Vertically symmetric alternating sign matrices and a multivariate Laurent polynomial identity

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A conjecture

Symmetrizer: \( \text{Sym} p(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \)

Conjecture (F., Riegler). For integers \( s, t \geq 0 \), consider the following rational function in \( z_1, \ldots, z_{s+t-1} \)

\[
P_{s,t} = \prod_{i=1}^{s} z_i^{2s-2i-t+1}(1 - z_i^{-1})^{i-1} \prod_{i=s+1}^{s+t-1} z_i^{2i-2s-t}(1 - z_i^{-1})^s
\]

\[
\times \prod_{1 \leq p < q \leq s+t-1} \frac{1 - z_p + z_p z_q}{z_q - z_p}
\]

and let \( R_{s,t}(z_1, \ldots, z_{s+t-1}) := \text{Sym} P_{s,t}(z_1, \ldots, z_{s+t-1}) \). If \( s \leq t \) then

\[
R_{s,t}(z_1, \ldots, z_i, \ldots, z_{s+t-1}) = R_{s,t}(z_1, \ldots, z_i-1, z_i^{-1}, z_i+1, \ldots, z_{s+t-1})
\]

for all \( i \in \{1, 2, \ldots, s + t - 1\} \).
Example: $s = 1, t = 3$

\[
P_{1,3} = z_1^{-2}z_2^{-2}(z_2 - 1)(z_3 - 1) \times \frac{(1 - z_1 + z_1z_2)(1 - z_1 + z_1z_3)(1 - z_2 + z_2z_3)}{(z_2 - z_1)(z_3 - z_1)(z_3 - z_2)} \]

\[
= 3 + z_1^{-2} - 4z_1^{-1} + z_2^{-2} + \ldots 32 \text{ terms} \ldots + z_2z_3^3 \frac{(z_2 - z_1)(z_3 - z_1)(z_3 - z_2)}{(z_2 - z_1)(z_3 - z_1)(z_3 - z_2)}
\]

\[
R_{1,3} = -3 + z_1 + z_1^{-1} + z_2 + z_2^{-1} + z_3 + z_3^{-1}
\]
Outline

- How did we come up with the conjecture: a refined enumeration of vertically symmetric alternating sign matrices.

- Partial result: it suffices to consider the cases $s = t$ and $s+1 = t!$

- Some remarks on the case $s = 0$. 
**ASM=Alternating Sign Matrix**

Quadratic $0,1,-1$ matrix such that in each row and each column

- the non–zero entries appear with alternating signs and

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

- the sum of entries is 1, that is the first and the last non-zero entry is a 1.

**VSASM=Vertically symmetric ASM:** $a_{i,j} = a_{i,n+1-j}$

**VSASMs**
- exist only for odd dimensions and
- the middle column is always $(1, -1, 1, -1, \ldots, -1, 1)^T$. 

Enumeration of VSASMs

Theorem (Kuperberg, 2002). The number of $(2n + 1) \times (2n + 1)$ VSASMs is

$$\prod_{i=1}^{n} \frac{(3i - 1)(2i - 1)!(6i - 3)!}{(4i - 2)!(4i - 1)!}.$$ 

Conjecture (F., 2009). The number $B_{n,i}$ of $(2n + 1) \times (2n + 1)$ VSASMs where the first 1 in the second row is in columns $i$ is

$$\frac{\binom{2n+i-2}{2n-1} \binom{4n-i-1}{2n-1}}{\binom{4n-2}{2n-1}} \prod_{j=1}^{n} \frac{(3j - 1)(2j - 1)!(6j - 3)!}{(4j - 2)!(4j - 1)!}.$$
Refined enumeration with respect to the first column

