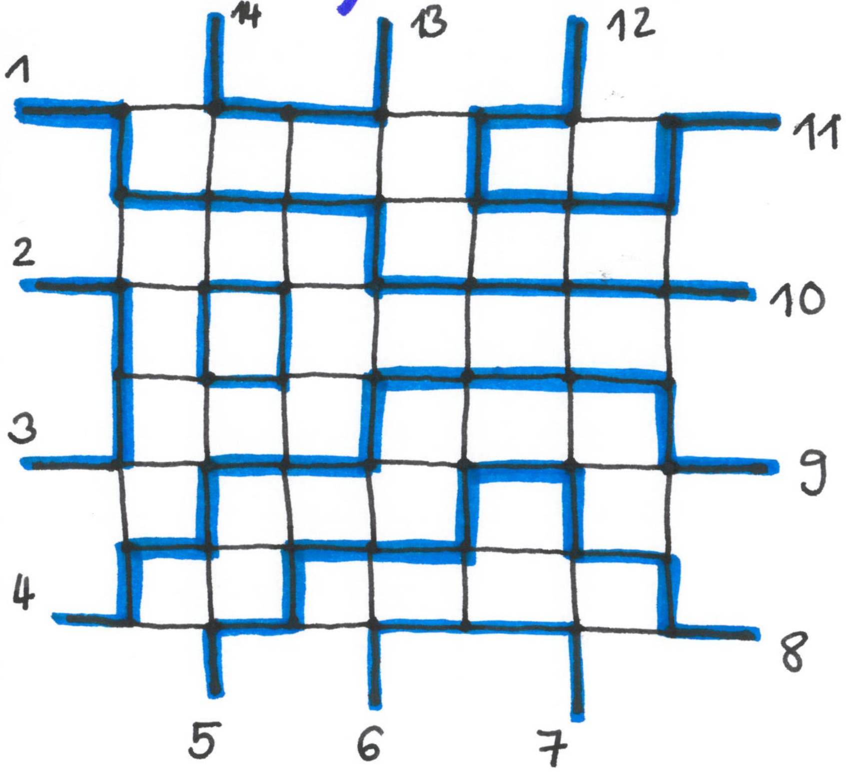
Fully Packed Loop configurations on $\triangle$s and Path Tangles

Ilse Fischer, Uni Wien

Joint work with Philippe Nadeau

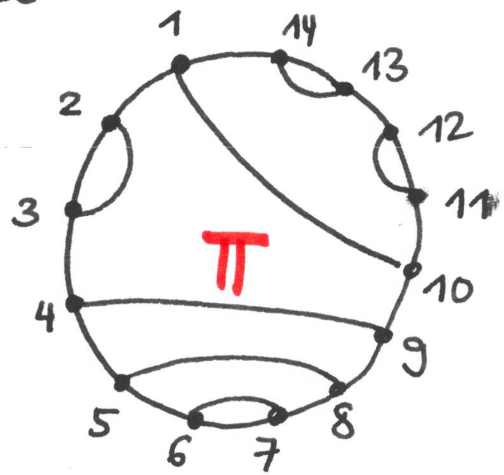
Fully Packed Loop configurations on $\square$s = FPL



Playground = Square grid G_n with n^2 vertices and $2n$ external edges chosen alternatively

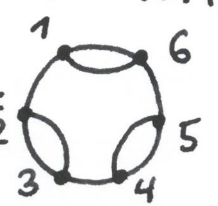
FPL = Subgraph of G_n such that each internal vertex is of degree 2 and contains all external edges

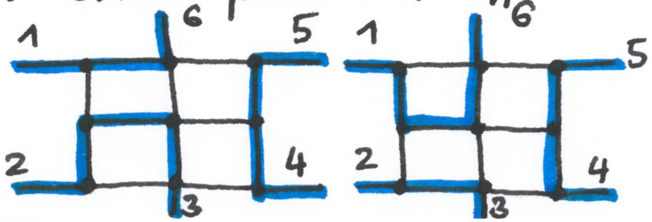
LINK PATTERN = set of n non-crossing chords between $2n$ points on a disk



A_π = # FPLS with link pattern π

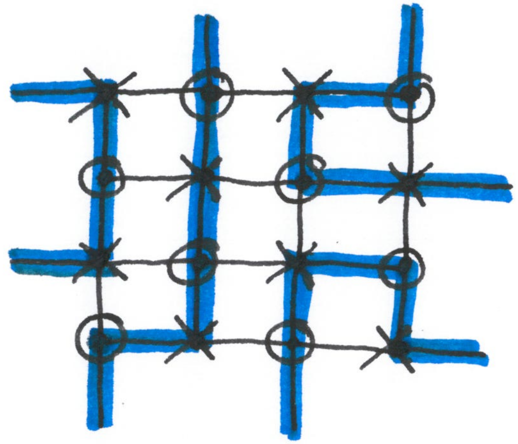
Ex: $\pi =$

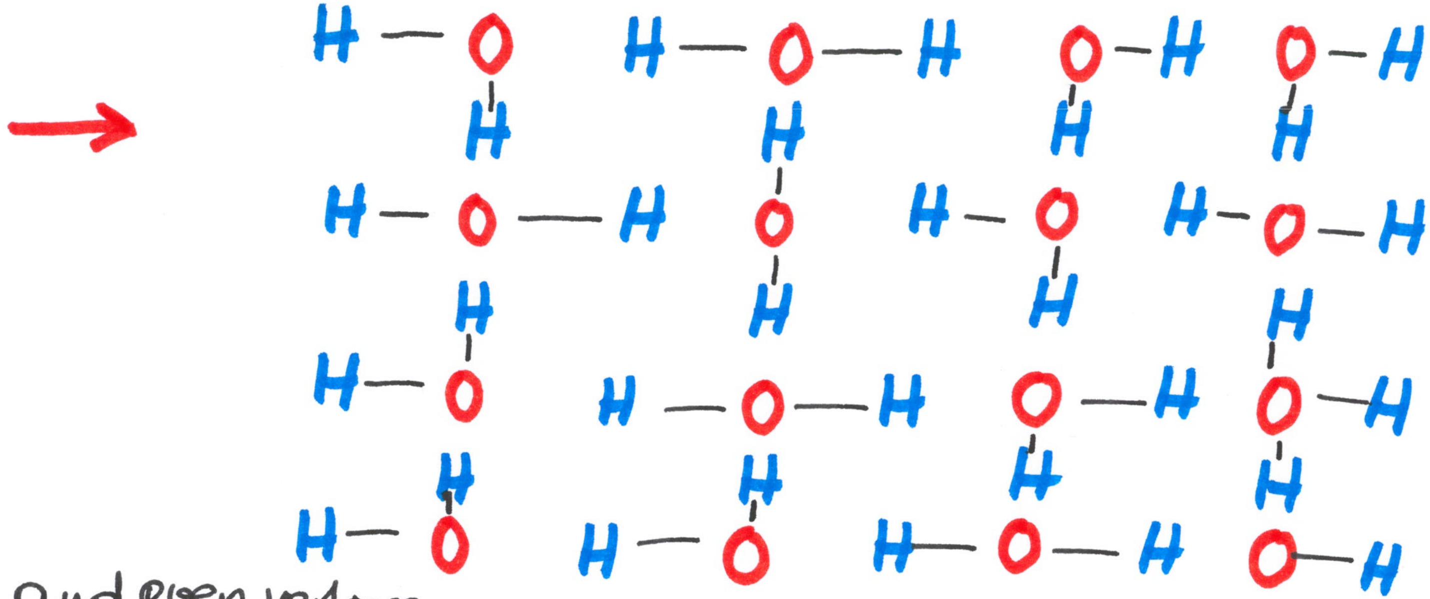




$\Rightarrow A_\pi = 2$

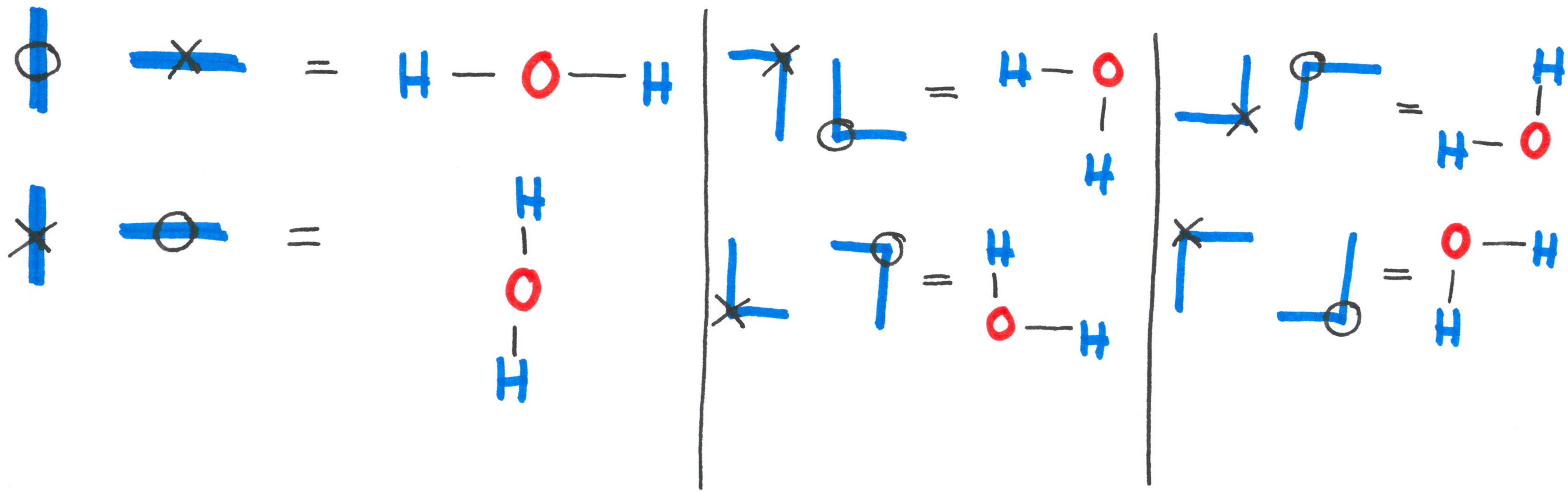
FPL = 2-dim arrangement of water molecules





