Alternating sign matrices and descending plane partitions:

$n + 3$ pairs of equivalent statistics

Ilse Fischer

Universität Wien

joint work with Florian Aigner (LaCIM, Montréal)

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Outline

I. Introduction

II. Arrowed monotone triangles

III. Set-valued balanced column strict plane partitions

IV. The main result

V. Specializing to the ASM-DPP case

VI. Schur expansion: Totally symmetric plane partitions !

I. Introduction: Four types of objects counted by $\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!}$

Alternating sign matrices = ASMs

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Square matrix with entries in $\{0, \pm 1\}$ such that in each row and each column

- the non–zero entries appear with alternating signs, and
- the sum of entries is 1.

How many?

n	1	2	3	4
	(1)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$3! + \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	42

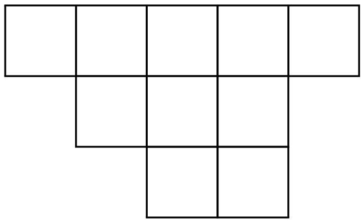
Conjecture (Mills, Robbins, Rumsey 1980s). The number of $n \times n$ alternating sign matrices is

$$\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} =: A_n.$$

Zeilberger gave the first proof (of a generalization including an additional parameter) in 1996. Kuperberg gave another proof (of the special case), using methods from statistical physics such as the Yang-Baxter equation.

Descending Plane Partitions = DPPs

• A strict partition is a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ with distinct parts, i.e., $\lambda_1 > \lambda_2 > \ldots > \lambda_l > 0$. The shifted Young diagram of shape $(5, 3, 2)$ is as follows.

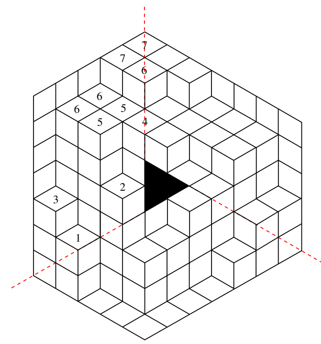


• A column strict shifted plane partition is a filling of a shifted Young diagram with positive integers such that rows decrease weakly and columns decrease strictly.

6	6	5	5	2
	5	4	4	
		3	1	

• A DPP is such a column strict shifted PP where the first part in each row is greater than the length of its row and less than or equal to the length of the previous row.

• Krattenthaler (2003) showed that they can be viewed as cyclically symmetric lozenge tilings of a hexagon with central triangular hole of size 2.



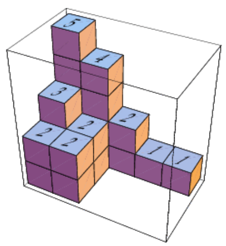
• DPPs with parts no greater than 3: $\emptyset$, 2, 3, $3\ \ 1$, $3\ \ 2$, $3\ \ 3$, $\begin{array}{cc}3 & 3\\ & 2\end{array}$

Theorem (Andrews 1979). The number of DPPs with parts no greater than n is (also) $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.

• In 2013 Behrend, Di Francesco and Zinn-Justin proved for **quadruples** of statistics on ASMs and on DPPs, respectively, that they have the same joint distribution (for certain "sub"-triples of statistics, this was already conjectured by Mills, Robbins and Rumsey 30 years earlier).

• Recently, Matjaž Konavlinka and I have constructed a (complicated) bijection between $\mathrm{ASM}_n \times \mathrm{DPP}_{n-1}$ and $\mathrm{ASM}_{n-1} \times \mathrm{DPP}_n$. It involves the involution principle.

Totally symmetric self-complementary plane partitions = TSSCPPs



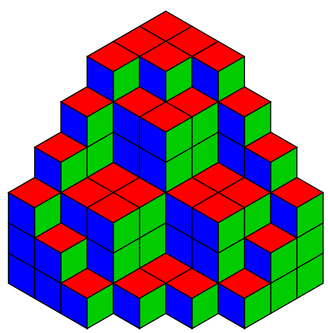
$a = 4, b = 3, c = 5$

A plane partition in an $a \times b \times c$ box is a subset

$$PP \subseteq \{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\} \times \{1, 2, \ldots, c\}$$

with

$$(i, j, k) \in PP \Rightarrow (i', j', k') \in PP \quad \forall (i', j', k') \leq (i, j, k).$$



- **Totally symmetric:**
$(i, j, k) \in PP \Rightarrow \sigma(i, j, k) \in PP \; \forall \sigma \in \mathcal{S}_3$
(MacMahon 1899, 1915/16)

- **Self-complementary:**
Equal to its complement in the $2n \times 2n \times 2n$ box
(Mills, Robbins and Rumsey 1986)

Theorem (Andrews 1994). The number of TSSCPPs in a $2n \times 2n \times 2n$ box is (again) $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.

Alternating sign triangles =ASTs

An AST of order n is a triangular array of 1's, -1's and 0's with n centered rows

$$\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & \bullet & \bullet & \bullet & \bullet & \bullet & \\
 & & \bullet & \bullet & \bullet & & \\
 & & & \bullet & & &
\end{array}$$

such that

(1) the non-zero entries alternate in each row and each column,

(2) all row sums are 1, and

(3) the topmost non-zero entry of each column is 1 (if such an entry exists).

Example:

$$\begin{array}{ccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & 1 & -1 & 1 & 0 & 0 & \\
 & & 1 & -1 & 1 & & \\
 & & & 1 & & &
\end{array}$$

Theorem (Ayyer, Behrend, and F., 2020). The number of ASTs with n rows is (again) $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.

II. Arrowed monotone triangles

ASMs $\longrightarrow$ monotone triangles

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\Rightarrow
\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & 1 & & 4 & & & \\
 & & 1 & & 2 & & 5 & & \\
 & 1 & & 2 & & 3 & & 5 & \\
1 & & 2 & & 3 & & 4 & & 5
\end{array}
$$

• A **monotone triangle** is a triangular array of integers with weak increase along $\nearrow$- and $\searrow$-diagonals, and strict increase along rows.

• Monotone triangles with bottom row $1, 2, \ldots, n$ are in easy bijective correspondence with $n \times n$ ASMs.

• **Arrowed monotone triangles** are monotone triangles where each entry is "decorated" with either $\nwarrow$, $\nearrow$ or $\nwarrow\!\!\!\!\!\nearrow$ such that two conditions are satisfied for each entry e:

  • If e is equal to its $\nwarrow$-neighbor, e must **not** carry $\nwarrow, \nwarrow\!\!\!\!\!\nearrow$ (and therefore carries $\nearrow$).

  • If e is equal to its $\nearrow$-neighbor, e must **not** carry $\nearrow, \nwarrow\!\!\!\!\!\nearrow$ (and therefore carries $\nwarrow$).

In other words: an arrow indicates a non-zero difference !

Arrowed monotone triangle

Example:

$$
\begin{array}{ccccccccccc}
 & & & & & \overset{\nearrow}{4} & & & & & \\
 & & & & \overset{\nwarrow\!\!\!\nearrow}{2} & & \overset{\nwarrow}{5} & & & & \\
 & & & \overset{\nwarrow}{2} & & \overset{\nearrow}{3} & & \overset{\nearrow}{5} & & & \\
 & & \overset{\nearrow}{1} & & \overset{\nwarrow}{3} & & \overset{\nwarrow\!\!\!\nearrow}{4} & & \overset{\nearrow}{6} & & \\
 & \overset{\nwarrow}{1} & & \overset{\nearrow}{2} & & \overset{\nearrow}{3} & & \overset{\nwarrow}{5} & & \overset{\nearrow}{6} & \\
\overset{\nwarrow}{1} & & \overset{\nwarrow}{2} & & \overset{\nwarrow}{3} & & \overset{\nwarrow\!\!\!\nearrow}{4} & & \overset{\nearrow}{5} & & \overset{\nearrow}{6}
\end{array}
$$

We associate the following weight to an arrowed monotone triangle with n rows,

$$u^{\# \nearrow} v^{\# \nwarrow} w^{\# \nwarrow\!\!\!\nearrow} \prod_{i=1}^{n} X_i^{(\text{sum of entries in row } i) - (\text{sum of entries in row } i-1) + (\# \nearrow \text{in row } i) - (\# \nwarrow \text{in row } i)}$$

where the sum of entries in row 0 is defined to be 0. In our example, we obtain

$$u^{10} v^8 w^3 X_1^5 X_2^2 X_3^4 X_4^5 X_5^4 X_6^3.$$

The exponents of the $u, v, w, X_1, \ldots, X_n$ are the $n+3$ statistics from the title...

Remark: Analogy with Schur polynomials

• A Gelfand-Tsetlin pattern is a triangular array (of the same shape as monotone triangles) with weak increase along $\nearrow$- and $\searrow$-diagonals (i.e., we drop the condition on the **strict increase along rows**).

