\ldots,b_{n-1}) \in \mathbb{Z}^{n-1}$, $x \in \mathbb{Z}$, *construct a sijection*

$$\rho = \rho_{\mathbf{a},\mathbf{b},x} \colon \bigsqcup_{\mathbf{l} \in \underline{[a_1,b_1]} \times \cdots \times \underline{[a_{n-1},b_{n-1}]}} \underline{\mathrm{GT}}(\mathbf{l}) \Longrightarrow \bigsqcup_{(l_1,\ldots,l_{n-1}) \in \underline{S}_1 \times \cdots \times \underline{S}_{n-1}} \underline{\mathrm{GT}}(l_1,\ldots,l_{n-1},x),$$

where $\underline{S}_i = (\{a_i\},\varnothing) \sqcup (\varnothing,\{b_i+1\})$.

Construction. Take a disjoint union (properly defined) of sijections β, and we obtain a sijection

$$\bigsqcup_{\mathbf{m}\in\underline{[a_1,b_1]}\times\cdots\times\underline{[a_{n-1},b_{n-1}]}} \underline{\mathrm{GT}}(\mathbf{m}) \Longrightarrow \bigsqcup_{\mathbf{m}\in\bigsqcup_{(l_1,\dots,l_{n-1})\in\underline{S}_1\times\cdots\times\underline{S}_{n-1}}\underline{[l_1,l_2]}\times\underline{[l_2,l_3]}\times\cdots\times\underline{[l_{n-2},l_{n-1}]}\times\underline{[l_{n-1},x]}} \underline{\mathrm{GT}}(\mathbf{m}).$$

By basic constructions, we get a sijection to

$$\bigsqcup_{(l_1,\dots,l_{n-1})\in\underline{S}_1\times\cdots\times\underline{S}_{n-1}} \bigsqcup_{\mathbf{m}\in\underline{[l_1,l_2]}\times\cdots\times\underline{[l_{n-2},l_{n-1}]}\times\underline{[l_{n-1},x]}} \underline{\mathrm{GT}}(\mathbf{m}),$$

and by definition of $\underline{\mathrm{GT}}$, this is equal to $\bigsqcup_{(l_1,\dots,l_{n-1})\in\underline{S}_1\times\cdots\times\underline{S}_{n-1}} \underline{\mathrm{GT}}(l_1,\dots,l_{n-1},x)$. □

After several such results concerning the signed sets of GT patterns, we can prove that the signed set of shifted GT patterns, denoted by $\underline{\mathrm{SGT}}(\mathbf{k})$, satisfy the same recursive identity as the signed set of generalized monotone triangles $\underline{\mathrm{MT}}(\mathbf{k})$. For monotone triangles with a strictly increasing bottom row, the recursion can be understood quite easily: if we delete the bottom row, say, $\mathbf{k} = (k_1,\dots,k_n)$ of a monotone triangle, then we obtain a monotone triangle with a new bottom row, say, $\mathbf{l} = (l_1,\dots,l_{n-1})$ where $k_1 \le l_1 \le k_2 \le l_2 \le \dots \le l_{n-1} \le k_n$ and $l_1 < l_2 < \dots < l_{n-1}$. It is also possible to write the resulting recursion more conveniently as a disjoint union over signed boxes. For $n = 3$, this would be

$$\underline{\mathrm{MT}}(k_1,k_2,k_3) = \bigsqcup_{(l_1,l_2)\in\underline{[k_1,k_2-1]}\times\underline{[k_2,k_3]}} \underline{\mathrm{MT}}(l_1,l_2) \sqcup \bigsqcup_{(l_1,l_2)\in\underline{[k_1,k_2]}\times\underline{[k_2+1,k_3]}} \underline{\mathrm{MT}}(l_1,l_2) \sqcup \left(- \bigsqcup_{(l_1,l_2)\in\underline{[k_1,k_2-1]}\times\underline{[k_2+1,k_3]}} \underline{\mathrm{MT}}(l_1,l_2) \right). \tag{7}$$

The difficult part is to show that the shifted GT patterns satisfy the same recursive identity, see [FK19a]; the construction uses many previously constructed sijections such as ρ. We omit the details here due to space limitations, but the resulting sijection is of the form

$$\Phi = \Phi_{\mathbf{k},x} \colon \bigsqcup_{\mu\in\underline{\mathrm{AR}}_n} \bigsqcup_{\mathbf{l}\in e(\mathbf{k},\mu)} \underline{\mathrm{SGT}}(\mathbf{l}) \Longrightarrow \underline{\mathrm{SGT}}(\mathbf{k}).$$

Here $\underline{\mathrm{AR}}_n$ is a certain (simple) signed set of *arrow rows*, and $e(\mathbf{k},\mu)$ is a certain deformation of $\mathbf{k}$. Together with the sijection that proves the same recursion for generalized monotone triangles, we obtain a sijection

$$\Gamma = \Gamma_{\mathbf{k},x} \colon \underline{\mathrm{MT}}(\mathbf{k}) \Longrightarrow \underline{\mathrm{SGT}}(\mathbf{k}) \tag{8}$$

by induction.