Dictionary:

Distinguish between odd and even vertices



Some classical results on the $\square$

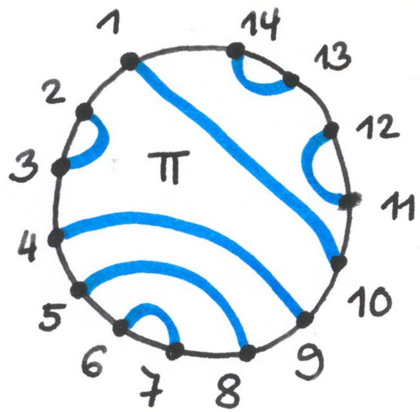
1) THEOREM (Zeilberger 1996):

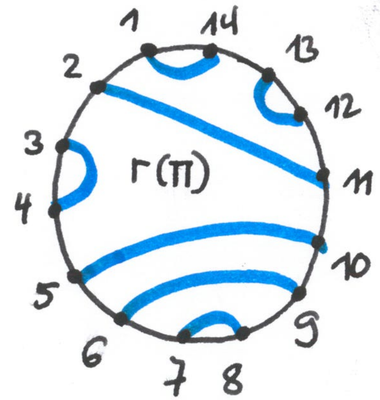
The total number of FPLs on a $n \times n$ grid is

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} \cong \left(\frac{3\sqrt{3}}{4}\right)^{n^2}.$$

$A_n = 1, 2, 7, 42, 429, 7436, 218348, \ldots.$

2) Rotational invariance:





$\pi : i \frown j \iff r(\pi) : i+1 \frown j+1 \mod 2n$

THEOREM (Ben Wieland, 2000):

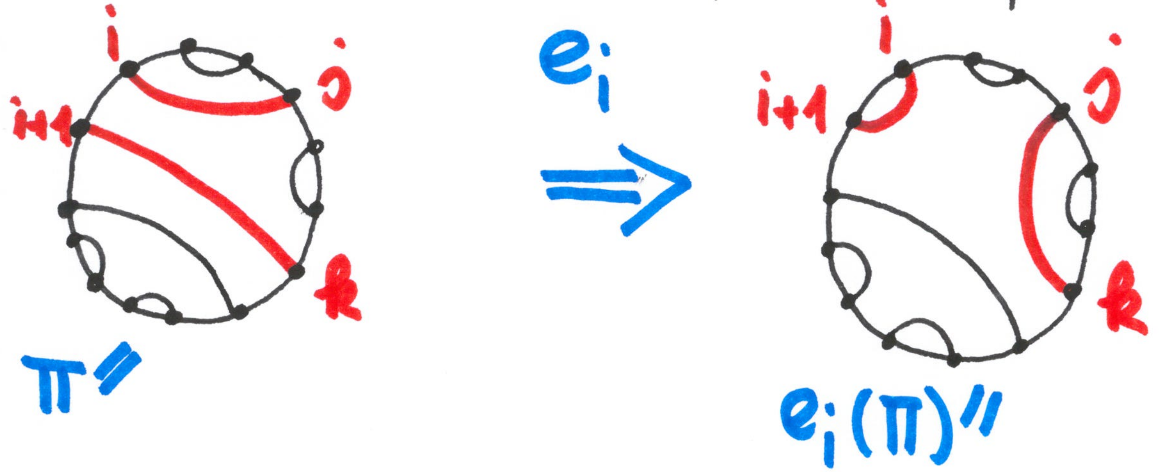
$\forall \pi : \quad A_\pi = A_{r(\pi)}$

Bijective proof:

Wieland gyration!

3) Razumov-Stroganov ex-Conjecture:

Define transformation e_i on link patterns:



$$\pi : i \frown j,\ i+1 \frown k$$

$$\Downarrow$$

$$e_i(\pi) : i \frown i+1,\ j \frown k$$

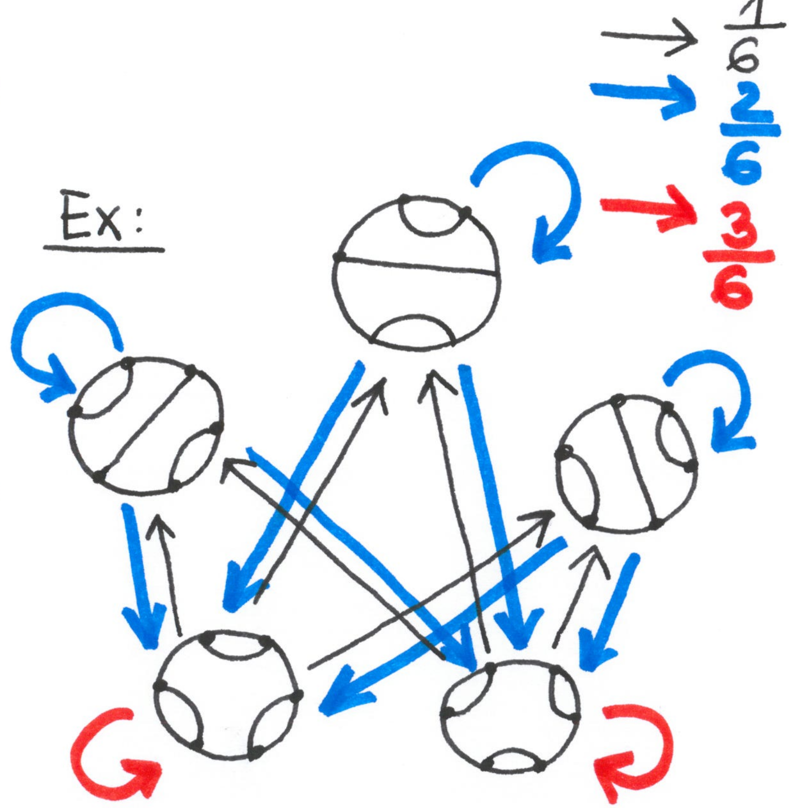
A MARKOV CHAIN ON LINK PATTERNS:

STATES = LINK PATTERNS OF SIZE n "LP_n"

Transition probabilities:

$$P(\pi_1 \to \pi_2) = \frac{\#\{i \in \{1,2,\dots,2n\} : e_i(\pi_1) = \pi_2\}}{2n}$$

Ex:



STATIONARY DISTRIBUTION

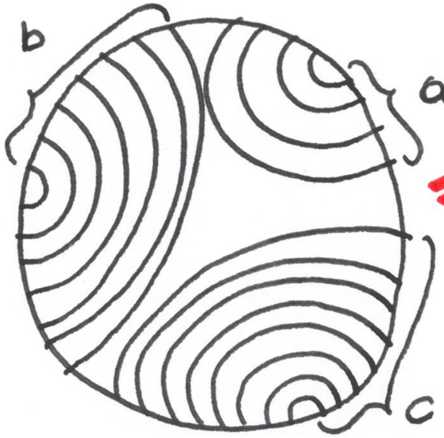
Perron-Frobenius $\Rightarrow$ Markov chain has a unique stationary distribution

Ex-Conjecture (Cantini / Sportiello, 2010):

The stationary distribution $(\Psi_\pi)_{\pi \in LP_n}$ is given by

$$\Psi_\pi = \frac{A_\pi}{A_n} . \qquad A_n = \sum_{\pi \in LP_n} A_\pi$$

4) A_π for special π's



$= \pi_{a,b,c}$

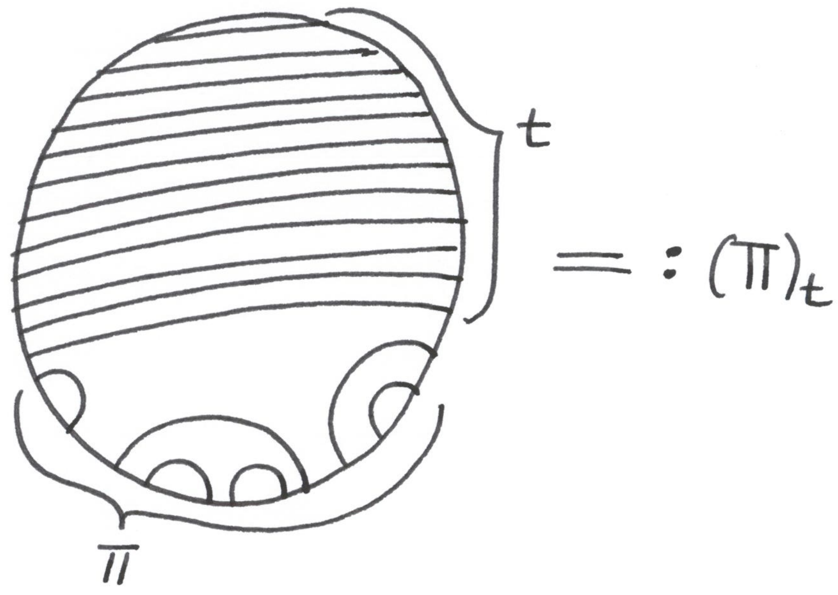
THEOREM (DiFrancesco, Zinn-Justin, Zuber, 2004):

$$A_{\pi_{a,b,c}} = \prod_{i=1}^{a} \frac{(c+i)(c+i+1)\cdot\ldots\cdot(c+i+b-1)}{i\cdot(i+1)\cdot\ldots\cdot(i+b-1)}$$

= # of plane partitions in an $a \times b \times c$-Box

A "qualitative" result for A_π

Link patterns with nested arcs:

