Where are the bijections ?

Plane Partitions and Alternating Sign Matrices

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joint work with Arvind Ayyer, Roger Behrend, and Matjaž Konvalinka

Outline

I. Alternating sign matrices

II. Cyclically symmetric lozenge tilings with a central triangular hole

III. Totally symmetric self-complementary plane partitions

IV. Alternating sign triangles

I. Alternating sign matrices (ASMs)

Alternating sign matrices = ASMs

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Square matrix with entries in $\{0, \pm 1\}$ such that in each row and each column

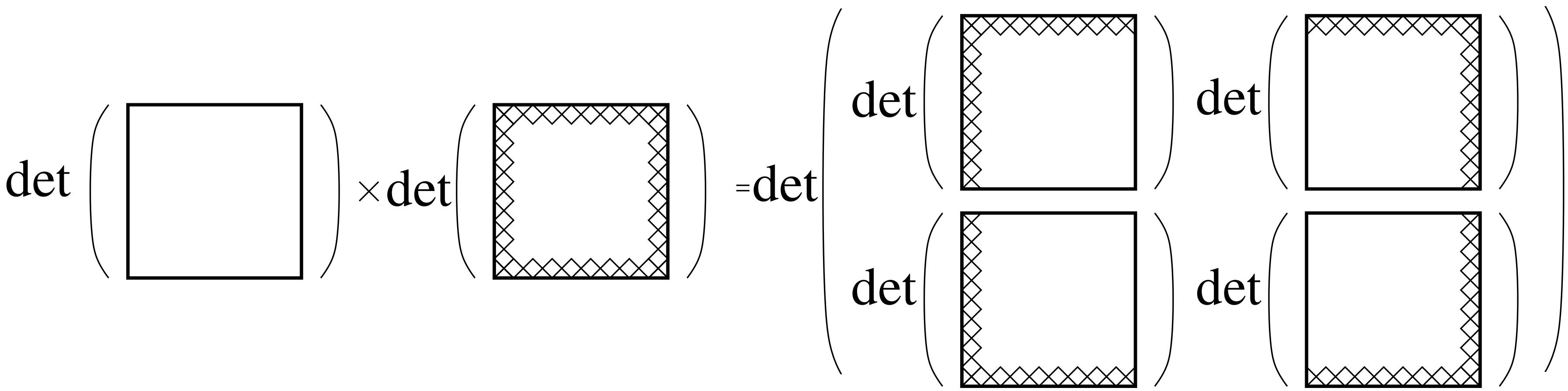
- the non–zero entries appear with alternating signs, and
- the sum of entries is 1.

How many?

n	1	2	3	4
	(1)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$3! + \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	42

Origin of ASMs: λ–determinant

The Desnanot–Jacobi identity:



Charles L. Dodgson (Lewis Carroll) used this to devise an algorithm for calculating determinants that required only 2×2 determinants. (Condensation of determinants, 1866)

3×3 **determinants**

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = \frac{1}{a_{2,2}}$$

$$\times \det\begin{pmatrix} \det\begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix} & \det\begin{pmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix} \\ \det\begin{pmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{pmatrix} & \det\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \end{pmatrix}$$

4×4 **determinants**

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} = \frac{1}{\det\begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix}}$$

$$\times \det\begin{pmatrix} \det\begin{pmatrix} a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} & \det\begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \\ \det\begin{pmatrix} a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix} & \det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \end{pmatrix}$$

3×3 determinants are expressible in terms of 2×2 determinants...and so are 4×4 determinants!

David P. Robbins and Howard Rumsey, Jr. in the 1980s: What happens if we generalize the definition of a 2×2 determinant to

$$\det_\lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} + \lambda a_{12}a_{21}$$

and, furthermore, use the previous observations to generalize the $n \times n$ determinant?

Theorem (Robbins and Rumsey, 1986). Let $A = (a_{i,j})$ be an $n \times n$ matrix, ASM_n the set of $n \times n$ alternating sign matrices, then

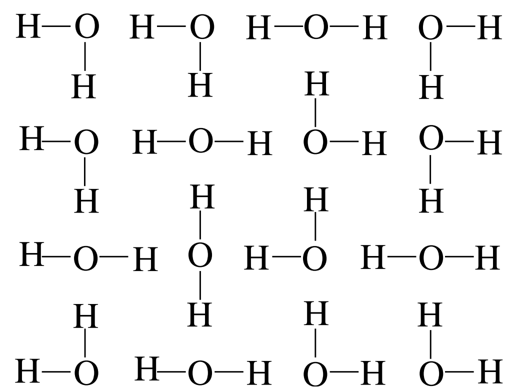
$$\det_\lambda(A) = \sum_{B \in \mathsf{ASM}_n} \lambda^{\mathcal{I}(B)} (1 + \lambda^{-1})^{\mathcal{N}(B)} \prod_{i,j=1}^{n} a_{i,j}^{B_{i,j}},$$

where $\mathcal{I}(B)$ the inversion number of B and $\mathcal{N}(B)$ the number of -1's in B.

ASMs in statistical physics

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

ASM

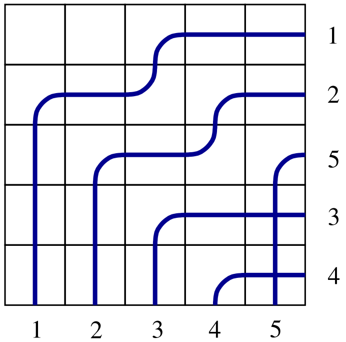


Square ice

The connection was established in 1992 by Elkies, Kuperberg, Larsen and Propp. The tools (e.g.,Yang-Baxter equation) physicists had developed turned out to be very useful in the following.

ASMs and Bumpless Pipe Dreams

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



In 1982, Grothendieck polynomials have been introduced to study the K-theory of the complete flag variety and they can be written as a certain generating function of bumpless pipe dreams (as revealed by Weigandt in 2020).

