Fully Packed Loop configurations on □s and △s

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Joint work with Philippe Nadeau (Lyon)

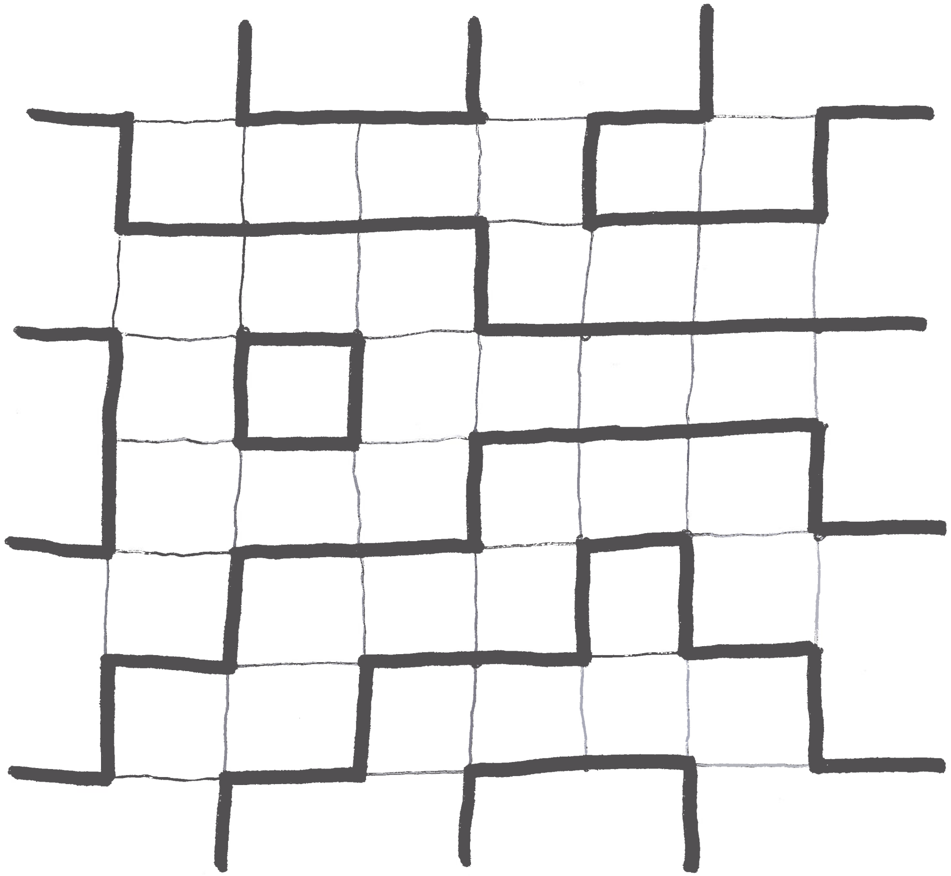
Outline

1) Definition

2) Some "classical" results on the $\square$

3) Some new results on the $\triangle$

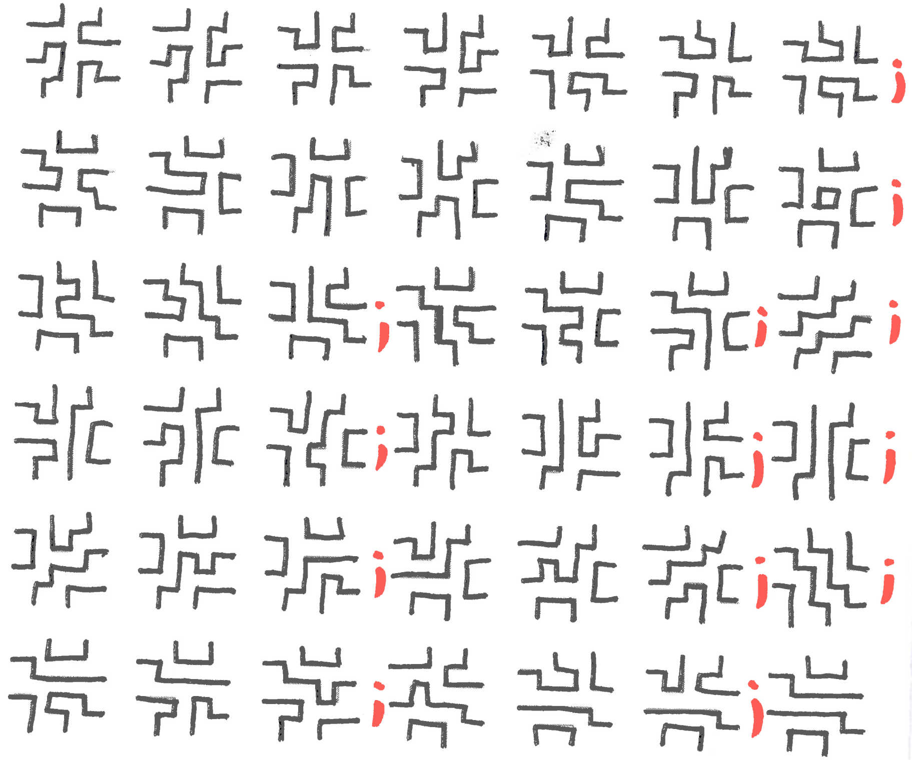
Fully Packed Loop Configurations = FPL



Playground: Square grid G_n with n^2 vertices and $2n$ external edges chosen alternatively

FPL: Subgraph of G_n which has two edges incident to each internal vertex and contains all external edges

The 42 FPLs of size 4



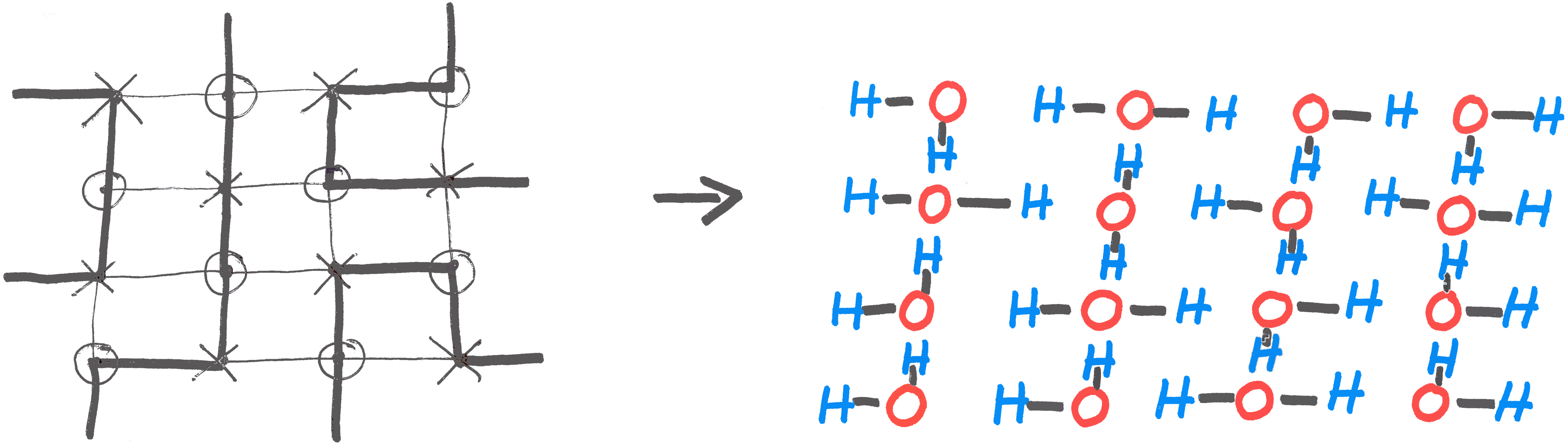
Theorem (Zeilberger 96):
The number of FPLs of size n is

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

$$A_n = 1, 2, 7, 42, 429, 7436, 218348, \ldots$$

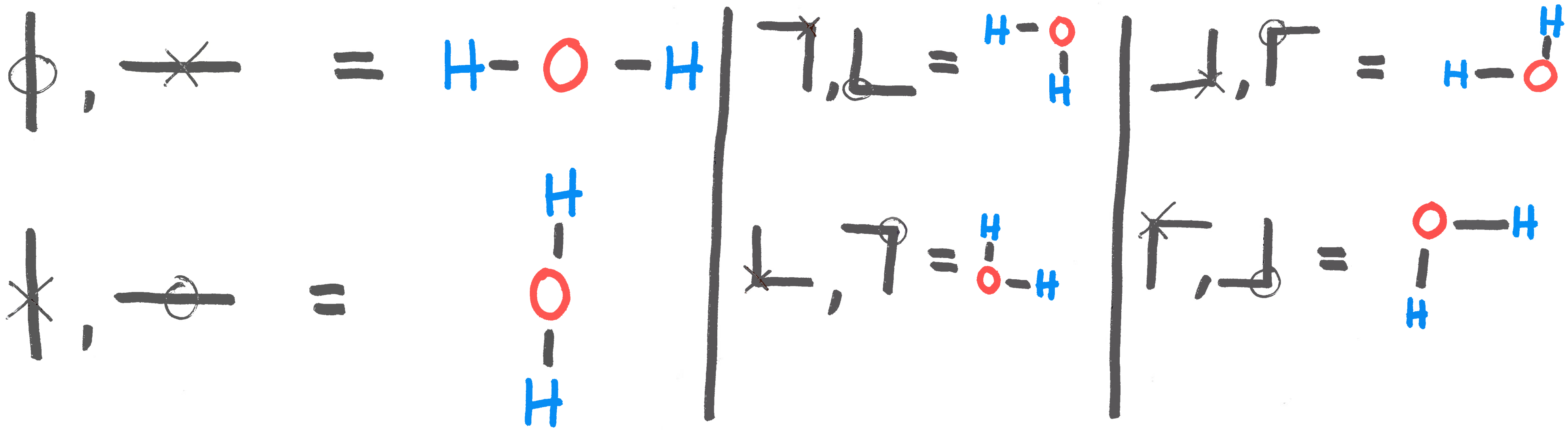
Asymptotics : $A_n \simeq \left(\frac{3\sqrt{3}}{4}\right)^{n^2}$