Theorem (Razumov, Stroganov, 2004). The number of \((2n + 1) \times (2n + 1)\) VSASMs where the first column’s unique 1 is located in row \(i\) is

\[
\prod_{j=1}^{n-1} \frac{(3j - 1)(2j - 1)!(6j - 3)!}{(4j - 2)!(4j - 1)!}
\times \sum_{r=1}^{i-1} (-1)^{i+r-1} \frac{(2n+r-2)}{2n-1} \frac{(4n-r-1)}{2n-1} \frac{2n-1}{4n-2} \frac{2n-1}{2n-1} =: B^*_n,i,
\]

\(i = 1, 2, \ldots, 2n + 1.\)

Relation: \(B_n,i = B^*_n,i + B^*_n,i+1, \quad i = 1, 2, \ldots, n\)
Bijective proof?

\[
\begin{array}{c}
i \\
1 \ -1 \\
1 \\
-1 \\
1
\end{array}
= \begin{array}{c}
i \ 1 \\
1 \\
-1 \\
1 \\
-1 \\
1
\end{array} + \begin{array}{c}
i+1 \ 1 \\
1 \\
-1 \\
1 \\
-1 \\
1
\end{array}
\]
Approach to attack the conjecture on the refined enumeration of VSASMs

Alternative proof of the Refined Alternating sign matrix theorem:
\( A_{n,i} = \# \text{ of } n \times n \text{ ASMs with } a_{1,i} = 1 \)

The vector \((A_{n,i})_{1 \leq i \leq n}\) is uniquely determined by the following linear equation system:

\[
A_{n,i} = \sum_{j=i}^{n} \binom{2n-i-1}{j-i} (-1)^{j+n} A_{n,j}, \quad i = 1, \ldots, n
\]

\[
A_{n,i} = A_{n,n+1-i}, \quad i = 1, \ldots, n
\]
Computer experiments suggest...

...that there is a similar linear equation system for $B_{n,i}$:

$$B_{n,n-i+1} = \sum_{j=i}^{n} \binom{3n - i - 1}{j - i} (-1)^{j+n} B_{n,n-j+1}, \quad -n + 1 \leq i \leq n,$$

$$B_{n,n-i+1} = B_{n,n+i}, \quad -n + 1 \leq i \leq n.$$  

But:  

$$(B_{n,n-i+1})_{-n+1 \leq i \leq n} = (B_{n,1}, \ldots, B_{n,n}, B_{n,n+1}, \ldots, B_{n,2n})$$

$i =$ position of the first 1 in the second row
• We have extended the combinatorial interpretation of $B_{n,i}$ to $i = n + 1, n + 2, \ldots, 2n$.

• In fact, we have two combinatorial extensions.

• If the conjecture on the symmetrized rational functions were true then we would know that the number of objects is the same for the two different combinatorial extensions...

• ...and this would conclude our proof of the refined enumeration of VSASMs.
First half of the linear equation system

Theorem (F., Riegler).

\[ C^{(d)}_{n,i} = \# \text{ of partial montone triangles of the following shape:} \]

Then

\[ C^{(d)}_{n,i+1} = \sum_{j=i}^{n} \binom{(d + 1)n - i - 1}{j - i} (-1)^{j + n} C^{(d)}_{n,j+1}, \quad i = 1, 2, \ldots, n. \]
Partial result

Recall the conjecture: \( P_{s,t}(z_1, \ldots, z_{s+t-1}) \) rational function, \( R_{s,t} = \text{Sym} P_{s,t} \) then

\[
R_{s,t}(z_1, \ldots, z_i, \ldots, z_{s+t-1}) = R_{s,t}(z_1, \ldots, z_i^{-1}, \ldots, z_{s+t-1})
\]

if \( 0 \leq s \leq t \).

However, to prove the formula for the refined enumeration of VSASMs, it suffices to show

\[
R_{s,t}(z_1, \ldots, z_{s+t-1}) = R_{s,t}(z_1^{-1}, \ldots, z_{s+t-1}^{-1})
\]

if \( 1 \leq s \leq t \).

