New Littlewood-type identities and the sequence $1, 4, 60, 3328 \ldots$

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Outline

I. Unbounded Littlewood-type identities related to alternating sign matrices

II. Where do they come from: AST(Z)s and PPs

III. Combinatorial interpretation of the LHS

IV. Bounded identities

V. Combinatorial interpretation of the RHS

VI. $1, 4, 60, 3328, 678912 \ldots$

I. Unbounded Littlewood-type identities related to alternating sign matrices

The classical (unbounded) Littlewood identity

$$\sum_{\lambda} s_\lambda(X_1,\ldots,X_n) = \prod_{i=1}^{n} \frac{1}{1-X_i} \prod_{1\le i<j\le n} \frac{1}{1-X_iX_j},$$

Proof: RSK and exploiting its symmetry.

We rewrite the classical Littlewood identity:

$$s_{(\lambda_1,\ldots,\lambda_n)}(X_1,\ldots,X_n) = \frac{\det_{1\le i,j\le n}\left(X_i^{\lambda_j+n-j}\right)}{\prod_{1\le i<j\le n}(X_i-X_j)} = \frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{i=1}^{n} X_i^{\lambda_i+n-i}\right]}{\prod_{1\le i<j\le n}(X_i-X_j)},$$

with $\mathbf{ASym}_{X_1,\ldots,X_n} f(X_1,\ldots,X_n) = \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}\sigma \cdot f(X_{\sigma(1)},\ldots,X_{\sigma(n)})$

Change of variables: $\lambda_1 \ge \cdots \ge \lambda_n \ge 0 \Rightarrow \underbrace{\lambda_1+n-1}_{k_n} > \underbrace{\lambda_2+n-2}_{k_{n-1}} > \ldots > \underbrace{\lambda_n}_{k_1} \ge 0$

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\sum_{0\le k_1<k_2<\ldots<k_n} X_1^{k_1}X_2^{k_2}\cdots X_n^{k_n}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)} = \prod_{i=1}^{n} \frac{1}{1-X_i} \prod_{1\le i<j\le n} \frac{1}{1-X_iX_j}$$

Littlewood-type identity related to ASMs

In two of my papers from 2019:

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i<j\le n}(1+X_j+X_iX_j)\sum_{0\le k_1<k_2<\ldots<k_n} X_1^{k_1}X_2^{k_2}\cdots X_n^{k_n}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)}$$

$$=\prod_{i=1}^{n}\frac{1}{1-X_i}\prod_{1\le i<j\le n}\frac{1+X_i+X_j}{1-X_iX_j}$$

Since then Hans Höngesberg and I realized that we can introduce two additional parameters:

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i<j\le n}(Q+(Q+r)X_i+X_j+X_iX_j)\sum_{0\le k_1<k_2<\ldots<k_n}\prod_{i=1}^{n}\left(\frac{X_i(1+X_i)}{Q+X_i}\right)^{k_i}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)}$$

$$=\prod_{i=1}^{n}\frac{Q+X_i}{Q-X_i^2}\frac{\prod_{1\le i<j\le n}Q(1+X_i)(1+X_j)+rX_iX_j}{\prod_{1\le i<j\le n}(Q-X_iX_j)}$$

Set $Q=1$ and $r=-1$ to obtain the previous identity.

II. Where do they come from: AST(Z)s and PPs

Alternating sign triangles = ASTs

An AST of order n is a triangular array of 1's, -1's and 0's with n centered rows

$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \\ & & \bullet & \bullet & \bullet & & \\ & & & \bullet & & & \end{array}$$

such that

(1) the non-zero entries alternate in each row and each column,

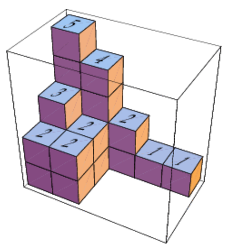
(2) all row sums are 1, and

(3) the topmost non-zero entry of each column is 1 (if such an entry exists).