Origin in statistical physics

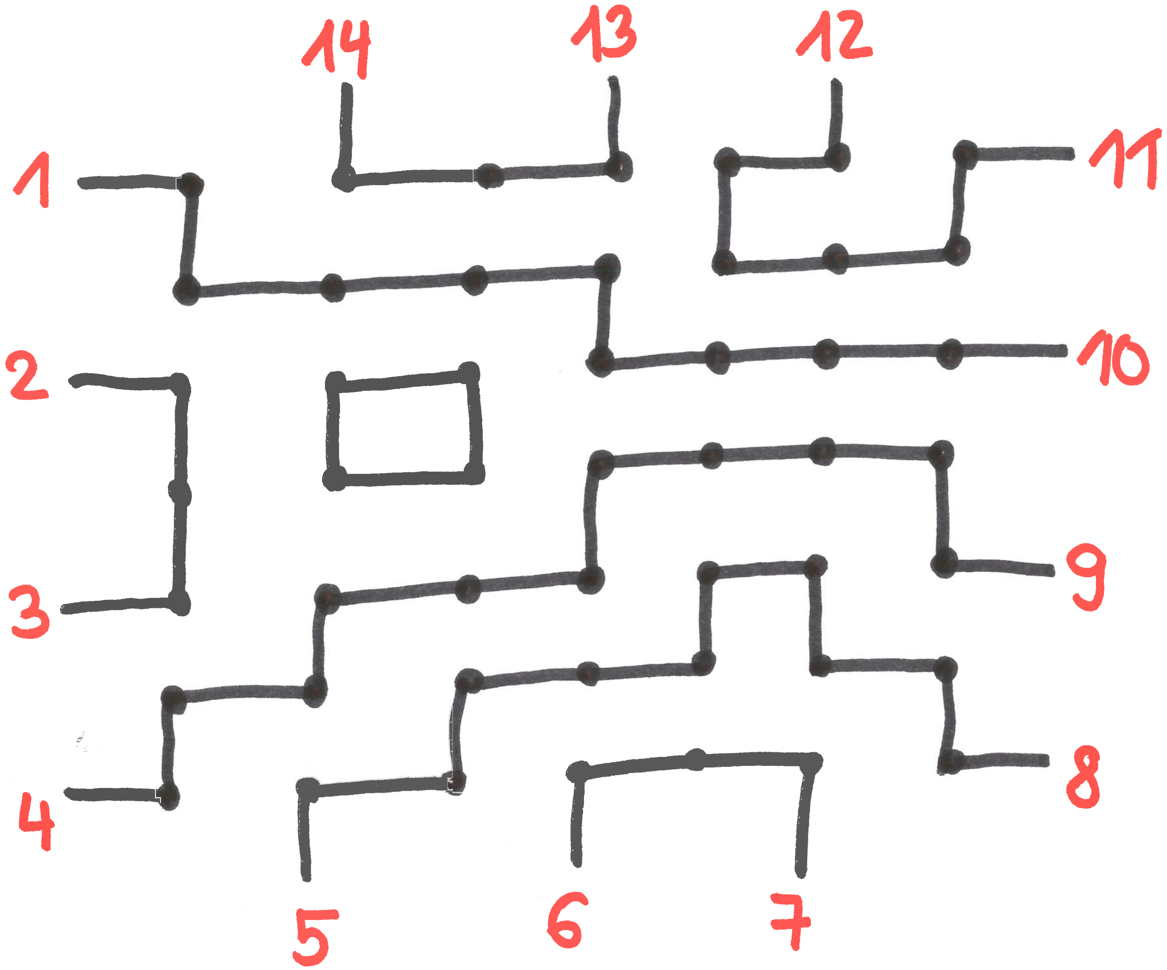


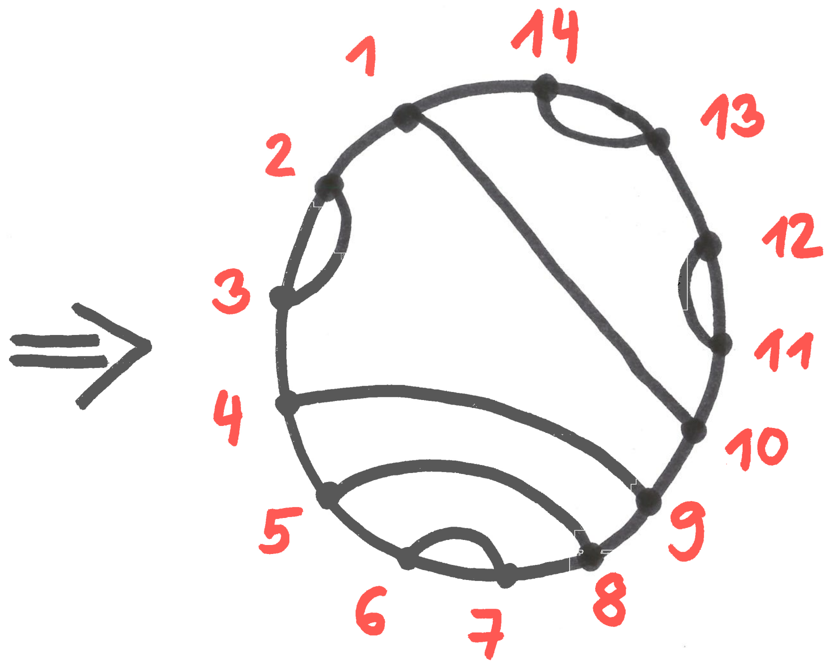
Dictionary:

Distinguish between odd and even vertices



LINK PATTERNS OF FPLs





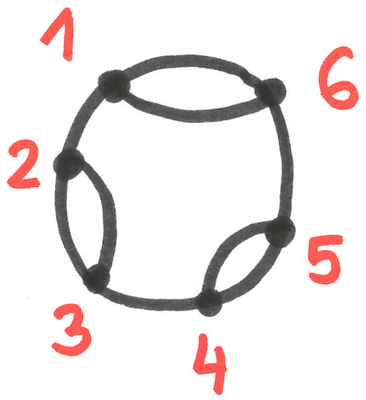
A link pattern π of size n is a set of n non-crossing chords between $2n$ points on a disk.

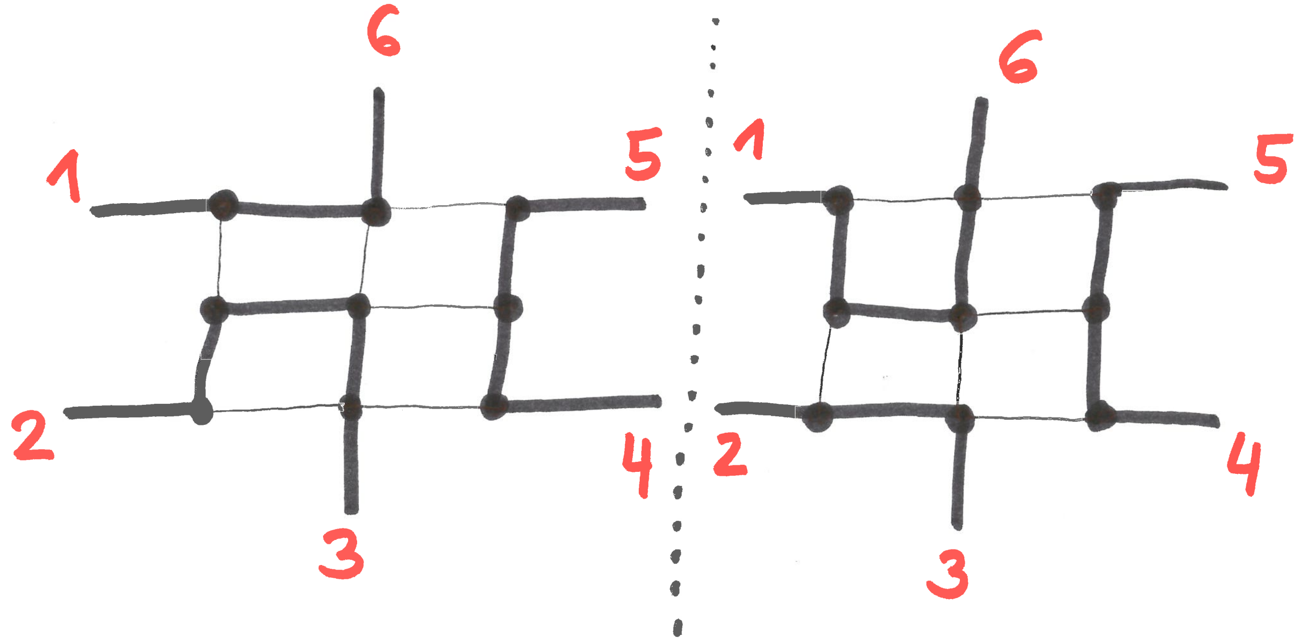
Study FPLs with fixed link pattern!

DEF: π link pattern

$$A_\pi := \# \text{ FPLs with link pattern } \pi$$

Ex:

$\pi =$ 

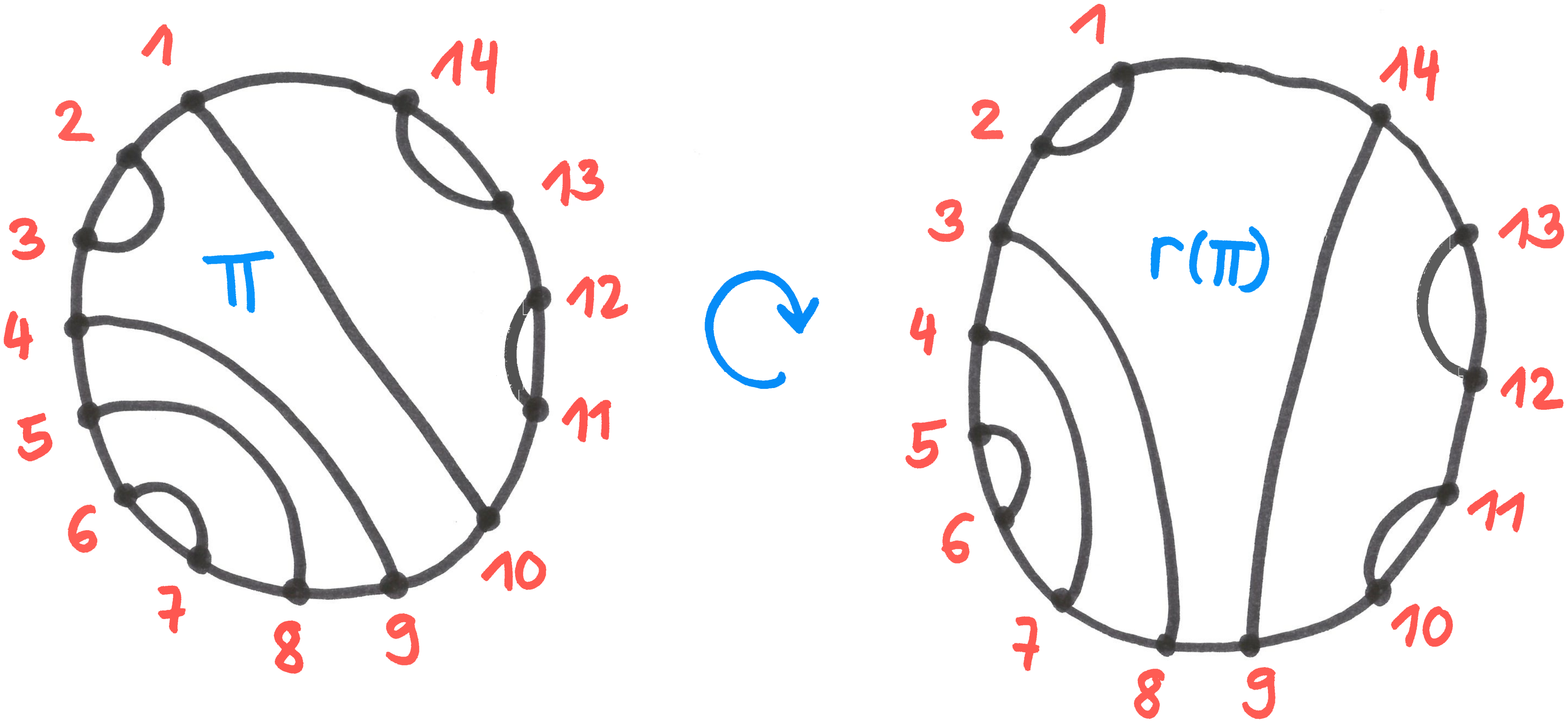


$$\Longrightarrow \quad A_\pi = 2$$

Some "classical" results on the $\square$

1) Rotational invariance of A_π
2) Razumov-Stroganov ex-Conjecture
3) A_π for special π's

1) Rotational invariance



$\pi : i \frown j \iff r(\pi) : i-1 \frown j-1 \mod 2n$

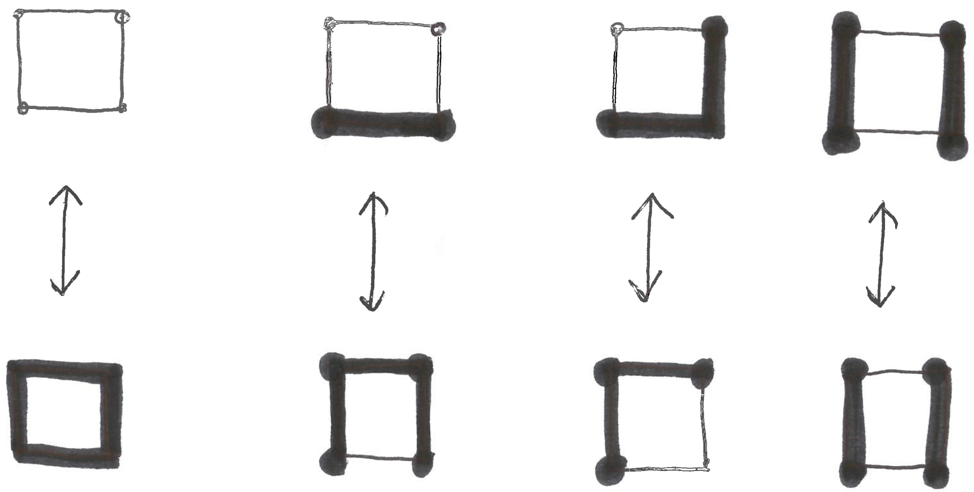
A brilliant contribution by an undergraduate

THEOREM (Ben Wieland, 2000):

