An alternative approach to alternating sign matrices

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Outline

- ASMs: definition, λ-determinant, square ice
- Monotone triangles with prescribed bottom row
- ASMs with general boundary conditions
- Linear relations for refined enumerations of ASMs
- Directed triangles ?

Alternating sign matrices

Square $0, 1, -1$ matrices such that in each row and each column

- the non–zero entries appear with alternating signs and
- the sum of entries is 1, i.e. the first and the last non-zero entry is a 1.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Theorem (Zeilberger 1996). The number of $n \times n$ alternating sign matrices is

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} =: A_n.$$

The origin of ASMs: λ–determinant

Leibniz formula for the determinant of a matrix $M = (m_{i,j})_{1 \le i,j \le n}$:

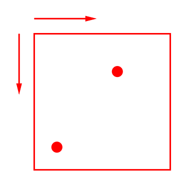
$$\det(M) = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma \, m_{1,\sigma(1)} m_{2,\sigma(2)} \cdots m_{n,\sigma(n)}$$

or

$$\det(M) = \sum_{B \in \mathcal{P}_n} (-1)^{\mathcal{I}(B)} \prod_{i,j=1}^{n} m_{i,j}^{B_{i,j}}$$

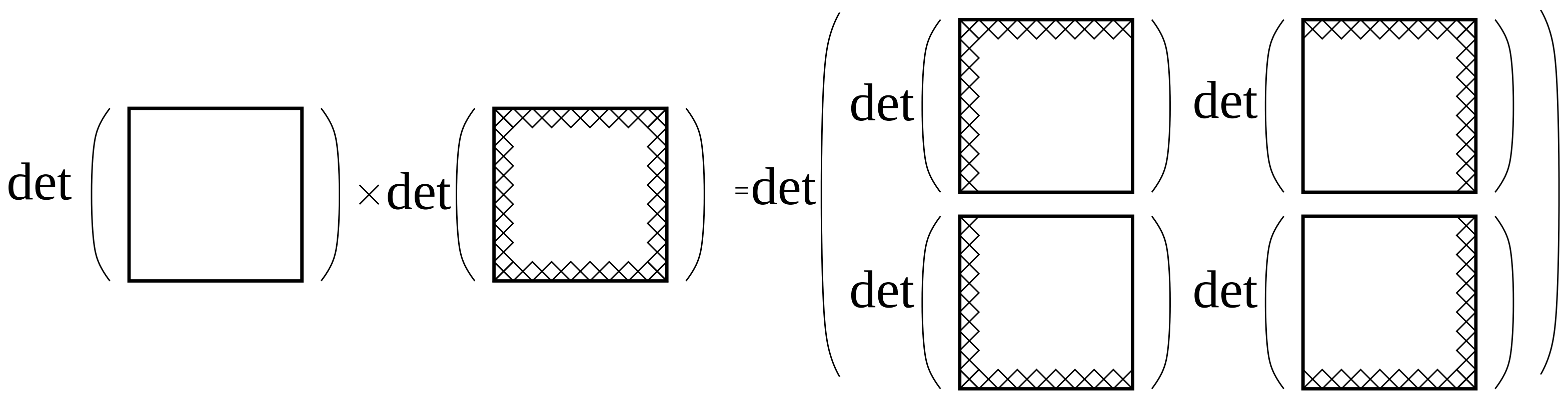
where $\mathcal{P}_n$ is the set of $n \times n$ permutation matrices and

$$\mathcal{I}(B) = \sum_{\substack{(i_1,j_1),(i_2,j_2) \\ i_1 > i_2, j_1 < j_2}} b_{i_1,j_1} b_{i_2,j_2}$$



is the inversion number of B.

The Desnanot–Jacobi identity...



...can be used to see that $n \times n$ determinants are expressible in terms of 2×2 determinants. (Condensation of determinants, Charles L. Dodgson, 1866)

3×3 determinants

$$\det\begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{pmatrix} = \frac{1}{m_{2,2}}$$

$$\times \det\begin{pmatrix} \det\begin{pmatrix} m_{2,2} & m_{2,3} \\ m_{3,2} & m_{3,3} \end{pmatrix} & \det\begin{pmatrix} m_{2,1} & m_{2,2} \\ m_{3,1} & m_{3,2} \end{pmatrix} \\ \det\begin{pmatrix} m_{1,2} & m_{1,3} \\ m_{2,2} & m_{2,3} \end{pmatrix} & \det\begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix} \end{pmatrix}$$

4×4 determinants

$$\det\begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{pmatrix} = \frac{1}{\det\begin{pmatrix} m_{2,2} & m_{2,3} \\ m_{3,2} & m_{3,3} \end{pmatrix}}$$
$$\times \det\begin{pmatrix} \det\begin{pmatrix} m_{2,2} & a_{2,3} & m_{2,4} \\ m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,2} & m_{4,3} & m_{4,4} \end{pmatrix} & \det\begin{pmatrix} m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \\ m_{4,1} & m_{4,2} & m_{4,3} \end{pmatrix} \\ \det\begin{pmatrix} m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,2} & m_{3,3} & m_{3,4} \end{pmatrix} & \det\begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{pmatrix} \end{pmatrix}$$

3×3 determinants are expressible in terms of 2×2 determinants...and so are 4×4 determinants!

Robbins and Rumsey in the 1980s: what happens if we generalize the definition of a 2×2 determinant to

$$\det_\lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} + \lambda a_{12}a_{21}$$

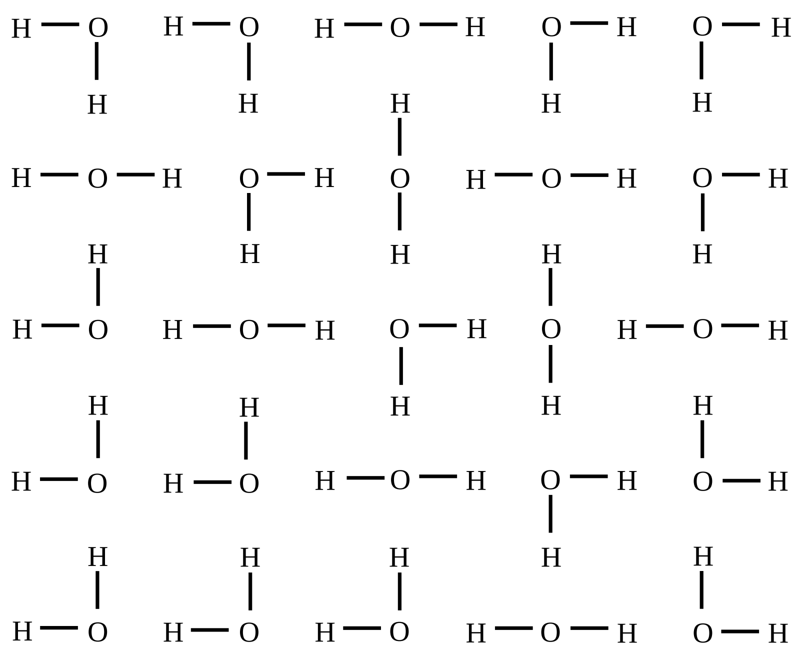
and, furthermore, use the previous observation to generalize the $n \times n$ determinant?