The number of $n \times n$ ASMs

Conjecture (Mills, Robbins, Rumsey 1980s). The number of $n \times n$ alternating sign matrices is

$$\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} =: A_n.$$

David Robbins, 1991: "*These conjectures are of such compelling simplicity that it is hard to know how any mathematician can bear the pain of living without understanding why they are true.*"

Double quotient: $\dfrac{\left(\frac{A_{n+2}}{A_{n+1}}\right)}{\left(\frac{A_{n+1}}{A_n}\right)} = \frac{3}{4}\frac{(3n+2)(3n+4)}{(2n+1)(2n+3)}$

The conjecture was presented by David Robbins at an Oberwolfach workshop in 1982. Doron Zeilberger was in the audience:

"So Dave was the first (and as far as I know only) person to give two hour-talks at the same Oberwolfach combinatorics meeting. I remember these talks like they were given yesterday. They were definitely in the top ten talks that I have ever heard. What is so captivating about Dave's lecture style is that unlike the rest of us, that try to state things in the most general setting (thereby completely obscuring the ideas), Dave went the other way, and made things as concrete as possible and actually had numbers in his talk, not general formulas. The formulas only came at the end, after the ideas and concepts were internalized."

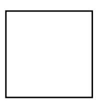
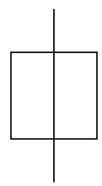
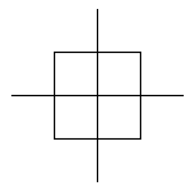
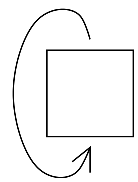
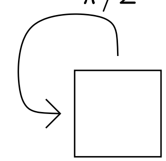
• Zeilberger then provided the first proof of the conjecture in 1996. The paper has 84 pages and an army of "proof checkers" was required before the proof was believed to be true.

• In the same year, Greg Kuperberg came up with another proof that is based on techniques developed by physicists.

• About ten years later, I gave yet another proof. The most concise version (9 pages) can be found in the 50th anniversary issue of JCT-A (2016).

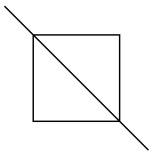
Symmetry classes of ASMs

8 symmetry classes; Robbins conjectured product formulas for $5\frac{1}{2}$ classes.

No symmetry	$\mathcal{V} \sim \mathcal{H}$	$\mathcal{V}$ and $\mathcal{H}$	$\mathcal{R}_{\pi}$	$\mathcal{R}_{\pi/2}$
				
Zeilberger 1996	Kuperberg 2002	Okada 2004	n even: Kuperberg 2002 n odd: Razumov & Stroganov 2005	n even: Kuperberg 2002 n odd: Razumov & Stroganov 2005

Many cases were dealt with by Greg Kuperberg in his 2002 Ann. Math. paper.

$\mathcal{D} \sim \mathcal{A}$

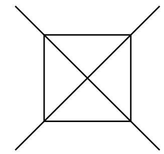


n	#
1	1
2	2
3	5
4	2^4
5	67
6	$2^4 \cdot 23$
7	$2 \cdot 5 \cdot 263$
8	$2^3 \cdot 11 \cdot 277$

All symmetries

n	#
1	1
2	0
3	1
4	0
5	1
6	0
7	2
8	0
9	2^2
10	0
11	13
12	0
13	$2 \cdot 23$
14	0
15	$2^3 \cdot 31$
16	0
17	$2^2 \cdot 379$

$\mathcal{D}$ and $\mathcal{A}$



	n	#
$\rightarrow$	1	1
	2	2
$\rightarrow$	3	3
	4	2^3
$\rightarrow$	5	$3 \cdot 5$
	6	$2^2 \cdot 13$
$\rightarrow$	7	$2 \cdot 3^2 \cdot 7$
	8	$2^3 \cdot 71$
$\rightarrow$	9	$2 \cdot 3^4 \cdot 11$
	10	$2^2 \cdot 2609$
$\rightarrow$	11	$3^3 \cdot 11^2 \cdot 13$
	12	$2^3 \cdot 31 \cdot 1303$
$\rightarrow$	13	$2 \cdot 3^2 \cdot 11 \cdot 13^2 \cdot 17$

Behrend, F., and Konvalinka 2016; Robbins prize 2019.

II. **Cyclically symmetric lozenge tilings with a central triangular hole**

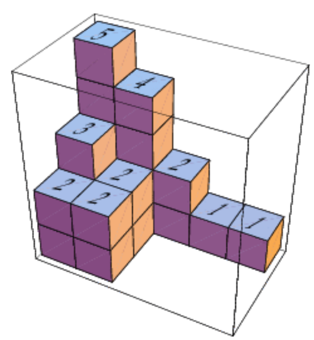
What is a plane partition?

A plane partition in an $a \times b \times c$ box is a subset

$$PP \subseteq \{1,2,\ldots,a\} \times \{1,2,\ldots,b\} \times \{1,2,\ldots,c\}$$

with

$$(i,j,k) \in PP \Rightarrow (i',j',k') \in PP \quad \forall (i',j',k') \leq (i,j,k).$$



$a = 4, b = 3, c = 5$

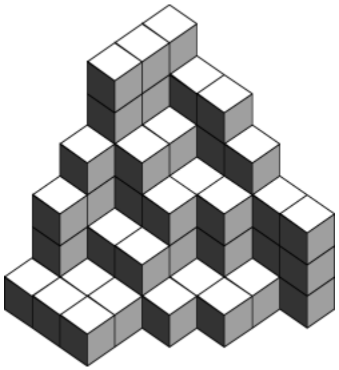
Cyclically symmetric plane partitions = CSPPs

An $n \times n \times n$ plane partition PP is cyclically symmetric if

$$(i, j, k) \in PP \Rightarrow (j, k, i) \in PP.$$

In 1979, George Andrews proved that the number of $n \times n \times n$ cyclically symmetric plane partitions is

$$\prod_{i=0}^{n-1} \frac{(3i+2)(3i)!}{(n+i)!}.$$



A determinant in Andrews' proof

In his proof, Andrews shows that the number of CSPPs of order n is given by the following determinant

$$\det_{0\le i,j\le n-1}\left(\delta_{i,j}+\binom{i+j}{i}\right)$$

and then proves that

$$\det_{0\le i,j\le n-1}\left(\delta_{i,j}+\binom{i+j}{i}\right)=\prod_{i=0}^{n-1}\frac{(3i+2)(3i)!}{(n+i)!}.$$