We sketch the proof of the following result:

If the latter identity is true for \( t = s \) and \( t = s + 1 \) then it is true for all \( s, t \) with \( s \leq t \).
Two rational functions:

\[
S_{s,t}(z; z_1, \ldots, z_{s+t-2}) := z^{2s-t-1} \frac{\prod_{i=1}^{s+t-2} (1 - z + z_i z)(1 - z_i^{-1})}{(z_i - z)} (z_i - z),
\]

\[
T_{s,t}(z; z_1, \ldots, z_{s+t-2}) := (1 - z^{-1})^s z^{t-2} \frac{\prod_{i=1}^{s+t-2} 1 - z_i + z_i z}{(z - z_i) z_i}.
\]

Two operators \(PS_{s,t}, PT_{s,t}\) on functions \(f\) in \(s + t - 2\) variables:

\[
PS_{s,t}[f] := S_{s,t}(z_1; z_2, \ldots, z_{s+t-1}) \cdot f(z_2, \ldots, z_{s+t-1}),
\]

\[
PT_{s,t}[f] := T_{s,t}(z_{s+t-1}; z_1, \ldots, z_{s+t-2}) \cdot f(z_1, \ldots, z_{s+t-2}).
\]

Recursions:

\[
P_{s,t} = PS_{s,t}[P_{s-1,t}] \quad \text{and} \quad P_{s,t} = PT_{s,t}[P_{s,t-1}].
\]
Two related operators on functions in $s + t - 2$ variables:

$$QS_{s,t}[f] := S_{s,t}(z_{s+t-1}; z_{s+t-2}, z_{s+t-3}, \ldots, z_1) \cdot f(z_1, \ldots, z_{s+t-2}),$$
$$QT_{s,t}[f] := T_{s,t}(z_1^{-1}; z_{s+t-1}, z_{s+t-2}, \ldots, z_2^{-1}) \cdot f(z_2, \ldots, z_{s+t-1}).$$

We set $Q_{s,t}(z_1, \ldots, z_{s+t-1}) = P_{s,t}(z_{s+t-1}^{-1}, \ldots, z_1^{-1})$. The recursions from the previous transparency immediately imply

$$Q_{s,t} = QS_{s,t}[Q_{s-1,t}] \quad \text{and} \quad Q_{s,t} = QT_{s,t}[Q_{s,t-1}].$$

We have to show

$$\text{Sym } P_{s,t}(z_1, \ldots, z_{s+t-1}) = \text{Sym } Q_{s,t}(z_1, \ldots, z_{s+t-1}).$$
Consider words $w$ over the “operator-alphabet” $\mathcal{A} = \{PS, PT, QS, QT\}$ and depict them as labelled lattice paths with starting point in $(1,1)$, step set $\{(1,0), (0,1)\}$ and labels $P, Q$.

**Example:** $w = (PT, PS, QT, PT, QS, QT)$

![Lattice diagram](image)

The letters $PS, QS$ correspond to $(1,0)$ steps, while the letters $PT, QT$ correspond to $(0,1)$ steps.
The endpoint of the path is \((|w|_S, |w|_T)\), where

\[
|w|_S = \# \text{ of occurrences of } PS, QS + 1,
\]
\[
|w|_T = \# \text{ of occurrences of } PT, QT + 1.
\]

**Def.** To a word \(w\) of length \(n\), we assign a function \(F_w(z_1, \ldots, z_{n+1})\) as follows: For instance, if

\[
w = (PT, PS, QT, PT, QS, QT)
\]

then

\[
F_w(z_1, \ldots, z_7) = QT_{3,5} \circ QS_{3,4} \circ PT_{2,4} \circ QT_{2,3} \circ PS_{2,2} \circ PT_{1,2}[1],
\]

i.e. apply the operators in reverse order; the indices are the integer points of the lattice path (except for the starting point).