Theorem (Robbins and Rumsey). Let $M = (m_{i,j})$ be an $n \times n$ matrix, $\mathcal{A}_n$ the set of $n \times n$ alternating sign matrices, $\mathcal{I}(B)$ the inversion number of B and $\mathcal{N}(B)$ the number of -1s in B then

$$\det_\lambda(M) = \sum_{B \in \mathcal{A}_n} \lambda^{\mathcal{I}(B)}(1+\lambda^{-1})^{\mathcal{N}(B)} \prod_{i,j=1}^{n} m_{i,j}^{B_{i,j}}.$$

ASMs in statistical physics: square ice

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



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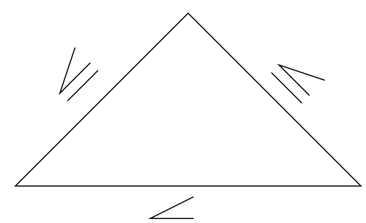
Monotone triangles

Triangular arrays of integers with monotonicity requirements:

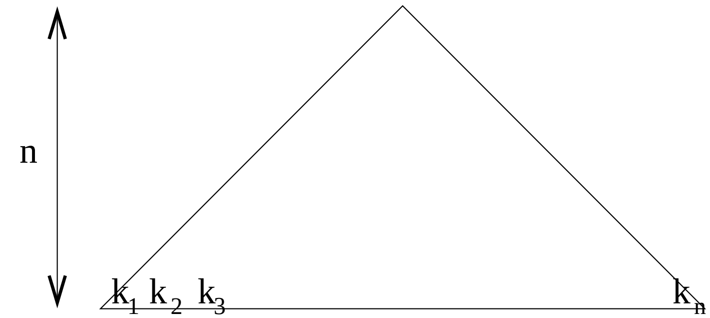


$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 2 & & 5 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Monotone triangles with bottom row $1, 2, \ldots, n \Leftrightarrow n \times n$ ASMs

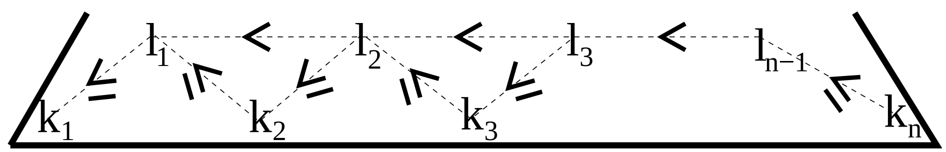
Monotone triangles with arbitrary strictly increasing bottom row

$$\begin{array}{ccccccc} & & & 6 & & & \\ & & 4 & & 7 & & \\ & 2 & & 6 & & 8 & \\ 1 & & 3 & & 7 & & 8 \end{array}$$



$\alpha(n; k_1, \ldots, k_n) = \#$ of monotone triangles with bottom row $(k_1, \ldots, k_n)$

Advantage: recursive structure!



$$\alpha(n; k_1, k_2, \ldots, k_n) = \sum_{\substack{k_1 \le l_1 \le k_2 \le l_2 \le k_3 \le \ldots \le l_{n-1} \le k_n \\ l_i < l_{i+1}}} \alpha(n-1; l_1, \ldots, l_{n-1})$$

Operator formula for the number of monotone triangles

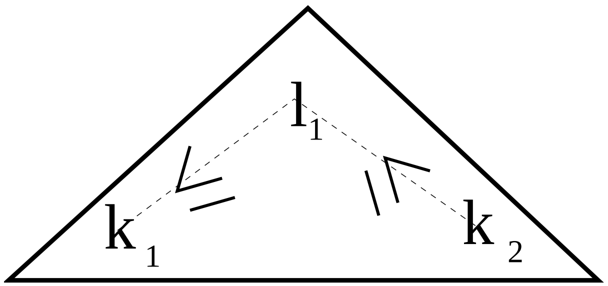
Theorem (2006). The number of monotone triangles with bottom row $(k_1, \ldots, k_n)$ is

$$\prod_{1 \le p < q \le n} (E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1}) \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i} =: \alpha(n; k_1, \ldots, k_n),$$

where $E_x p(x) = p(x+1)$.

How to use the formula:

- Treat k_i as a variable.
- Apply the product of operators to the polynomial $\prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}$.
- This gives a polynomial in $(k_1, \ldots, k_n)$ in which you may replace k_i by the desired integer.

Example. $n = 2$: 

$$(E_{k_1} + E_{k_2}^{-1} - E_{k_1}E_{k_2}^{-1})(k_2 - k_1 + 1)$$
$$= E_{k_1}(k_2 - k_1 + 1) + E_{k_2}^{-1}(k_2 - k_1 + 1) - E_{k_1}E_{k_2}^{-1}(k_2 - k_1 + 1)$$
$$= k_2 - k_1 + 1 = \# \text{ of integers between } k_1 \text{ and } k_2$$

Proof by induction

Need to show:

$$\prod_{1\le p<q\le n}(E_{k_p}+E_{k_q}^{-1}-E_{k_p}E_{k_q}^{-1})\prod_{1\le i<j\le n}\frac{k_j-k_i+j-i}{j-i}$$
$$=\sum_{\substack{k_1\le l_1\le k_2\le l_2\le k_3\le l_3\le\ldots\le l_{n-1}\le k_n\\ l_i<l_{i+1}}}\prod_{1\le p<q\le n-1}(E_{l_p}+E_{l_q}^{-1}-E_{l_p}E_{l_q}^{-1})\prod_{1\le i<j\le n-1}\frac{l_j-l_i+j-i}{j-i}$$

Summation is the discrete equivalent of integration!

Discrete equivalent of the fundamental theorem of calculus. Let $f:\mathbb{Z}\to\mathbb{C}$. If $F:\mathbb{Z}\to\mathbb{C}$ is a function such that $f(i)=\Delta_i F(i):=F(i+1)-F(i)$ then

$$\sum_{i=a}^{b} f(i)=F(b+1)-F(a).$$

A multivariate discrete equivalent – related to the recursion

Let $a : \mathbb{Z}^{n-1} \to \mathbb{C}$. If $A : \mathbb{Z}^{n-1} \to \mathbb{C}$ is a function with

$$a(l_1, \ldots, l_{n-1}) = \Delta_{l_1} \cdots \Delta_{l_{n-1}} A(l_1, \ldots, l_{n-1})$$

and, for all $i \in \{1, 2, \ldots, n-2\}$,

$$\begin{aligned} A(l_1, \ldots, l_{i-1}, l_i - 1, l_i - 1, l_{i+2}, \ldots, l_{n-1}) \\ + A(l_1, \ldots, l_{i-1}, l_i, l_i, l_{i+2}, \ldots, l_{n-1}) \\ - A(l_1, \ldots, l_{i-1}, l_i - 1, l_i, l_{i+2}, \ldots, l_{n-1}) = 0 \end{aligned}$$

then

$$\begin{aligned} \sum_{\substack{k_1 \le l_1 \le k_2 \le l_2 \le k_3 \le \ldots \le l_{n-1} \le k_n \\ l_i < l_{i+1}}} a(l_1, \ldots, l_{n-1}) = A(k_2+1, k_3+1, \ldots, k_n+1) \\ - A(k_1, k_3+1, \ldots, k_n+1) + A(k_1, k_2, k_4+1, \ldots, k_n+1) \\ - \ldots + (-1)^{n-1} A(k_1, k_2, \ldots, k_{n-1}). \end{aligned}$$

How to compute the discrete antiderivative of a polynomial

Trick: use the binomial basis $\left(\binom{x}{k}\right)_{k\geq 0}$ instead of $\left(x^k\right)_{k\geq 0}$:

$$p(x) = a_0\binom{x}{0} + a_1\binom{x}{1} + a_2\binom{x}{2} + \ldots + a_n\binom{x}{n}$$

Antiderivative of $p(x)$:

$$P(x) = a_0\binom{x}{1} + a_1\binom{x}{2} + a_2\binom{x}{3} + \ldots + a_n\binom{x}{n+1}$$

Proof:

$$\Delta_x\binom{x}{k+1} = \binom{x+1}{k+1} - \binom{x}{k+1} = \binom{x}{k}$$

A generalization: the number of -1s in an ASM

How is a -1 in the ASM reflected in the monotone triangle?