$\forall \pi \quad : \qquad A_{\pi} = A_{r(\pi)}$

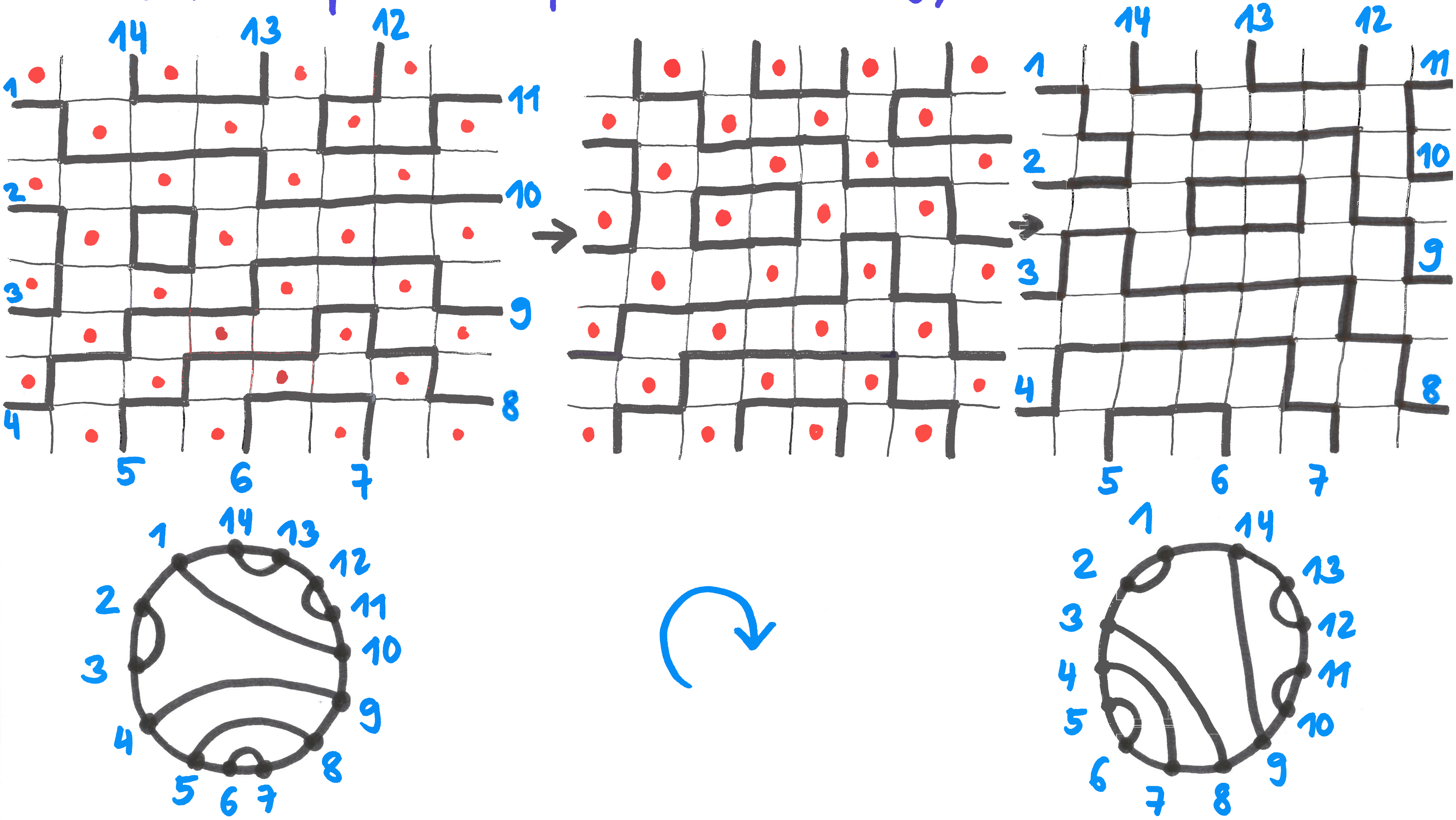
Conjectured by Cohn and Propp / Bosley and Fidkowski

Bijective proof : Wieland gyration

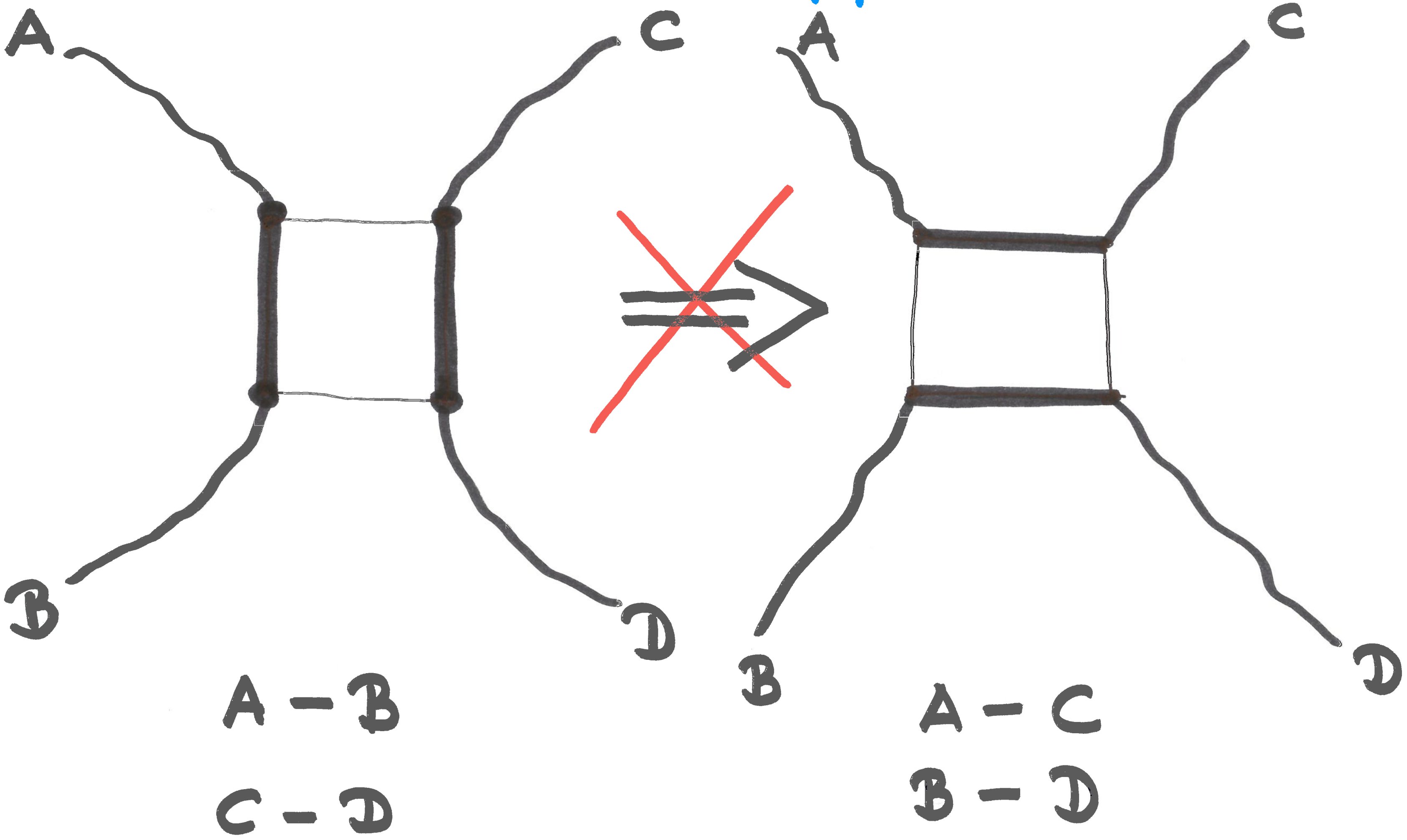


up to rotation

Example of Wieland gyration

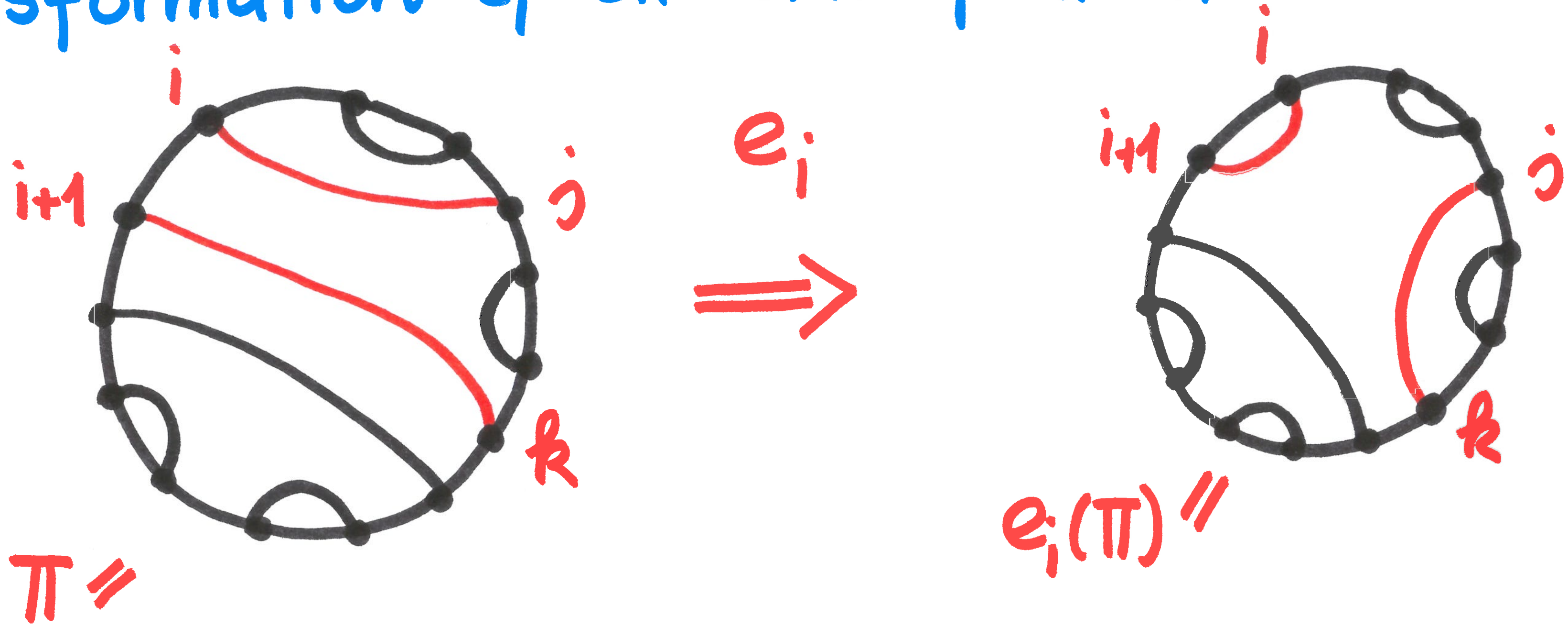


Crucial fact: Wieland gyration preserves the connectivity of path endpoints in each internal cell it is applied to!



2) Razumov-Stroganov ex-Conjecture

Transformation e_i on link patterns :



$$\pi : i \frown j \,,\; i{+}1 \frown k \quad \Rightarrow \quad e_i(\pi) : i \frown i{+}1 \,,\; j \frown k$$

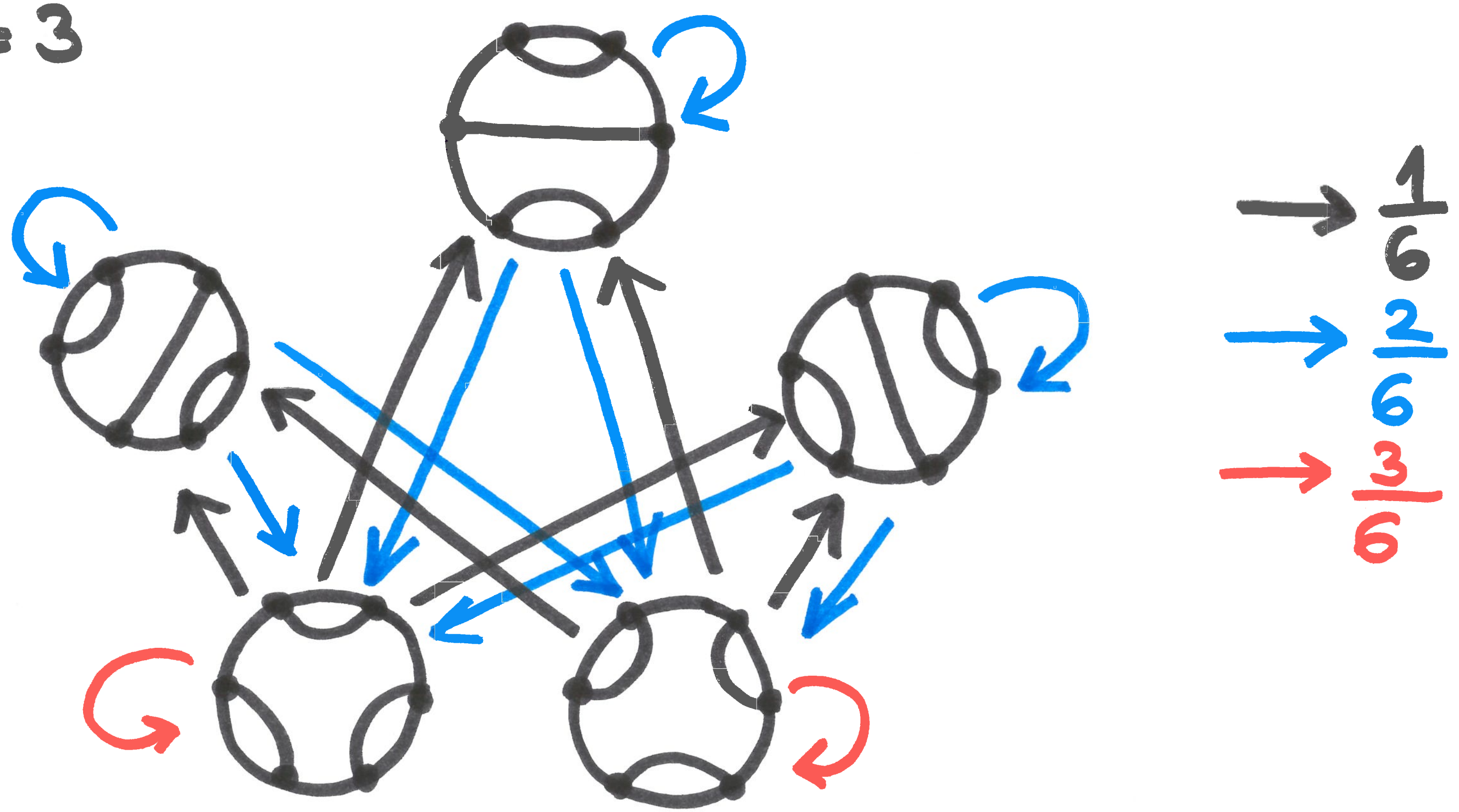
A MARKOV CHAIN ON LINK PATTERNS

STATES = LINK PATTERNS OF SIZE $n =: LP_n$

Transition probabilities:

$$P(\pi_1 \to \pi_2) = \frac{\#\, i \in \{1,2,\dots,2n\} : e_i(\pi_1) = \pi_2}{2n}$$

Ex: $n=3$



STATIONARY DISTRIBUTION

Perron-Frobenius $\Rightarrow$ Markov chain has a unique stationary distribution

Razumov-Stroganov ex-Conjecture (2001):

The stationary distribution $(\psi_\pi)_{\pi \in LP_n}$ is given by $\psi_\pi = \dfrac{A_\pi}{\sum_{\pi \in LP_n} A_\pi}$.