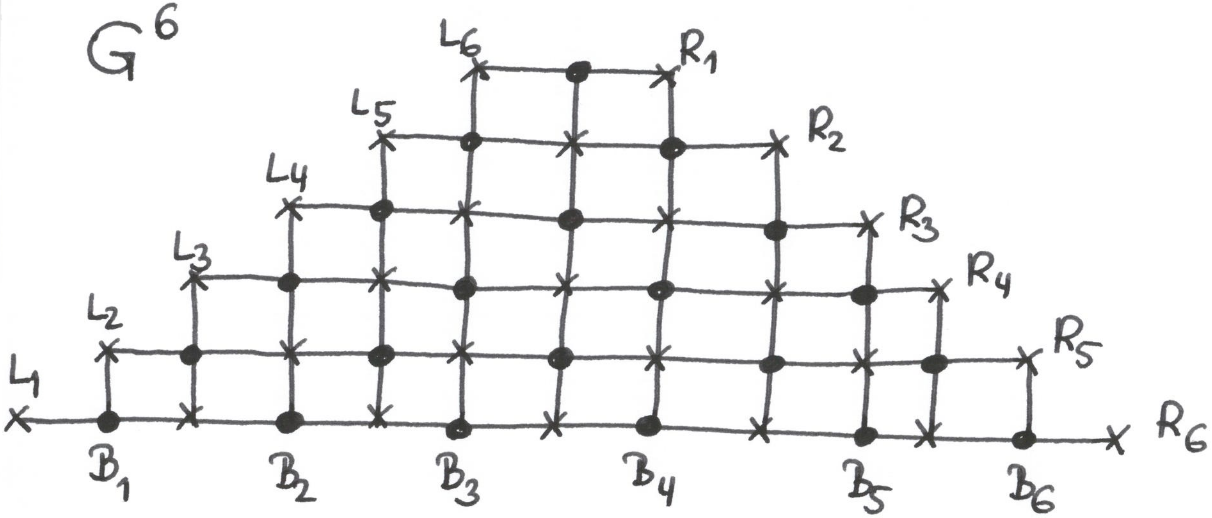


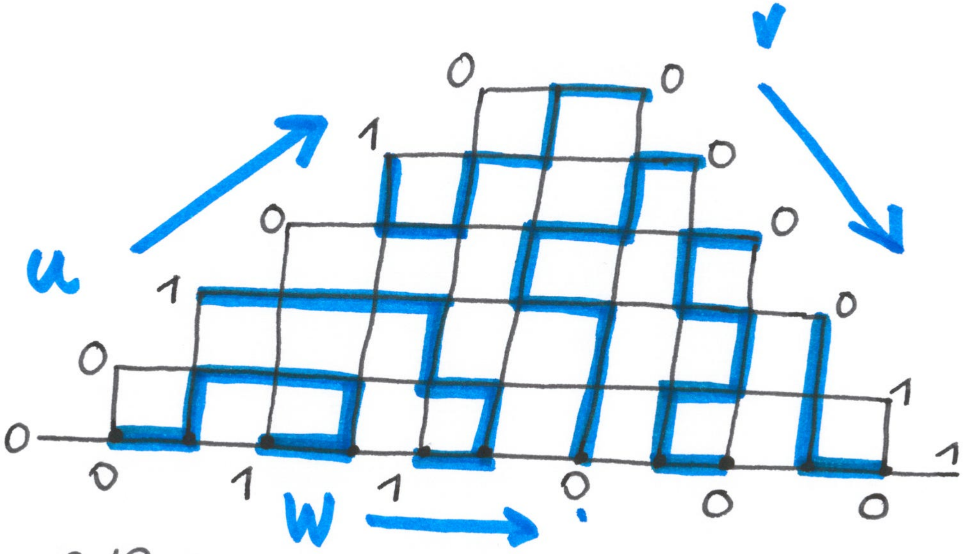
THEOREM (Casselli, Krattenthaler, Lass, Nadeau, 2005):

The quantity $A_{(\pi)_t}$ is a polynomial function in t for fixed link pattern π.

Fully Packed Loops in a triangle were invented for the proof!

FPLs on Δs = TFPLs



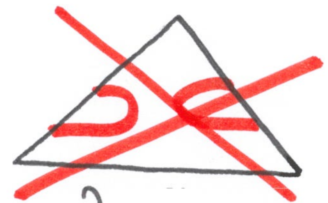


<u>DEF:</u> A TFPL f is a subgraph of G^N such that:

(1) $\deg(L_i) \in \{0,1\}$, $\deg(R_i) \in \{0,1\}$

(2) $\deg(B_i) = 1$

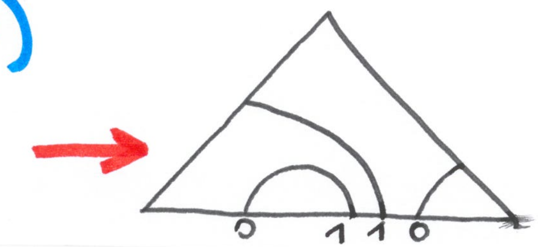
(3) All other vertices have degree 2.



(4) A path in f cannot join two vertices in $\{L_1, L_2, \ldots, L_N\}$, nor two vertices in $\{R_1, \ldots, R_N\}$.

Boundary: $u = (0,0,1,0,1,0)$, $v = (0,0,0,0,1,1)$

$w = (0,1,1,0,0,0)$

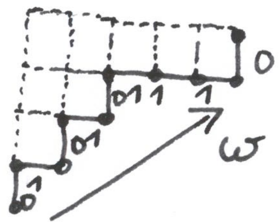


Q: Does there exist a TFPL with boundary (u,v,w) for any choice of 01-words of the same length?

No!

DEFs:

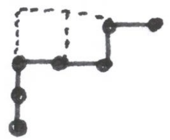
1) $w = 010101110 \iff$ FERRERS DIAGRAM $\lambda(w)$

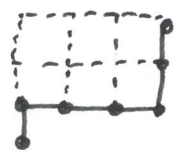


2) $d(w)$ = # of cells in $\lambda(w)$ = # of "inversions" = 8

3) Partial order on 01-words: $u \leq v \iff \lambda(u) \subseteq \lambda(v)$

Ex: $001101 \leq 011100$

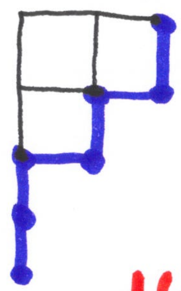




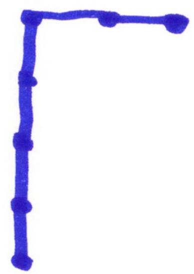
4) $|w|_1$ = # of 1s = 5 , $|w|_0$ = # of 0s = 4

In our example.....

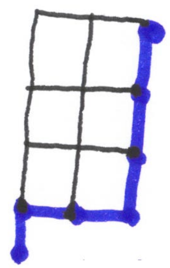
$u=(0,0,1,0,1,0)$ $\quad$ $v=(0,0,0,0,1,1)$ $\quad$ $w=(0,1,1,0,0,0)$



$d(u)=3$



$d(v)=0$



$d(w)=6$

Observations :

1) $|u|_1 = |v|_1 = |w|_1$
2) $u \leq w$ and $v \leq w$
3) $d(u)+d(v) \leq d(w)$

This is true in general!

THEOREM: Let u, v, w be 01-words of the same length.
There can only exists a TFPL with boundary $(u, v; w)$
if the following conditions are fulfilled:

(1) $|u|_1 = |v|_1 = |w|_1$

(2) $u \leq w$ and $v \leq w$

(3) $d(u) + d(v) \leq d(w)$

---

u, v, w Dyck words:

(2) First proof by Caselli, Krattenthaler, Lass, Nadeau, 2005
Philippe: "tedious argument"

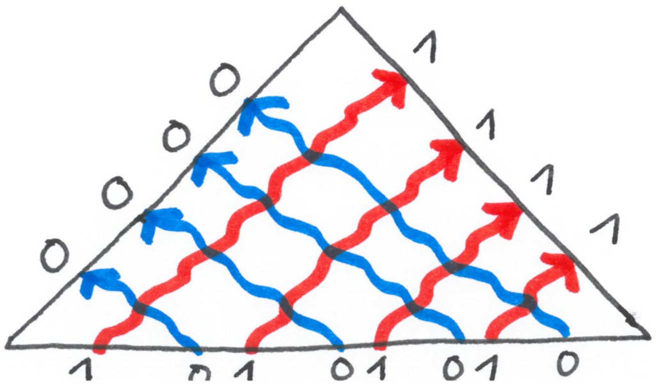
(3) First proof by Thapper, 2007: very nice proof involving a polynomial argument and Wieland gyration.

BUT: It should be possible to read off such constraints "directly" from the objects.

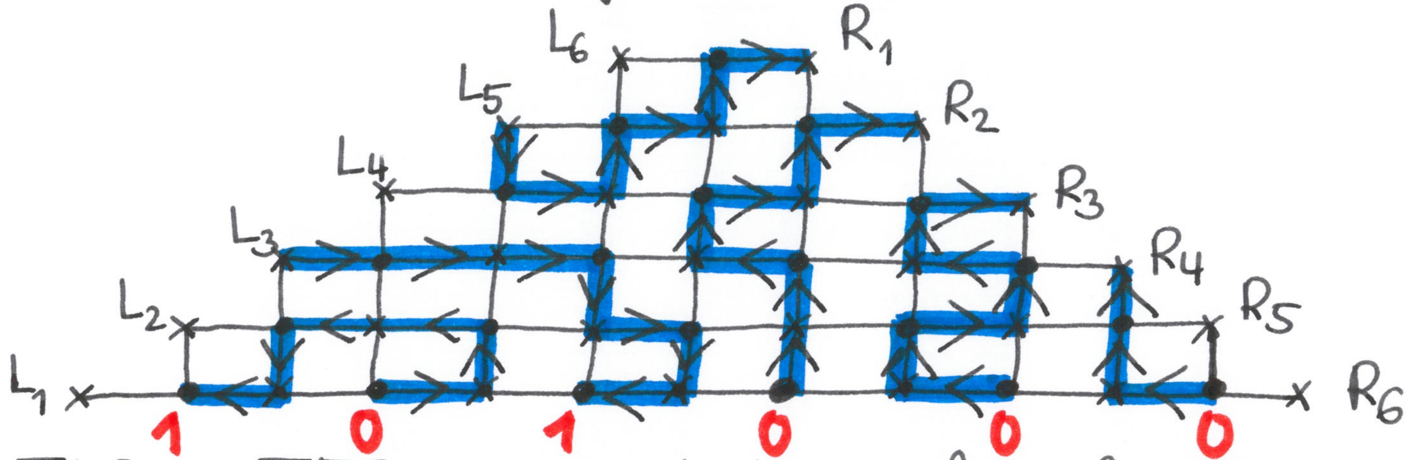
However, this does not seem to be the case for TFPLS

THEREFORE: translate into other objects where the constraints can be read off directly!