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 2 & & 4 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Weight: $Q^{\#\text{ of }(-1)\text{-triangles}}$

Theorem (2009). The weighted enumeration of monotone triangles with bottom row $(k_1, k_2, \ldots, k_n)$ is given by

$$\prod_{1 \le p < q \le n} \left(E_{k_p} + E_{k_q}^{-1} + (Q-2) E_{k_p} E_{k_q}^{-1} \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}.$$

A variation: vertically symmetric alternating sign matrices

Symmetry with respect to the vertical axis:

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 & 0 & -1 & 1 \\
0 & 0 & 1 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\Leftrightarrow
\begin{array}{ccccccccccccc}
 & & & & & & 4 & & & & & & \\
 & & & & & 2 & & 6 & & & & & \\
 & & & & 1 & & 4 & & 7 & & & & \\
 & & & 1 & & 3 & & 5 & & 7 & & & \\
 & & 1 & & 2 & & 4 & & 6 & & 7 & & \\
 & 1 & & 2 & & 3 & & 5 & & 6 & & 7 & \\
1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7
\end{array}
$$

Operator formula for halved monotone triangles

A halved monotone triangle is a triangular array of integers of shape

$$\begin{array}{cccccc}
 & & & & & a_{1,1} \\
 & & & & a_{2,1} & \\
 & & & a_{3,1} & & a_{3,2} \\
 & & a_{4,1} & & a_{4,2} & \\
 & a_{5,1} & & a_{5,2} & & a_{5,3} \\
a_{6,1} & & a_{6,2} & & a_{6,3} &
\end{array}$$

which is monotone increasing in $\nearrow$-direction and $\searrow$-direction and strictly increasing along rows.

Theorem (2007). The number of halved monotone triangles with n rows, bottom row $(k_1, \ldots, k_{\lceil n/2 \rceil})$ and where no entry exceeds x is equal to

$$\prod_{1 \le p < q \le (n+1)/2} \left(E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1} \right) \left(E_{k_p}^{-1} + E_{k_q}^{-1} - \mathrm{id} \right)$$

$$\prod_{1 \le i < j \le (n+1)/2} \frac{(k_j - k_i + j - i)(2x - 1 - k_i - k_j - i - j)}{(j-i)(j+i-1)}$$

if n is odd and equal to

$$\prod_{1 \le p < q \le n/2} \left(E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1} \right) \left(E_{k_p}^{-1} + E_{k_q}^{-1} - \mathrm{id} \right)$$

$$\prod_{1 \le i < j \le n/2} \frac{(k_j - k_i + j - i)(2x - k_i - k_j - i - j)}{(j-i)(j+i)} \prod_{i=1}^{n/2} \frac{x - k_i - i}{i}$$

if n is even.

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What does $\alpha(n; k_1, \ldots, k_n)$ count in terms of ASMs?

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 6 & & & & & & & \\
 & & & & & & 4 & & 7 & & & & & & \\
 & & & & & 2 & & 6 & & 8 & & & & & \\
 & & & & 1 & & 3 & & 7 & & 8 & & & & \\
 & & & 1 & & 2 & & 3 & & 7 & & 8 & & & \\
 & & 1 & & 2 & & 3 & & 4 & & 7 & & 8 & & \\
 & 1 & & 2 & & 3 & & 4 & & 5 & & 7 & & 8 & \\
1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8
\end{array}
\Rightarrow
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \\
1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

$n \times m$ matrices with entries in $\{0, 1, -1\}$ such that

- non-zero entries alternate in each row and column
- row sums are equal to 1, i.e. first/last non-zero entries are 1
- in each column the first non-zero entry is 1
- the last non-zero entry is 1 precisely in the columns $k_1, k_2, \ldots, k_n$

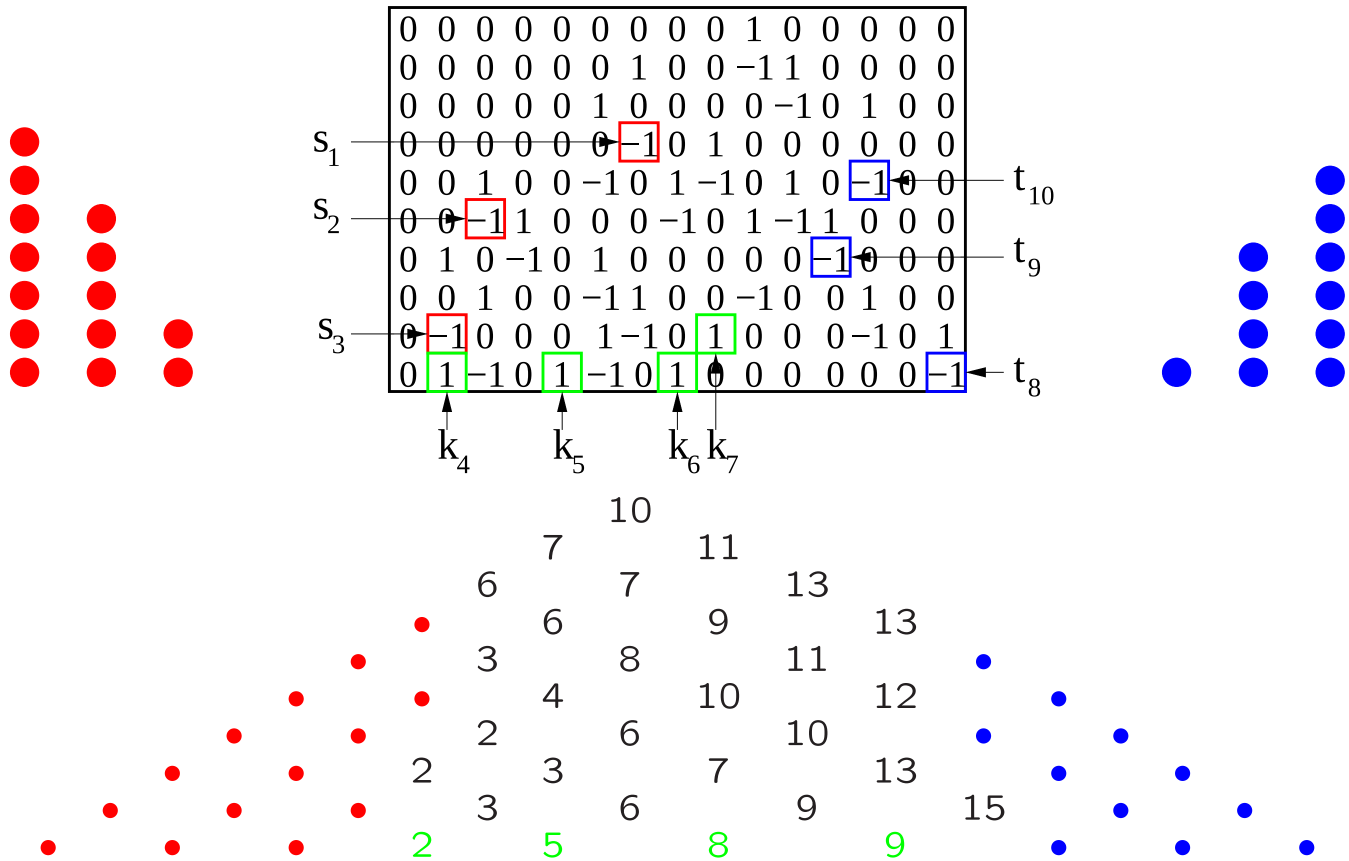
General boundary conditions: left, right and bottom

Loosen the requirement that the first/last non-zero entry in each row/column is 1 . . .