• We associate the following weight to Gelfand-Tsetlin patterns:

$$\prod_{i=1}^{n} X_i^{(\text{sum of entries in row } i)-(\text{sum of entries in row } i-1)}$$

• The Schur polynomial $s_{(\lambda_1,\ldots,\lambda_l)}(X_1,\ldots,X_n)$ is the generating function of Gelfand-Tsetlin patterns with bottom row

$$(0^{n-l},\lambda_l,\lambda_{l-1},\ldots,\lambda_1)$$

and with respect to the weight just defined.

Arrowed monotone triangles with bottom row $(1, 2)$

In the following, we use $\nwarrow e \nearrow$ instead of $\overset{\nwarrow\nearrow}{e}$ in our arrowed monotone triangles.

<table>
<tr><th>AMT</th><th>W</th><th>SBCSPP</th><th>AMT</th><th>W</th><th>SBCSPP</th></tr>
<tr><td>$\begin{array}{ccc} & \nwarrow 1 & \\ \nwarrow 1 & & \nwarrow 2 \end{array}$</td><td>$v^3$</td><td>$\emptyset$</td><td>$\begin{array}{ccc} & \nwarrow 2 & \\ \nwarrow 1 \nearrow & & 2 \nearrow \end{array}$</td><td>$uvwX_1X_2^2$</td><td>$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & \\ \cline{1-1} \end{array}$</td></tr>
<tr><td>$\begin{array}{ccc} & \nwarrow 1 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 \end{array}$</td><td>$v^2wX_1$</td><td>$\begin{array}{|c|}\hline 1 \\ \hline \end{array}$</td><td>$\begin{array}{ccc} & \nwarrow 1 \nearrow & \\ \nwarrow 1 & & 2 \nearrow \end{array}$</td><td>$uvwX_1X_2^2$</td><td>$\begin{array}{|c|c|}\hline 2,1 & 1 \\ \hline \end{array}$</td></tr>
<tr><td>$\begin{array}{ccc} & 1 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 \end{array}$</td><td>$uv^2X_1^2$</td><td>$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}$</td><td>$\begin{array}{ccc} & 2 \nearrow & \\ \nwarrow 1 & & 2 \nearrow \end{array}$</td><td>$u^2vX_1^3X_2$</td><td>$\begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline 1 & & \\ \cline{1-1} \end{array}$</td></tr>
<tr><td>$\begin{array}{ccc} & \nwarrow 1 & \\ \nwarrow 1 & & \nwarrow 2 \nearrow \end{array}$</td><td>$v^2wX_2$</td><td>$\begin{array}{|c|}\hline 2 \\ \hline \end{array}$</td><td>$\begin{array}{ccc} & \nwarrow 2 \nearrow & \\ \nwarrow 1 \nearrow & & 2 \nearrow \end{array}$</td><td>$uw^2X_1^2X_2^2$</td><td>$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array}$</td></tr>
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<tr><td>$\begin{array}{ccc} & \nwarrow 1 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 \nearrow \end{array}$</td><td>$vw^2X_1X_2$</td><td>$\begin{array}{|c|}\hline 2,1 \\ \hline \end{array}$</td><td>$\begin{array}{ccc} & 2 \nearrow & \\ \nwarrow 1 \nearrow & & 2 \nearrow \end{array}$</td><td>$u^2wX_1^3X_2^2$</td><td>$\begin{array}{|c|c|c|}\hline 2 & 2 & 1 \\ \hline 1 & 1 & \\ \cline{1-2} \end{array}$</td></tr>
<tr><td>$\begin{array}{ccc} & \nwarrow 1 & \\ \nwarrow 1 & & 2 \nearrow \end{array}$</td><td>$uv^2X_2^2$</td><td>$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}$</td><td>$\begin{array}{ccc} & \nwarrow 2 & \\ 1 \nearrow & & 2 \nearrow \end{array}$</td><td>$u^2vX_1X_2^3$</td><td>$\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline 1 & & \\ \cline{1-1} \end{array}$</td></tr>
<tr><td>$\begin{array}{ccc} & \nwarrow 2 \nearrow & \\ \nwarrow 1 & & 2 \nearrow \end{array}$</td><td>$uvwX_1^2X_2$</td><td>$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1} \end{array}$</td><td>$\begin{array}{ccc} & \nwarrow 2 \nearrow & \\ 1 \nearrow & & 2 \nearrow \end{array}$</td><td>$u^2wX_1^2X_2^3$</td><td>$\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline 1 & 1 & \\ \cline{1-2} \end{array}$</td></tr>
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</table>

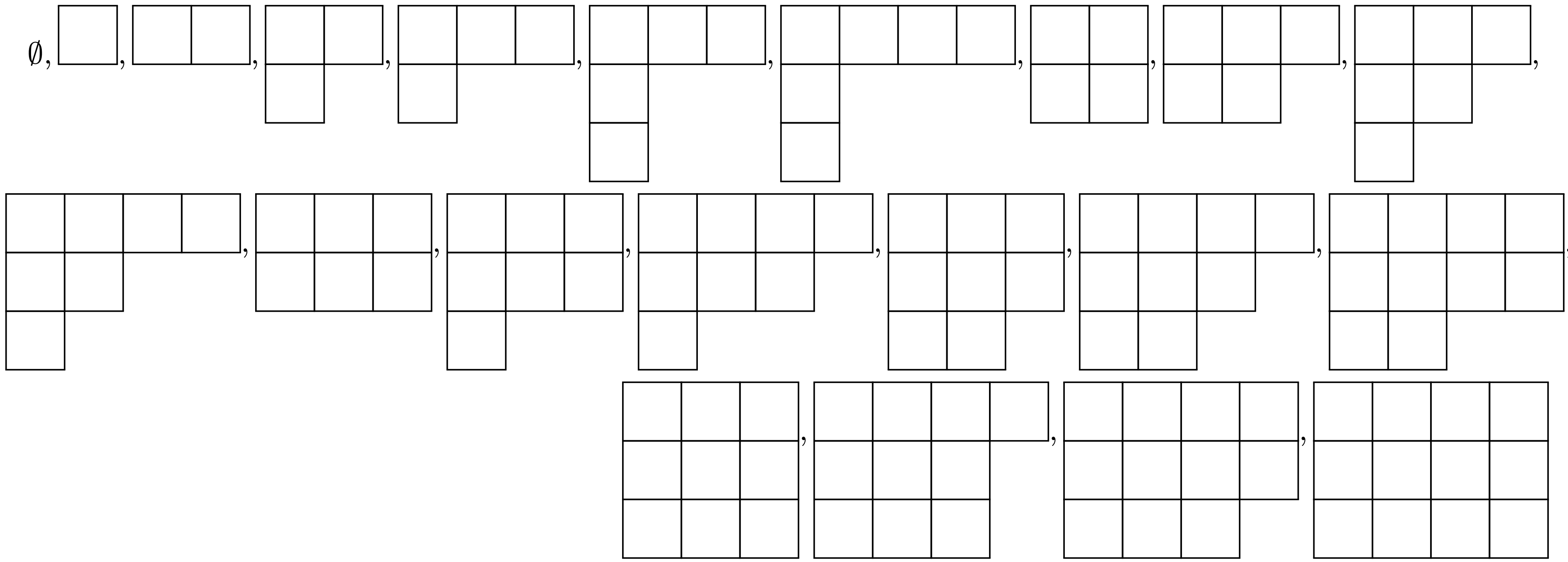
Arrowed monotone triangles with bottom row $(1,2,3)$

$\begin{array}{ccccc} & & {}^*1^* & & \\ & \nwarrow 1 & & {}^*2^* & \\ \nwarrow 1 & & \nwarrow 2 & & {}^*3^* \end{array}$	$v^3 X_1 X_2 X_3 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)(uX_3 + vX_3^{-1} + w)$
$\begin{array}{ccccc} & & {}^*1^* & & \\ & \nwarrow 1 & & {}^*3^* & \\ \nwarrow 1 & & {}^*2^* & & 3\nearrow \end{array}$	$uv^2 X_1 X_2^2 X_3^2 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)(uX_3 + vX_3^{-1} + w)$
$\begin{array}{ccccc} & & {}^*2^* & & \\ & {}^*1^* & & 2\nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & {}^*3^* \end{array}$	$uv^2 X_1^2 X_2^2 X_3 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)(uX_3 + vX_3^{-1} + w)$
$\begin{array}{ccccc} & & {}^*2^* & & \\ & {}^*1^* & & {}^*3^* & \\ \nwarrow 1 & & {}^*2^* & & 3\nearrow \end{array}$	$uv X_1^2 X_2^2 X_3^2 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)^2(uX_3 + vX_3^{-1} + w)$
$\begin{array}{ccccc} & & {}^*2^* & & \\ & \nwarrow 2 & & {}^*3^* & \\ {}^*1^* & & 2\nearrow & & 3\nearrow \end{array}$	$u^2 v X_1^2 X_2^2 X_3^3 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)(uX_3 + vX_3^{-1} + w)$
$\begin{array}{ccccc} & & {}^*3^* & & \\ & {}^*1^* & & 3\nearrow & \\ \nwarrow 1 & & {}^*2^* & & 3\nearrow \end{array}$	$u^2 v X_1^3 X_2^2 X_3^2 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)(uX_3 + vX_3^{-1} + w)$
$\begin{array}{ccccc} & & {}^*3^* & & \\ & {}^*2^* & & 3\nearrow & \\ {}^*1^* & & 2\nearrow & & 3\nearrow \end{array}$	$u^3 X_1^3 X_2^3 X_3^3 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)(uX_3 + vX_3^{-1} + w)$