**Remark.**

- If \(w\) is a word over \(\{PS, PT\}\) then \(F_w = P|w|_S, |w|_T\).
- If \(w\) is a word over \(\{QS, QT\}\) then \(F_w = Q|w|_S, |w|_T\).
Swapping letters

Key Lemma.

1. $F_{w_1} = F_{w_2}$ if $w_1 = w_L PS PT w_R$ and $w_2 = w_L PT PS w_R$.

2. $F_{w_1} = F_{w_2}$ if $w_1 = w_L QS QT w_R$ and $w_2 = w_L QT QS w_R$.

3. $F_{w_1} = F_{w_2}$ if $w_1 = w_L PT QT w_R$ and $w_2 = w_L QT PT w_R$. 
We prove the following more general statement: suppose \( w_1, w_2 \) are two words whose labelled paths have the same endpoint and are both prefixes of (rotated) Dyck paths. Then

\[
\text{Sym} F_{w_1} = \text{Sym} F_{w_2}.
\]

Induction with respect to the length of the word; nothing to prove for the empty word.

**Case 1.** The last letters of \( w_1 \) and \( w_2 \) coincide. W.l.o.g. \( w_i = w'_i PS, \ i = 1, 2 \). Then

\[
\text{Sym} F_{w_i} = \text{Sym} PS_{s,t}[F_{w'_i}] = \text{Sym} S_{s,t}(z_1; z_2, \ldots, z_{s+t-1}) F_{w'_i}(z_2, \ldots, z_{s+t-1})
\]

\[
= \sum_{j=1}^{s+t-1} \sum_{\sigma \in S_n : \sigma(1) = j} S_{s,t}(z_j; z_1, \ldots, \widehat{z}_j, \ldots, z_{s+t-1}) F_{w'_i}(z_{\sigma(2)}, \ldots, z_{\sigma(s+t-1)})
\]

\[
= \sum_{j=1}^{s+t-1} S_{s,t}(z_j; z_1, \ldots, \widehat{z}_j, \ldots, z_{s+t-1}) \text{Sym} F_{w'_i}(z_1, \ldots, \widehat{z}_j, \ldots, z_{s+t-1})
\]

and, by the induction hypothesis, \( \text{Sym} F_{w'_1} = \text{Sym} F_{w'_2} \).
Case 2. The last letters of $w_1$ and $w_2$ differ.

Endpoint: $(|w_i|_S, |w_i|_T) =: (s, t)$

$t = s, s + 1$: use $\text{Sym } P_{s,s} = \text{Sym } Q_{s,s}$ and $\text{Sym } P_{s,s+1} = \text{Sym } Q_{s,s+1}$. 
$s + 1 < t$: The last letter of $w_i$, $i = 1, 2$, is w.l.o.g.in $\{PT, QT\}$: Suppose $w_i = w'_i PS$ and choose $w''_i$ such that the path of $w''_i PT$ has the same endpoint as the path of $w'_i$. By Case 1 and the Key Lemma,

$$\text{Sym } F_{w_i} \overset{\text{Def}}{=} \text{Sym } F_{w'_i PS} \overset{\text{C.1}}{=} \text{Sym } F_{w''_i PT PS} \overset{\text{K. L.}}{=} \text{Sym } F_{w''_i PS PT}.$$
W.l.o.g. $w_1 = w'_1 PT$ and $w_2 = w'_2 QT$. Choose $w''_1$ such that the path of $w''_1 QT$ has the same endpoint as the path of $w'_1$. By Case 1 and the Key Lemma,

$$\text{Sym } F_{w_1} \overset{\text{Def}}{=} \text{Sym } F_{w'_1 PT} \overset{\text{C.1}}{=} \text{Sym } F_{w''_1 QT PT} \overset{\text{K.L.}}{=} \text{Sym } F_{w''_1 PT QT} \overset{\text{C.1}}{=} \text{Sym } F_{w_2}.$$
Some remarks on the case $s = 0$

$$P_{0,n+1} = \prod_{1 \leq i < j \leq n} \frac{z_i^{-1} + z_j - 1}{1 - z_i z_j^{-1}}$$

**Question:** Are there also other rational functions $T(x, y)$ such that symmetrizing $\prod_{1 \leq i < j \leq n} T(z_i, z_j)$ leads to a Laurent polynomial that is invariant under replacing $z_i$ by $z_i^{-1}$?