$$
\begin{array}{c|ccccccccccccccc|c}
\cline{2-16}
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & \\
 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & \\
s_1 \rightarrow & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{-1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & \boxed{-1} & 0 & 0 & \leftarrow t_{10} \\
s_2 \rightarrow & 0 & 0 & \boxed{-1} & 1 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & \\
 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \boxed{-1} & 0 & 0 & 0 & \leftarrow t_9 \\
 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & \\
s_3 \rightarrow & 0 & \boxed{-1} & 0 & 0 & 0 & 1 & -1 & 0 & \boxed{1} & 0 & 0 & 0 & -1 & 0 & 1 & \\
 & 0 & \boxed{1} & -1 & 0 & \boxed{1} & -1 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{-1} & \leftarrow t_8 \\
\cline{2-16}
\multicolumn{1}{c}{} & & k_4 & & & k_5 & & & k_6 & k_7 & & & & & & & \multicolumn{1}{c}{}
\end{array}
$$

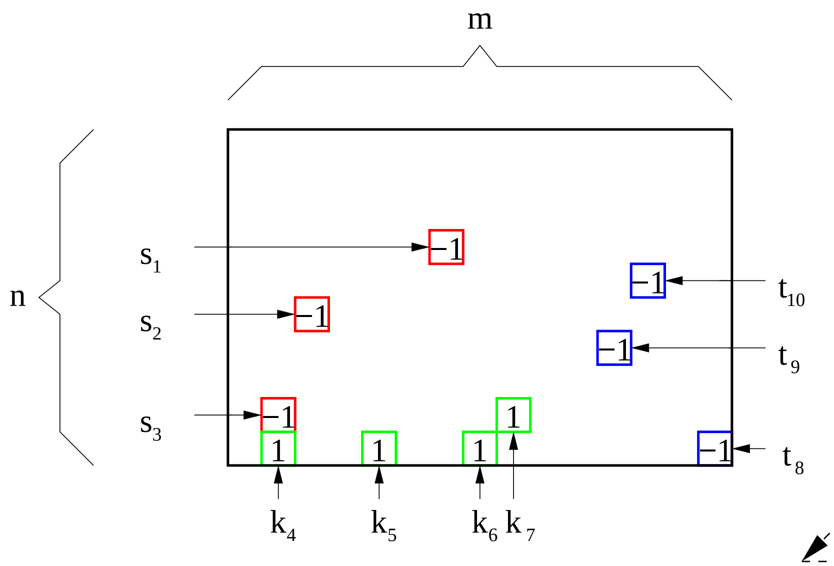
. . . with the exception of the first non-zero entry in the columns.

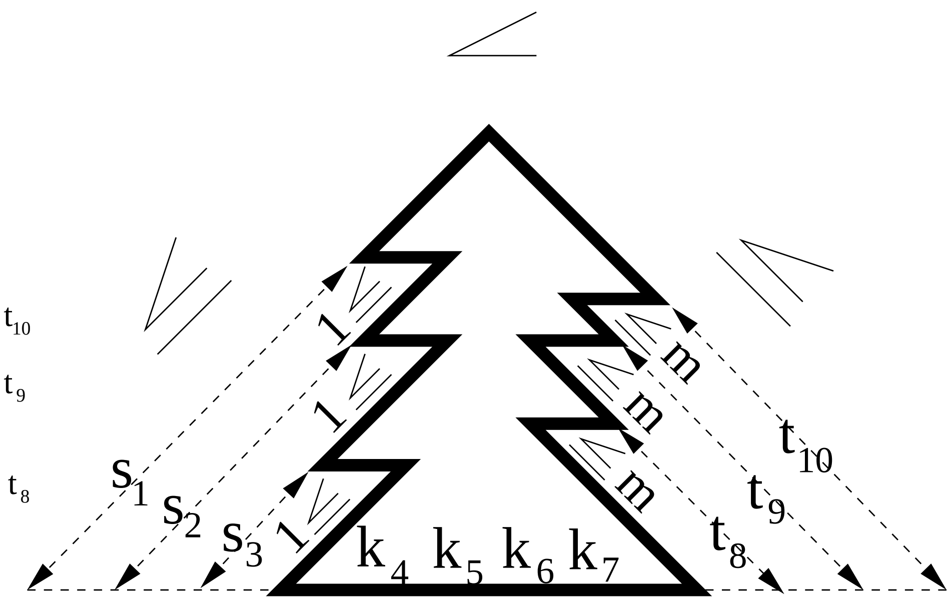
In terms of partial monotone triangles



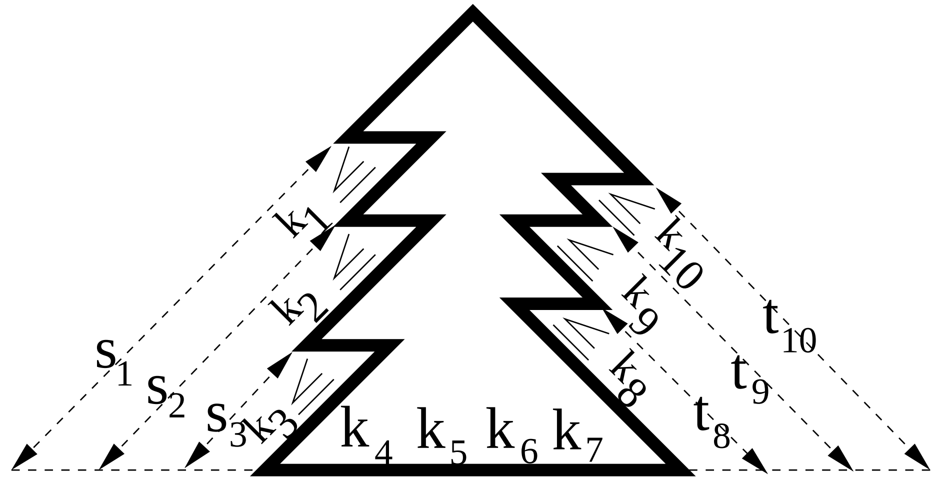
Rows are counted from the bottom!