III. Set-valued balanced column strict plane partitions (SBCSPPs)

Almost self-conjugate partitions

Balanced shapes with at most 3 rows:

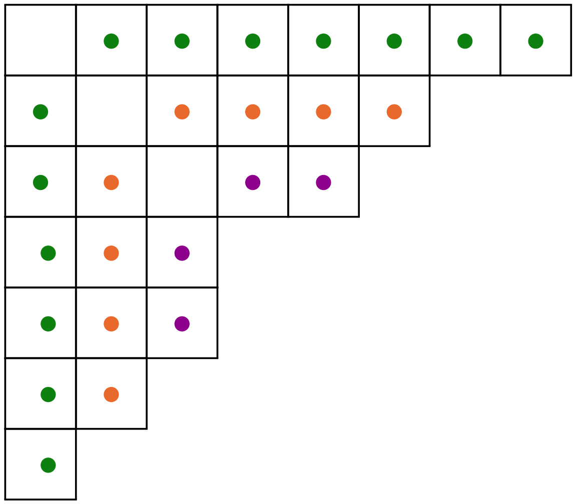


Definition of balanced shapes

Let $\lambda = (a_1, \ldots, a_l | b_1, \ldots, b_l)$ be a partition in Frobenius notation, i.e., a_i is the number of cells right of the diagonal cell (i,i) in the same row, while b_i is the number of cells below (i,i) in the same column. We say that λ is balanced if, for all i, either $a_i = b_i$ or $a_i = b_i + 1$. The weight is

$$\mathsf{W}(\lambda) = w^{l+\sum_{i=1}^{l}(b_i - a_i)}.$$

Example:



Set-valued balanced column strict plane partitions

A **set-valued balanced column strict plane partition (SBCSPP)** D of shape λ and order n is a filling of a balanced shape with non-empty subsets of $\{1, 2, \ldots, n\}$ such that strictly above the diagonal the subsets are singletons, and

1. rows decrease weakly in the sense that the maxima of the sets form a decreasing sequence if read from left to right, and

2. columns decrease strictly in the sense that for two adjacent cells in a column, all elements in the top cell are strictly greater than all elements in the bottom cell.

The weight of D is as follows

$$\mathrm{W}(D) = \mathrm{W}(\lambda) \cdot u^{\text{#of cells strictly above the main diagonal}} \cdot v^{\binom{n+1}{2} - \text{#of entries on and below the main diagonal}} \cdot w^{\text{#of entries} - \text{#of cells}} \cdot \prod_{i=1}^{n} X_i^{\text{# of } i \text{ in D}}.$$

Again: The exponents of the $u, v, w, X_1, \ldots, X_n$ are the $n+3$ statistics from the title...

Example:

8	8	8	7	7	6	4	1
7	7	7	6	5	5		
6	6	5	4	4	4		
5	4	3	$3,2$	3	2		
3	2	$2,1$	1				
2	1						
1							

Letting $n = 9$, the weight is

$$u^{16}v^{26}w^{3}X_1^5X_2^5X_3^4X_4^5X_5^4X_6^4X_7^5X_8^3.$$

The case $n = 2$

AMT	W	SBCSPP	AMT	W	SBCSPP
$\begin{array}{ccc} & \nwarrow 1 & \\ \nwarrow 1 & & \nwarrow 2 \end{array}$	v^3	$\emptyset$	$\begin{array}{ccc} & \nwarrow 2 & \\ \nwarrow 1 \nearrow & & 2 \nearrow \end{array}$	$uvwX_1X_2^2$	$\begin{ytableau} 2 & 2 \\ 1 \end{ytableau}$
$\begin{array}{ccc} & \nwarrow 1 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 \end{array}$	v^2wX_1	$\begin{ytableau} 1 \end{ytableau}$	$\begin{array}{ccc} & \nwarrow 1 \nearrow & \\ \nwarrow 1 & & 2 \nearrow \end{array}$	$uvwX_1X_2^2$	$\begin{ytableau} 2,1 & 1 \end{ytableau}$
$\begin{array}{ccc} & 1 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 \end{array}$	$uv^2X_1^2$	$\begin{ytableau} 1 & 1 \end{ytableau}$	$\begin{array}{ccc} & 2 \nearrow & \\ \nwarrow 1 & & 2 \nearrow \end{array}$	$u^2vX_1^3X_2$	$\begin{ytableau} 2 & 1 & 1 \\ 1 \end{ytableau}$
$\begin{array}{ccc} & \nwarrow 1 & \\ \nwarrow 1 & & \nwarrow 2 \nearrow \end{array}$	v^2wX_2	$\begin{ytableau} 2 \end{ytableau}$	$\begin{array}{ccc} & \nwarrow 2 \nearrow & \\ \nwarrow 1 \nearrow & & 2 \nearrow \end{array}$	$uw^2X_1^2X_2^2$	$\begin{ytableau} 2 & 2 \\ 1 & 1 \end{ytableau}$
$\begin{array}{ccc} & \nwarrow 2 & \\ \nwarrow 1 & & 2 \nearrow \end{array}$	$uv^2X_1X_2$	$\begin{ytableau} 2 & 1 \end{ytableau}$	$\begin{array}{ccc} & 1 \nearrow & \\ \nwarrow 1 & & 2 \nearrow \end{array}$	$u^2vX_1^2X_2^2$	$\begin{ytableau} 2 & 2 & 1 \\ 1 \end{ytableau}$
$\begin{array}{ccc} & \nwarrow 1 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 \nearrow \end{array}$	$vw^2X_1X_2$	$\begin{ytableau} 2,1 \end{ytableau}$	$\begin{array}{ccc} & 2 \nearrow & \\ \nwarrow 1 \nearrow & & 2 \nearrow \end{array}$	$u^2wX_1^3X_2^2$	$\begin{ytableau} 2 & 2 & 1 \\ 1 & 1 \end{ytableau}$
$\begin{array}{ccc} & \nwarrow 1 & \\ \nwarrow 1 & & 2 \nearrow \end{array}$	$uv^2X_2^2$	$\begin{ytableau} 2 & 2 \end{ytableau}$	$\begin{array}{ccc} & \nwarrow 2 & \\ 1 \nearrow & & 2 \nearrow \end{array}$	$u^2vX_1X_2^3$	$\begin{ytableau} 2 & 2 & 2 \\ 1 \end{ytableau}$
$\begin{array}{ccc} & \nwarrow 2 \nearrow & \\ \nwarrow 1 & & 2 \nearrow \end{array}$	$uvwX_1^2X_2$	$\begin{ytableau} 2 & 1 \\ 1 \end{ytableau}$	$\begin{array}{ccc} & \nwarrow 2 \nearrow & \\ 1 \nearrow & & 2 \nearrow \end{array}$	$u^2wX_1^2X_2^3$	$\begin{ytableau} 2 & 2 & 2 \\ 1 & 1 \end{ytableau}$
$\begin{array}{ccc} & 1 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 \nearrow \end{array}$	$uvwX_1^2X_2$	$\begin{ytableau} 2,1 & 1 \end{ytableau}$	$\begin{array}{ccc} & 2 \nearrow & \\ 1 \nearrow & & 2 \nearrow \end{array}$	$u^3X_1^3X_2^3$	$\begin{ytableau} 2 & 2 & 2 \\ 1 & 1 & 1 \end{ytableau}$

IV. The main result

Theorem. The generating function of arrowed monotone triangles with bottom row $1, 2, \ldots, n$ is equal to the generating function of set-valued balanced column-strict plane partitions with parts in $\{1, 2, \ldots, n\}$.

But why should we care?