⇒ PATH TANGLES



ORIENTED TFPLs : How to get rid of the two nasty "global" conditions in the definition of TFPLs

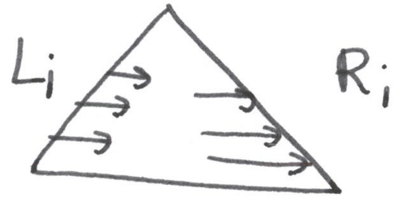


ORIENTED TFPL = TFPL + orientation of each edge such that:

(1) Each vertex of degree 2 has one incoming and one outgoing edge.

(2) The edges of R_i are incoming.

(3) The edges of L_i are outgoing.



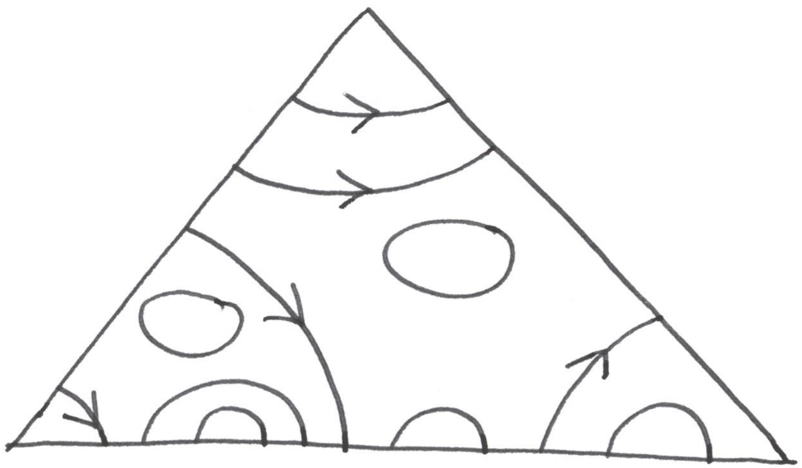
Cond. 4 can be omitted!

Boundary: $u = (0,0,1,0,1,0)$

$v = (0,0,0,0,1,1)$

$w = (1,0,1,0,0,0)$ ← "local" definition!

CONNECTION BETWEEN TFPLs and oriented TFPLs



- ORIENTATION OF PATHS THAT START/END on LEFT/RIGHT BOUNDARY IS FORCED
- TWO POSSIBLE ORIENTATIONS FOR LOOPS
- THE BOUNDARY WORDS OF non-oriented and oriented TFPL coincide iff all paths are oriented from left to right

$\Rightarrow$ If there exists a TFPL with boundary (u,v,w) then there also exists an oriented TFPL with boundary (u,v,w).

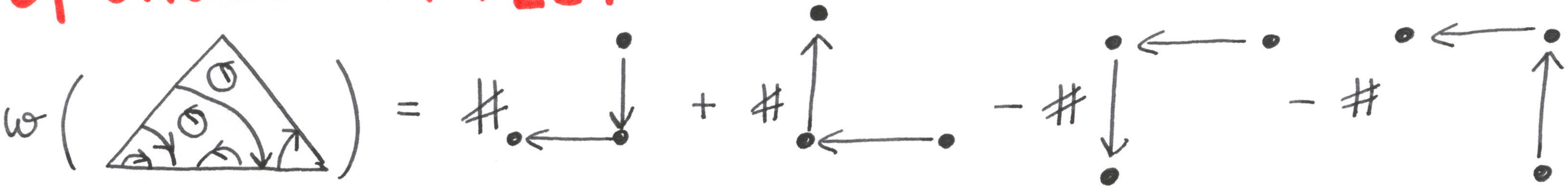
! It suffices to show the constraints on the boundary for oriented TFPLs !

CLOSER CONNECTION

$t^{w}_{u,v}$ = # of non-oriented TFPLs with boundary (u,v,w)

$\vec{t}^{\,w}_{u,v}$ = # of oriented TFPLs with boundary (u,v,w)

It suffices to consider oriented TFPLs, since $t^{w}_{u,v}$ can be derived from a certain weighted enumeration of oriented TFPLs.



$$\vec{t}^{\,w}_{u,v}(q) := \sum_{\substack{f \text{ oriented TFPL} \\ \text{with boundary } (u,v,w)}} q^{w(f)}$$

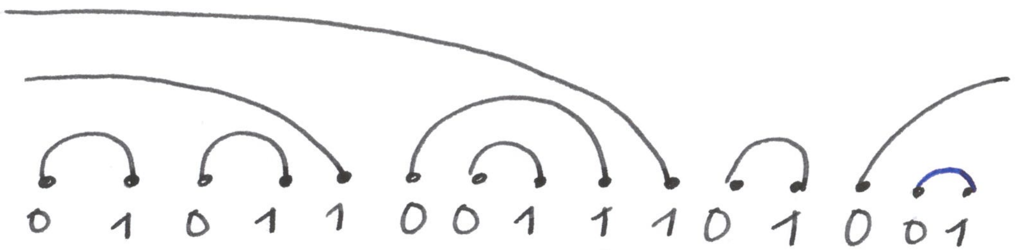
THE MATRIX $M(N_0,N_1)$

$N_0, N_1 \geq 0$ integers; $M(N_0,N_1)$ = Matrix whose rows and columns are indexed by 01-words with N_0 zeros and N_1 ones.

$$M(N_0,N_1)_{w,w'} = \begin{cases} q^{RL(w,w')} & w' \text{ feasible for } w \\ 0 & \text{else} \end{cases}$$

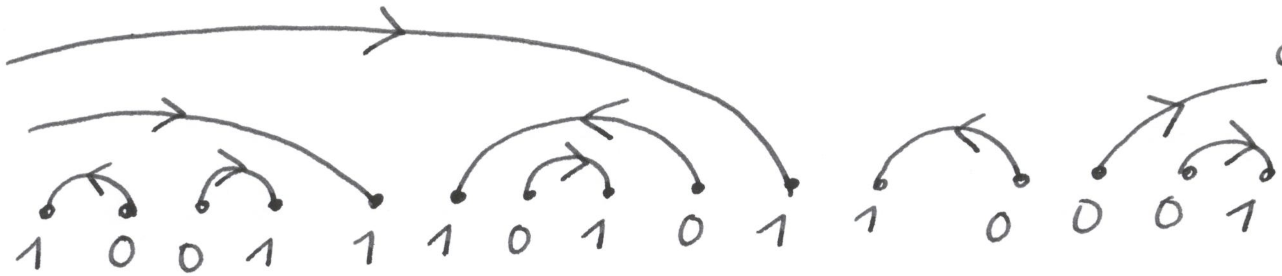
FEASIBILITY:

$w' = 010110011101001 \Rightarrow \pi =$



extended link pattern

$\Rightarrow$



$\Rightarrow w = 100111010110$

source-sink-word

directed extended link pattern

$RL(w,w')$ = # of right-left-arcs = 3

THEOREM (F & NADEAU):

u, v, w 01-words of length N_0+N_1 and with $|u|_0=|v|_0=|w|_0=N_0$; $M=M(N_0,N_1)$. Then the following relation holds

$$t_{u,v}^{w} = \sum_{w'} (M^{-1})_{w,w'} \vec{t}_{u,v}^{\,w'}(\rho)$$

where $\rho = \frac{1}{2} + \frac{i\sqrt{3}}{2}$.

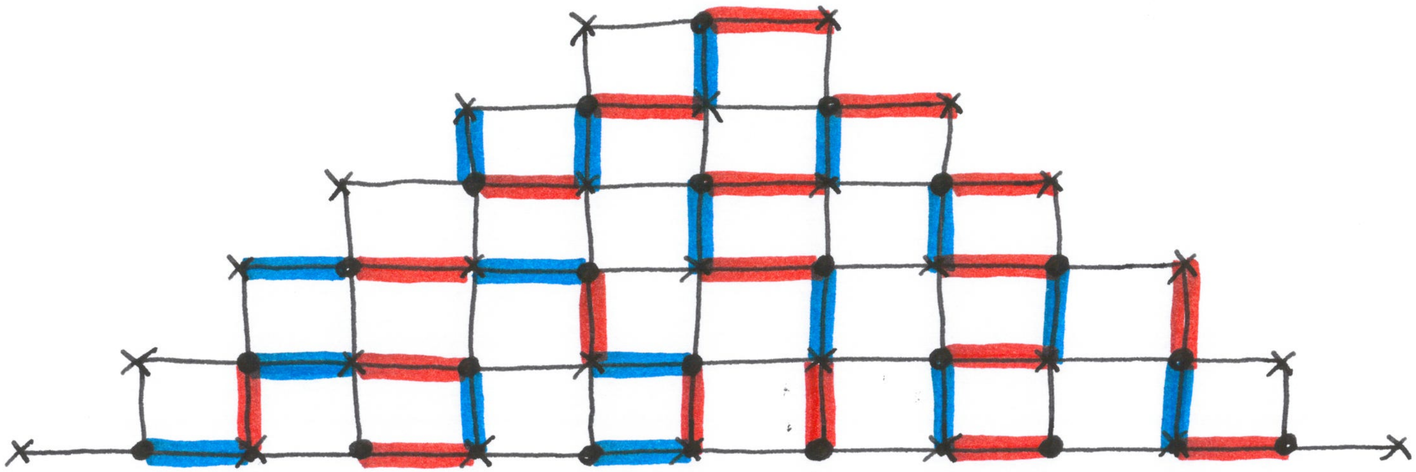
REM:
- $M(N_0,N_1)$ was studied by Kenyon and Wilson, 2011, and Jang Soo Kim, 2012: $(M^{-1})_{w,w'}$ is the generating function of certain "Dyck tilings" of the skew shape $\lambda(w)/\lambda(w')$
- Independently: Shigechi and Zinn-Justin, 2012, show that $M(N_0,N_1)$ encodes a change of basis in certain representations of the Hecke algebra. Also provide the Dyck tiling interpretation and relate it to an older combinatorial rule of Lascoux and Schützenberger.