General boundary conditions: left, right and bottom





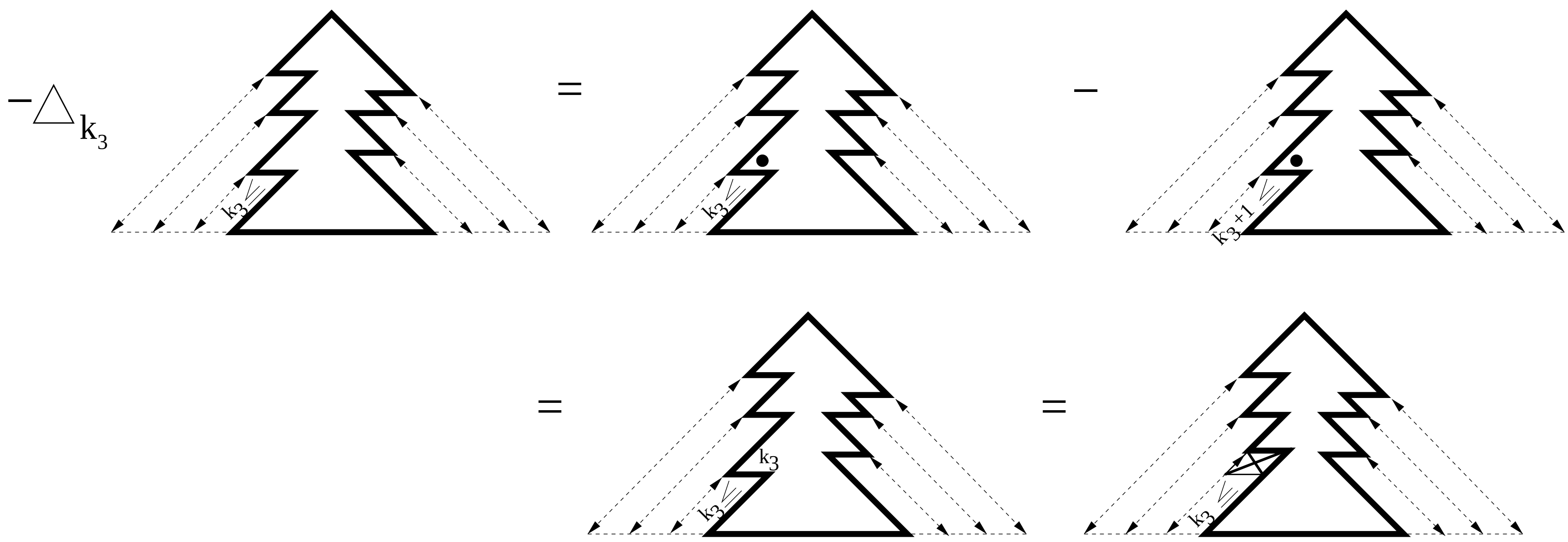
More general partial monotone triangles



Lower bound of the i-th NE-diagonal: $1 \to k_i$

Upper bound of the i-th SE-diagonal: $m \to k_i$

The negative forward difference $-\Delta_{k_i}$ truncates the i-th NE-diagonal.



Similar: the backward difference δ_{k_i}, where $\delta_x p(x) = p(x) - p(x-1)$, truncates the i-th SE-diagonal.

To obtain a formula for more general partial monotone triangles:

- Start with a full monotone triangle
- Δ_{k_i} "eats" the NE-diagonals
- δ_{k_i} "eats" the SE-diagonals

Therefore: the number of more general partial monotone triangles is

$$(-\Delta_{k_1})^{s_1-1}(-\Delta_{k_2})^{s_2-1}\dots(-\Delta_{k_l})^{s_l-1} \\ \delta_{k_{n-r+1}}^{t_{n-r+1}-1}\delta_{k_{n-r+2}}^{t_{n-r+2}-1}\dots\delta_{k_n}^{t_n-1}\alpha(n;k_1,\dots,k_n).$$

General boundary conditions: left, right and bottom

Theorem (2010). The number of ASMs with boundary conditions

$$(s_1, \ldots, s_l), (k_{l+1}, \ldots, k_{n-r}), (t_{n-r+1}, \ldots, t_n)$$

is

$$\Delta_{k_1}^{s_1-1} \Delta_{k_2}^{s_2-1} \ldots \Delta_{k_l}^{s_l-1} \delta_{k_{n-r+1}}^{t_{n-r+1}-1} \delta_{k_{n-r+2}}^{t_{n-r+2}-1} \ldots \delta_{k_n}^{t_n-1} (-1)^{s_1+s_2+\ldots+s_l+l}$$
$$\times \alpha(n; k_1, \ldots, k_n)|_{k_1=k_2=\ldots=k_l=1, k_{n-r+1}=k_{n-r+2}=\ldots=k_n=m}.$$

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Strategy

Use identities for $\alpha(n; k_1, \ldots, k_n)$ to deduce identities for enumerations of ASMs with general boundary conditions = refined enumerations of ASMs

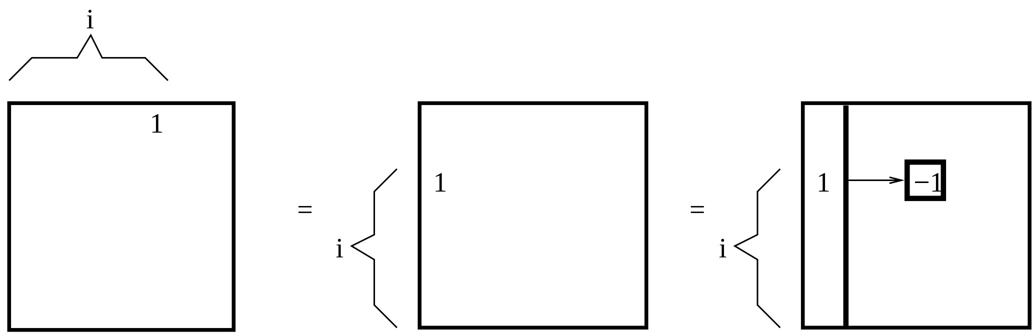
Identity 1.

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{n-1} \alpha(n; k_2, \ldots, k_n, k_1 - n)$$

Refined alternating sign matrix theorem

Theorem (Zeilberger 1996). The number of $n \times n$ ASMs with a 1 in position $(1, i)$ is

$$\binom{n+i-2}{n-1}\frac{(2n-i-1)!}{(n-i)!}\prod_{j=0}^{n-2}\frac{(3j+1)!}{(n+j)!} =: A_{n,i}.$$



Two expressions for $A_{n,i}$:

$$A_{n,i} = (-1)^{i-1}\Delta_{k_1}^{i-1}\alpha(n;k_1,\ldots,k_n)\Big|_{(k_1,\ldots,k_n)=(1,1,2,\ldots,n-1)}$$
$$= \delta_{k_n}^{i-1}\alpha(n;k_1,\ldots,k_n)\Big|_{(k_1,\ldots,k_n)=(1,2,3,\ldots,n-1,n-1)}$$

Use Identity 1 to express one in terms of the other:

$$A_{n,i} = (-1)^{i-1}\Delta_{k_1}^{i-1}\alpha(n;k_1,\ldots,k_n)\Big|_{(k_1,\ldots,k_n)=(1,1,2,\ldots,n-1)}$$
$$= (-1)^{i+n}\Delta_{k_1}^{i-1}\alpha(n;k_2,\ldots,k_n,k_1-n)\Big|_{(k_1,\ldots,k_n)=(1,1,2,\ldots,n-1)}$$
$$= (-1)^{i+n}E_{k_1}^{-2n+2}\Delta_{k_1}^{i-1}\alpha(n;k_2,\ldots,k_n,k_1)\Big|_{(k_2,\ldots,k_n,k_1)=(1,2,\ldots,n-1,n-1)}$$

Now $\Delta_x = E_x - \mathrm{id} = E_x(\mathrm{id} - E_x^{-1}) = E_x\delta_x$, which gives

$$(-1)^{i+n}E_{k_n}^{-2n+1+i}\delta_{k_n}^{i-1}\alpha(n;k_1,k_2,\ldots,k_n)\Big|_{(k_1,\ldots,k_{n-1},k_n)=(1,2,\ldots,n-1,n-1)}.$$

Getting rid of the shift: binomial theorem

$$E_x^{-m} = (\mathrm{id} - (\mathrm{id} - E_x^{-1}))^m = (\mathrm{id} - \delta_x)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \delta_x^j$$

Therefore

$$\begin{aligned} A_{n,i} &= \sum_{j=0}^{2n-1-i} \binom{2n-1-i}{j} (-1)^{i+j+n} \delta_{k_n}^{i+j-1}\, \alpha(n; k_1, k_2, \ldots, k_n)|_{(k_1,\ldots,k_{n-1},k_n)=(1,2,\ldots,n-1,n-1)} \\ &= \sum_{j=0}^{2n-1-i} \binom{2n-1-i}{j} (-1)^{i+j+n} A_{n,i+j} = \sum_{k=i}^{n} \binom{2n-1-i}{k-i} (-1)^{k+n} A_{n,k} \\ &= \sum_{k=i}^{n} \binom{2n-1-i}{k-i} (-1)^{k+n} A_{n,n+1-k} \end{aligned}$$

for $1 \leq i \leq n$.

System of linear equations for $(A_{n,i})_{1 \leq i \leq n}$ – determines the numbers uniquely!