Then, probably out of curiousity, he also considered the following more general determinant:

$$\det_{0\le i,j\le n-1}\left(\delta_{i,j}+\binom{k+i+j}{i}\right):=D_n(k)$$

$D_n(k)$ for small values of n

$$2$$

$$k+5$$

$$(k+4)(k+5)$$

$$\frac{1}{12}(k+4)^2(k+9)(k+11)$$

$$\frac{1}{72}(k+4)^2(k+6)(k+9)(k+11)^2$$

$$\frac{(k+4)^2(k+6)^2(k+11)^2(k+13)(k+15)(k+17)}{8640}$$

$$\frac{(k+4)^2(k+6)^2(k+8)(k+10)(k+11)(k+13)(k+15)^2(k+17)^2}{518400}$$

$$\frac{(k+4)^2(k+6)^2(k+8)^2(k+10)^2(k+15)^2(k+17)^3(k+19)(k+21)(k+23)}{870912000}$$

$$\frac{(k+4)^2(k+6)^2(k+8)^2(k+10)^3(k+12)(k+15)(k+17)^3(k+19)^2(k+21)^2(k+23)^2}{731566080000}$$

Double quotient:

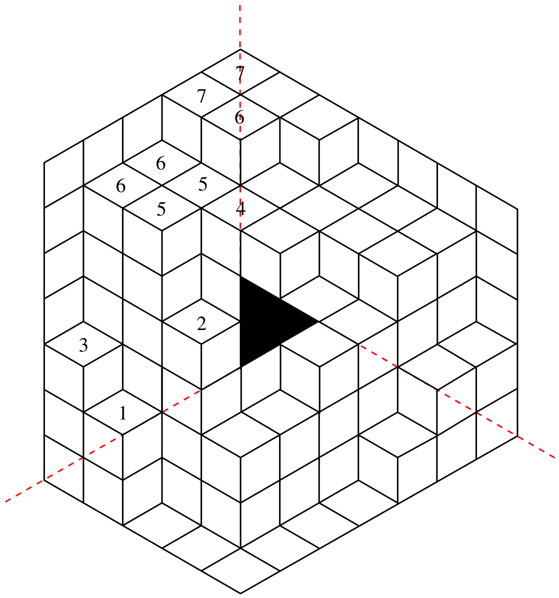
$$\frac{\left(\frac{D_{n+2}(k)}{D_{n+1}(k)}\right)}{\left(\frac{D_{n+1}(k)}{D_n(k)}\right)} = \begin{cases} \frac{(3n+k+1)(3n+k+3)(3n+k+5)}{4(n+1)(2n+k+1)(2n+k+3)} & n \text{ even} \\ \frac{(3n+2k+1)(3n+2k+3)(3n+2k+5)}{4(n+k+1)(2n+k+1)(2n+k+3)} & n \text{ odd} \end{cases}$$

This can be used to derive a (conjectural) product formula for $D_n(k)$.

Big surprise (Richard Stanley?):

$$D_n(2) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

Combinatorial interpretation for $D_n(2)$



Cyclically symmetric lozenge tiling of a hexagon with side lengths $n+2, n, n+2, n, n+2, n$ with a central triangular hole of size 2.

To obtain the combinatorial interpretation for any k, replace 2 by k!

Summary: There is the same number of $n \times n$ alternating sign matrices as there is of cyclically symmetric lozenge tilings of a hexagon with side lengths $n+2, n, n+2, n, n+2, n$ with a central triangular hole of size 2.

Where is the bijection ?

Stanley 2009: *"This is one of the most intriguing open problems in the area of bijective proofs."*

Two bijections (Fischer and Konvalinka, 2020)

ASM_n = set of $n \times n$ ASMs

$\mathrm{ASM}_{n,i}$ = subset of ASM_n of matrices $(a_{p,q})_{1 \le p,q \le n}$ with $a_{1,i} = 1$

$\mathrm{CSLT}(2)_n$ = cyclically symmetric lozenge tilings of a hexagon with side lengths $n+2, n, n+2, n, n+2, n$ with a central triangular hole of size 2.

$\mathrm{CSLT}(2)_{n,i}$ = subset of $\mathrm{CSLT}(2)_n$ of tilings with i horizontal lozenges along the NW side.

We have constructed a bijection between the following sets:

$$\mathrm{CSLT}(2)_{n-1} \times \mathrm{ASM}_{n,i} \longrightarrow \mathrm{ASM}_{n-1} \times \mathrm{CSLT}(2)_{n,i}$$

- Once such a bijection is constructed, it follows that

$$|\,\mathrm{CSLT}(2)_{n-1}| \cdot |\,\mathrm{ASM}_{n,i}\,| = |\,\mathrm{ASM}_{n-1}\,| \cdot |\,\mathrm{CSLT}(2)_{n,i}|.$$

- By induction, we can assume $|\,\mathrm{CSLT}(2)_{n-1}| = |\,\mathrm{ASM}_{n-1}\,|$ and so $|\,\mathrm{ASM}_{n,i}\,| = |\,\mathrm{CSLT}(2)_{n,i}|$.
- Summing this over all i implies $|\,\mathrm{CSLT}(2)_n| = |\,\mathrm{ASM}_n\,|$.