**Computer experiments:**

$$T(x, y) = \frac{[a(x^{-1} + y) + c][b(x + y^{-1}) + c]}{1 - xy^{-1}} + abx^{-1}y + d, \ a, b, c, d \in \mathbb{C}.$$
Some special cases are easy...for instance:

\[ \text{Sym} \prod_{1 \leq i < j \leq n} \frac{z_i^{-1} + z_j}{1 - z_i z_j^{-1}} = \ldots \]

\[
= \prod_{1 \leq i < j \leq n} (1 + z_i z_j) \prod_{i=1}^{n} z_i^{-n+1} \text{Sym} \frac{\prod_{i=1}^{n} z_i^{2i-2}}{\prod_{1 \leq i < j \leq n} (z_j - z_i)} \]

\[
= \prod_{1 \leq i < j \leq n} (1 + z_i z_j) \prod_{i=1}^{n} z_i^{-n+1} \frac{\text{det} \left( \left( z_i^2 \right)^{j-1} \right)}{\prod_{1 \leq i < j \leq n} (z_j - z_i)} = \ldots \]

\[
= \prod_{1 \leq i < j \leq n} (1 + z_i z_j) (z_i + z_j) \prod_{i=1}^{n} z_i^{-n+1} \]
Two final theorems

\[ R_{0,n+1} = \text{Sym} \prod_{1 \leq i < j \leq n} \frac{z_i^{-1} + z_j - 1}{1 - z_i z_j} =: \text{VSASM}(1; z_1, \ldots, z_n) \prod_{i=1}^{n} z_i^{1-n+1} \]

**Computer experiments:** $R_{0,n+1}(1,1,\ldots,1)$ is the number of $(2n+1) \times (2n+1)$ VSASMs.

**Theorem.** Let $\text{VSASM}(X; z_1) = 1$ and, for $n > 1$,

\[
\text{VSASM}(X; z_1, \ldots, z_n) = \sum_{j=1}^{n} z_j^{2n-2} \prod_{1 \leq i \leq n, i \neq j} \frac{1 + z_i(X-2) + z_i z_j}{z_j - z_i} \times \text{VSASM}(X; z_1, \ldots, \hat{z_j}, \ldots, z_n). 
\]

Then the coefficient of $z_i^i X^j$ in $\text{VSASM}(X; z, 1, 1, \ldots, 1)$ is the number of $(2n+1) \times (2n+1)$ VSASMs with $a_{i,1} = 1$ and $j$ occurrences of $-1$ in the first $n$ columns.
Theorem. Let \( \text{ASM}(X; z_1) = 1 \) and, for \( n > 1 \),

\[
\text{ASM}(X; z_1, \ldots, z_n) = \sum_{j=1}^{n} z_j^{n-1} \prod_{1 \leq i \leq n, i \neq j} \frac{1 + z_i(X - 2) + z_i z_j}{z_j - z_i} \times \text{ASM}(X; z_1, \ldots, \hat{z_j}, \ldots, z_n).
\]

Then the coefficient of \( z_i^i X^j \) in \( \text{ASM}(X; z, 1, 1, \ldots, 1) \) is the number of \( n \times n \) ASMs with \( a_{1,i} = 1 \) and \( j \) occurrences of \(-1\).

To reprove the alternating sign matrix theorem, it would suffice to show that

\[
\text{ASM}(1; 1, 1, \ldots, 1) = \prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n + j)!}.
\]
Thank you!