The case $n = 3$

Multiplicity 1:

AMT	W	SBCSPP	AMT	W	SBCSPP
$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & \nwarrow 2 & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	v^6	$\emptyset$	$\begin{array}{ccccc} & & \nwarrow 1 \nearrow & & \\ & \nwarrow 1 & & \nwarrow 2 & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	$v^5 w X_1$	$\begin{array}{c} 1 \end{array}$
$\begin{array}{ccccc} & & 1 \nearrow & & \\ & \nwarrow 1 & & \nwarrow 2 & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	$uv^5 X_1^2$	$\begin{array}{cc} 1 & 1 \end{array}$	$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & \nwarrow 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	$v^5 w X_2$	$\begin{array}{c} 2 \end{array}$
$\begin{array}{ccccc} & & \nwarrow 1 \nearrow & & \\ & \nwarrow 1 & & \nwarrow 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	$v^4 w^2 X_1 X_2$	$\begin{array}{c} 21 \end{array}$	$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	$uv^5 X_2^2$	$\begin{array}{cc} 2 & 2 \end{array}$
$\begin{array}{ccccc} & & 1 \nearrow & & \\ & \nwarrow 1 & & 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	$u^2 v^4 X_1^2 X_2^2$	$\begin{array}{ccc} 2 & 2 & 1 \\ 1 & & \end{array}$	$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & \nwarrow 2 & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \nearrow \end{array}$	$v^5 w X_3$	$\begin{array}{c} 3 \end{array}$
$\begin{array}{ccccc} & & \nwarrow 1 \nearrow & & \\ & \nwarrow 1 & & \nwarrow 2 & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \nearrow \end{array}$	$v^4 w X_1 X_3$	$\begin{array}{c} 31 \end{array}$	$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & \nwarrow 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \nearrow \end{array}$	$v^4 w^2 X_2 X_3$	$\begin{array}{c} 32 \end{array}$
$\begin{array}{ccccc} & & \nwarrow 1 \nearrow & & \\ & \nwarrow 1 & & \nwarrow 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \nearrow \end{array}$	$v^3 w^3 X_1 X_2 X_3$	$\begin{array}{c} 321 \end{array}$	$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & \nwarrow 2 & \\ \nwarrow 1 & & \nwarrow 2 & & 3 \nearrow \end{array}$	$uv^5 X_3^2$	$\begin{array}{cc} 3 & 3 \end{array}$
$\begin{array}{ccccc} & & 1 \nearrow & & \\ & \nwarrow 1 & & \nwarrow 2 & \\ \nwarrow 1 & & \nwarrow 2 & & 3 \nearrow \end{array}$	$u^2 v^4 X_1^2 X_3^2$	$\begin{array}{ccc} 3 & 3 & 1 \\ 1 & & \end{array}$	$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & 3 \nearrow \end{array}$	$u^2 v^4 X_2^2 X_3^2$	$\begin{array}{ccc} 3 & 3 & 2 \\ 2 & & \end{array}$

AMT	W	SBCSPP	AMT	W	SBCSPP
$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & \nwarrow 3 & \\ \nwarrow 1 & & \nwarrow 2 & & 3 \nearrow \end{array}$	$uv^5X_2X_3$	$\begin{array}{ll} \boxed{3} & \boxed{2} \end{array}$	$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & 3 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & 3 \nearrow \end{array}$	$u^2v^4X_2^3X_3$	$\begin{array}{lll} \boxed{3} & \boxed{2} & \boxed{2} \\ \boxed{2} & & \end{array}$
$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & \nwarrow 3 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 \nearrow & & 3 \nearrow \end{array}$	$uv^3w^2X_2^2X_3^2$	$\begin{array}{ll} \boxed{3} & \boxed{3} \\ \boxed{2} & \boxed{2} \end{array}$	$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & 3 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 \nearrow & & 3 \nearrow \end{array}$	$uv^3wX_2^3X_3^2$	$\begin{array}{lll} \boxed{3} & \boxed{3} & \boxed{2} \\ \boxed{2} & \boxed{2} & \end{array}$
$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & \nwarrow 3 & \\ \nwarrow 1 & & \nwarrow 2 \nearrow & & 3 \nearrow \end{array}$	$u^2v^4X_2X_3^3$	$\begin{array}{lll} \boxed{3} & \boxed{3} & \boxed{3} \\ \boxed{2} & & \end{array}$	$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & \nwarrow 3 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 \nearrow & & 3 \nearrow \end{array}$	$u^2v^3wX_2^2X_3^3$	$\begin{array}{lll} \boxed{3} & \boxed{3} & \boxed{3} \\ \boxed{2} & \boxed{2} & \end{array}$
$\begin{array}{ccccc} & & \nwarrow 1 & & \\ & \nwarrow 1 & & 3 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 \nearrow & & 3 \nearrow \end{array}$	$u^3v^3X_2^3X_3^3$	$\begin{array}{lll} \boxed{3} & \boxed{3} & \boxed{3} \\ \boxed{2} & \boxed{2} & \boxed{2} \end{array}$			

AMT	W	SBCSPP	AMT	W	SBCSPP
$\begin{array}{ccccc} & & \nwarrow 2 & & \\ & \nwarrow 1 & & 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	$uv^5X_1X_2$	$\begin{array}{ll} \boxed{2} & \boxed{1} \end{array}$	$\begin{array}{ccccc} & & 2 \nearrow & & \\ & \nwarrow 1 & & 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	$u^2v^4X_1^3X_2$	$\begin{array}{lll} \boxed{2} & \boxed{1} & \boxed{1} \\ \boxed{1} & & \end{array}$
$\begin{array}{ccccc} & & \nwarrow 2 \nearrow & & \\ & \nwarrow 1 \nearrow & & 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	$uv^3w^2X_1^2X_2^2$	$\begin{array}{ll} \boxed{2} & \boxed{2} \\ \boxed{1} & \boxed{1} \end{array}$	$\begin{array}{ccccc} & & 2 \nearrow & & \\ & \nwarrow 1 \nearrow & & 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	$u^2v^3wX_1^3X_2^2$	$\begin{array}{lll} \boxed{2} & \boxed{2} & \boxed{1} \\ \boxed{1} & \boxed{1} & \end{array}$
$\begin{array}{ccccc} & & \nwarrow 2 & & \\ & 1 \nearrow & & 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	$u^2v^4X_1X_2^3$	$\begin{array}{lll} \boxed{2} & \boxed{2} & \boxed{2} \\ \boxed{1} & & \end{array}$	$\begin{array}{ccccc} & & \nwarrow 2 \nearrow & & \\ & 1 \nearrow & & 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	$u^2v^3wX_1^2X_2^3$	$\begin{array}{lll} \boxed{2} & \boxed{2} & \boxed{2} \\ \boxed{1} & \boxed{1} & \end{array}$
$\begin{array}{ccccc} & & 2 \nearrow & & \\ & 1 \nearrow & & 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 \end{array}$	$u^3v^3X_1^3X_2^3$	$\begin{array}{lll} \boxed{2} & \boxed{2} & \boxed{2} \\ \boxed{1} & \boxed{1} & \boxed{1} \end{array}$			

AMT	W	SBCSPP	AMT	W	SBCSPP
$\nwarrow 2$ $\nwarrow 1 \quad \nwarrow 3$ $\nwarrow 1 \quad \nwarrow 2 \quad 3 \nearrow$	$uv^5X_1X_3$	3 1	$2 \nearrow$ $\nwarrow 1 \quad \nwarrow 3$ $\nwarrow 1 \quad \nwarrow 2 \quad 3 \nearrow$	$u^2v^4X_1^3X_3$	3 1 1 1
$\nwarrow 2 \nearrow$ $\nwarrow 1 \quad \nwarrow 3$ $\nwarrow 1 \quad \nwarrow 2 \nearrow \quad 3 \nearrow$	$uv^3w^2X_1^2X_3^2$	3 3 1 1	$2 \nearrow$ $\nwarrow 1 \quad \nwarrow 3$ $\nwarrow 1 \quad \nwarrow 2 \nearrow \quad 3 \nearrow$	$u^2v^3wX_1^3X_3^2$	3 3 1 1 1
$\nwarrow 2$ $\nwarrow 1 \quad \nwarrow 3$ $\nwarrow 1 \quad 2 \nearrow \quad 3 \nearrow$	$u^2v^4X_1X_3^3$	3 3 3 1	$\nwarrow 2 \nearrow$ $\nwarrow 1 \quad \nwarrow 3$ $\nwarrow 1 \quad 2 \nearrow \quad 3 \nearrow$	$u^2v^3wX_1^2X_3^3$	3 3 3 1 1
$2 \nearrow$ $\nwarrow 1 \quad \nwarrow 3$ $\nwarrow 1 \quad 2 \nearrow \quad 3 \nearrow$	$u^3v^3X_1^3X_3^3$	3 3 3 1 1 1			

AMT	W	SBCSPP
$\nwarrow 2 \nearrow$ $\nwarrow 1 \nearrow \quad \nwarrow 3 \nearrow$ $\nwarrow 1 \quad \nwarrow 2 \nearrow \quad 3 \nearrow$	$uvw^4X_1^2X_2^2X_3^2$	3 3 21 21