Example $\mathrm{CSLT}(2)_3 \times \mathrm{ASM}_{4,2} \longrightarrow \mathrm{ASM}_3 \times \mathrm{CSLT}(2)_{4,2}$ for $x = 0$

$\left(\emptyset, \begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&2&1\end{smallmatrix}\right)$	$\left(\emptyset, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&1&1\end{smallmatrix}\right)$	$\left(\emptyset, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&1\end{smallmatrix}\right)$	$\left(\emptyset, \begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&2\end{smallmatrix}\right)$
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$\left(2, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&2\end{smallmatrix}\right)$	$\left(2, \begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3\end{smallmatrix}\right)$	$\left(2, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3\end{smallmatrix}\right)$	$\left(2, \begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3\end{smallmatrix}\right)$
$\left(2, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&2\end{smallmatrix}\right)$	$\left(2, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&2\end{smallmatrix}\right)$	$\left(2, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3\end{smallmatrix}\right)$	$\left(2, \begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3\end{smallmatrix}\right)$
$\left(2, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&2\end{smallmatrix}\right)$	$\left(2, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&2\end{smallmatrix}\right)$	$\left(2, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&1\end{smallmatrix}\right)$	$\left(2, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&1\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&2&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&2&2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&3\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&3\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, 4\right)$	$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&3\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&3\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&3\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&2&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&1&1\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3&3\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3&3\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&2\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&2&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&2&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&2&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3&2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&2&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&2&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, 4\right)$	$\left(\begin{smallmatrix}3&1\end{smallmatrix}, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, 4\right)$
$\left(3, \begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&1\end{smallmatrix}\right)$	$\left(3, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&1\end{smallmatrix}\right)$	$\left(3, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&2&1\end{smallmatrix}\right)$	$\left(3, \begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&1\end{smallmatrix}\right)$
$\left(3, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&1\end{smallmatrix}\right)$	$\left(3, \begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&1\end{smallmatrix}\right)$	$\left(3, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&2&1\end{smallmatrix}\right)$	$\left(3, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&2&1\end{smallmatrix}\right)$
$\left(3, \begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3&1\end{smallmatrix}\right)$	$\left(3, \begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&3&1\end{smallmatrix}\right)$	$\left(3, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&2&1\end{smallmatrix}\right)$	$\left(3, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&2&1\end{smallmatrix}\right)$
$\left(3, \begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&1&1\end{smallmatrix}\right)$	$\left(3, \begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, \begin{smallmatrix}4&1&1\end{smallmatrix}\right)$		

Bijective Proof of the Product Formula

B_n = set of $(2n-1)$-subsets of $[3n-2] = \{1, 2, \dots, 3n-2\}$; $|\,\mathrm{B}_n\,| = \binom{3n-2}{2n-1}$

$\mathrm{B}_{n,i}$ = set of elements of B_n whose median is $n+i-1$; $|\,\mathrm{B}_{n,i}\,| = \binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}$

We have constructed a bijection between the following sets:

$$\mathrm{CSLT}(2)_{n-1} \times \mathrm{B}_{n,1} \times \mathrm{ASM}_{n,i} \longrightarrow \mathrm{CSLT}(2)_{n-1} \times \mathrm{ASM}_{n-1} \times \mathrm{B}_{n,i}$$

Then we also have a bijection

$$\mathrm{CSLT}(2)_{n-1} \times \mathrm{B}_{n,1} \times \mathrm{ASM}_{n} \longrightarrow \mathrm{CSLT}(2)_{n-1} \times \mathrm{ASM}_{n-1} \times \mathrm{B}_{n}\,.$$

Iterating this, we obtain a bijection

$$\mathrm{CSLT}(2)_0 \times \cdots \times \mathrm{CSLT}(2)_{n-1} \times \mathrm{B}_{1,1} \times \cdots \times \mathrm{B}_{n,1} \times \mathrm{ASM}_n \longrightarrow \\ \mathrm{CSLT}(2)_0 \times \cdots \times \mathrm{CSLT}(2)_{n-1} \times \mathrm{B}_1 \times \cdots \times \mathrm{B}_n\,.$$

Example: $\mathrm{CSLT}(2)_2 \times \mathrm{B}_{3,1} \times \mathrm{ASM}_{3,2} \longrightarrow \mathrm{CSLT}(2)_2 \times \mathrm{ASM}_2 \times \mathrm{B}_{3,2}$ **for** $x = 0$

$\left(\emptyset, 12345, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23457\right)$	$\left(\emptyset, 12345, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23456\right)$	$\left(\emptyset, 12345, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23456\right)$
$\left(\emptyset, 12346, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13457\right)$	$\left(\emptyset, 12346, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13456\right)$	$\left(\emptyset, 12346, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13456\right)$
$\left(\emptyset, 12347, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12457\right)$	$\left(\emptyset, 12347, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 12456\right)$	$\left(\emptyset, 12347, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12456\right)$
$\left(\emptyset, 12356, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13456\right)$	$\left(\emptyset, 12356, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 12456\right)$	$\left(\emptyset, 12356, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12456\right)$
$\left(\emptyset, 12357, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13457\right)$	$\left(\emptyset, 12357, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 12457\right)$	$\left(\emptyset, 12357, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12457\right)$
$\left(\emptyset, 12367, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13467\right)$	$\left(\emptyset, 12367, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 12467\right)$	$\left(\emptyset, 12367, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12467\right)$
$\left(2, 12345, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23467\right)$	$\left(2, 12345, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23467\right)$	$\left(2, 12345, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23457\right)$
$\left(2, 12346, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13467\right)$	$\left(2, 12346, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13467\right)$	$\left(2, 12346, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13457\right)$
$\left(2, 12347, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12467\right)$	$\left(2, 12347, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 12467\right)$	$\left(2, 12347, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\emptyset, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 12457\right)$
$\left(2, 12356, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23456\right)$	$\left(2, 12356, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23456\right)$	$\left(2, 12356, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13456\right)$
$\left(2, 12357, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23457\right)$	$\left(2, 12357, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23457\right)$	$\left(2, 12357, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13457\right)$
$\left(2, 12367, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23467\right)$	$\left(2, 12367, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23467\right)$	$\left(2, 12367, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) \leftrightarrow \left(2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13467\right)$

The python code is available at `https://www.fmf.uni-lj.si/~konvalinka/asmcode.html`.

Our approach

- We translate some non-bijective ("computational") proofs into combinatorics!

- All arithmetic operations accept for division can be "modelled": Addition through disjoint unions, subtraction through signed sets, multiplication through Cartesian products.

- The fact that we cannot deal with division explains the "redundant" factors in our bijections.