ORIENTED TFPLs ⟹ PATH TANGLES

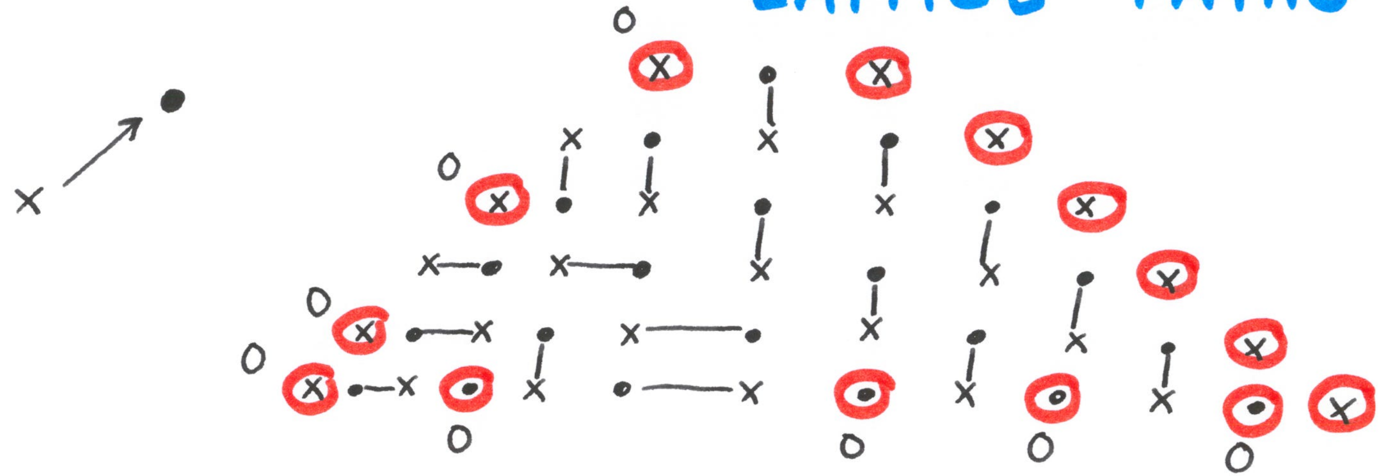
An oriented TFPL is uniquely determined by two disjoint matchings:

1. Matching:

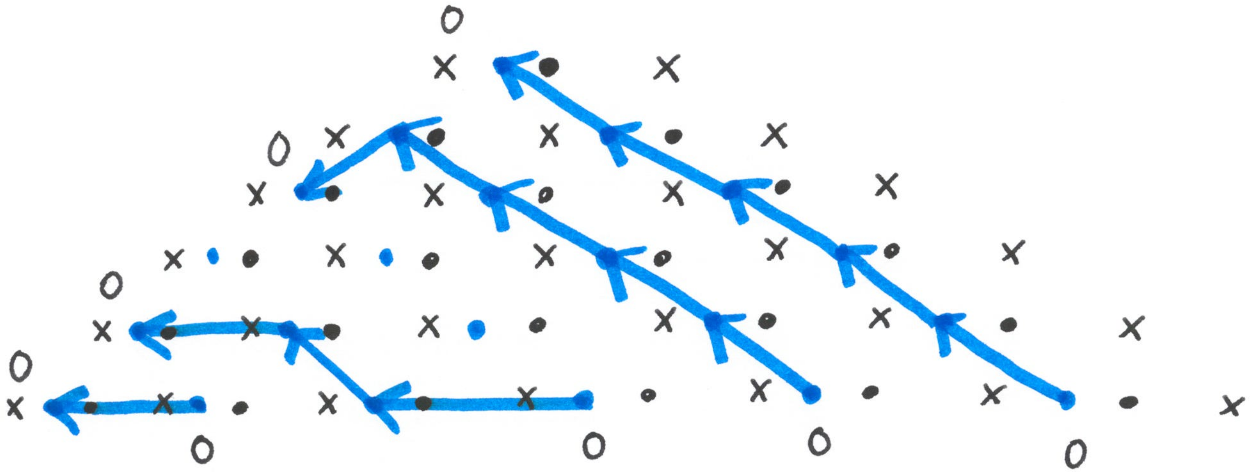
2. Matching:



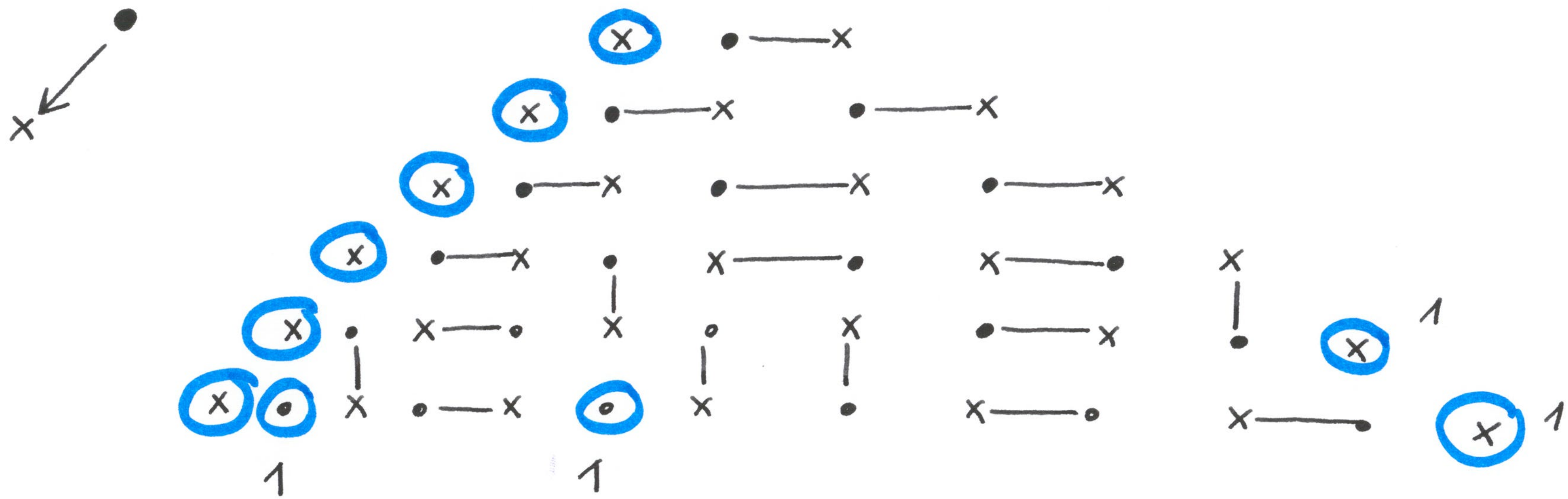
FIRST MATCHING ⇒ BLUE FAMILY OF NON-INTERSECTING LATTICE PATHS



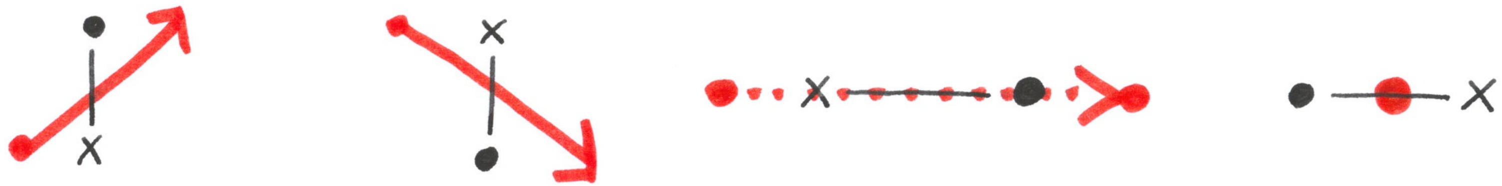
Introduce • right of x and perform the following:

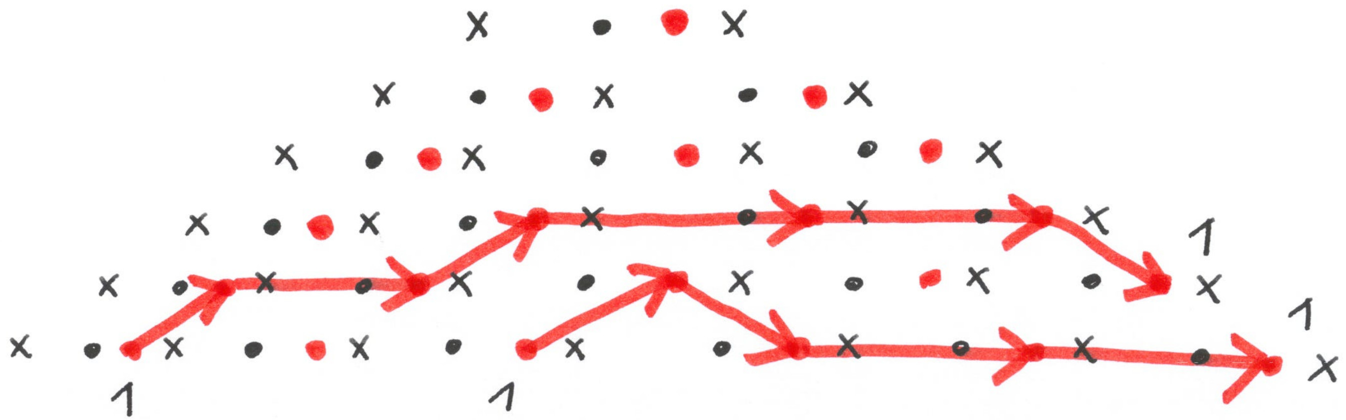


SECOND MATCHING ⇒ RED FAMILY OF NON-INTERSECTING LATTICE PATHS



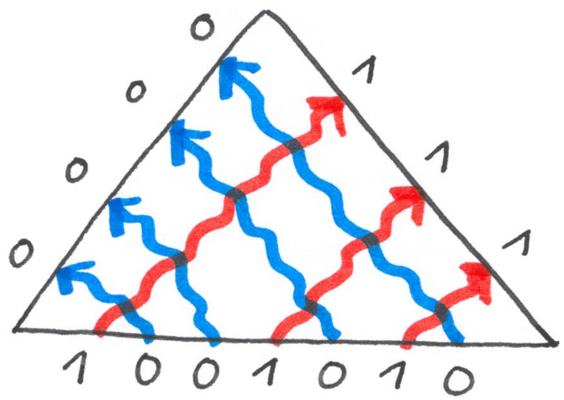
Introduce $\color{red}\bullet$ left of $\times$ and perform the following:





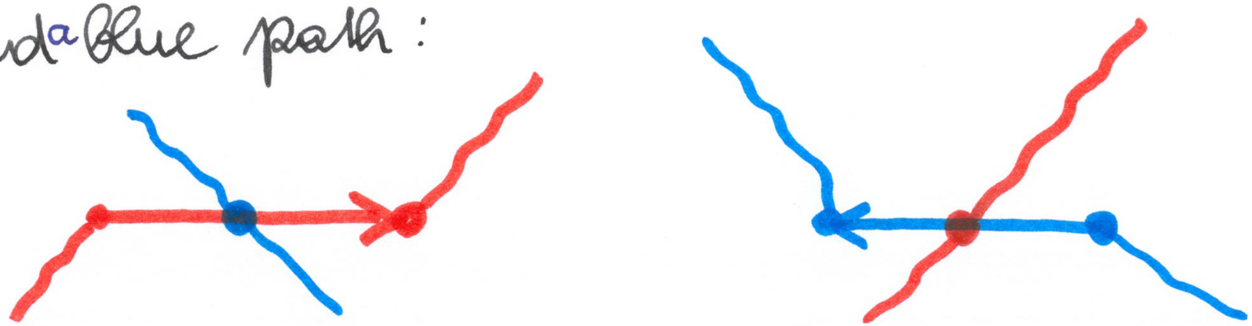
PATH TANGLES

In one picture:



The fact that the two matchings are disjoint translates into the fact that the following crossings do not occur:

and that each horizontal step is involved in a crossing of a red and a blue path:

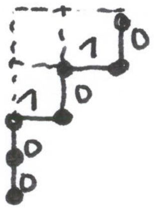


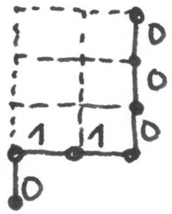
COMING BACK TO THE CONDITIONS ON (u,v,w)

1) $|u|_0 = |w|_0$ and $|v|_1 = |w|_1$

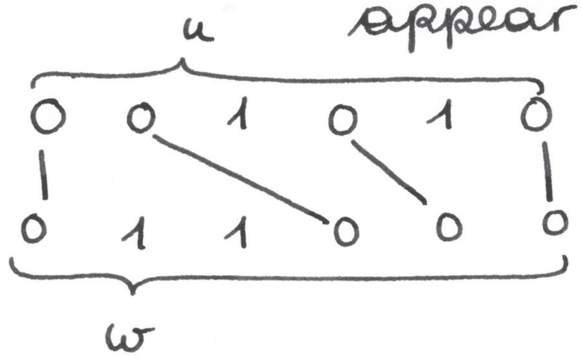
2) $\lambda(u) \subseteq \lambda(w)$ [by symmetry $\Rightarrow \lambda(v) \subseteq \lambda(w)$]

Ex: $u = 001010$ $w = 011000$

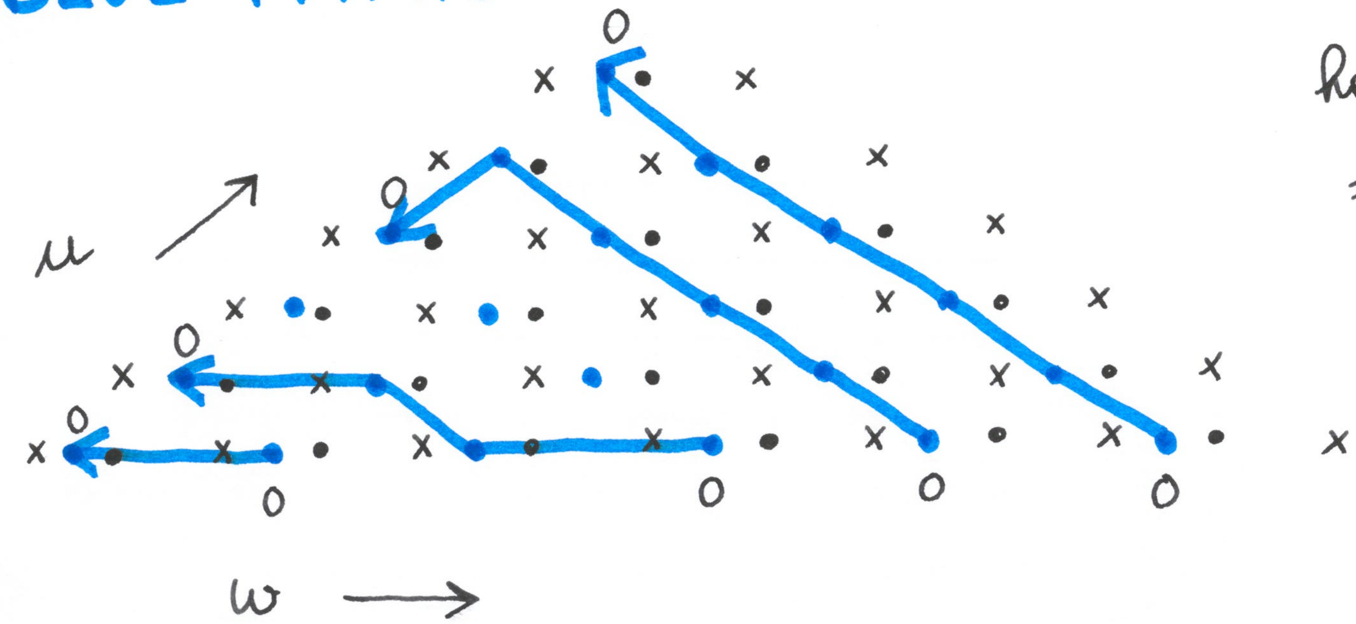




$\lambda(u) \subseteq \lambda(w) \Leftrightarrow$ corresponding zeros appear later in w



BLUE PATHS:



how much later $=$

$\# \leftarrow + \# \swarrow =$

$d(w) - d(u)$

Proof of $d(u)+d(v) \leq d(w)$

1) $d(w)$ = # inversions in w = # pairs of red and blue paths that cross

$$\leq \# \leftarrow \; + \; \# \rightarrow$$

↓ every crossing involves a horizontal step

2) $d(w) - d(u) = \# \swarrow \; + \; \# \leftarrow$

3) $d(w) - d(v) = \# \searrow \; + \; \# \rightarrow$

Now: $d(w) - d(u) - d(v) = (d(w) - d(u)) + (d(w) - d(v)) - d(w) \geq$

$$\geq (\swarrow + \leftarrow) + (\searrow + \rightarrow) - (\leftarrow + \rightarrow)$$

$$= \swarrow + \searrow \geq 0$$

COMBINATORIAL INTERPRETATION FOR d(w)-d(u)-d(v)

A more careful analysis and translating it back to TFPLs leads to:

THEOREM (F & Nadeau):

For any oriented TFPL with boundary (u,v,w), one has the following formula:

$$d(w)-d(u)-d(v) := \left(\# \text{ of } \begin{smallmatrix}\times\\ \downarrow\\ \bullet\end{smallmatrix}\right) + \left(\# \text{ of } \begin{smallmatrix}\times\\ \uparrow\\ \bullet\end{smallmatrix}\right)$$

$$+ \left(\times \leftarrow \bullet \leftarrow \times\right) + \left(\bullet \leftarrow \times \leftarrow \bullet\right)$$

$$+ \left(\begin{smallmatrix}\times & \\ \uparrow & \\ \bullet & \leftarrow \times\end{smallmatrix}\right) + \left(\begin{smallmatrix}\bullet \leftarrow & \times\\ & \uparrow\\ & \bullet\end{smallmatrix}\right) + \left(\begin{smallmatrix} & \times\\ & \downarrow\\ \times \leftarrow & \bullet\end{smallmatrix}\right) + \left(\begin{smallmatrix}\times & \leftarrow \bullet\\ \downarrow & \\ \bullet & \end{smallmatrix}\right)$$

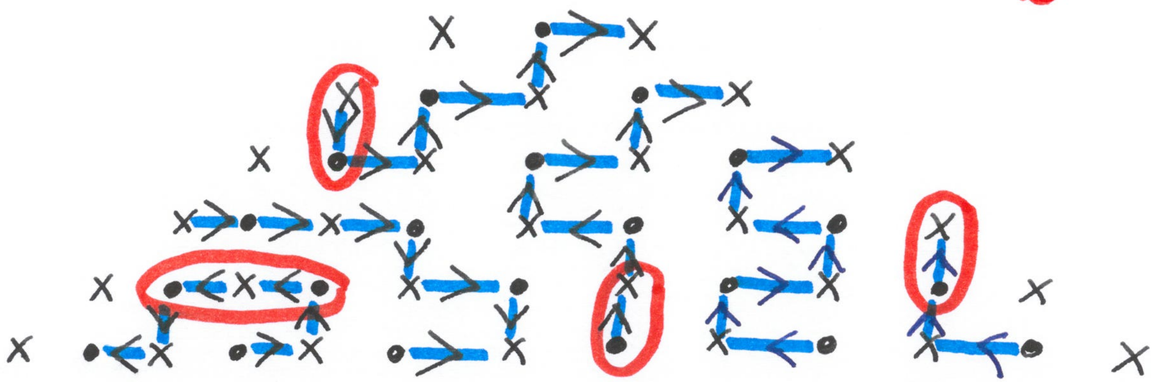
Ex:

$u = (0,0,1,0,1,0)$

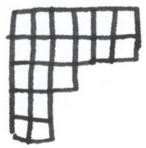
$v = (0,0,0,0,1,1)$

$w = (1,0,1,0,0,0)$

$d(w)-d(u)-d(v) = 4$



Extreme case : $d(u)+d(v)=d(w)$

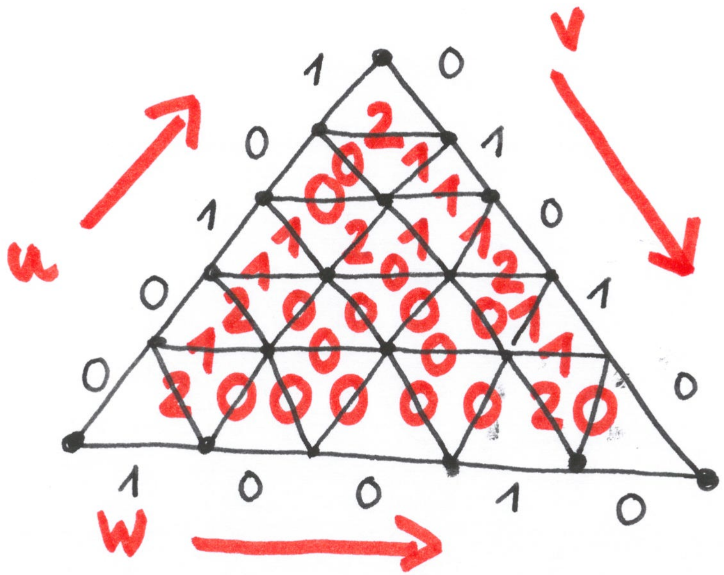
u 01 word $\Rightarrow$ $\lambda(u)$ =  $\Rightarrow$ Schur polynomial $S_{\lambda(u)}(x_1,\dots,x_n)=S_u$

LR-coefficients : u,v 01-words

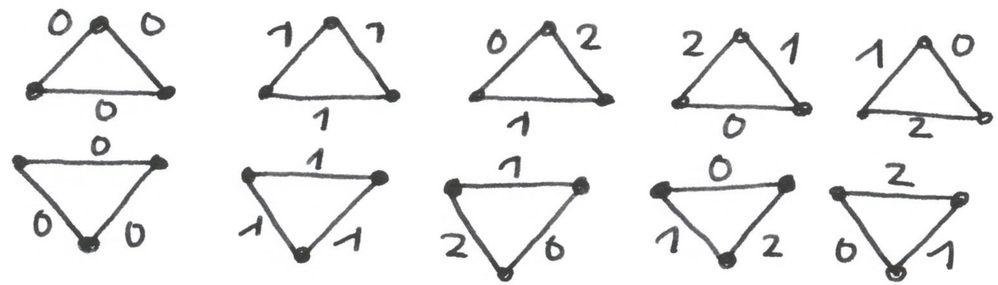
$$s_u \cdot s_v = \sum_w \boxed{c_{u,v}^{w}}\; s_w$$

COMBINATORIAL MODEL FOR LR-Coefficients:

Knutson-Tao-Puzzles



Puzzle Pieces:



THEOREM (Nadeau): u, v, w with $d(u)+d(v)=d(w)$

$$\Rightarrow \quad t_{u,v}^{w} = c_{u,v}^{w}$$

Proof (F. & Nadeau):

$d(w)-d(u)-d(v)=0 \Rightarrow$ the following "local patterns" do not occur:

$$w(\triangle) = \#\,{\leftarrow}\!{\downarrow} + \#\,{\uparrow}\!{\leftarrow} - \#\,{\downarrow}\!{\leftarrow} - \#\,{\leftarrow}\!{\uparrow} = 1$$

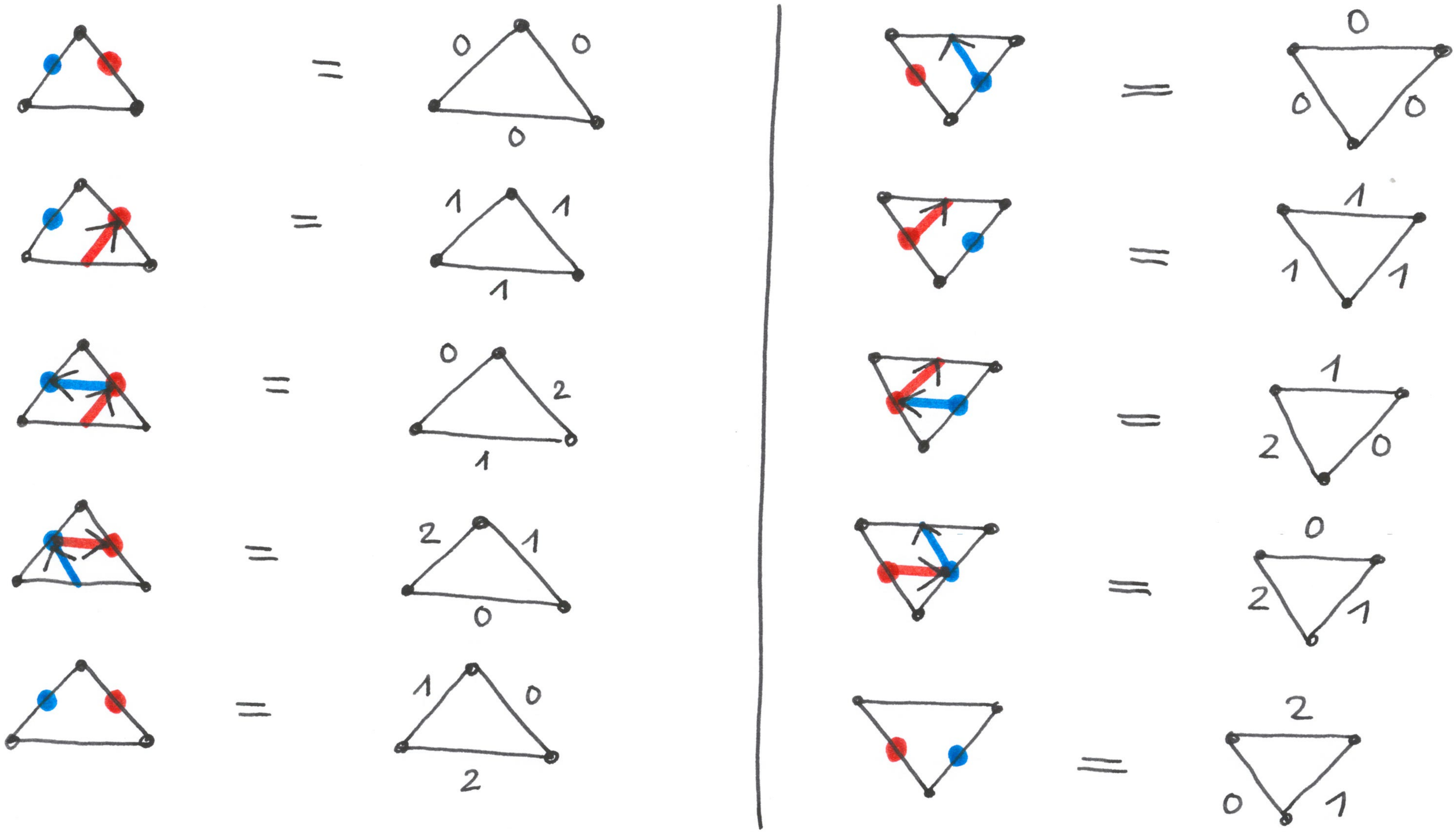
$$t_{u,v}^{w} = \sum_{w'} (M^{-1})_{ww'}\, \vec{t}_{u,v}^{\,w'}(\rho) = \sum_{w' \le w} \underbrace{(M^{-1})_{ww'}}_{\text{=0 except for the case when } w' \le w} \underbrace{\vec{t}_{u,v}^{\,w'}(\rho)}_{\text{=0 if } w'<w} = \vec{t}_{u,v}^{\,w}$$

=0 except for the case when $w' \le w$

=0 if $w'<w$ since then: $d(w') < d(w) = d(u)+d(v)$

Superimpose triangular grid on path tangle

Dictionary :



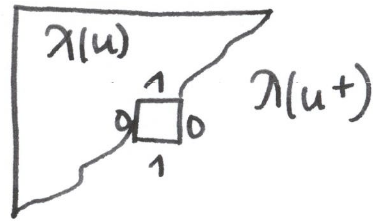
(Zinn-Justin)

Next case : $d(w)-d(u)-d(v)=1$

Message in short : $t_{u,v}^{w}$ can also be expressed in terms of Littlewood-Richardson coefficients (Conjectured by Zinn-Justin '22)

DEFs: u^+ covers u if there exist u_L, u_R with :

$$u = u_L\, 01\, u_R$$
$$u^+ = u_L\, 10\, u_R$$



$$L_0(u,u^+) = |u_L|_0$$
$$L_1(u,u^+) = |u_L|_1$$
$$L(u,u^+) = L_0(u,u^+) + L_1(u,u^+) + 1$$

THEOREM (F. & Nadeau)

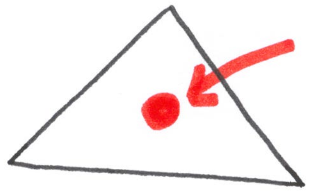
Let u, v, w be words of the same length, $|u|_0 = |v|_0 = |w|_0$ and $d(w) - d(u) - d(v) = 1$

(1) $$\vec{t}_{u,v}^{\,w} = \sum_{u^+ : u \to u^+} \left(|u|_1 + L_1(u, u^+) \right) c_{u^+, v}^{w} + \sum_{v^+ : v \to v^+} \left(L(v, v^+) + L_1(v, v^+) + 1 \right) \cdot c_{u, v^+}^{w} - 2 \sum_{w^- : w^- \to w} L_1(w^-, w)\, c_{u,v}^{w^-}$$

(2) $$t_{u,v}^{w} = \sum_{v^+ : v \to v^+} \left(|v|_1 + L(v, v^+) + 1 \right) c_{u, v^+}^{w} - \sum_{w^- : w^- \to w} L_1(w^-, w)\, c_{u,v}^{w^-}$$

Idea of the proof

- The gap $d(w)-d(u)-d(v)=1$ can be "realized" as a certain $\underbrace{\text{local configuration}}_{\text{defect}}$ in the TFPL.