AMT	W	SBCSPP	AMT	W	SBCSPP
↖2 1↗ 3↗ ↖1 ↖2 3↗	$u^3v^3X_1X_2^4X_3$	3 2 2 2 2 1	↖2↗ 1↗ 3↗ ↖1 ↖2 3↗	$u^3v^2wX_1^2X_2^4X_3$	3 2 2 2 2 1 1
2↗ 1↗ 3↗ ↖1 ↖2 3↗	$u^4v^2X_1^3X_2^4X_3$	3 2 2 2 2 1 1 1	↖2 1↗ 3↗ ↖1 ↖2↗ 3↗	$u^3v^2wX_1X_2^4X_3^2$	3 3 2 2 2 2 1
↖2↗ 1↗ 3↗ ↖1 ↖2↗ 3↗	$u^3vw^2X_1^2X_2^4X_3^2$	3 3 2 2 2 21 1	↖2 1↗ 3↗ ↖1 2↗ 3↗	$u^4v^2X_1X_2^4X_3^3$	3 3 3 2 2 2 2 1
2↗ 1↗ 3↗ ↖1 2↗ 3↗	$u^5vX_1^3X_2^4X_3^3$	3 3 3 2 2 2 2 1 1 1			

AMT	W	SBCSPP	AMT	W	SBCSPP
↖2 ↖2 ↖3 1↗ 2↗ 3↗	$u^3v^3X_1X_2X_3^4$	3 3 3 3 2 1	↖2↗ ↖2 ↖3 1↗ 2↗ 3↗	$u^3v^2wX_1^2X_2X_3^4$	3 3 3 3 2 1 1
2↗ ↖2 ↖3 1↗ 2↗ 3↗	$u^4v^2X_1^3X_2X_3^4$	3 3 3 3 2 1 1 1	↖2 ↖2 ↖3↗ 1↗ 2↗ 3↗	$u^3v^2wX_1X_2^2X_3^4$	3 3 3 3 2 2 1
↖2↗ ↖2 ↖3↗ 1↗ 2↗ 3↗	$u^3vw^2X_1^2X_2^2X_3^4$	3 3 3 3 2 21 1	↖2 ↖2 3↗ 1↗ 2↗ 3↗	$u^4v^2X_1X_2^3X_3^4$	3 3 3 3 2 2 2 1
2↗ ↖2 3↗ 1↗ 2↗ 3↗	$u^5vX_1^3X_2^3X_3^4$	3 3 3 3 2 2 2 1 1 1			

AMT	W	SBCSPP	AMT	W	SBCSPP
$3\nearrow$ $\nwarrow 1$ $3\nearrow$ $\nwarrow 1$ $\nwarrow 2$ $3\nearrow$	$u^3v^3X_1^4X_2X_3$	3 1 1 1 2 1	$3\nearrow$ $\nwarrow 1\nearrow$ $3\nearrow$ $\nwarrow 1$ $\nwarrow 2$ $3\nearrow$	$u^3v^2wX_1^4X_2^2X_3$	3 2 1 1 2 1 1
$3\nearrow$ $1\nearrow$ $3\nearrow$ $\nwarrow 1$ $\nwarrow 2$ $3\nearrow$	$u^4v^2X_1^4X_2^3X_3$	3 2 2 1 2 1 1 1	$3\nearrow$ $\nwarrow 1$ $3\nearrow$ $\nwarrow 1$ $\nwarrow 2\nearrow$ $3\nearrow$	$u^3v^2wX_1^4X_2X_3^2$	3 3 1 1 2 1 1
$3\nearrow$ $\nwarrow 1\nearrow$ $3\nearrow$ $\nwarrow 1$ $\nwarrow 2\nearrow$ $3\nearrow$	$u^3vw^2X_1^4X_2^2X_3^2$	3 3 1 1 2 21 1	$3\nearrow$ $\nwarrow 1$ $3\nearrow$ $\nwarrow 1$ $2\nearrow$ $3\nearrow$	$u^4v^2X_1^4X_2X_3^3$	3 3 3 1 2 1 1 1
$3\nearrow$ $1\nearrow$ $3\nearrow$ $\nwarrow 1$ $2\nearrow$ $3\nearrow$	$u^5vX_1^4X_2^3X_3^3$	3 3 3 2 2 2 1 1 1 1			

AMT	W	SBCSPP	AMT	W	SBCSPP
3↗ ↖2 3↗ ↖1 2↗ 3↗	$u^4v^2X_1^4X_2^2X_3^2$	$\begin{array}{cccc}3&3&2&1\\2&1&1\\1\end{array}$	↖3 2↗ 3↗ ↖1 2↗ 3↗	$u^4v^2X_1^2X_2^4X_3^2$	$\begin{array}{cccc}3&3&2&2\\2&2&1\\1\end{array}$
3↗ 2↗ 3↗ ↖1 2↗ 3↗	$u^5vX_1^4X_2^4X_3^2$	$\begin{array}{cccc}3&3&2&2\\2&2&1&1\\1&1\end{array}$	↖3↗ ↖2↗ 3↗ ↖1↗ 2↗ 3↗	$u^3w^3X_1^3X_2^3X_3^3$	$\begin{array}{ccc}3&3&3\\2&2&2\\1&1&1\end{array}$
3↗ ↖2↗ 3↗ ↖1↗ 2↗ 3↗	$u^4w^2X_1^4X_2^3X_3^3$	$\begin{array}{cccc}3&3&3&1\\2&2&2\\1&1\end{array}$	↖3↗ 2↗ 3↗ ↖1↗ 2↗ 3↗	$u^4w^2X_1^3X_2^4X_3^3$	$\begin{array}{cccc}3&3&3&2\\2&2&2\\1&1&1\end{array}$
3↗ 2↗ 3↗ ↖1↗ 2↗ 3↗	$u^5wX_1^4X_2^4X_3^3$	$\begin{array}{cccc}3&3&3&2\\2&2&2&1\\1&1&1\end{array}$	↖3 ↖2 3↗ 1↗ 2↗ 3↗	$u^4v^2X_1^2X_2^2X_3^4$	$\begin{array}{cccc}3&3&3&3\\2&2&1\\1\end{array}$
3↗ ↖2 3↗ 1↗ 2↗ 3↗	$u^5vX_1^4X_2^2X_3^4$	$\begin{array}{cccc}3&3&3&3\\2&2&1&1\\1&1\end{array}$	↖3↗ ↖2↗ 3↗ 1↗ 2↗ 3↗	$u^4w^2X_1^3X_2^3X_3^4$	$\begin{array}{cccc}3&3&3&3\\2&2&2\\1&1&1\end{array}$
3↗ ↖2↗ 3↗ 1↗ 2↗ 3↗	$u^5wX_1^4X_2^3X_3^4$	$\begin{array}{cccc}3&3&3&3\\2&2&2&1\\1&1&1\end{array}$	↖3 2↗ 3↗ 1↗ 2↗ 3↗	$u^5vX_1^2X_2^4X_3^4$	$\begin{array}{cccc}3&3&3&3\\2&2&2&2\\1&1\end{array}$
↖3↗ 2↗ 3↗ 1↗ 2↗ 3↗	$u^5wX_1^3X_2^4X_3^4$	$\begin{array}{cccc}3&3&3&3\\2&2&2&2\\1&1&1\end{array}$	3↗ 2↗ 3↗ 1↗ 2↗ 3↗	$u^6X_1^4X_2^4X_3^4$	$\begin{array}{cccc}3&3&3&3\\2&2&2&2\\1&1&1&1\end{array}$

In total: 71

The case $n = 3$

Multiplicity 2: $\ldots 31 \ldots$

Multiplicity 3: $\ldots 6 \ldots$

Multiplicity 4: $\ldots 14 \ldots$

Multiplicity 5: $\ldots 6 \ldots$

Multiplicity 6:

<table>
<tr><th colspan="2">AMTs</th><th>W</th><th>SBCSPPs</th></tr>
<tr><td>1↗
↖1 ↖3↗
↖1 ↖2↗ 3↗</td><td>↖2↗
↖1↗ 2↗
↖1 ↖2 3↗</td><td rowspan="3">$u^2v^2w^2X_1^2X_2^2X_3^2$</td><td>$\begin{array}{|c|c|c|}\hline 3&3&1\\ \hline 21&2\\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 32&3&2\\ \hline 1&1\\ \cline{1-2}\end{array}$</td></tr>
<tr><td>↖2↗
↖1 3↗
↖1 ↖2↗ 3↗</td><td>↖2↗
1↗ ↖3
↖1 ↖2↗ 3↗</td><td>$\begin{array}{|c|c|c|}\hline 3&3&1\\ \hline 2&21\\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 3&3&2\\ \hline 21&1\\ \cline{1-2}\end{array}$</td></tr>
<tr><td>↖2↗
↖2 ↖3↗
↖1 2↗ 3↗</td><td>↖3
↖1↗ 3↗
↖1 ↖2↗ 3↗</td><td>$\begin{array}{|c|c|c|}\hline 3&3&1\\ \hline 2&2\\ \cline{1-2} 1\\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 3&3&2\\ \hline 2&1\\ \cline{1-2} 1\\ \cline{1-1}\end{array}$</td></tr>
</table>

V. Specializing to the ASM-DPP case

Specializing arrowed monotone triangles

Claim: When setting $u = v = 1, w = -1$ and $(X_1, \ldots, X_n) = (1, \ldots, 1)$ in the generating function of arrowed monotone triangles with bottom row $(k_1, \ldots, k_n)$, we obtain the number of monotone triangles with bottom row $(k_1, \ldots, k_n)$.