This allows one to prove statements for monotone triangles via GT patterns, which are much more accessible. For example, a crucial step in [FK19b] is the sijection

$$\underline{\mathrm{MT}}(\mathbf{k}) \Longrightarrow (-1)^{n-1}\,\underline{\mathrm{MT}}(\mathrm{rot}(\mathbf{k})),$$

where $\mathrm{rot}(\mathbf{k}) = (k_2,\dots,k_n,k_1-n)$. See [FK19b, Problem 16]. Note that the construction is far from easy, even assuming that we have the map Γ. See [FK19b, §6] for a proof.

Following several other constructions, we arrive at the following system of "linear equalities", i.e. sijections

$$\bigsqcup_{j=1}^{n} (-1)^{j+1} \binom{[2n-i-1]}{n-i-j+1} \times \mathrm{ASM}_{n,j} \Longrightarrow \mathrm{ASM}_{n,i} \tag{9}$$

for $i \in \{1,\dots,n\}$. See [FK19b, Problem 22].

To complete the construction, we need, among some other results, a few ingredients from "bijective linear algebra". Denote by $\underline{\mathfrak{S}}_m$ the signed set of permutations (with the usual sign). Given signed sets $\underline{P}_{i,j}$, $1 \le i,j \le m$, define the *determinant* of $\underline{\mathcal{P}} = [\underline{P}_{ij}]_{i,j=1}^m$ as the signed set $\det(\underline{\mathcal{P}}) = \bigsqcup_{\pi \in \underline{\mathfrak{S}}_m} \underline{P}_{1,\pi(1)} \times \cdots \times \underline{P}_{m,\pi(m)}$. Among other classical properties, we have the following version of Cramer's rule.

Problem 4. [FK19b, Problem 9] *Given* $\underline{\mathcal{P}} = [\underline{P}_{p,q}]_{p,q=1}^m$*, signed sets* $\underline{X}_i, \underline{Y}_i$ *and sijections* $\bigsqcup_{q=1}^m \underline{P}_{i,q} \times \underline{X}_q \Rightarrow \underline{Y}_i$ *for all* $i \in [m]$*, construct sijections*

$$\det(\underline{\mathcal{P}}) \times \underline{X}_j \Longrightarrow \det(\underline{\mathcal{P}}^j),$$

where $\underline{\mathcal{P}}^j = [\underline{P}^j_{p,q}]_{p,q=1}^m$*,* $\underline{P}^j_{p,q} = \underline{P}_{p,q}$ *if* $q \neq j$*,* $\underline{P}^j_{p,j} = \underline{Y}_p$*, for all* $j \in [m]$*.*

Essentially, sijections like the one in Problem 4 tell us that "linear equalities" for sijections like (9) can be used to find sijections on the signed sets involved. As a result, we get a sijection (and hence a bijection) between $\mathrm{DPP}_{n-1} \times \mathrm{ASM}_{n,j}$ and $\mathrm{ASM}_{n-1} \times \mathrm{DPP}_{n,j}$. By induction, that implies that $\mathrm{ASM}_{n,j}$ and $\mathrm{DPP}_{n,j}$ have the same number of elements, so we have indeed constructed the first bijective proof of this result. Similar considerations lead to the bijection (6).

We expect that one can use similar techniques to find other elusive bijective proofs, both for results related to alternating sign matrices, and in other areas of enumerative combinatorics. We intend to use them to connect ASMs and DPPs with the remaining two classes of objects mentioned here, TSSCPPs and ASTs.

Acknowledgments

We thank Matija Pretnar, Alex Simpson and Doron Zeilberger for interesting conversations and helpful suggestions. We are also grateful to the referees for reading the manuscript so carefully, and for a number of suggested improvements.

References

[ABF16] A. Ayyer, R. Behrend, and I. Fischer. Extreme diagonally and antidiagonally symmetric alternating sign matrices of odd order, 2016. preprint.

[And79] G. Andrews. Plane partitions. III. The weak Macdonald conjecture. *Invent. Math.*, 53(3):193–225, 1979.

[And86] George E. Andrews. *q-series: their development and application in analysis, number theory, combinatorics, physics, and computer algebra*, volume 66 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.

[And94] G. Andrews. Plane partitions. V. The TSSCPP conjecture. *J. Combin. Theory Ser. A*, 66(1):28–39, 1994.

[Bau19] A. Bauer. What is an explicit bijection? FPSAC 2019 invited lecture, available at `http://videolectures.net/FPSAC2019_bauer_explicit_bijection/`, 2019.

[BC16] Philippe Biane and Hayat Cheballah. Gog and GOGAm pentagons. *J. Combin. Theory Ser. A*, 138:133–154, 2016.

[BDFZJ13] R. Behrend, P. Di Francesco, and P. Zinn-Justin. A doubly-refined enumeration of alternating sign matrices and descending plane partitions. *J. Combin. Theory Ser. A*, 120(2):409–432, 2013.