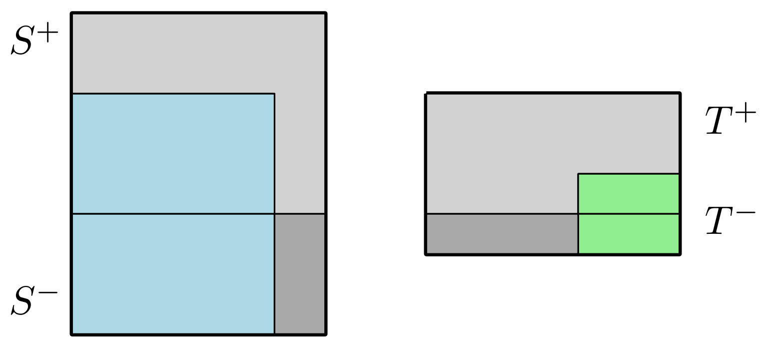
- In the original proofs, signs are unavoidable and this makes it necessary to work with signed sets.

- Is there a non-bijective proof that avoids signs? Is there a bijective proof that avoids signed sets (and can this proof be translated into a simpler computation)?

Sijections

A signed set is a pair of disjoint finite sets: $\underline{S} = (S^+, S^-)$ with $S^+ \cap S^- = \emptyset$. The size of a signed set $\underline{S}$ is $|\underline{S}| = |S^+| - |S^-|$.

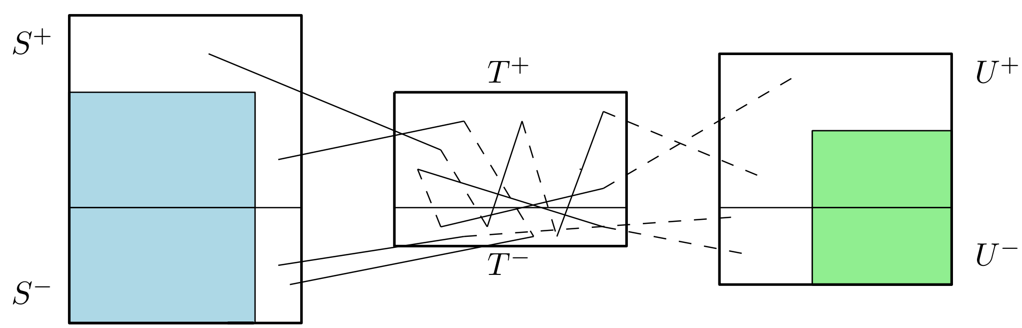
The role of bijections for signed sets is played by "signed bijections", which we call sijections: A sijection φ from $\underline{S}$ to $\underline{T}$, $\varphi\colon \underline{S} \Rightarrow \underline{T}$, is an involution on the set $(S^+ \cup S^-) \sqcup (T^+ \cup T^-)$ with $\varphi(S^+ \sqcup T^-) = S^- \sqcup T^+$.



This implies: $|\underline{S}| = |S^+| - |S^-| = |T^+| - |T^-| = |\underline{T}|$

Composition of sijections

Suppose $\underline{S}, \underline{T}, \underline{U}$ are signed sets and $\varphi : \underline{S} \Rightarrow \underline{T}$, $\psi : \underline{T} \Rightarrow \underline{U}$, then we can construct a sijection $\psi \circ \varphi : \underline{S} \to \underline{U}$ by alternating applications of φ (solid lines) and ψ (dashed lines) as sketched next.



The special case $S^- = U^- = \emptyset$ is the **Garsia-Milne involution principle.**

III. Totally symmetric self-complementary plane partitions (TSSCPPs)

Totally symmetric self-complementary plane partitions

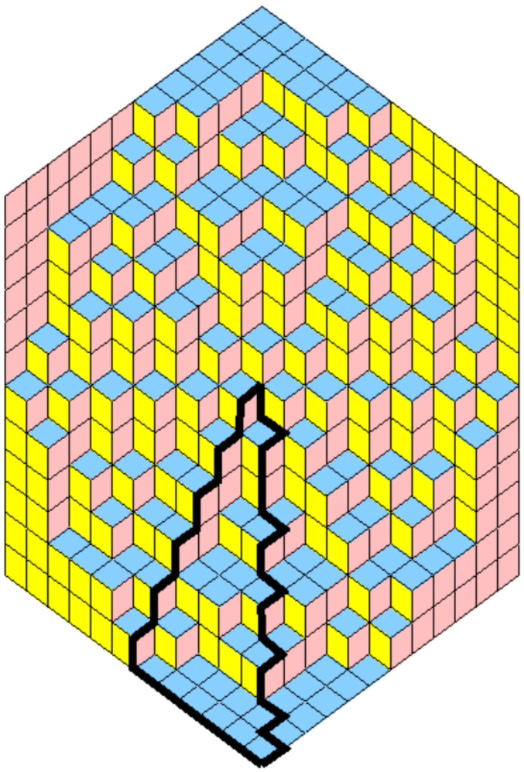


Figure by Di Francesco / Zinn-Justin

- **Totally symmetric:**
$(i, j, k) \in PP \Rightarrow \sigma(i, j, k) \in PP \; \forall \sigma \in \mathcal{S}_3$
(MacMahon 1899, 1915/16)

- **Self-complementary:**
Equal to its complement in the $2n \times 2n \times 2n$ box

It was again David Robbins who introduced this new operation of complementation.

Another surprise!

The number of TSSCPPs in a $2n \times 2n \times 2n$ box is (again) $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.

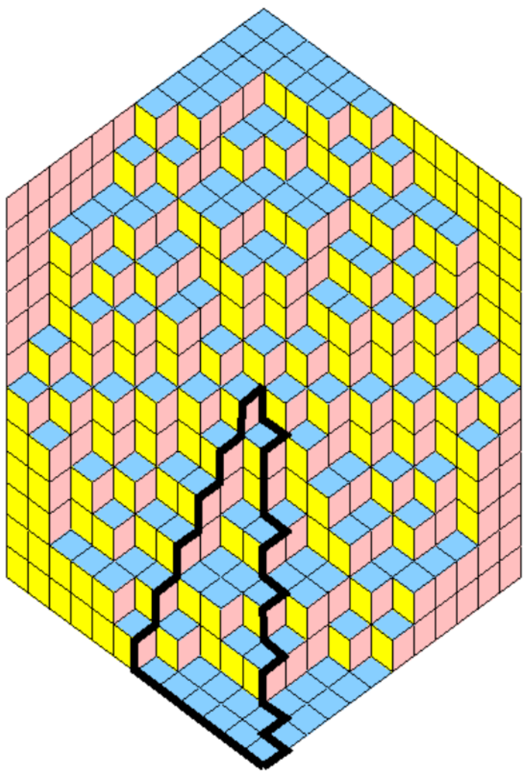
Conjectured by Mills, Robbins and Rumsey in 1986, proved by Andrews in 1994.