Proof by Cantini/Sportiello, 2010.

Physic language:

stationary distribution = ground-state coeff. in the even-length dense $O(1)$ loop model

A_π for special π's

$\pi_{a,b,c}$

$\Vert$

THEOREM (Di Franceso, Zinn-Justin, Zuber, 2004):

$$A_{\pi_{a,b,c}} = \prod_{i=1}^{a} \frac{(c+i)(c+i+1)(c+i+2)\cdot\ldots\cdot(c+i+b-1)}{i\cdot(i+1)\cdot(i+2)\cdot\ldots\cdot(i+b-1)}$$

Percy MacMahon, 1916: This is also the number of Plane Partitions in an $a \times b \times c$-Box!

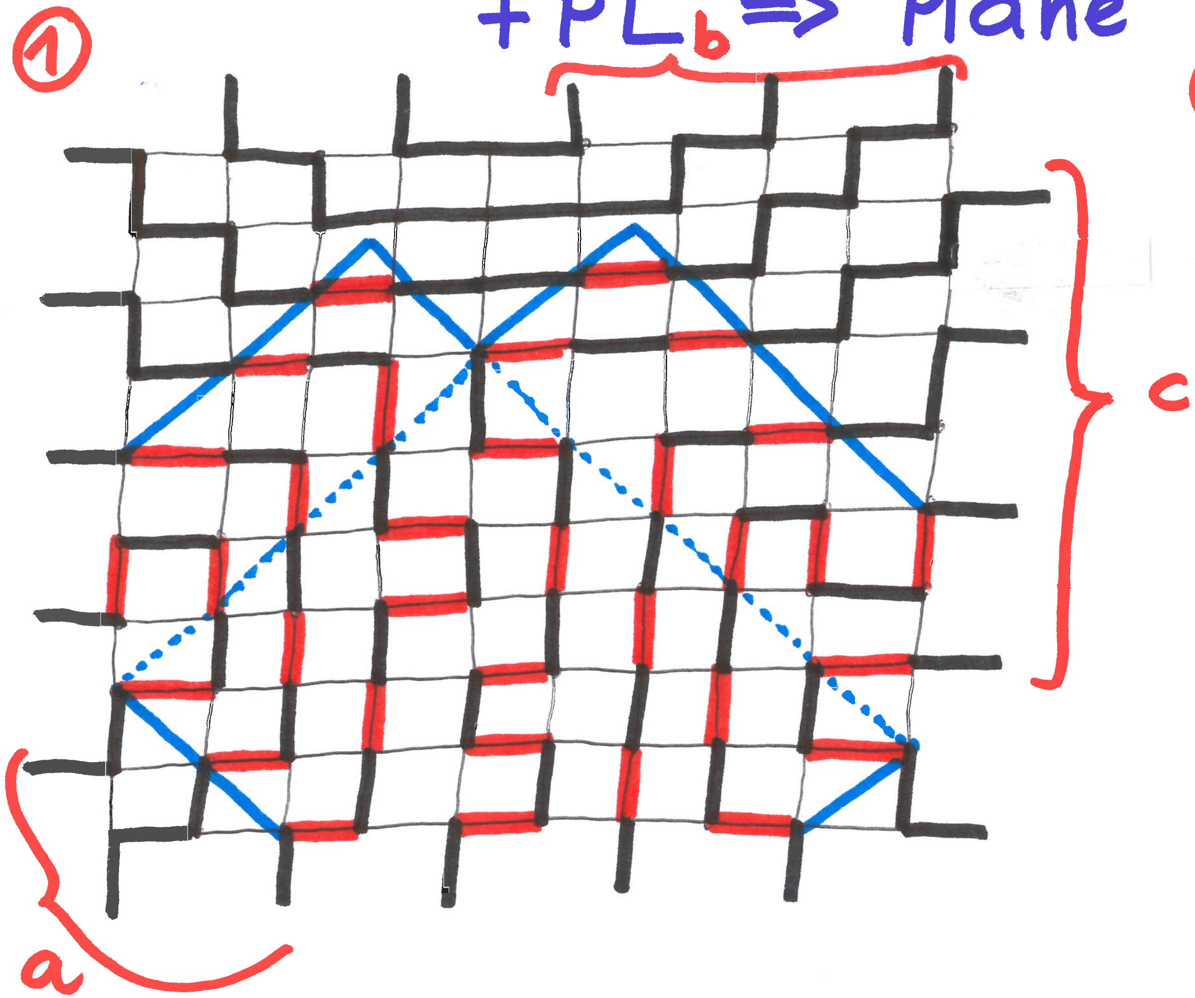
Plane Partitions in an $a \times b \times c$-Box

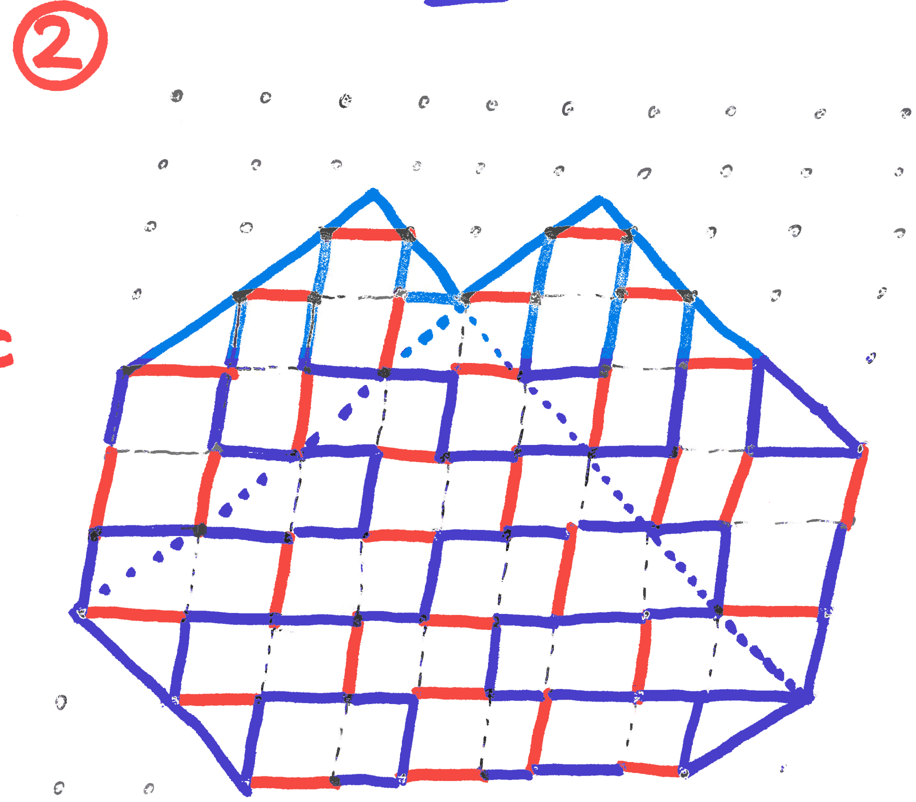
... are $a \times b$-matrices with entries in $\{0,1,2,\ldots,c\}$ such that rows and columns are weakly decreasing.

Ex: $a=3,\ b=3,\ c=4$

$$\begin{array}{cccc} & & \geq & \\ & 4 & 3 & 1 \\ \geq & 4 & 3 & 0 \\ & 4 & 1 & 0 \end{array}$$