A doubly refined enumeration of ASMs: $\overline{\overline{A}}_{n,i,j}$

Situation for the two top rows of an ASM:

$$\begin{array}{c} \begin{array}{cccccc} & \mathrm{i} & & \mathrm{k} & \mathrm{j} & \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ & & & \vdots & & \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{cccccc} & \mathrm{i{=}k} & & & \mathrm{j} & \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ & & & \vdots & & \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{cccccc} & \mathrm{i} & & & \mathrm{k{=}j} & \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ & & & \vdots & & \end{pmatrix} \end{array}$$

$\Rightarrow i \le k \le j$; the choice of k has no influence on the rest of the matrix.

Let $\overline{\overline{A}}_{n,i,j}$ denote the number of $n \times n$ ASMs where the two top rows are fixed and i, j are as above.

Theorem (F. and Romik, 2009). There exists an extension of $\overline{\overline{A}}_{n,i,j}$ to all $1 \leq i, j \leq n$ such that

$$\overline{\overline{A}}_{n,i,j} = \sum_{p,q=1}^{n} (-1)^{p+q} \binom{2n-i-2}{p-i} \binom{2n-j-2}{q-j} \overline{\overline{A}}_{n,n+1-p,n+1-q} + r_{n,i,j},$$

where

$$r_{n,i,j} = A_{n-1}(-1)^n \left([i=1] \left(\binom{2n-2-j}{n-1-j} - (2n-3)\binom{2n-2-j}{n-j} \right) \right.$$
$$\left. -[i=2]\binom{2n-2-j}{n-j} \right)$$

and [statement] $=1$ iff the statement is true and zero otherwise.

Conjecture. The system of linear equations together with $\overline{\overline{A}}_{n,i,n} = A_{n-1,i}$ determines the numbers $\overline{\overline{A}}_{n,i,j}$ uniquely.

Extends to refined enumerations with respect to the d top rows and the c bottom rows!

Linear relations between different types of refined enumerations of ASMs

Identity 2. If $j \geq 1$ then

$$e_j(\Delta_{k_1}, \ldots, \Delta_{k_n}) \alpha(n; k_1, \ldots, k_n) = 0,$$

where

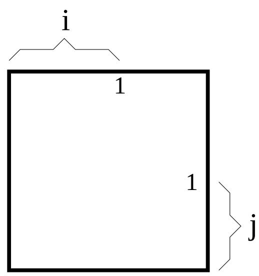
$$e_j(X_1, \ldots, X_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq n} X_{i_1} X_{i_2} \cdots X_{i_p}$$

is the j-th elementary symmetric function.

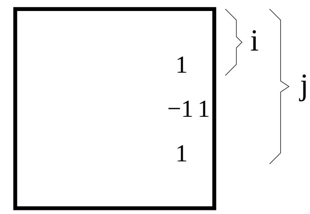
Example. $j = 1, n = 2$:

$$e_1(\Delta_{k_1}, \Delta_{k_2}) \alpha(2; k_1, k_2) = (\Delta_{k_1} + \Delta_{k_2})(k_2 - k_1 + 1) = 0$$

Two doubly refined enumerations of ASMs



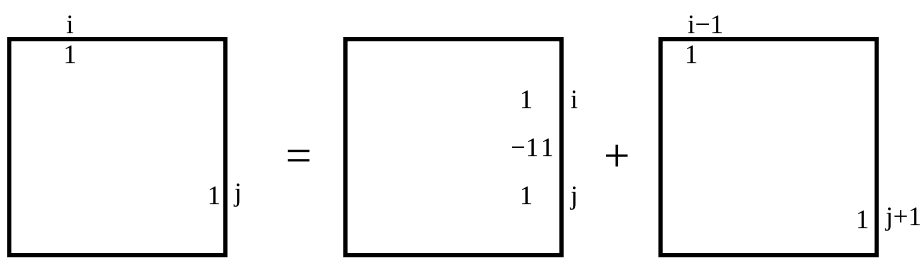
$\overline{A|}_{n,i,j}$



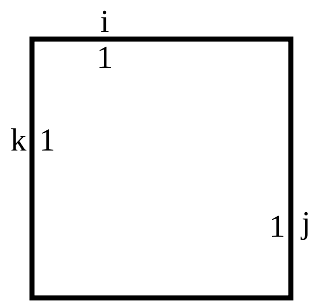
$A||_{n,i,j} = \overline{\overline{A}}_{n,i,j}$

Theorem (2010). If $j \neq 1$ then

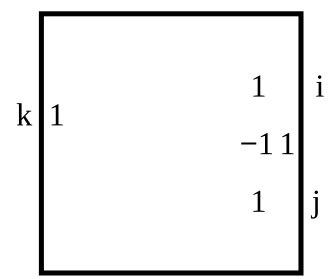
$$\overline{A|}_{n,i,j} = A||_{n,i,j} + \overline{A|}_{n,i-1,j+1}$$



Two triply refined enumerations of ASMs



$\overline{|A|}_{n,k,i,j}$



$|A||_{n,k,i,j}$

Theorem (2010). If $j, k \neq 1$ then

$$\overline{|A|}_{n,k+1,i,j} + \overline{|A|}_{n,k+1,i,j+1} + \overline{|A|}_{n,k,i+1,j}$$
$$= |A||_{n,k,i,j} + \overline{|A|}_{n,k+1,i-1,j+1} + \overline{|A|}_{n,k,i,j+1} + \overline{|A|}_{n,k+1,i+1,j}$$

Outline

- ASMs: definition, λ-determinant, square ice
- Monotone triangles with prescribed bottom row
- ASMs with general boundary conditions
- Linear relations for refined enumerations of ASMs
- Directed triangles !

Less computations, more combinatorics!

Combinatorial proof of

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{n-1} \alpha(n; k_2, \ldots, k_n, k_1 - n)?$$

Problem: The right hand side has no combinatorial meaning if $k_1 < k_2 < \ldots < k_n$ as $k_2, k_3, \ldots, k_n, k_1 - n$ is not strictly increasing in this case.

Combinatorial interpretation of $\alpha(n; k_1, \ldots, k_n)$ for all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$

Example.

$$\begin{array}{ccccccccc}
 & & & & \overset{\leftrightarrow}{5} & & & & \\
 & & & \overset{\leftarrow}{5} & & \overset{\rightarrow}{6} & & & \\
 & & \overset{\leftarrow}{4} & & \overset{\leftarrow}{6} & & \overset{\leftrightarrow}{7} & & \\
 & \overset{\leftarrow}{2} & & \overset{\leftarrow}{6} & & \overset{\rightarrow}{7} & & \overset{\rightarrow}{5} & \\
\overset{\rightarrow}{3} & & \overset{\leftrightarrow}{1} & & \overset{\rightarrow}{8} & & \overset{\rightarrow}{5} & & \overset{\rightarrow}{4}
\end{array}$$

- $\begin{smallmatrix} & y & \\ x & & z \end{smallmatrix}$: each entry y lies (numerically) in between its SW-neighbor x and its SE-neighbor z.

- "Lying in between" is not precise: the arrows indicate whether the inequalities are strict or not.