Where are the bijections ?

• ... between TSSCPPs and ASMs.

• ... between TSSCPPs and cyclically symmetric lozenge tilings of a hexagon with a central triangular hole of size 2.

TSSCPPs and triangular arrays of numbers



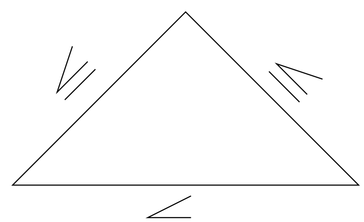
$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 2 & & & & \\
 & & & 1 & & 1 & & 3 & & & \\
 & & 1 & & 1 & & 3 & & 3 & & \\
 & 1 & & 1 & & 2 & & 3 & & 4 & \\
1 & & 1 & & 2 & & 3 & & 4 & & 6
\end{array}
$$

Triangular arrays of positive integers such that
- the entries increase weakly along $\nearrow$-diagonals,
- the entries increase weakly along $\searrow$-diagonals, and
- the entries in the i-th diagonal, counted from the left, are bounded by i.

= Magog triangles

ASMs and triangular arrays of numbers

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 2 & & 5 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

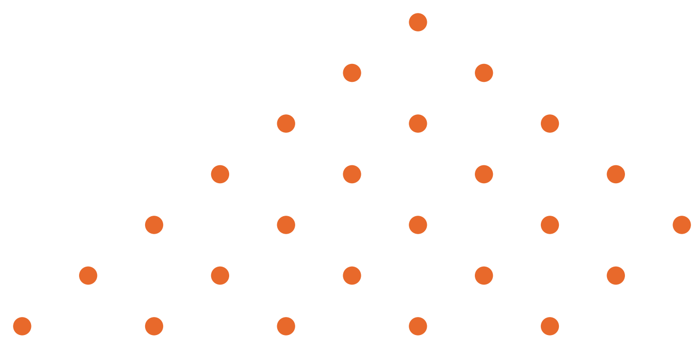
- Monotonicity requirements: 

- Bottom row: $1, 2, \ldots, n$.

$=$ Gog triangles (or monotone triangles).

Generalization: Gog trapezoids

(n,k)-**Gog trapezoid:** arrangement of positive integers of the following form



$n =$ number of rows $= 7$

$k =$ number of $\nearrow$-diagonals $= 5$

such that

- $\nearrow$- and $\searrow$-diagonals are weakly increasing,
- rows are strictly increasing,
- entries in the i-th $\searrow$-diagonal are bounded from above by i.

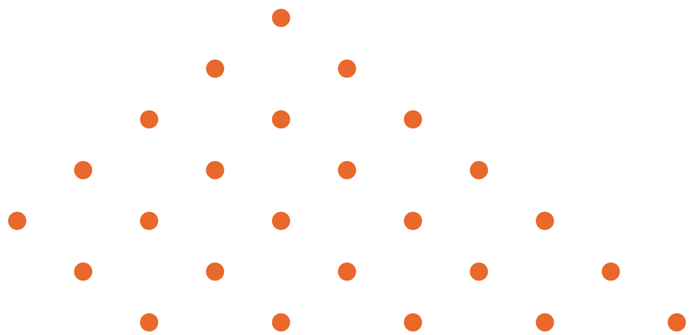
$n = k$: Gog triangles

Example: $(7, 5)$-Gog trapezoid

						4				
					3		5			
				3		4		5		
			2		3		4		5	
		1		2		3		5		6
	1		2		3		4		6	
1		2		3		4		5		

Generalization: Magog trapezoids

(n,k)-**Magog trapezoid:** arrangement of positive integers of the following form



$n =$ number of rows $= 7$

$k =$ number of $\searrow$-diagonals $= 5$

such that

- $\nearrow$- and $\searrow$-diagonals are weakly increasing,
- entries in the i-th $\nearrow$-diagonal are bounded from above by i.

$n = k$: Magog triangles

Example: $(7,5)$-Magog trapezoid

$$
\begin{array}{ccccccccccc}
 & & & & 1 & & & & & & \\
 & & & 1 & & 2 & & & & & \\
 & & 1 & & 2 & & 3 & & & & \\
 & 1 & & 1 & & 2 & & 3 & & & \\
1 & & 1 & & 2 & & 3 & & 4 & & \\
 & 1 & & 2 & & 3 & & 4 & & 6 & \\
 & & 2 & & 3 & & 3 & & 5 & & 6
\end{array}
$$

Generalized conjecture

Conjecture (Mills, Robbins, Rumsey, 1986). There is the same number of (n,k)-Gog trapezoids as there is of (n,k)-Magog trapezoids.

• Zeilberger actually proved this conjecture. His proof is non-bijective and involves complicated computations.

• Kuperberg "only" provided a (non-bijective) proof for the special case $n = k$.

Statistics on Gogs and Magogs

(n, k)-Gog trapezoids:

• A minimum is an entry equal to 1.

• A maximum is an entry in the k-th $\nearrow$-diagonal that is equal to the upper bound for the entries in its $\searrow$-diagonal.

(n, k)-Magog trapezoids:

• A minimum is an entry equal to 1 that is located in the leftmost $\searrow$-diagonal.

• A maximum is an entry in the rightmost $\searrow$-diagonal that is equal to the upper bound for the entries in its $\nearrow$-diagonal.

Example

Gog:

```
            4
          3   5
        3   4   5
      2   3   4   5
    1   2   3   5   6
  1   2   3   4   6
1   2   3   4   5
```

Magog:

```
            1
          1   2
        1   2   3
      1   1   2   3
    1   1   2   3   4
      1   2   3   4   6
        2   3   3   5   6
```

Minima and Maxima

Refined generalized conjecture

Conjecture (Mills, Robbins, Rumsey, 1986). The number of (n,k)-Gog trapezoids with p minima and q maxima is equal to the number of (n,k)-Magog trapezoids with p maxima and q minima.