$FPL_b \Rightarrow$ Plane Partition I



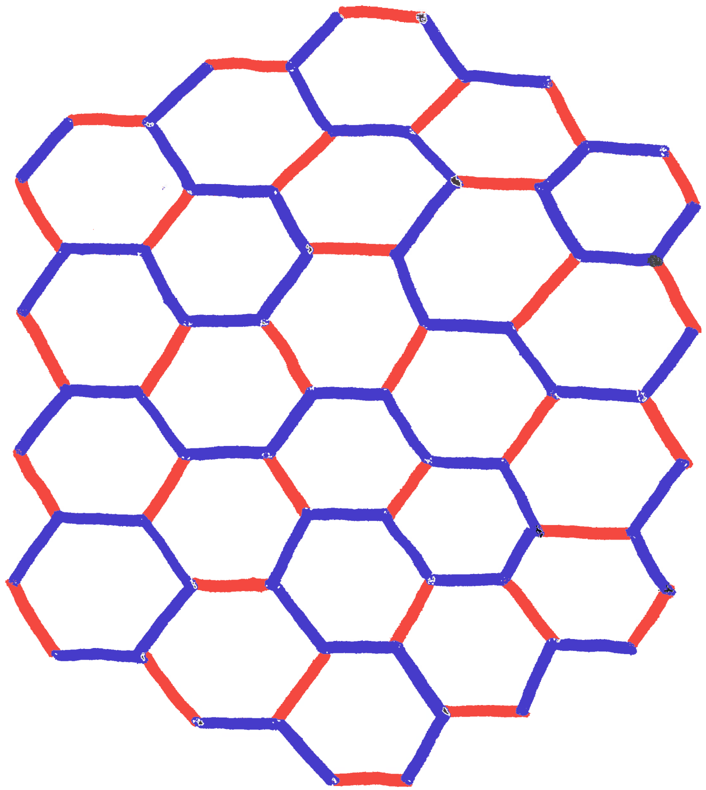


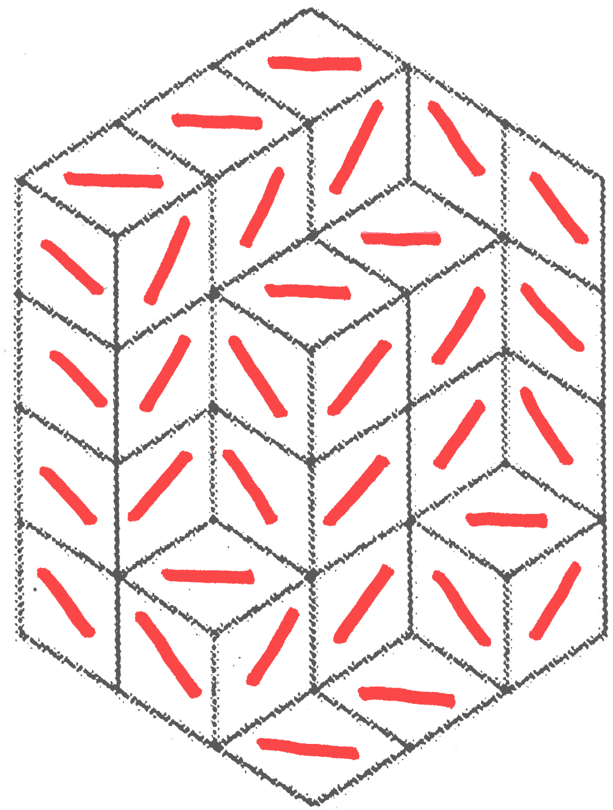
fixed edges

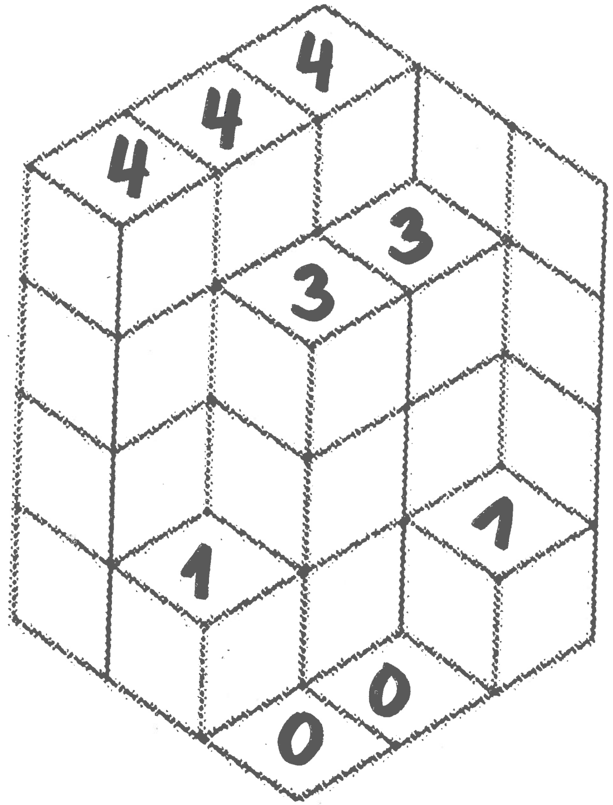
fixed edges

non-edges

FPL ⇒ Plane Partition II







Perfect matching of the hexagonal grid

Rhombus Tiling

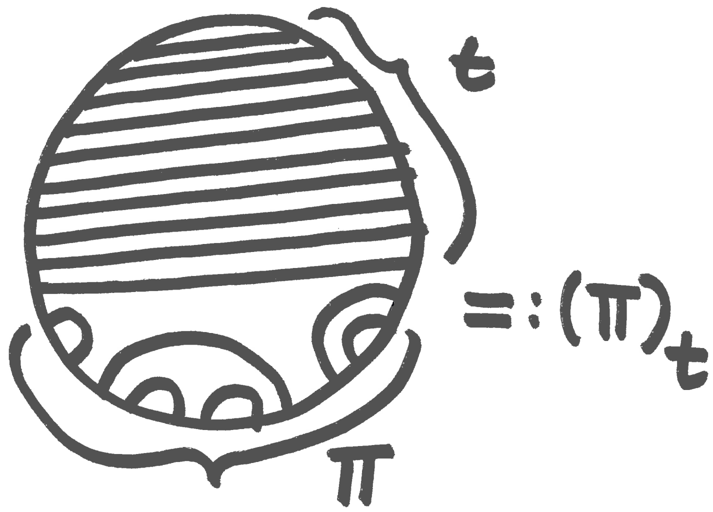
$$\begin{matrix} 4 & 3 & 1 \\ 4 & 3 & 0 \\ 4 & 1 & 0 \end{matrix}$$

3) Some new results on the Δ

- Definition and origin
- Restrictions on the boundary – including a nasty inequality
- Matchings and Path Tangles
- "Littlewood-Richardson" - configurations

A qualitative result for A_π

Link patterns with nested arcs:

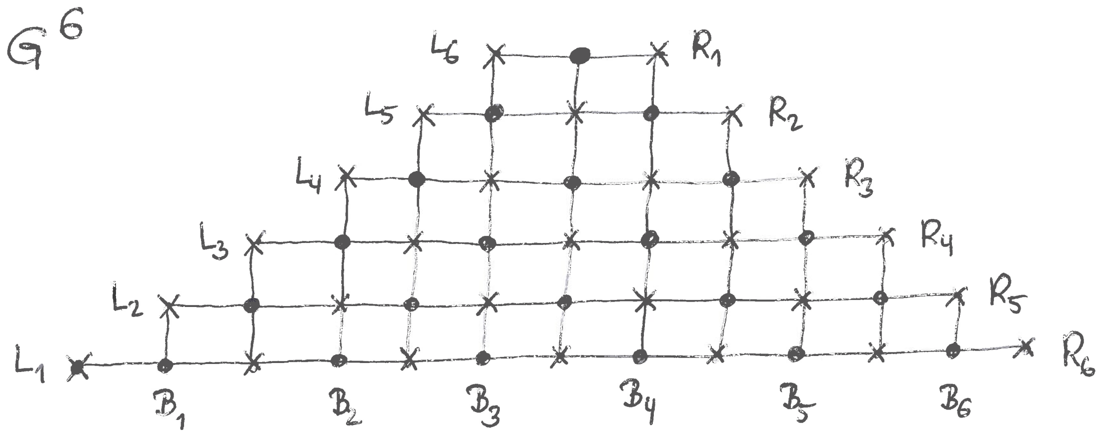


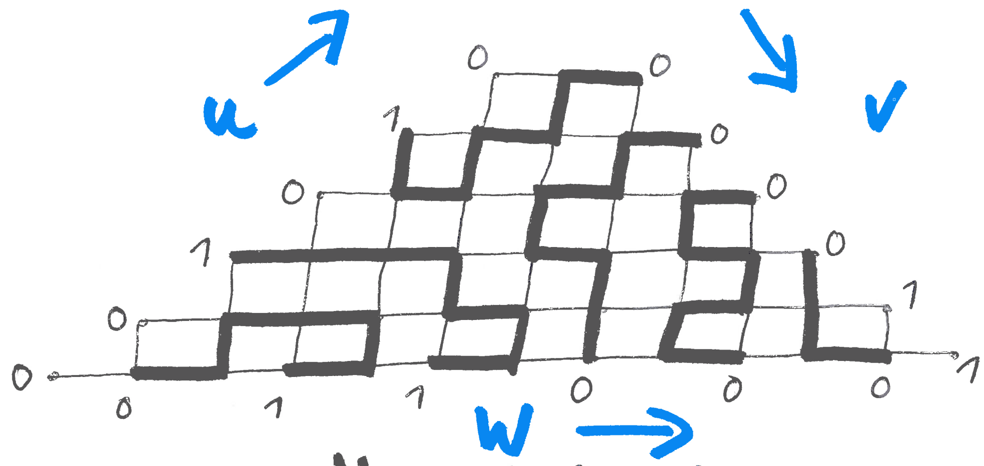
THEOREM (Casselli, Krattenthaler, Lass, Nadeau, 2005):

For fixed π, the quantity $A_{(\pi)_t}$ is a polynomial function in t.

Fully Packed Loops in a triangle were invented for the proof!

FPLs on Δs = TFPLs





DEF: A TFPL f is a subgraph of G^N such that:

(1) $\deg(L_i) \in \{0,1\}$, $\deg(R_i) \in \{0,1\}$

(2) $\deg(B_i) = 1$

(3) All other vertices have degree 2.

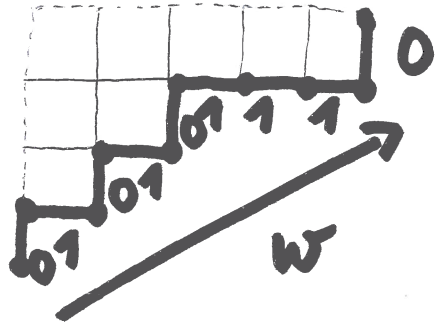
(4) A path in f cannot join two vertices in $\{L_1, L_2, \dots, L_N\}$, nor two vertices in $\{R_1, \dots, R_N\}$.

Boundary: $u = (0,0,1,0,1,0)$ $\quad w = (0,1,1,0,0,0)$

$v = (0,0,0,0,1,1)$

Necessary conditions on u,v,w for the existence of TFPLs

1) $w = 010101110 \Longleftrightarrow$ FERRERS DIAGRAM $\lambda(w)$

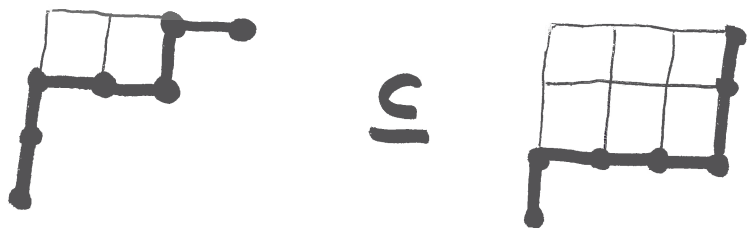


2) $d(w)$ = # of cells in $\lambda(w)$ = # of "inversions" = 8

3) PARTIAL ORDER ON 01-words:

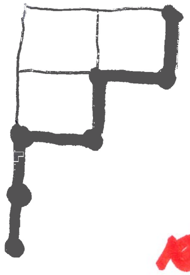
$$u \leq v \Longleftrightarrow \lambda(u) \subseteq \lambda(v)$$

Ex: $001101 \leq 011100$



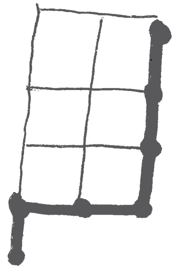
In our example...

$u = (0,0,1,0,1,0)$ $\quad$ $v = (0,0,0,0,1,1)$ $\quad$ $w = (0,1,1,0,0,0)$



$d(u) = 3$

$d(v) = 0$



$d(w) = 6$

Observations:

1) $|u|_1 = |v|_1 = |w|_1$
2) $u \leq w$ and $v \leq w$
3) $d(u) + d(v) \leq d(w)$

DEF: $t_{u,v}^{w}$ = # of TFPLs with boundary (u,v,w)

THEOREM: Let u,v,w be 01-words of the same length. Then $t_{u,v}^{w} > 0$ implies the following three constraints:

(1) $|u|_1 = |v|_1 = |w|_1$

(2) $u \le w$ and $v \le w$

(3) $d(u)+d(v) \le d(w)$

---

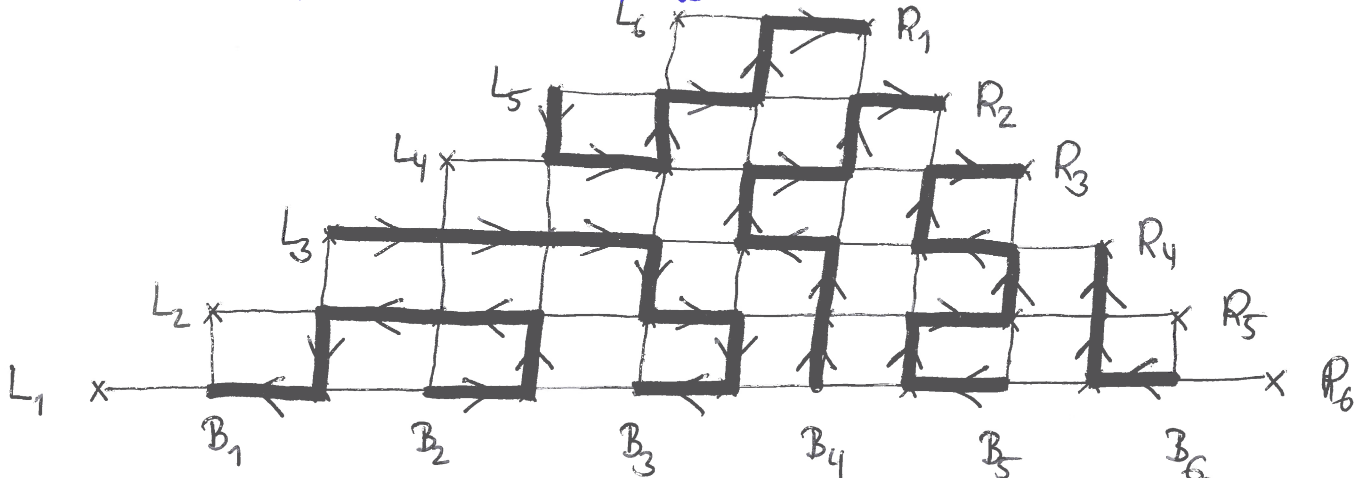
u,v,w Dyck words:

(2) First proof by Caselli, Krattenthaler, Lass, Nadeau, 2005

Philippe: "tedious argument"

(3) First proof by Thapper, 2007: algebraic proof based on Wieland gyration.

ORIENTED TFPLs



ORIENTED TFPL = TFPL + Orientation of each edge such that:

(1) Each vertex of degree 2 has one incoming and one outgoing edge.

(2) The edges of L_i are outgoing.

(3) The edges of R_i are incoming.

(Cond. 4 can be omitted.)