[BFK17] R. Behrend, I. Fischer, and M. Konvalinka. Diagonally and antidiagonally symmetric alternating sign matrices of odd order. *Adv. Math.*, 315:324–365, 2017.

[BP99] D. Bressoud and J. Propp. How the alternating sign matrix conjecture was solved. *Notices Amer. Math. Soc.*, 46(6):637–646, 1999.

[Bre99] D. Bressoud. *Proofs and confirmations. The story of the alternating sign matrix conjecture*. MAA Spectrum. Mathematical Association of America and Cambridge University Press, Washington, DC and Cambridge, 1999.

[Fis05] I. Fischer. A method for proving polynomial enumeration formulas. *J. Combin. Theory Ser. A*, 111:37–58, 2005.

[Fis06] I. Fischer. The number of monotone triangles with prescribed bottom row. *Adv. in Appl. Math.*, 37(2):249–267, 2006.

[Fis07] I. Fischer. A new proof of the refined alternating sign matrix theorem. *J. Combin. Theory Ser. A*, 114(2):253–264, 2007.

[Fis10] I. Fischer. The operator formula for monotone triangles—simplified proof and three generalizations. *J. Combin. Theory Ser. A*, 117(8):1143–1157, 2010.

[Fis11] I. Fischer. Refined enumerations of alternating sign matrices: monotone (d, m)-trapezoids with prescribed top and bottom row. *J. Algebraic Combin.*, 33(2):239–257, 2011.

[Fis12a] I. Fischer. Linear relations of refined enumerations of alternating sign matrices. *J. Combin. Theory Ser. A*, 119(3):556–578, 2012.

[Fis12b] I. Fischer. Sequences of labeled trees related to gelfand-tsetlin patterns. *Adv. in Appl. Math.*, 49:165–195, 2012.

[Fis16] I. Fischer. Short proof of the asm theorem avoiding the six-vertex model. *J. Combin. Theory Ser. A*, 144:139–156, 2016.

[FK19a] I. Fischer and M. Konvalinka. A bijective proof of the ASM theorem, part I: the operator formula. preprint, arXiv: 1910.04198, 2019.

[FK19b] I. Fischer and M. Konvalinka. A bijective proof of the ASM theorem, part II: ASM enumeration and ASM-DPP relation. preprint, arXiv: 1912.01354, 2019.

[FZJ08] T. Fonseca and P. Zinn-Justin. On the doubly refined enumeration of alternating sign matrices and totally symmetric self-complementary plane partitions. *Electron. J. Combin.*, 15:Research Paper 81, 35 pp. (electronic), 2008.

[GM81a] A. M. Garsia and S. C. Milne. Method for constructing bijections for classical partition identities. *Proc. Nat. Acad. Sci. U.S.A.*, 78(4, part 1):2026–2028, 1981.

[GM81b] A. M. Garsia and S. C. Milne. A Rogers-Ramanujan bijection. *J. Combin. Theory Ser. A*, 31(3):289–339, 1981.

[Kra96] C. Krattenthaler. A gog-magog conjecture, 1996.

[Kra16] C. Krattenthaler. Plane partitions in the work of Richard Stanley and his school. In *The mathematical legacy of Richard P. Stanley*, pages 231–261. Amer. Math. Soc., Providence, RI, 2016.

[Kup96] G. Kuperberg. Another proof of the alternating-sign matrix conjecture. *Internat. Math. Res. Notices*, (3):139–150, 1996.

[MRR82] W. Mills, D. Robbins, and H. Rumsey, Jr. Proof of the Macdonald conjecture. *Invent. Math.*, 66(1):73–87, 1982.

[MRR83] W. Mills, D. Robbins, and H. Rumsey, Jr. Alternating sign matrices and descending plane partitions. *J. Combin. Theory Ser. A*, 34(3):340–359, 1983.

[MRR86] W. Mills, D. Robbins, and H. Rumsey, Jr. Self-complementary totally symmetric plane partitions. *J. Combin. Theory Ser. A*, 42(2):277–292, 1986.

[RR86] David P. Robbins and Howard Rumsey, Jr. Determinants and alternating sign matrices. *Adv. in Math.*, 62(2):169–184, 1986.

[Voe17] V. Voevodsky. C-systems defined by universe categories: presheaves. arXiv = 1706.03620, 2017.

[Zei96a] D. Zeilberger. Proof of the alternating sign matrix conjecture. *Electron. J. Combin.*, 3(2):Research Paper 13, 84 pp. (electronic), 1996. The Foata Festschrift.

[Zei96b] D. Zeilberger. Proof of the refined alternating sign matrix conjecture. *New York J. Math.*, 2:59–68 (electronic), 1996.

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090, Vienna, Austria
Email address: `ilse.fischer@univie.ac.at`

Department of Mathematics, University of Ljubljana & Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
Email address: `matjaz.konvalinka@fmf.uni-lj.si`