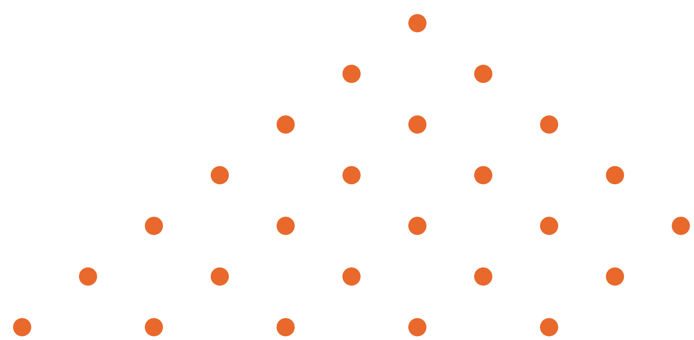
Where is the bijection that switches the number of maxima and minima ?

• So far there is not even a “computational” proof of this conjecture.

• In a recent article, I have provided constant term formulas for the quantities involved. To prove the conjecture, it would suffice to show that the constant terms of two expressions are the same.

Krattenthaler's additional parameter m

(m, n, k)-Gog trapezoid: arrangement of positive integers of the following form



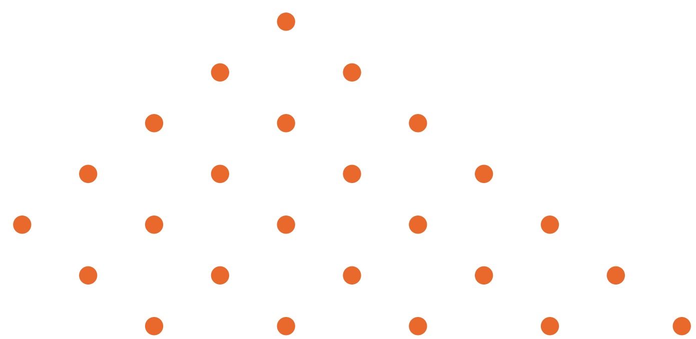
$n =$ number of rows $= 7$

$k =$ number of $\nearrow$-diagonals $= 5$

such that

- $\nearrow$- and $\searrow$-diagonals are weakly increasing,
- rows are strictly increasing,
- entries in the i-th $\searrow$-diagonal are bounded from above by $m + i$.

(m, n, k)-**Magog trapezoid:** arrangement of positive integers of the following form



$n =$ number of rows $= 7$

$k =$ number of $\searrow$-diagonals $= 5$

such that

- $\nearrow$- and $\searrow$-diagonals are weakly increasing,
- entries in the i-th $\nearrow$-diagonal are bounded from above by $m + i$.

Generalization of the refined generalized conjecture

Conjecture (Krattenthaler, 1997). The number of (m, n, k)-Gog trapezoids with p minima and q maxima is equal to the number of (m, n, k)-Magog trapezoids with p maxima and q minima.

- For $k = 1$, it is not difficult to find a bijection that switches the number of maxima and minima.
- 2011: Biane and Chebellah provided an explicit bijection for $m = 0$ and $k = 2$.
- 2015: Bettinelli presented another explicit bijection for $m = 0$ and $k = 2$.
- The parameter m can also be included in my constant term formulas.
- 2014: Biane and Chebellah found further exciting conjectures.

IV. Alternating sign triangles

Alternating sign triangles =ASTs

An AST of order n is a triangular array of 1's, -1's and 0's with n centered rows

$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \\ & & \bullet & \bullet & \bullet & & \\ & & & \bullet & & & \end{array}$$

such that

(1) the non-zero entries alternate in each row and each column,

(2) all row sums are 1, and

(3) the topmost non-zero entry of each column is 1 (if such an entry exists).

Example:

$$\begin{array}{ccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 1 & -1 & 1 & 0 & 0 & \\ & & 1 & -1 & 1 & & \\ & & & 1 & & & \end{array}$$

Theorem (Ayyer, Behrend, and F., 2016). There is the same number of $n \times n$ ASMs as there is of ASTs with n rows.

Back to Andrews' determinant

$$D_n(k) = \det_{0 \le i,j \le n-1} \left(\delta_{i,j} + \binom{k+i+j}{i} \right)$$

Recall:

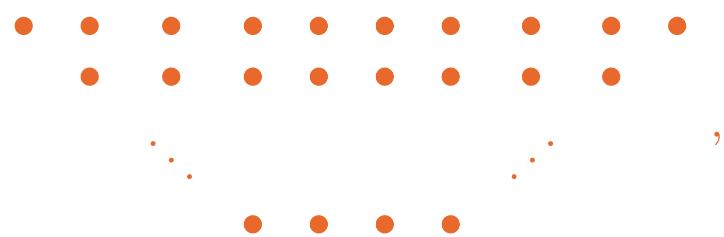
• $D_n(2)$ is the number of $n \times n$ ASMs as well as the number of ASTs with n rows.

• $D_n(k)$ is the number of cyclically symmetric lozenge tilings of a hexagon with central triangular hole of size k.

Is there a combinatorial realization of $D_n(k)$ in terms of ASM-like objects ?

Alternating sign trapezoids

For $n \geq 1, l \geq 2^*$, an *(n, l)-alternating sign trapezoid* is an array of 1's, -1's and 0's with n centered rows and l elements in the bottom row, arranged as follows



such that the following conditions are satisfied.

(1) In each row and column, the non-zero entries alternate.

(2) All row sums are 1.

(3) The topmost non-zero entry in each column is 1.

(4) The column sums are 0 for the middle $l - 2$ columns.

*Can be extended to $l = 1$.