Boundary: $u = (0,0,1,0,1,0)$

$v = (0,0,0,0,1,1)$

$w = (1,0,1,0,0,0)$

DEF: $\vec{t}_{u,v}^{\,w}$ = # of oriented TFPLs with boundary (u,v,w)

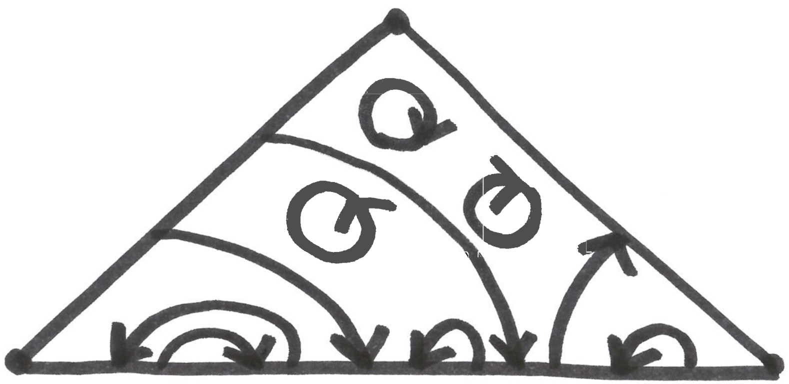
$$t_{u,v}^{w} > 0 \Rightarrow \vec{t}_{u,v}^{\,w} > 0$$

! Above mentioned theorem holds for $\vec{t}_{u,v}^{\,w}$!

Proof: Translation into equivalent objects where necessary conditions can be read off directly.

SHORT INTERMEZZO: Also in general it suffices to consider oriented TFPLS, since $t_{u,v}^{w}$ can be derived from a certain weighted enumeration of oriented TFPLs.

THE WEIGHT OF ORIENTED TFPLs

$f =$ 

$RL(f) = \#$ of paths oriented from right to left $= 3$

$N^{\circlearrowright}(f) = \#$ of clockwise oriented loops $= 2$

$N^{\circlearrowleft}(f) = \#$ of counterclockwise oriented loops $= 1$

$$\vec{t}^{\,w}_{u,v}(q) := \sum_{\substack{f \text{ oriented TFPL} \\ \text{with boundary } (u,v,w)}} q^{RL(f) + N^{\circlearrowleft}(f) - N^{\circlearrowright}(f)}$$

THE MATRIX $M(N_0,N_1)$

$N_0,N_1 \geq 0$ integers; $M(N_0,N_1)$: Matrix whose rows and columns are indexed by 01-words with N_0 zeros and N_1 ones.

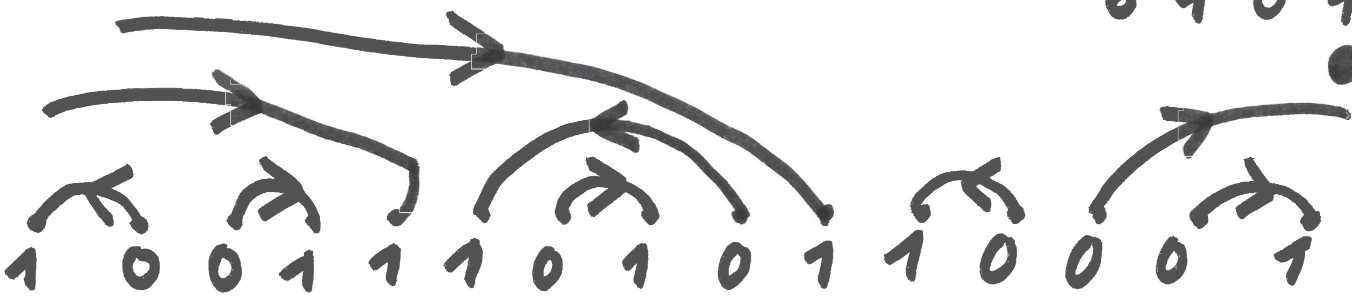
$$M(N_0,N_1)_{w,w'} = \begin{cases} q^{RL(w,w')} & w' \text{ feasible for } w \\ 0 & \text{else} \end{cases}$$

Feasibility:

$w' = 0101100111\,01\,001 \Rightarrow \pi =$

extended link pattern

$\Rightarrow$



$\Rightarrow w = 1001110101\,0110$

source-sink-word

directed link pattern

$RL(w,w') = \#$ of right-left-arcs $= 3$

CONNECTION BETWEEN TFPLs and oriented TFPLs

THEOREM (F. & NADEAU):

u, v, w 01-words of length N_0+N_1 and with $|u|_0=|v|_0=|w|_0=N_0$

$M = M(N_0, N_1)$

$$t_{u,v}^{w} = \sum_{w'} (M^{-1})_{w,w'} \, \vec{t}_{u,v}^{\,w'}(\rho)$$

where $\rho = \frac{1}{2} + \frac{i\sqrt{3}}{2}$.

Combinatorial interpretation for $d(w)-d(u)-d(v)$

THEOREM (F. & Nadeau):

For any oriented TFPL with boundary (u,v,w), one has the following formula:

$$d(w)-d(u)-d(v) := \left(\#\text{ of } \begin{smallmatrix}\times\\ \downarrow\\ \bullet\end{smallmatrix}\right) + \left(\#\text{ of } \begin{smallmatrix}\times\\ \uparrow\\ \bullet\end{smallmatrix}\right) + (\times \leftarrow \bullet \leftarrow \times) + (\bullet \leftarrow \times \leftarrow \bullet)$$

$$+ \left(\begin{smallmatrix}\times & & \\ \uparrow & & \\ \bullet & \leftarrow & \times\end{smallmatrix}\right) + \left(\begin{smallmatrix}\bullet & \leftarrow & \times\\ & & \uparrow\\ & & \bullet\end{smallmatrix}\right) + \left(\begin{smallmatrix} & & \times\\ & & \downarrow\\ \times & \leftarrow & \bullet\end{smallmatrix}\right) + \left(\begin{smallmatrix}\times & \leftarrow & \bullet\\ \downarrow & & \\ \bullet & & \end{smallmatrix}\right)$$

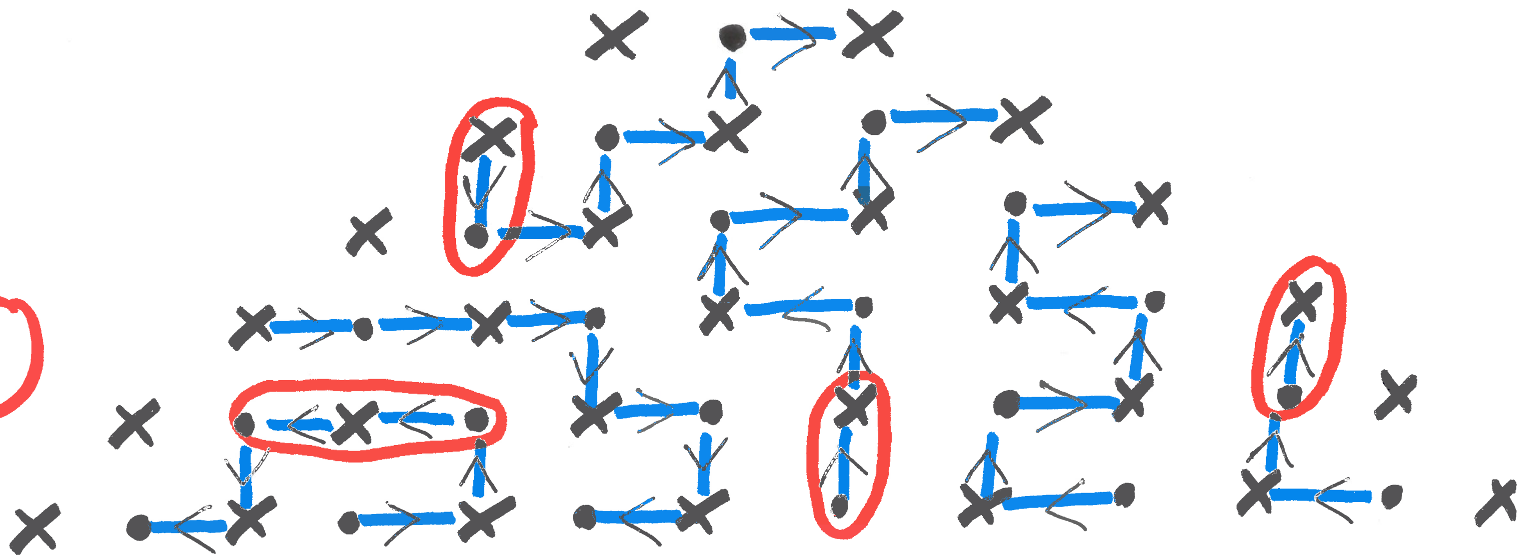
Example:

$u=(0,0,1,0,1,0)$

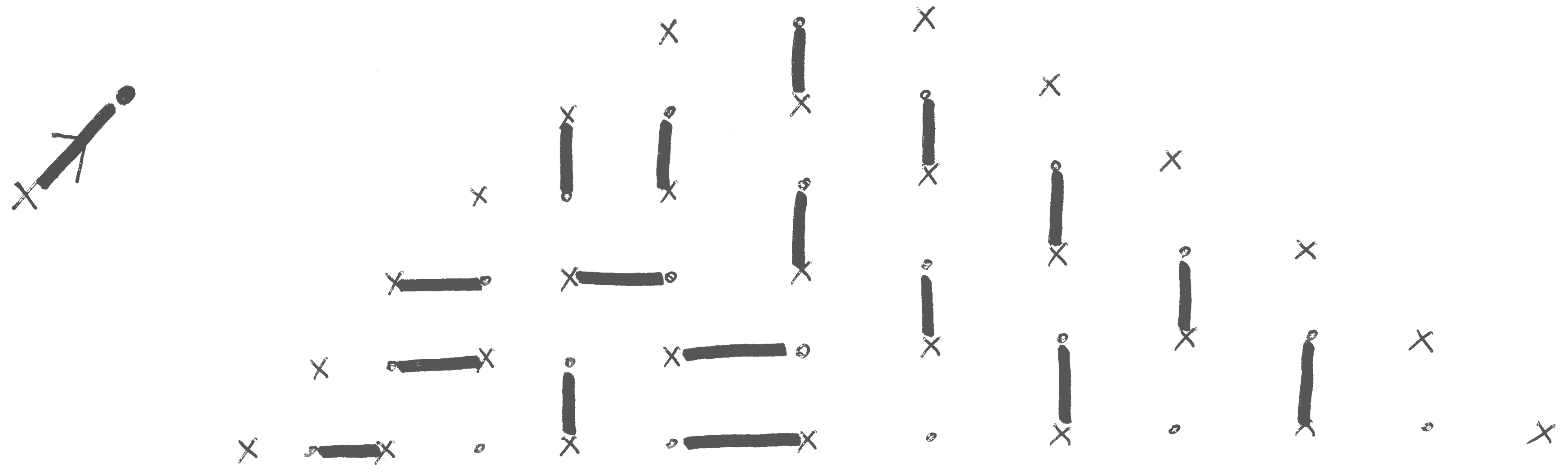
$v=(0,0,0,0,1,1)$

$w=(1,0,1,0,0,0)$

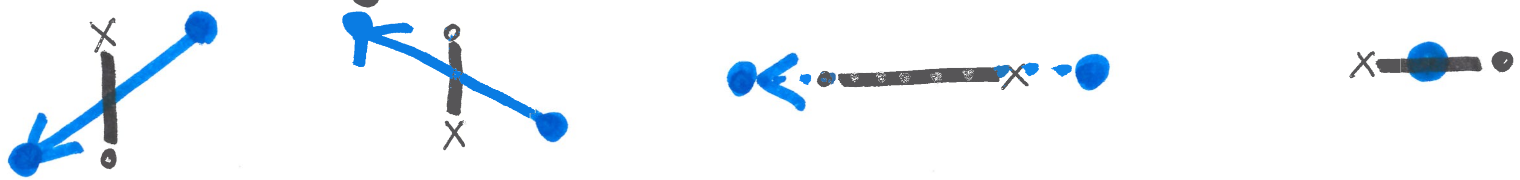
$d(w)-d(u)-d(v)=4$

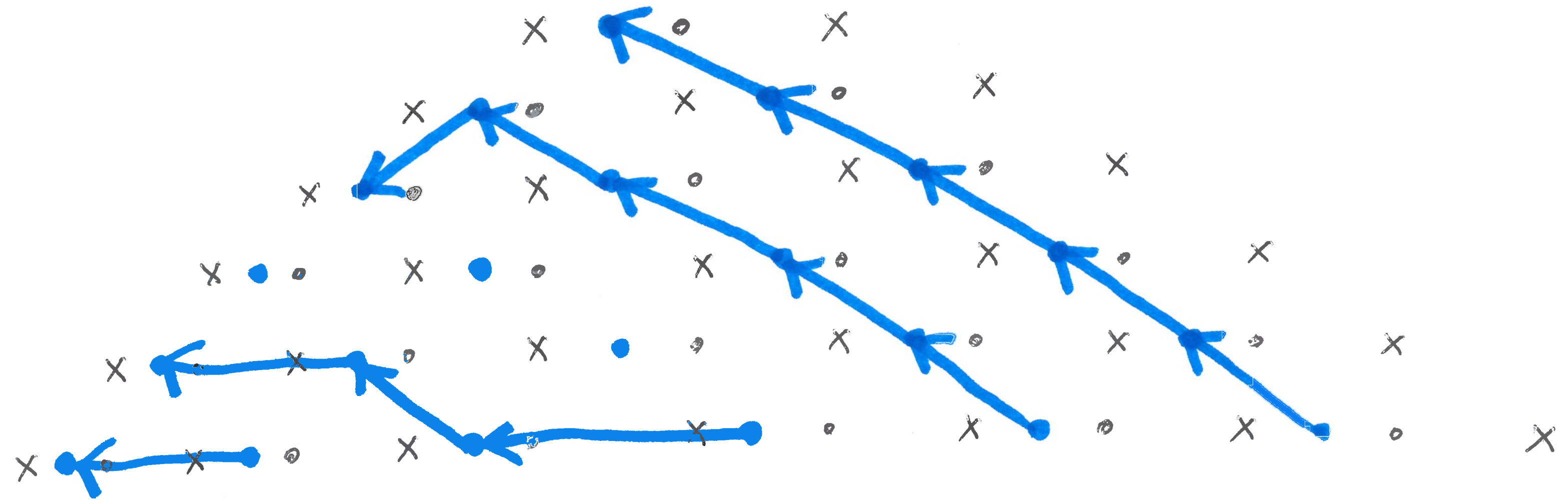


Tool: Matchings and Paths

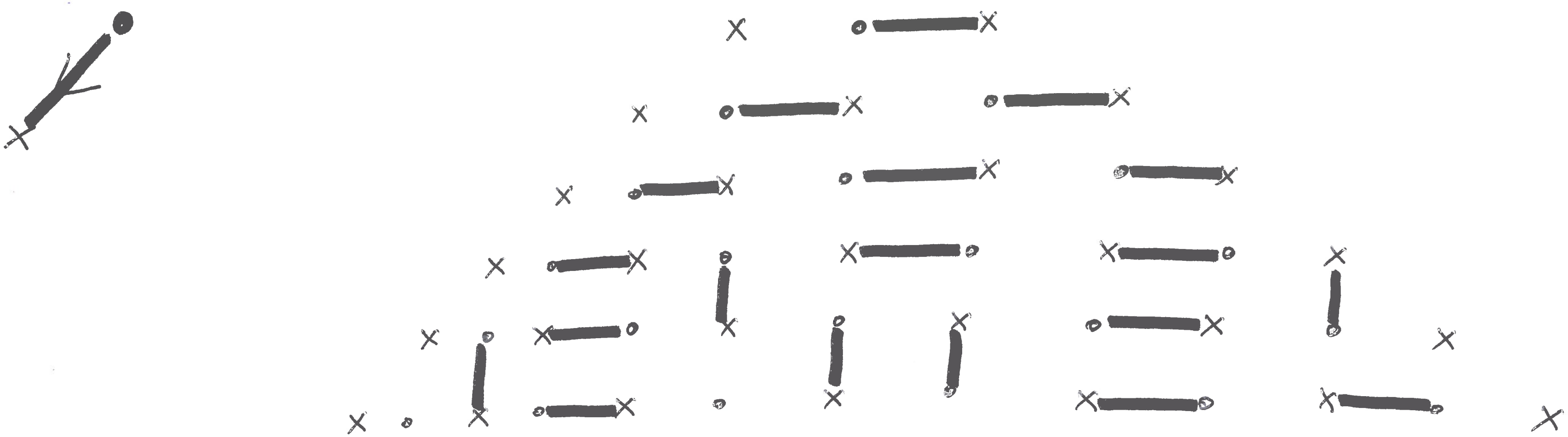


Introduce ● right of × and perform the following:

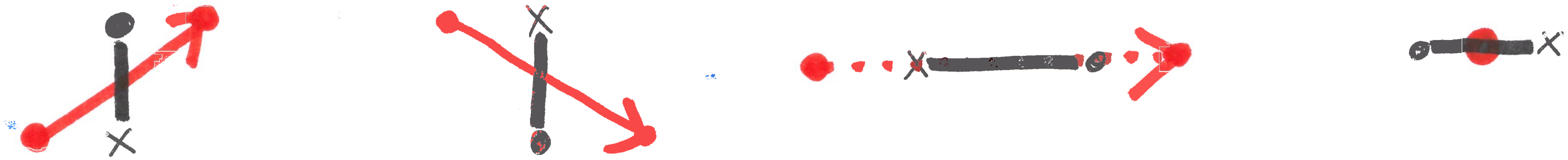


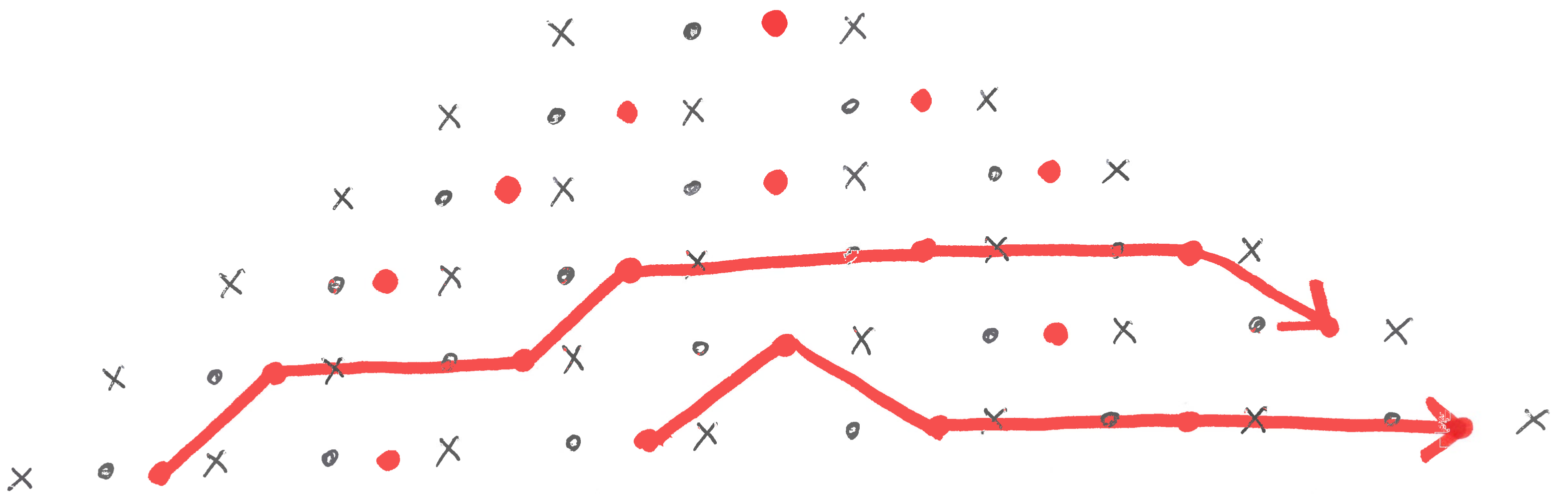


Second matching



Introduce ● left of × and perform the following:

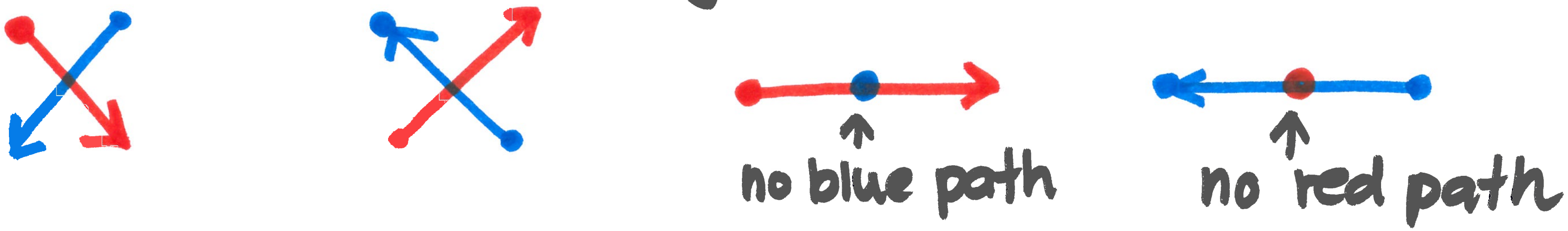




PATH TANGLES

In one picture:

The fact that the ~~two~~ matchings are disgoint translates into the fact that the following local patterns do not occur:



Important facts: – CROSSINGS of blue and red paths always involve horizontal steps.

[– Each horizontal step is involved in a crossing.]

Proof of $d(u) + d(v) \leq d(w)$

1) $d(w) = \#$ inversions in $w = \#$ crossing pairs $\leq \# \leftarrow + \# \rightarrow$

2) $d(w) - d(u) = \# \swarrow + \# \leftarrow$

3) $d(w) - d(v) = \# \searrow + \# \rightarrow$

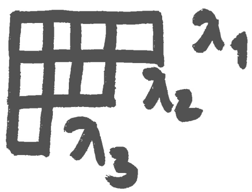
Now: $d(w) - d(u) - d(v) = (d(w) - d(u)) + (d(w) - d(v)) - d(w) \geq$

$$\geq (\swarrow + \leftarrow) + (\searrow + \rightarrow) - (\leftarrow + \rightarrow)$$

$$= \swarrow + \searrow \geq 0$$

Extreme case : $d(u)+d(v)=d(w)$

Schur function : u 01-word

$\lambda(u) =$ 

$= (\lambda_1, \lambda_2, \ldots, \lambda_n)$

$$S_u(x_1,\ldots,x_n) = \frac{\det\limits_{1\le i,j\le n} x_i^{\lambda_j+n-j}}{\prod\limits_{1\le i<j\le n} (x_j - x_i)}$$

- Important basis of symmetric polynomials
- Representation theory : Characters of the irreducible representations of the general linear group

Littlewood-Richardson coefficients

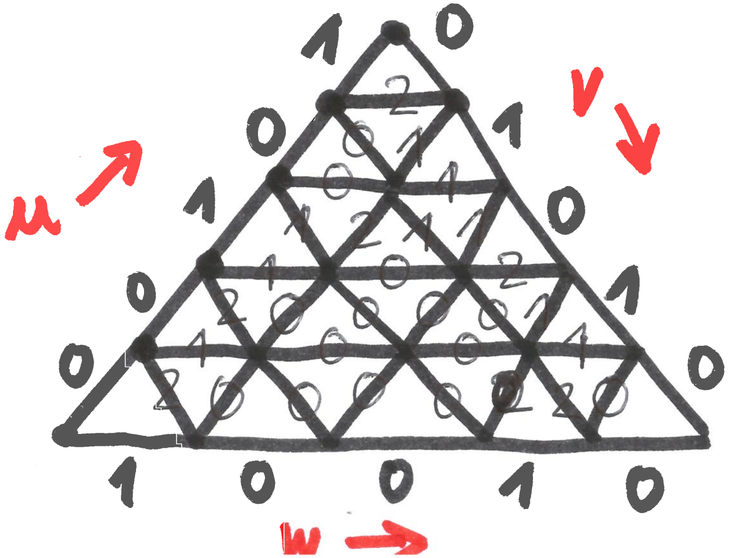
u, v 01-words :

$$s_u \cdot s_v = \sum_w \boxed{c_{u,v}^w}\, s_w$$

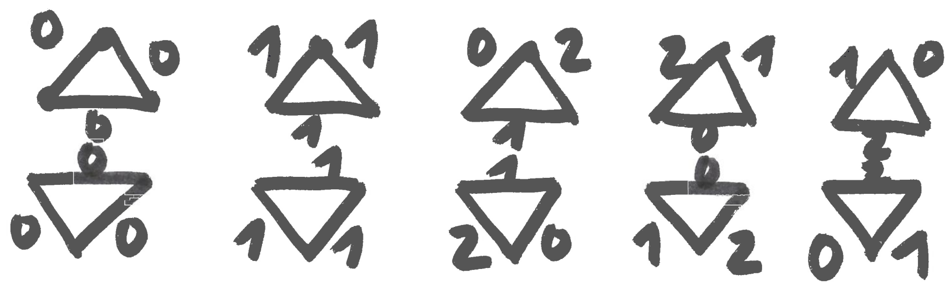
LR-coefficient; non-negative integer

COMBINATORIAL MODEL FOR LR-coefficients :

Knutson-Tao-Puzzles



Puzzle Pieces :