What do we need to do?

• For a given arrowed monotone triangle, let $(-1)^{\# \nwarrow\!\!\!\!\nearrow}$ be its sign.

• We identify a set of **positive** arrowed monotone triangles that are in bijective correspondence with monotone triangles.

• On the remaining set of arrowed monotone triangles, we define a **sign-reversing involution.**

• In an (arrowed) monotone triangle, an entry is said to be **free**, if it is different from its $\nwarrow$-neighbour and its $\nearrow$-neighbour. In the example, the free entries are indicated in green.

$$
\begin{array}{ccccccccccc}
 & & & & & 4\nearrow & & & & & \\
 & & & & \nwarrow 2\nearrow & & \nwarrow 5 & & & & \\
 & & & \nwarrow 2 & & 3\nearrow & & 5\nearrow & & & \\
 & & 1\nearrow & & \nwarrow 3 & & \nwarrow 4\nearrow & & 6\nearrow & & \\
 & \nwarrow 1 & & 2\nearrow & & 3\nearrow & & \nwarrow 5 & & 6\nearrow & \\
\nwarrow 1 & & \nwarrow 2 & & \nwarrow 3 & & \nwarrow 4\nearrow & & 5\nearrow & & 6\nearrow
\end{array}
$$

• Free entries can be decorated with $\nwarrow$, $\nearrow$ or $\nwarrow\!\!\!\!\nearrow$, for all other entries, there is a unique choice (either $\nwarrow$ or $\nearrow$).

• The set of positive arrowed monotone triangles that correspond to monotone triangles is a follows: In a monotone triangle, decorate all free entries with $\nwarrow$ and all other entries with the unique eligible element in $\{\nwarrow, \nearrow\}$.

• We have the following **sign-reversing involution** on the remaining arrowed monotone triangles: Take the topmost and leftmost free entry that is not decorated with $\nwarrow$. Change its decoration from $\nearrow$ to $\nwarrow\!\!\!\!\nearrow$, or vice-versa. This clearly changes the sign.

Specializing SBCSPPs

Claim: When setting $u = v = 1, w = -1$ and $(X_1, \ldots, X_n) = (1, \ldots, 1)$ in the generating function of SBCSPPs of order n, we obtain the number of DPPs of order n.

What do we need to do?

• For a given SBCSPP of shape $(a_1, \ldots, a_l | b_1, \ldots, b_l)$, let its sign be

$$(-1)^{l+\sum_{i=1}^{l}(b_i-a_i)+(\#\text{ of entries})-(\#\text{ of cells})} = (-1)^{\#\text{ of entries}}$$

• We define **two sign-reversing involutions** to "cancel" certain subsets.

• The remaining set will be a set of **positive** SBCSPPs that is in **easy bijective correspondence with the set of DPPs.**

The first sign-reversing involution

• A **principal SBCSPP** has singletons in each cell. We can associate a principal SBCSPP to each SBCSPP by just keeping the **maximum** in each cell.

9	8	8	7	7	5	3	2
7	6	5	4	3	3		
6	5	4	3	2			
4	4	3	1				
3	3						
2	1						
1							

$9_{,8}$	8	8	7	7	5	3	2
7	6	5	4	3			
$6_{,5}$	5	4	3	2			
4	4	$3_{,2,1}$	1				
3	$3_{,2}$						
2	1						
1							

• If for a fixed principal SBCSPP with **more than one** SBCSPP associated with it, there is the following sign-reversing involution: Fix the topmost and leftmost cell c that can contain **more than one entry**, and let e be the minimal possible entry for this cell (i.e., $e-1$ is in the cell below). If c contains e remove it, otherwise add it.

In the example: $c=(1,1)$ and $e=8$.

The second sign-reversing involution

• Principal SBCSPPs that have no other SBCSPP associated with it are characterized as follows: for each diagonal entry d, the entries below in the same column are $d-1, d-2, \ldots, 1$.

7	7	7	7	7	5	3	2
6	5	5	4	3	3		
5	4	2	2	2			
4	3	1	1				
3	2						
2	1						
1							

7	7	7	7	7	5	1	1
6	5	5	4	3	3		
5	4	2	2	2			
4	3	1	1				
3	2						
2	1						
1							

• We define a sign-reversing involution on the subset of the remaining SBCSPPs for which at least one of the following is satisfied: the SBCSPP contains a 1 **strictly above the diagonal** or $a_i \neq b_i + 1$ for an i.

• If $a_i \neq b_i + 1$ for an i, choose the minimal such i. If there is no 1 in row $1, \ldots, i-1$, add a 1 at the end of row i. Otherwise remove the topmost and rightmost 1.

In our examples, we have $a_2 \neq b_2 + 1$. In the left example, we add a 1 to the second row, while in the other example, we delete the last 1 from the first row.

Analyzing the positive remainder

What remains are SBCSPPs such that (1) all cells contain a single element, (2) $a_i = b_i + 1$, (3) weakly below a diagonal entry we have consecutive integers ending with 1, and (4) there are no 1's above the diagonal. All such SBCSPPs have weight 1.

$$
\begin{array}{cccccccc}
7 & 6 & 6 & 5 & 5 & 5 & 3 & 2 \\
6 & 4 & 4 & 4 & 3 & 3 & & \\
5 & 3 & 2 & 2 & 2 & & & \\
4 & 2 & 1 & & & & & \\
3 & 1 & & & & & & \\
2 & & & & & & & \\
1 & & & & & & &
\end{array}
\overset{(1)}{\Longrightarrow}
\begin{array}{cccccccc}
7 & 6 & 6 & 5 & 5 & 5 & 3 & 2 \\
 & 4 & 4 & 4 & 3 & 3 & & \\
 & & 2 & 2 & 2 & & &
\end{array}
\overset{(2)}{\Longrightarrow}
\begin{array}{cccccccc}
6 & 5 & 5 & 4 & 4 & 4 & 2 & 1 \\
 & 3 & 3 & 3 & 2 & 2 & & \\
 & & 1 & 1 & 1 & & &
\end{array}
$$

(1) **Remove all cells strictly below the main diagonal** and obtain a column strict shifted plane partition (CSSPP). From $a_i = b_i + 1$, it follows that the **first part of each row is one less than the length.**

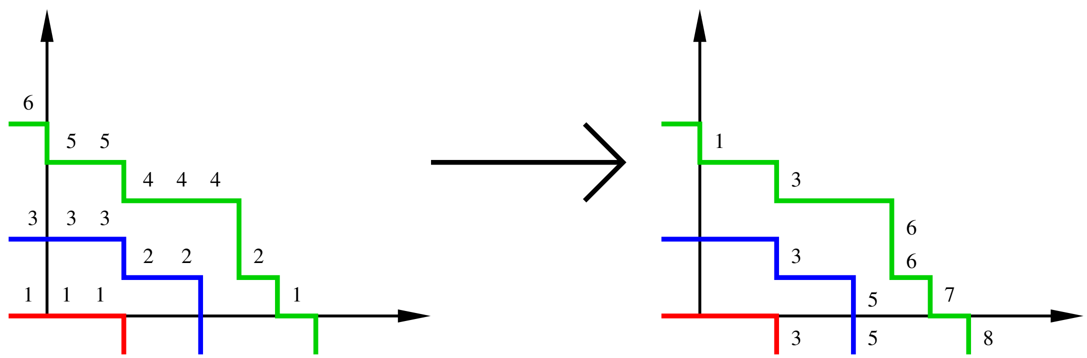
(2) Since there is no 1 in the plane partition, we may **subtract** 1 from each entry and obtain a column strict shifted plane partition such that the **first part of each row is two less than the length of its row.**

(3) By **conjugating the partition in each row**, such CSSPPs with parts no greater than $n-1$ are in easy bijective correspondence with CSSPPs with parts no greater than $n+1$ such that **the first part of each row exceeds its length by precisely** 2**.**

(4) To obtain the corresponding DPP, we **subtract** 1 **from each entry and remove all** 0**s.**

$$\begin{array}{cccccccc} 6 & 5 & 5 & 4 & 4 & 4 & 2 & 1 \\ & 3 & 3 & 3 & 2 & 2 & & \\ & & 1 & 1 & 1 & & & \end{array} \quad \overset{(3)}{\Rightarrow} \quad \begin{array}{cccccc} 8 & 7 & 6 & 6 & 3 & 1 \\ & 5 & 5 & 3 & & \\ & & 3 & & & \end{array} \quad \overset{(4)}{\Rightarrow} \quad \begin{array}{ccccc} 7 & 6 & 5 & 5 & 2 \\ & 4 & 4 & 2 & \\ & & 2 & & \end{array}$$

Lattice paths help in understanding that (3) leads to another CSSPP.