Example

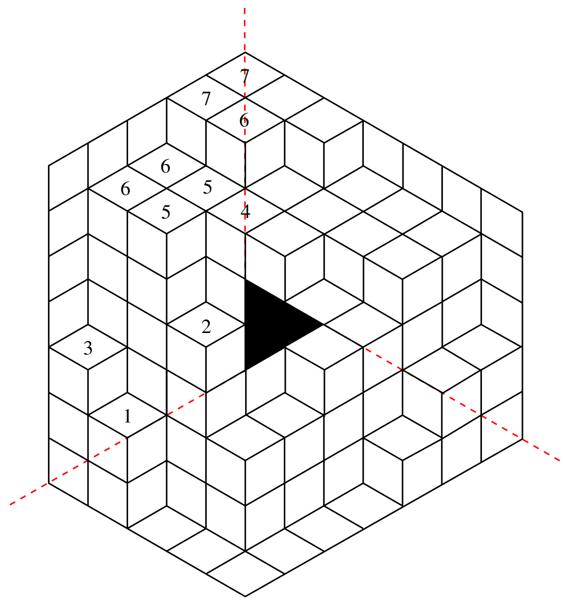
A $(5,4)$-alternating sign trapezoid.

$$\begin{array}{ccccccccccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 1 & & 0 & & 0 & & 0 & & 0 \\
& 0 & & 0 & & 0 & & 0 & & 1 & & 0 & & -1 & & 1 & & 0 & & 0 & \\
& & 0 & & 1 & & 0 & & -1 & & 0 & & 1 & & -1 & & 1 & & & & \\
& & & 0 & & 0 & & 0 & & 1 & & -1 & & 1 & & & & & & & \\
& & & & 1 & & 0 & & -1 & & 1 & & & & & & & & & &
\end{array}$$

ASTs with n rows are $(n-1,3)$-alternating sign trapezoids. (Delete the bottom row.)

Alternating sign trapezoids and cyclically symmetric lozenge tilings of a holey hexagon

Theorem (Behrend, F. 2018). There is the same number of (n,l)-alternating sign trapezoids as there is of cyclically symmetric lozenge tilings of a hexagon with side lengths $n+l-1, n, n+l-1, n, n+l-1, n$ that has a central triangular hole of size $l-1$.



Thank you!

Three statistics on alternating sign trapezoids

- A 1-column is a column with sum 1.
- A 10-column is a 1-column whose bottom element is 0.

Simple fact: An (n,l)-alternating sign trapezoid has n 1-columns

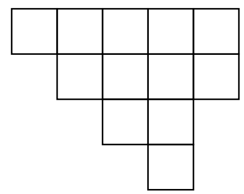
The statistics on (n,l)-alternating sign trapezoids T:

$$\begin{aligned}
\mathsf{p}(T) &= \# \text{ of 10-columns among the } n \text{ leftmost columns},\\
\mathsf{q}(T) &= \# \text{ of 10-columns among the } n \text{ rightmost columns},\\
\mathsf{r}(T) &= \# \text{ of 1-columns among the } n \text{ leftmost columns}.
\end{aligned}$$

In the example above, we have $\mathsf{p}(T)=1, \mathsf{q}(T)=0, \mathsf{r}(T)=2$.

Column strict shifted plane partitions of a fixed class aka CSLTs with a central triangular hole

With each strict partition (= partition with distinct parts), we associate a shifted Ferrers diagram. The shifted Ferrers diagram of $(5, 4, 2, 1)$ is



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A column strict shifted plane partition is a filling of a shifted Ferrers diagram with positive integers such that the rows are weakly decreasing and the columns are strictly decreasing.

Example.

$$\begin{array}{ccccc} 7 & 7 & 6 & 6 & 3 \\ & 6 & 5 & 5 & 1 \\ & & 4 & 2 & \end{array}$$

A column strict shifted plane partition is of class k if the first part of each row exceeds the length of the row by precisely k.

It is easy to construct a bijection between column strict shifted plane partitions of class k where the length of the top row does not exceed n and CSLTs with a central triangular whole of size k.

Three statistics on column strict shifted plane partitions

For $d \in \{1, \dots, k\}$ and a column strict shifted plane partition C of class k, we define

$$\begin{aligned} \mathsf{p}_d(C) &= \# \text{ of parts } j - i + d \text{ where } i \text{ is the row and } j \text{ is the column,} \\ \mathsf{q}(C) &= \# \text{ of 1's,} \\ \mathsf{r}(C) &= \# \text{ of rows.} \end{aligned}$$

In the example above, we have $\mathsf{p}_1(C) = 1, \mathsf{q}(C) = 1, \mathsf{r}(C) = 3$.

Theorem (F. 2018). The number of (n, l)-alternating sign trapezoids T with $\mathsf{p}(T) = p, \mathsf{q}(T) = q, \mathsf{r}(T) = r$ is equal to the number of column strict shifted plane partitions of class $l - 1$ with $\mathsf{p}_d(C) = p, \mathsf{q}(C) = q, \mathsf{r}(C) = r$, where the length of the first row does not exceed n.

The case of no statistic was conjectured first by Behrend and then by Aigner. The three statistics were conjectured independently by Behrend.

The case $n = 2, l = 4$

Alternating sign trapezoids:

$$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & \\ \multicolumn{6}{c}{(0,0,2)} \end{array} \qquad \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & \\ \multicolumn{6}{c}{(0,0,1)} \end{array} \qquad \begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & \\ \multicolumn{6}{c}{(1,0,1)} \end{array} \qquad \begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ & 1 & -1 & 0 & 1 & \\ \multicolumn{6}{c}{(0,0,1)} \end{array}$$

$$\begin{array}{cccccc} 0 & 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & -1 & 1 & \\ \multicolumn{6}{c}{(0,0,1)} \end{array} \qquad \begin{array}{cccccc} 0 & 0 & 0 & 0 & 1 & 0 \\ & 1 & 0 & 0 & 0 & \\ \multicolumn{6}{c}{(0,1,1)} \end{array} \qquad \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 0 & \\ \multicolumn{6}{c}{(0,0,1)} \end{array} \qquad \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 1 & \\ \multicolumn{6}{c}{(0,0,0)} \end{array}$$

Column strict shifted plane partitions:

	$\emptyset$	4	5 1	5 2	5 3	5 4	5 5	$\begin{smallmatrix}5&5\\&4\end{smallmatrix}$
$d = 1$	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(1,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,2)$
$d = 2$	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(0,0,1)$	$(1,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,2)$
$d = 3$	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(0,0,1)$	$(0,0,1)$	$(1,0,1)$	$(0,0,1)$	$(0,0,2)$

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