Directed triangles

Triangular arrays of integers of the following shape

$$
\begin{array}{ccccccccccccc}
 & & & & & & a_{1,1} & & & & & & \\
 & & & & & a_{2,1} & & a_{2,2} & & & & & \\
 & & & & \dots & & \dots & & \dots & & & & \\
 & & & a_{n-2,1} & & \dots & & \dots & & a_{n-2,n-2} & & & \\
 & a_{n-1,1} & & & a_{n-1,2} & & \dots & & \dots & & a_{n-1,n-1} & & \\
a_{n,1} & & a_{n,2} & & & a_{n,3} & & \dots & & \dots & & & a_{n,n}
\end{array}
$$

together with a function $f : \{a_{i,j}\} \to \{\leftarrow, \rightarrow, \leftrightarrow\}$, such that for all entries y that are not situated in the bottom row the following is fulfilled:

Four cases:

1. $\underset{x}{\leftarrow} \quad {}^{y} \quad \underset{z}{\leftarrow,\leftrightarrow}$: $x \leq y < z$ or $x > y \geq z$

2. $\underset{x}{\leftarrow} \quad {}^{y} \quad \underset{z}{\rightarrow}$: $x \leq y \leq z$ or $x > y > z$

3. $\underset{x}{\leftrightarrow,\rightarrow} \quad {}^{y} \quad \underset{z}{\leftarrow,\leftrightarrow}$: $x < y < z$ or $x \geq y \geq z$

4. $\underset{x}{\leftrightarrow,\rightarrow} \quad {}^{y} \quad \underset{z}{\rightarrow}$: $x < y \leq z$ or $x \geq y > z$

Rules:

- if (x, z) are "in order" and an arrow is pointing towards y then the respective inequality is strict.
- if (x, z) are not in order then

$$\leq \ \leftrightarrow \ > \quad \text{and} \quad \geq \ \leftrightarrow \ < .$$

Signed enumeration

Whenever (x, z) are not "in order" (blue cases) then y is said to be an inversion. The sign of a directed triangle is

$$(-1)^{\#\text{ of inversions}+\#\text{ of }\leftrightarrow}.$$

Theorem (2011). The signed enumeration of directed triangles with bottom row $k_1, \ldots, k_n$ is

$$\prod_{1 \le p < q \le n} (E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1}) \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}.$$

Thank you for your attention!

Remark

If $k_1 < k_2 < \ldots < k_n$ then a directed triangle is not a monotone triangle!

But it is obvious that the signed enumeration of directed triangles gives the number of monotone triangles in this case:

- All rows are strictly increasing in this case.
- Situation for y:

$$\begin{array}{ccc} x & & z \\ & y & \end{array}$$

If $f(y) = \leftarrow$ then $x < y \leq z$.
If $f(y) = \rightarrow$ then $x \leq y < z$.
If $f(y) = \leftrightarrow$ then $x < y < z$.

Precise definition of generalized ASMs

Boundary conditions:
- $S = \{s_1, s_2, \ldots, s_l\} \subseteq \{1, 2, \ldots, n\}$, strictly decreasing
- $K = \{k_{l+1}, \ldots, k_{n-r}\} \subseteq \{1, 2, \ldots, m\}$, strictly incerasing
- $T = \{t_{n-r+1}, \ldots, t_n\} \subseteq \{1, 2, \ldots, n\}$, strictly increasing

rows in S are of type $-1, 1, -1, \ldots$; the others are of type $1, -1, 1, \ldots$
rows in T are of type $\ldots, -1, 1, -1$; the others are of type $\ldots, 1, -1, 1$
columns in K are of type $1, -1, \ldots, -1, 1$; the others are of type $1, -1, \ldots, 1, -1$

$n \times m$ matrices with entries in $\{0, 1, -1\}$, non-zero entries in rows and columns alternate in sign s.t.
- the first non-zero entry in each column is 1
- column sums are 1 precisely for the columns in K
- row sums are 1 for the rows in $\{1, 2, \ldots, n\} \setminus S \cup T$
- row sums are -1 for the rows in $S \cap T$
- row sums are 0 and the first non-zero entry is -1 for the rows in S

Open problem

Bijective proof of

$$\alpha(n;k_1,\ldots,k_n)=\prod_{1\le p<q\le n}(E_{k_p}+E_{k_q}^{-1}-E_{k_p}E_{k_q}^{-1})\prod_{1\le i<j\le n}\frac{k_j-k_i+j-i}{j-i}$$

if $k_1<k_2<\ldots<k_n$.

Gelfand-Tsetlin pattern: triangular array of integers of the following shape

$$
\begin{array}{ccccccccccccc}
 & & & & & & a_{1,1} & & & & & & \\
 & & & & & a_{2,1} & & a_{2,2} & & & & & \\
 & & & & \dots & & \dots & & \dots & & & & \\
 & & a_{n-2,1} & & & \dots & & \dots & & a_{n-2,n-2} & & & \\
 & a_{n-1,1} & & a_{n-1,2} & & & \dots & & \dots & & a_{n-1,n-1} & & \\
a_{n,1} & & a_{n,2} & & & a_{n,3} & & \dots & & \dots & & & a_{n,n}
\end{array}
$$

with weak increase in $\nearrow$-direction and in $\searrow$-direction, i.e. $a_{i+1,j} \le a_{i,j} \le a_{i+1,j+1}$ for all i, j.

\# of Gelfand-Tsetlin patterns with bottom row $(k_1, k_2, \dots, k_n)$

$$= \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}$$

Direction pattern: $p : \{(i, j)|1 \leq j \leq i \leq n\} \to \{\leftarrow, \to, \leftrightarrow\}$

$$
\begin{array}{ccccccccc}
 & & & & \overset{\leftrightarrow}{(1,1)} & & & & \\
 & & & \overset{\leftarrow}{(2,1)} & & \overset{\rightarrow}{(2,2)} & & & \\
 & & \overset{\leftarrow}{(3,1)} & & \overset{\leftarrow}{(3,2)} & & \overset{\leftrightarrow}{(3,3)} & & \\
 & \overset{\leftarrow}{(4,1)} & & \overset{\leftarrow}{(4,2)} & & \overset{\rightarrow}{(4,3)} & & \overset{\rightarrow}{(4,4)} & \\
\overset{\rightarrow}{(5,1)} & & \overset{\leftrightarrow}{(5,2)} & & \overset{\rightarrow}{(5,3)} & & \overset{\rightarrow}{(5,4)} & & \overset{\rightarrow}{(5,5)}
\end{array}
$$

$$\mathrm{sgn}(p) := (-1)^{\#\text{ of } \leftrightarrow}$$

Left hand side

Gelfand-Tsetlin pattern respecting a given direction pattern:

1. $\underset{a_{i+1,j}}{\leftarrow} \quad \overset{a_{i,j}}{} \quad \underset{a_{i+1,j+1}}{\leftarrow,\leftrightarrow}$: $a_{i+1,j} \leq a_{i,j} < a_{i+1,j+1}$

2. $\underset{a_{i+1,j}}{\leftarrow} \quad \overset{a_{i,j}}{} \quad \underset{a_{i+1,j+1}}{\rightarrow}$: $a_{i+1,j} \leq a_{i,j} \leq a_{i+1,j+1}$

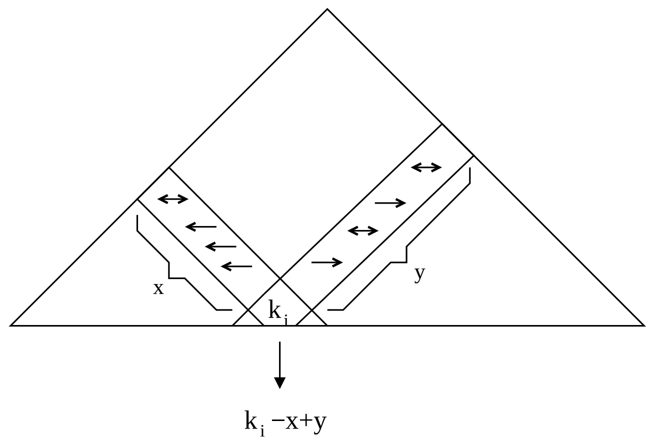
3. $\underset{a_{i+1,j}}{\leftrightarrow,\rightarrow} \quad \overset{a_{i,j}}{} \quad \underset{a_{i+1,j+1}}{\leftarrow,\leftrightarrow}$: $a_{i+1,j} < a_{i,j} < a_{i+1,j+1}$