Would a weight-preserving bijection between arrowed monotone triangles and SBCSPPs imply an ASM-DPP bijection?

"Natural" approach: Consider the SBCSPPs that are left after the two sign-reversing involutions (they are equinumerous with DPPs), take the corresponding arrowed monotone triangles in the weight-preserving bijection and delete the arrows to obtain monotone triangles and thus ASMs.

Case $n = 2$.

AMT	W	SBCSPP	AMT	W	SBCSPP
$\nwarrow 1$ $\nwarrow 1 \quad \nwarrow 2$	v^3	$\emptyset$	$\nwarrow 2$ $\nwarrow 1 \nearrow \quad 2 \nearrow$	$uvwX_1X_2^2$	2 2 1
$\nwarrow 1 \nearrow$ $\nwarrow 1 \quad \nwarrow 2$	v^2wX_1	1	$\nwarrow 1 \nearrow$ $\nwarrow 1 \quad 2 \nearrow$	$uvwX_1X_2^2$	2,1 1
$1 \nearrow$ $\nwarrow 1 \quad \nwarrow 2$	$uv^2X_1^2$	1 1	$2 \nearrow$ $\nwarrow 1 \quad 2 \nearrow$	$u^2vX_1^3X_2$	2 1 1 1
$\nwarrow 1$ $\nwarrow 1 \quad \nwarrow 2 \nearrow$	v^2wX_2	2	$\nwarrow 2 \nearrow$ $\nwarrow 1 \nearrow \quad 2 \nearrow$	$uw^2X_1^2X_2^2$	2 2 1 1
$\nwarrow 2$ $\nwarrow 1 \quad 2 \nearrow$	$uv^2X_1X_2$	2 1	$1 \nearrow$ $\nwarrow 1 \quad 2 \nearrow$	$u^2vX_1^2X_2^2$	2 2 1 1
$\nwarrow 1 \nearrow$ $\nwarrow 1 \quad \nwarrow 2 \nearrow$	$vw^2X_1X_2$	2,1	$2 \nearrow$ $\nwarrow 1 \nearrow \quad 2 \nearrow$	$u^2wX_1^3X_2^2$	2 2 1 1 1
$\nwarrow 1$ $\nwarrow 1 \quad 2 \nearrow$	$uv^2X_2^2$	2 2	$\nwarrow 2$ $1 \nearrow \quad 2 \nearrow$	$u^2vX_1X_2^3$	2 2 2 1
$\nwarrow 2 \nearrow$ $\nwarrow 1 \quad 2 \nearrow$	$uvwX_1^2X_2$	2 1 1	$\nwarrow 2 \nearrow$ $1 \nearrow \quad 2 \nearrow$	$u^2wX_1^2X_2^3$	2 2 2 1 1
$1 \nearrow$ $\nwarrow 1 \quad \nwarrow 2 \nearrow$	$uvwX_1^2X_2$	2,1 1	$2 \nearrow$ $1 \nearrow \quad 2 \nearrow$	$u^3X_1^3X_2^3$	2 2 2 1 1 1

The two SBCSPPs that remain after applying the two sign-reversing involutions are $\emptyset$ and 2 2 2 / 1. If we ignore the arrows in the corresponding arrowed monotone triangles, we obtain the bijection to monotone triangles.

Already for $n=3$, this can't work!

For $n=3$, the following 7 SBCSPP are left after applying the two sign-reversing involutions (we also provide the weights):

$$(\emptyset, v^6),\ \left(\begin{array}{ccc} 2&2&2\\ 1&& \end{array}, u^2v^4X_1X_2^3\right),\ \left(\begin{array}{cccc} 3&2&2&2\\ 2&&&\\ 1&&& \end{array}, u^3v^3X_1X_2^4X_3\right),\ \left(\begin{array}{cccc} 3&3&2&2\\ 2&&&\\ 1&&& \end{array}, u^3v^3X_1X_2^3X_3^2\right),$$

$$\left(\begin{array}{cccc} 3&3&3&2\\ 2&&&\\ 1&&& \end{array}, u^3v^3X_1X_2^2X_3^3\right),\ \left(\begin{array}{cccc} 3&3&3&3\\ 2&&&\\ 1&&& \end{array}, u^3v^3X_1X_2X_3^4\right),\ \left(\begin{array}{cccc} 3&3&3&3\\ 2&2&2&2\\ 1&1&& \end{array}, u^5vX_1^2X_2^4X_3^4\right).$$

Crucial observation 1: Only for one of these SBCSPPs, the exponent of X_1 in the weight is greater than 1.

$\begin{array}{ccccc} & & *1* & & \\ & \nwarrow 1 & & *2* & \\ \nwarrow 1 & & \nwarrow 2 & & *3* \end{array}$	$v^3 X_1 X_2 X_3 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)(uX_3 + vX_3^{-1} + w)$
$\begin{array}{ccccc} & & *1* & & \\ & \nwarrow 1 & & *3* & \\ \nwarrow 1 & & *2* & & 3 \nearrow \end{array}$	$uv^2 X_1 X_2^2 X_3^2 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)(uX_3 + vX_3^{-1} + w)$
$\begin{array}{ccccc} & & *2* & & \\ & *1* & & 2 \nearrow & \\ \nwarrow 1 & & \nwarrow 2 & & *3* \end{array}$	$uv^2 X_1^2 X_2^2 X_3 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)(uX_3 + vX_3^{-1} + w)$
$\begin{array}{ccccc} & & *2* & & \\ & *1* & & *3* & \\ \nwarrow 1 & & *2* & & 3 \nearrow \end{array}$	$uv X_1^2 X_2^2 X_3^2 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)^2 (uX_3 + vX_3^{-1} + w)$
$\begin{array}{ccccc} & & *2* & & \\ & \nwarrow 2 & & *3* & \\ *1* & & 2 \nearrow & & 3 \nearrow \end{array}$	$u^2 v X_1^2 X_2^2 X_3^3 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)(uX_3 + vX_3^{-1} + w)$
$\begin{array}{ccccc} & & *3* & & \\ & *1* & & 3 \nearrow & \\ \nwarrow 1 & & *2* & & 3 \nearrow \end{array}$	$u^2 v X_1^3 X_2^2 X_3^2 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)(uX_3 + vX_3^{-1} + w)$
$\begin{array}{ccccc} & & *3* & & \\ & *2* & & 3 \nearrow & \\ *1* & & 2 \nearrow & & 3 \nearrow \end{array}$	$u^3 X_1^3 X_2^3 X_3^3 (uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)(uX_3 + vX_3^{-1} + w)$

Crucial observation 2: For two (out of the 7) monotone triangles, the exponents of X_1 in the weight of the associated arrowed monotone triangles is at least 2 (namely for those that have a 3 at the top).

Therefore, "forgetting arrows" can't work!

VI. Schur expansion:

Totally symmetric plane partitions !

Formula for the generating function

The generating function of arrowed monotone triangles with bottom row $1, 2, \ldots, n$ and of SBCSPPs of order n is

$$\prod_{i=1}^{n} X_i^n \frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le p\le q\le n}\left(uX_q + vX_p^{-1} + w\right)\right]}{\prod_{1\le i<j\le n}(X_j - X_i)} = \prod_{i=1}^{n} X_i^n \frac{\det_{1\le i,j\le n}\left((uX_i+w)^j - (-vX_i^{-1})^j\right)}{\prod_{1\le i<j\le n}(X_j - X_i)}.$$

This is obviously a symmetric function in $X_1, X_2, \ldots, X_n$.

Schur polynomial expansion of $\prod_{i=1}^{n} X_i^{n-1} \frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le p<q\le n}\left(uX_q+vX_p^{-1}+w\right)\right]}{\prod_{1\le i<j\le n}(X_j-X_i)}$?