defect

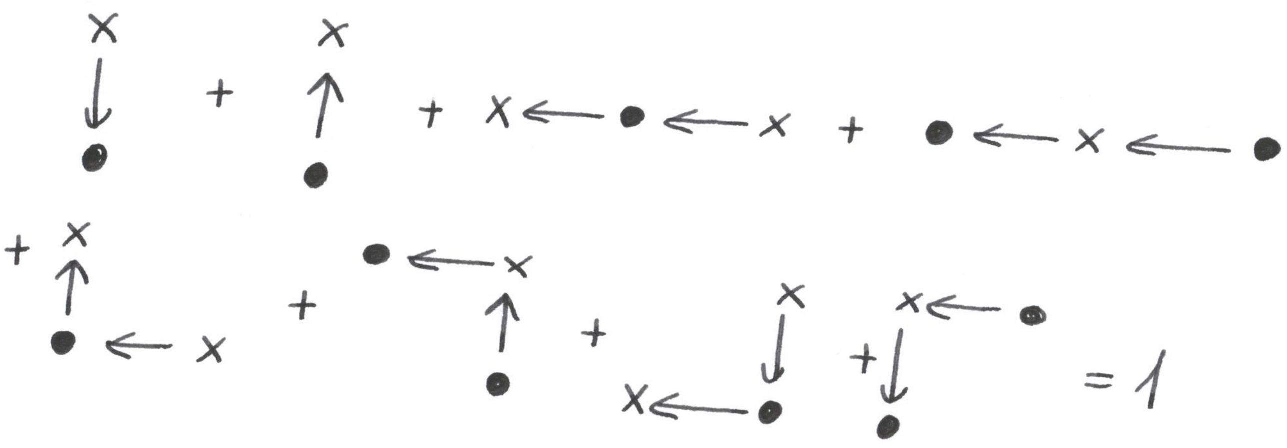
- Define operations to move the **defect.**
- Move **defect** to the boundary and remove it

$$\Longrightarrow \underbrace{\text{TFPL with } d(w)-d(u)-d(v)=0}_{\text{LR-TFPL}}$$

- To prove the formula the following question has to be answered: **How many TFPLs are mapped to the same LR-TFPL?**

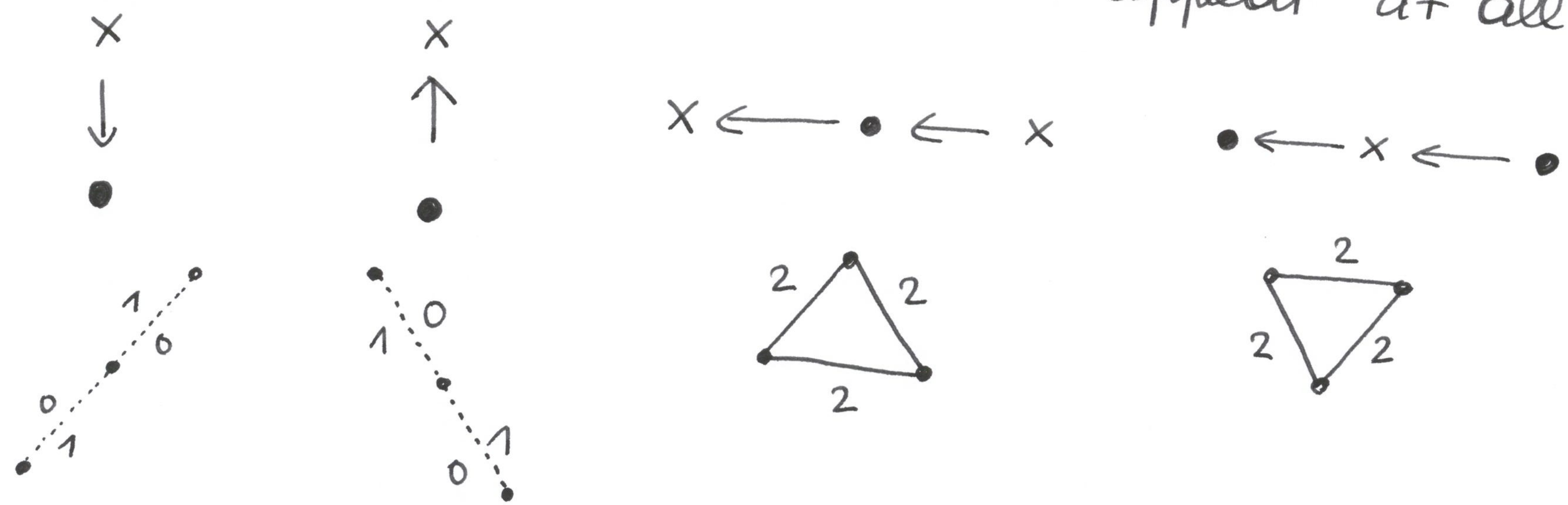
More concretely :

$d(w) - d(u) - d(v) =$

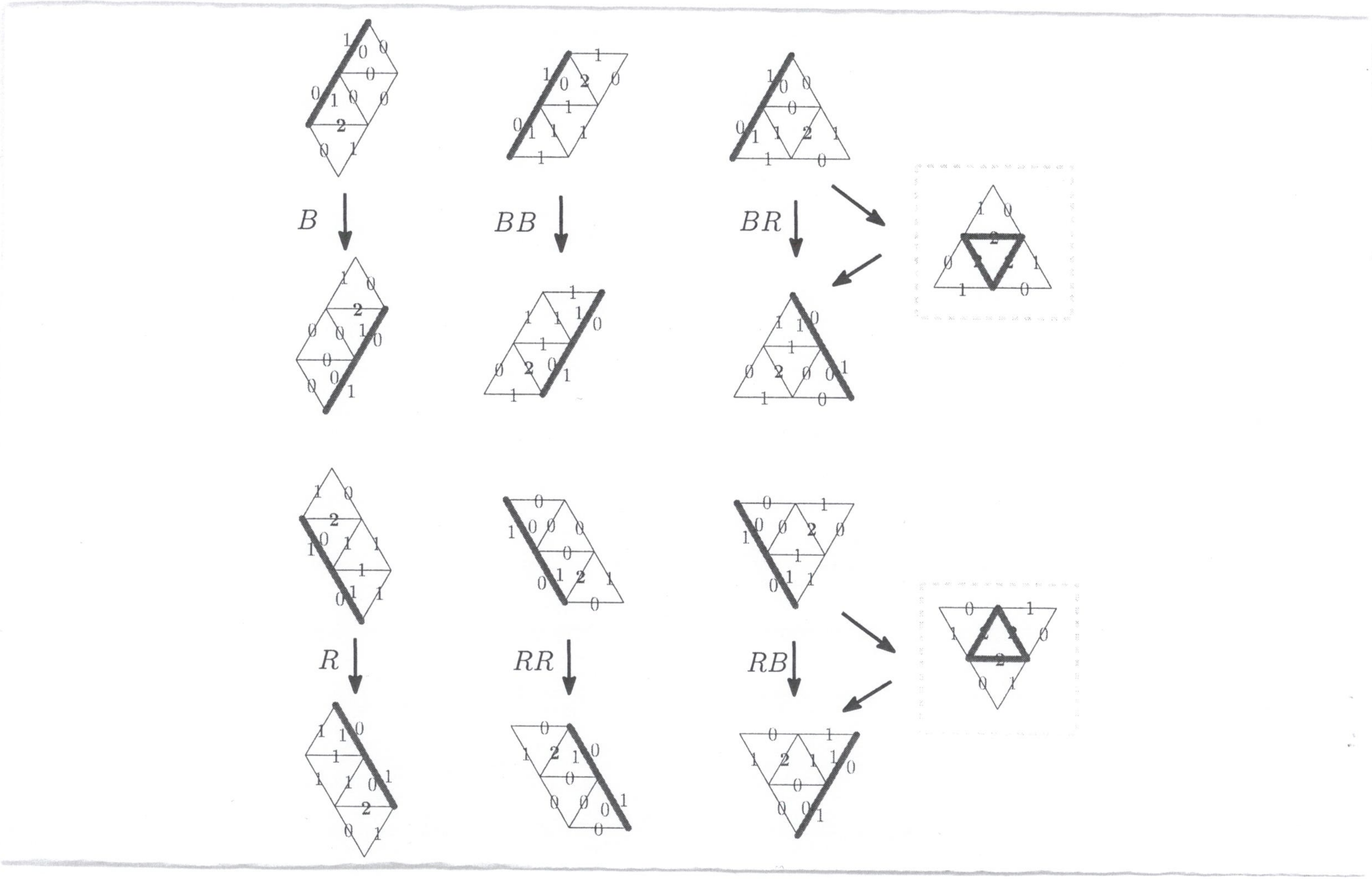


$= 1$

$\Longleftrightarrow$ There is one local configuration among the first four in this list that appears precisely once, while the other seven configurations in the list do not appear at all :



How to move the defect:



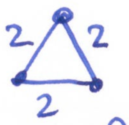
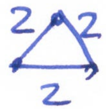
PERSPECTIVES

Obvious question: Is it possible to express $\vec{t}_{u,v}^{\,w}$ (or $t_{u,v}^{w}$) in terms of LR-coefficients if $d(w)-d(u)-d(v)>1$?

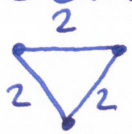
Situation: More than one defect. Usually it is possible to move them around. However, one might run into problems if two defects are too close.

Excess 1: defect $\in$ 

We have formulas for each type of defect!

For  this is a special case of a formula which follows from certain results in the K-theory of the Grassmannian. There, any number of -pieces are allowed.

Combinatorial proof?

-pieces?

PERSPECTIVES 2

Other applications of path tangles?

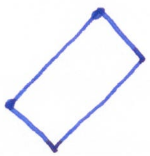
- Path tangles generalize LR-models: Can "LR-results" be transferred to TFPLs?

  Possible example: For LR-coefficients $c_{\lambda,\mu}^{\nu}$ Horn (1962) introduced certain inequalities for (λ,μ,ν) that are necessary for $c_{\lambda,\mu}^{\nu}$ to be positive. Knutson and Tao (1999) showed that these inequalities are also necessary.

  Is it possible to generalize these inequalities to $\vec{t}_{u,v}^{\,w}$ (or $t_{u,v}^{w}$)?

- In a forthcoming paper, Sabine Beil (PhD-student) studies Wieland gyration on TFPLs – the "defect-point-of-view" seems to be very helpful.

  BTW: She also transfers our results to different shapes

   (tilted rectangle).