4. $\underset{a_{i+1,j}}{\leftrightarrow,\rightarrow} \quad \overset{a_{i,j}}{} \quad \underset{a_{i+1,j+1}}{\rightarrow}$: $a_{i+1,j} < a_{i,j} \leq a_{i+1,j+1}$

$$\text{LHS} = \sum_{p \text{ direction pattern of order } n} \text{sgn}(p)$$
$$\times\,(\#\text{ of GTPs with bottom row } (k_1, \ldots, k_n) \text{ respecting direction pattern } p)$$

Right hand side

Given a direction pattern $p : \{(i,j)\} \to \{\leftarrow, \rightarrow, \leftrightarrow\}$, associate a deformation $p \cdot (k_1, \ldots, k_n)$ of the bottom row $(k_1, \ldots, k_n)$ as follows:



$$\text{RHS} = \sum_{p \text{ direction pattern of order } n} \text{sgn}(p)$$
$$\times\, (\#\text{ of GTPs with bottom row } p \cdot (k_1, \ldots, k_n))$$

Writing the recursion in terms of "simple" summations

$n = 3:$

$$\sum_{\substack{k_1 \le l_1 \le k_2 \le l_2 \le k_3 \\ l_1 < l_2}} a(l_1, l_2)$$

$$= \sum_{k_1 \le l_1 < k_2 \le l_2 \le k_3} a(l_1, l_2) + \sum_{k_1 \le l_1 \le k_2 < l_2 \le k_3} a(l_1, l_2) - \sum_{k_1 \le l_1 < k_2 < l_2 \le k_3} a(l_1, l_2)$$

$$= \sum_{l_1=k_1}^{k_2-1} \sum_{l_2=k_2}^{k_3} a(l_1, l_2) + \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2+1}^{k_3} a(l_1, l_2) - \sum_{l_1=k_1}^{k_2-1} \sum_{l_2=k_2+1}^{k_3} a(l_1, l_2)$$

Letting $V_{x,y} = E_x^{-1} + E_y - E_x^{-1} E_y$, this is equal to

$$\left. \left(V_{k_2, k_2'} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2'}^{k_3} a(l_1, l_2) \right) \right|_{k_2 = k_2'} .$$

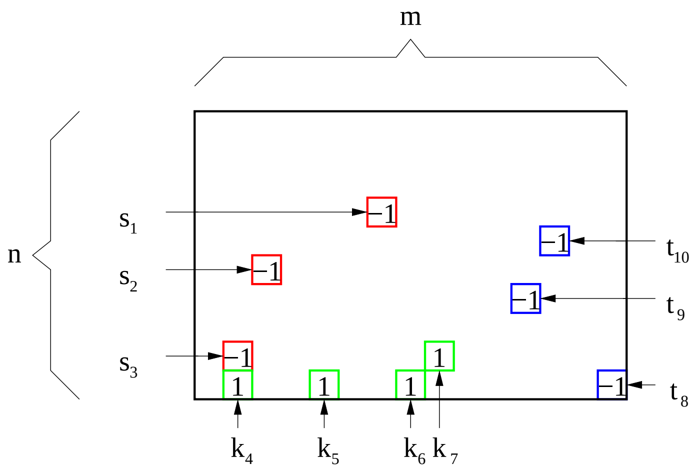
The recursion in terms of simple summations for arbitrary n

$$\sum_{\substack{k_1 \le l_1 \le k_2 \le l_2 \le k_3 \le \dots \le l_{n-1} \le k_n \\ l_i \neq l_{i+1}}} a(l_1, l_2, \dots, l_{n-1})$$

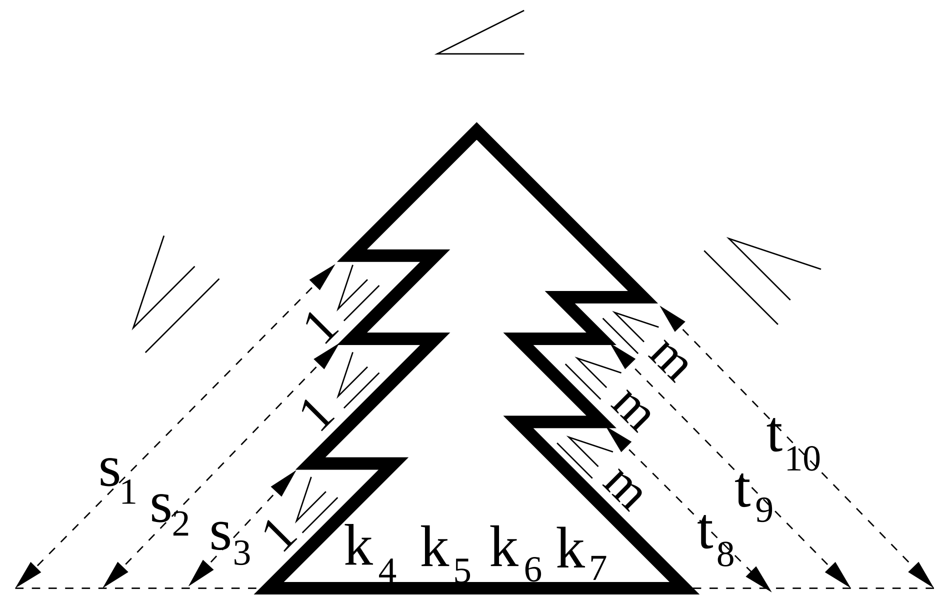
$$= \left(V_{k_2,k_2'} V_{k_3,k_3'} \cdots V_{k_{n-1},k_{n-1}'} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2'}^{k_3} \sum_{l_3=k_3'}^{k_4} \cdots \sum_{l_{n-1}=k_{n-1}'}^{k_n} a(l_1, \dots, l_{n-1}) \right) \Bigg|_{(k_2',\dots,k_{n-1}')=(k_2,\dots,k_{n-1})}$$

$$=: \sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} a(l_1, l_2, \dots, l_{n-1})$$

General boundary conditions: left, right and bottom



- the first non-zero entry is -1 precisely in the rows $s_1, s_2, \ldots, s_l$
- the last non-zero entry is -1 precisely in the rows $t_{n-r+1}, \ldots, t_n$
- the first non-zero entry is 1 in each column
- the column sums are 1 precisely for the columns $k_{l+1}, k_{l+2}, \ldots, k_{n-r}$



Partial monotone triangles with entries in $\{1, \ldots, m\}$ s. t.

- the i-th NE-diagonal is truncated by s_i elements for $1 \leq i \leq l$
- the i-th SE-diagonal is truncated by t_i elements for $n - r + 1 \leq i \leq n$ and
- the bottom row is $(k_{l+1}, k_{l+2}, \ldots, k_{n-r})$.

Application: proof of the operator formula

$$a(l_1,\ldots,l_{n-1})$$
$$= \prod_{1\le p<q\le n-1} (E_{l_p}+E_{l_q}^{-1}-E_{l_p}E_{l_q}^{-1}) \prod_{1\le i<j\le n-1} \frac{l_j-l_i+j-i}{j-i}$$
$$= \prod_{1\le p<q\le n-1} (E_{l_p}+E_{l_q}^{-1}-E_{l_p}E_{l_q}^{-1}) \det_{1\le i,j\le n-1} \left(\binom{l_i+i}{j-1}\right)$$

(Vandermonde determinant)

Then

$$A(l_1,\ldots,l_{n-1}) = \prod_{1\le p<q\le n-1} (E_{l_p}+E_{l_q}^{-1}-E_{l_p}E_{l_q}^{-1}) \det_{1\le i,j\le n-1} \left(\binom{l_i+i}{j}\right)$$

is the discrete antiderivative of $a(l_1,\ldots,l_{n-1})$ with the desired property.