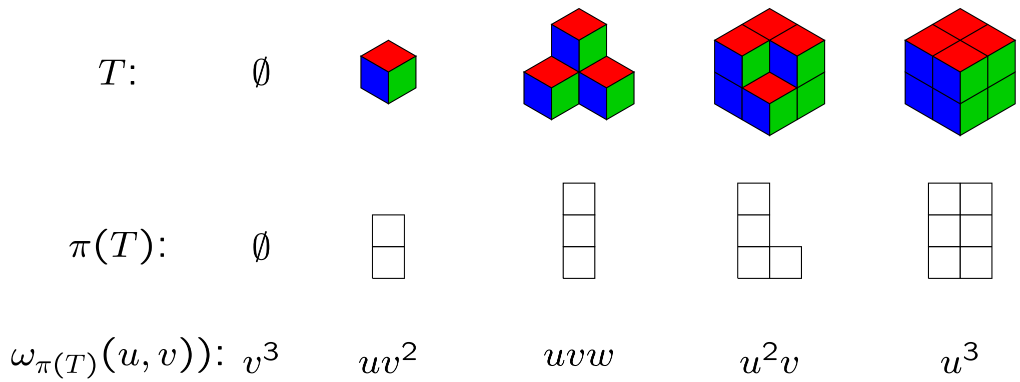
For $n = 3$:

$$v^3 + uv^2 s_{(1,1)}(x_1,x_2,x_3) + uvws_{(1,1,1)}(x_1,x_2,x_3) + u^2vs_{(2,1,1)}(x_1,x_2,x_3) + u^3 s_{(2,2,2)}(x_1,x_2,x_3).$$

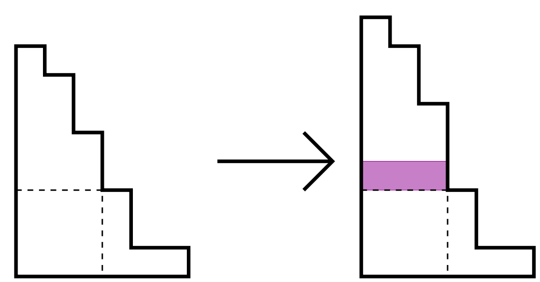
(BTW, this is the generating function of a natural variation of arrowed monotone triangles, which we call **down-arrowed monotone triangles**.)

The case $n = 3$

$$v^3 + uv^2 s_{(1,1)}(x_1, x_2, x_3) + uvw s_{(1,1,1)}(x_1, x_2, x_3) + u^2 v s_{(2,1,1)}(x_1, x_2, x_3) + u^3 s_{(2,2,2)}(x_1, x_2, x_3)$$



Here we see all totally symmetric plane partitions in a $2 \times 2 \times 2$ box, its slightly modified "profile" along the diagonal $y = x$ together with a certain weight.

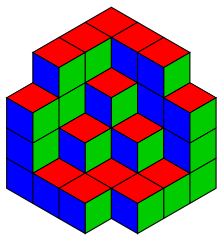


Thin partitions

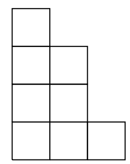
Definition. A partition in Frobenius notation $(a_1, \ldots, a_l | b_1, \ldots, b_l)$ is said to be thin (a.k.a. modified balanced) if $a_i < b_i$.

Thin partitions whose parts do not exceed $n-1$ are counted by the n-th Catalan number. That is why we see $C_3 = 5$ Schur polynomials in the expansion above.

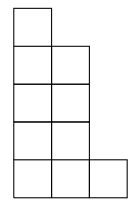
We consider **totally symmetric plane partitions.**



T



diag(T)



$\pi(T)$

Suppose diag$(T) = (a_1, \ldots, a_l | b_1, \ldots, b_l)$ is the diagonal profil of the totally symmetry plane partition T in Frobenius notation, then it is not terribly hard to see that $\pi(T) = (a_1, \ldots, a_l | b_1 + 1, \ldots, b_l + 1)$ is a thin partition.

The weight of a thin partition $\lambda = (a_1, \ldots, a_l | b_1 + 1, \ldots, b_l + 1)$ of order n is defined to be

$$\omega_\lambda(u, v) = u^{\sum_{i=1}^{l} (a_i + 1)} v^{\binom{n}{2} - \sum_{i=1}^{l} b_i} w^{\sum_{i=1}^{l} (b_i - a_i)}.$$

Theorem (F. Aigner, I. Fischer, M. Konvalinka, P. Nadeau, V. Tewari, FPSAC 2020). The generating function of down-arrowed monotone triangles of order n has the following Schur polynomial expansion.

$$\sum_{T \in \mathsf{TSPP}_{n-1}} \omega_{\pi(T)}(u,v) \cdot s_{\pi(T)}(X_1, \ldots, X_n)$$

Why is it so hard to find a (nice) ASM-DPP bijection?

• In order to have a significance increase in the number of equivalent statistics, it was necessary to extend the objects.

• The ASM-DPP relation follows from a certain (-1)-enumeration. Are signs unavoidable?

• It is not even clear that a bijection between arrowed monotone triangles and SBCSPPs will lead to an ASM-DPP bijection.

Merci !

A multivariate operator formula

Basics on signed sets

- A signed set is a pair of disjoint sets $\underline{S} = (S^+, S^-)$. The size of a signed set is

$$|\underline{S}| = |S^+| - |S^-|.$$

- More generally, for given weights on the elements of $\underline{S}$, the generating function is defined as

$$\sum_{s \in S^+} \mathsf{W}(s) - \sum_{s \in S^-} \mathsf{W}(s).$$

- We will use signed intervals: for $a, b \in \mathbb{Z}$, we let

$$\underline{[a,b]} = \begin{cases} ([a,b], \emptyset) & a \leq b \\ (\emptyset, [b+1, a-1]) & b+1 \leq a-1 \\ (\emptyset, \emptyset) & b+1 = a \end{cases}.$$

Generalized arrowed monotone triangles

• A generalized arrowed monotone triangle is a triangular array of the following form

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & a_{1,1} & & & & & & & \\
 & & & & & & a_{2,1} & & a_{2,2} & & & & & & \\
 & & & & & \dots & & \dots & & \dots & & & & & \\
 & & & & a_{n-2,1} & & \dots & & \dots & & a_{n-2,n-2} & & & & \\
 & & & a_{n-1,1} & & a_{n-1,2} & & \dots & & \dots & & a_{n-1,n-1} & & & \\
 & & a_{n,1} & & a_{n,2} & & a_{n,3} & & \dots & & \dots & & a_{n,n} & &
\end{array},
$$

where each $a_{i,j}$ is an integer decorated with an element from $\{{}^{s}\nwarrow\!\!\!\!\!\nearrow^{t} | s, t \in \mathbb{Z}\}$ and the following is satisfied for each integer a not in the bottom row: Suppose

$$
\begin{array}{ccc}
 & a & \\
b & & c
\end{array},
$$

and $\mathrm{decor}(b) = {}^{s_b}\nwarrow\!\!\!\!\!\nearrow^{t_b}$ and $\mathrm{decor}(c) = {}^{s_c}\nwarrow\!\!\!\!\!\nearrow^{t_c}$, then we require $a \in \underline{[b + t_b, c - s_c]}$.

• Arrowed monotone triangles are the subclass of generalized arrowed monotone triangles with possible decorations ${}^{s}\nwarrow\!\!\!\!\!\nearrow^{t}$ with $(s, t) \in \{(1, 0), (0, 1), (1, 1)\}$.

The weights and the theorem

Suppose we are given arbitrary weights $\omega({}^{s}\nwarrow\!\!\!\times\!\!\!\nearrow^{t})$ for all $s,t \in \mathbb{Z}$, and, moreover, we define $\alpha({}^{s}\nwarrow\!\!\!\times\!\!\!\nearrow^{t}) = t-s$. Then the weight of a given generalized arrowed monotone triangle $A = (a_{i,j})_{1 \le j \le i \le n}$ is

$$\mathrm{W}(A) = \prod_{p,q \in \mathbb{Z}} \omega({}^{p}\nwarrow\!\!\!\times\!\!\!\nearrow^{q})^{\#\text{ of } {}^{p}\nwarrow\!\!\!\times\!\!\!\nearrow^{q}} \prod_{i=1}^{n} X_i^{\sum_{j=1}^{i} a_{i,j} - \sum_{j=1}^{i-1} a_{i-1,j} + \sum_{j=1}^{i} \alpha(\mathrm{decor}(a_{i,j}))}.$$

Theorem. Suppose almost all weights $\omega({}^{p}\nwarrow\!\!\!\times\!\!\!\nearrow^{q})$, $p,q \in \mathbb{Z}$, vanish. The generating function of generalized arrowed monotone triangles with bottom row $k_1, \ldots, k_n$ is

$$\prod_{i=1}^{n} \sum_{s,t \in \mathbb{Z}} \omega({}^{s}\nwarrow\!\!\!\times\!\!\!\nearrow^{t}) X_i^{t-s} \prod_{1 \le p < q \le n} \left(\sum_{s,t \in \mathbb{Z}} \omega({}^{s}\nwarrow\!\!\!\times\!\!\!\nearrow^{t}) \mathrm{E}_{k_p}^{t} \mathrm{E}_{k_q}^{-s} \right) s_{(k_n, k_{n-1}, \ldots, k_1)}(X_1, \ldots, X_n).$$