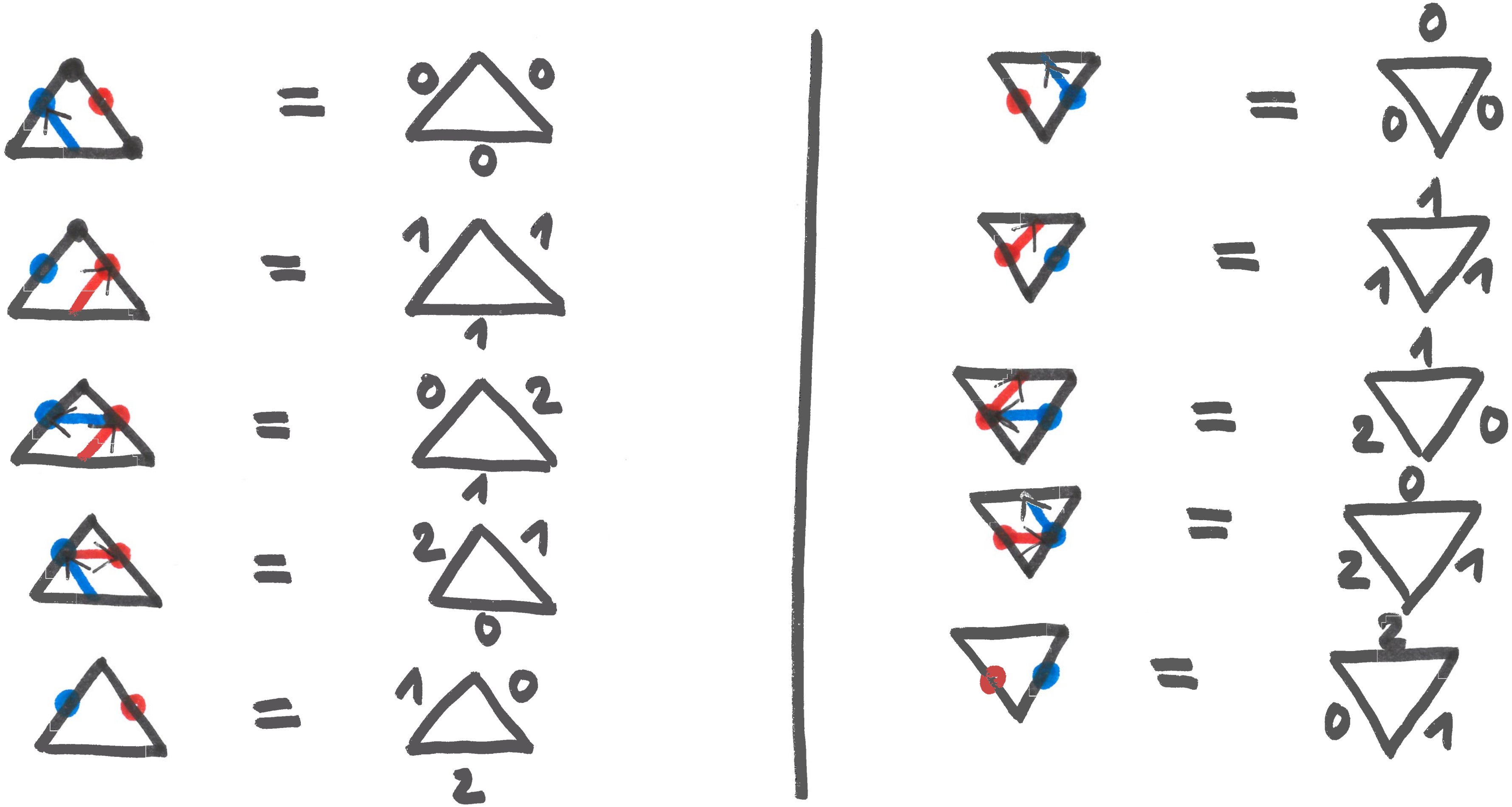


THEOREM: u, v, w with $d(u) + d(v) = d(w)$

$$\Rightarrow \quad t_{u,v}^{w} = c_{u,v}^{w}$$

Proof: Superimpose triangular grid on path tangle

Dictionary:

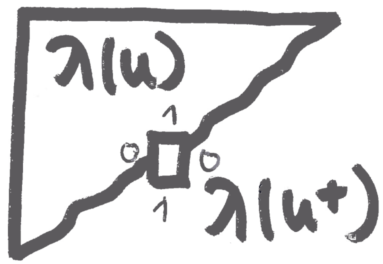


Next case: $d(w) - d(u) - d(v) = 1$

Message in short: Number can also be expressed in terms of Littlewood-Richardson coefficients

DEFs: u^+ covers u if there exist u_L, u_R with:

$$u = u_L\,01\,u_R$$
$$u^+ = u_L\,10\,u_R$$



$$L_0(u,u^+) = |u_L|_0$$
$$L_1(u,u^+) = |u_L|_1$$
$$L(u,u^+) = L_0(u,u^+) + L_1(u,u^+) + 1$$

THEOREM (F. & Nadeau)

Let u, v, w be words of the same length, $|u|_0 = |v|_0 = |w|_0$ and $d(w) - d(u) - d(v) = 1$.

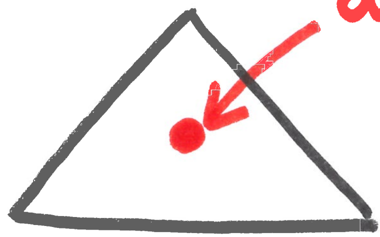
(1) $$\vec{t}^{\,w}_{u,v} = \sum_{u^+ : u \to u^+} \left(|u|_1 + L_1(u, u^+)\right) c^{w}_{u^+, v} + \sum_{v^+ : v \to v^+} \left(L(v, v^+) + L_1(v, v^+) + 1\right) c^{w}_{u, v^+}$$
$$- 2 \sum_{w^- : w^- \to w} L_1(w^-, w)\, c^{w^-}_{u,v}$$

(2) $$t^{w}_{u,v} = \sum_{v^+ : v \to v^+} \left(|v|_1 + L(v, v^+) + 1\right) c^{w}_{u, v^+} - \sum_{w^- : w^- \to w} L_1(w^-, w)\, c^{w^-}_{u,v}$$

Idea of the proof

- The gap $d(w)-d(u)-d(v)=1$ can be "realized" as a certain $\underbrace{\text{local configuration}}_{\text{defect}}$ in the TFPL

defect



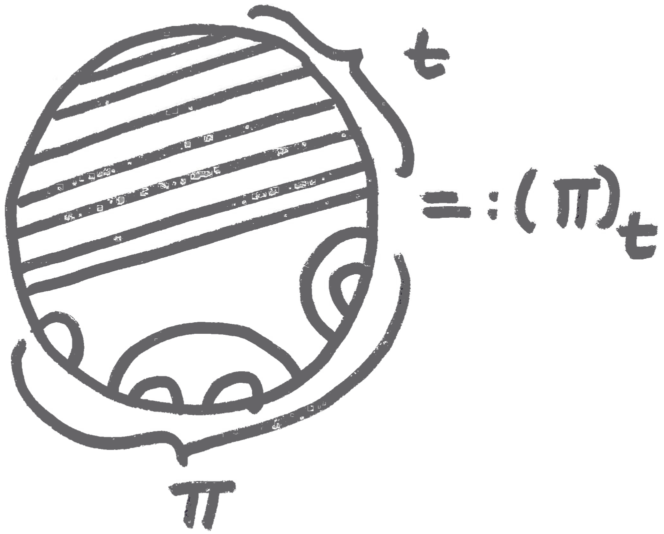
- Define operations to move the defect.
- Move defect to the boundary and remove it

$$\Longrightarrow \underbrace{\text{TFPL with } d(w)-d(u)-d(v)=0}_{\text{LR-TFPL}}$$

- To prove the formula the following question has to be answered: How many TFPLs are mapped to the same LR-TFPL?

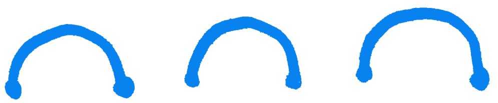
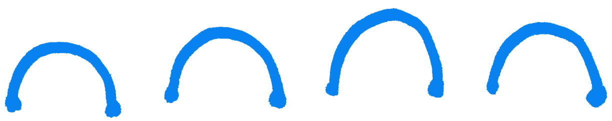
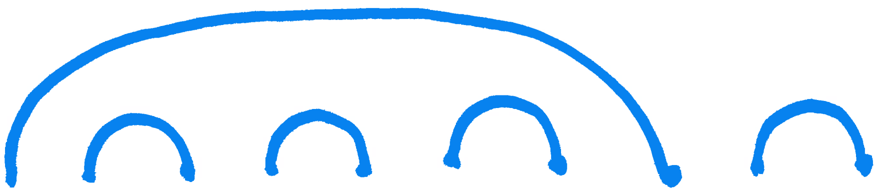
A conjecture by Fonseca and Nadeau

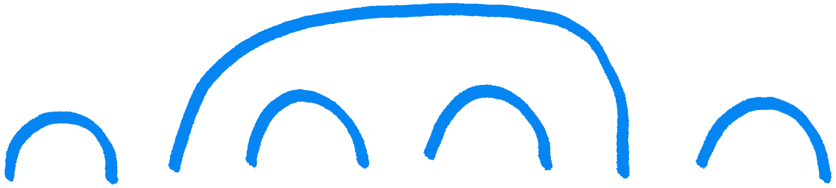
Recall:



THEOREM: $t \to A_{(\pi)_t}$ is a polynomial function in t.

DATA (J.-B. Zuber, 2003):

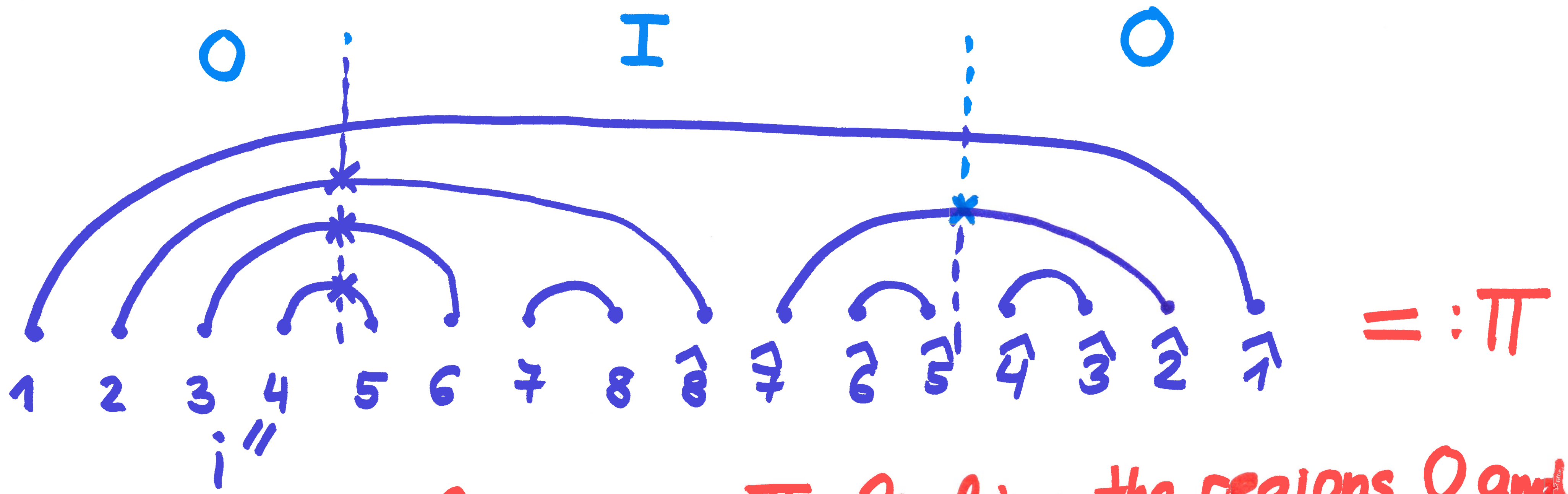
π	$A_{(\pi)_t}$
	$\frac{1}{6}(t+1)(2t^2+7t+12)$
	$\frac{1}{180}(t+1)(t+3)(4t^4+32t^3+155t^2+334t+420)$
	$\frac{1}{720}(t+1)(t+2)(t+4)(5t^4+62t^3+377t^2+1098t+1530)$

π	$A_{(\pi)_t}$
	$\frac{1}{20160}(t+1)(t+4)(45t^6+715t^5+5639t^4+24655t^3+64924t^2+91500t+70560)$
	$\frac{1}{2880}(t+1)(t+2)(t+3)(t+4)(5t^4+54t^3+335t^2+998t+1680)$
	$\frac{1}{2520}(t+1)(10t^6+165t^5+1228t^4+4932t^3+11623t^2+14802t+10080)$

LINEAR FACTORS OF $A_{(\pi)_t}$

ROOT CONJECTURE: All real roots of $A_{(\pi)_t}$ are negative integers. The multiplicity of $-i$ is $m_i(\pi)$.

$m_i(\pi)$: $i = 4$



$$m_i(\pi) = \frac{\#\text{ of arc in } \pi \text{ linking the regions O and I}}{2}$$