Example:

$$\begin{array}{ccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 1 & -1 & 1 & 0 & 0 & \\ & & 1 & -1 & 1 & & \\ & & & 1 & & & \end{array}$$

Totally symmetric self-complementary plane partitions = TSSCPPs



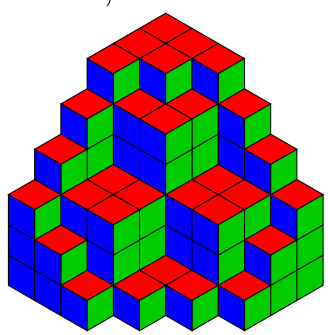
$a = 4, b = 3, c = 5$

A (boxed) plane partition in an $a \times b \times c$ box is a subset

$$PP \subseteq \{1,2,\ldots,a\} \times \{1,2,\ldots,b\} \times \{1,2,\ldots,c\}$$

with

$$(i,j,k) \in PP \Rightarrow (i',j',k') \in PP \quad \forall (i',j',k') \leq (i,j,k).$$



- **Totally symmetric:**
  $(i,j,k) \in PP \Rightarrow \sigma(i,j,k) \in PP \; \forall \sigma \in \mathcal{S}_3$
  (MacMahon 1899, 1915/16)

- **Self-complementary:**
  Equal to its complement in the $2n \times 2n \times 2n$ box
  (Mills, Robbins and Rumsey 1986)

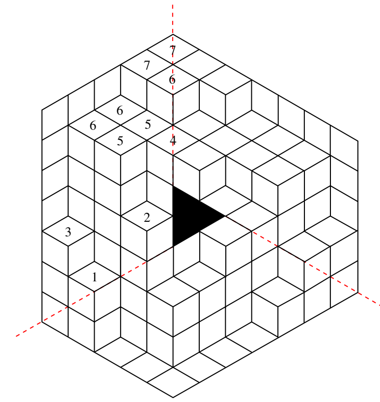
Now: "Our" first Littlewood-type identity was the crucial point in showing that there is the same number of ASTs with n rows as there is of TSSCPPs in a $2n \times 2n \times 2n$ box.

Second application: ASTZs and DPPs

0	0	0	0	0	0	0	1	0	0	0	0
	0	0	0	0	1	0	−1	1	0	0	
		0	1	0	−1	0	1	−1	1		
			0	0	0	1	−1	1			
				1	0	−1	1				

alternating sign trapezoids

$\Leftrightarrow$



cyclically symmetric lozenge tilings of a hexagon with a central triangular hole

- Central hole has size 2 $\rightarrow$ descending plane partition (DPP)

- The Q is necessary to take care of the numbers of -1's in the alternating sign trapezoids.

All proofs of these relations between alternating sign arrays and plane partition objects are very complicated. One of my motivations to study these Littlewood-type identities is to improve the combinatorial understanding of the relations.

III. Combinatorial interpretation of the LHS

Gelfand-Tsetlin patterns

A Gelfand-Tsetlin pattern is a triangular array of integers of the form

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & & a_{2,2} & & \\ & \cdots & & \cdots & & \cdots & \\ a_{n,1} & & \cdots & & \cdots & & a_{n,n} \end{array}$$

with weak increase in $\nearrow$- and $\searrow$-direction.

The weight of a Gelfand-Tsetlin pattern is $\prod_{i=1}^{n} X_i^{\sum_j a_{i,j} - \sum_j a_{i-1,j}}$ and $s_\lambda(X_1, \ldots, X_n)$ is the sum of weights of all Gelfand-Tsetlin patterns with bottom row $(0, \ldots, 0, \lambda_l, \ldots, \lambda_1)$.

Example:

$$\begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & 3 & & 5 & & & \\ & & 2 & & 4 & & 6 & & \\ & 1 & & 3 & & 4 & & 7 & \\ 1 & & 1 & & 5 & & 7 & & 8 \end{array}$$

Arrowed Gelfand-Tsetlin patterns

An arrowed Gelfand-Tsetlin pattern is a Gelfand-Tsetlin pattern where each entry is decorated with an element from $\{\nwarrow, \nearrow, \nwarrow\!\!\!\!\!\nearrow, \emptyset\}$ such that for the little triangles in the pattern

$$\begin{array}{ccc} & y & \\ x & & z \end{array}$$

we have the following:

- If $x = y$ and $\text{decor}(x) \in \{\nearrow, \nwarrow\!\!\!\!\!\nearrow\}$, then $z = y = x$ and $\text{decor}(z) \in \{\nwarrow, \nwarrow\!\!\!\!\!\nearrow\}$, and
- if $y = z$ and $\text{decor}(z) \in \{\nwarrow, \nwarrow\!\!\!\!\!\nearrow\}$, then $x = y = z$ and $\text{decor}(x) \in \{\nearrow, \nwarrow\!\!\!\!\!\nearrow\}$.

Both instances contribute -1 to the sign.

Summary: Arrows between diagonal neighbours indicate that the entries are different, except when we have two such occurrences appearing in a little triangle. In this case, we have a contribution of -1 to the sign.

Example:

$$\begin{array}{ccccccccc}
 & & & & \nwarrow 3 \nearrow & & & & \\
 & & & 3 & & \nwarrow 5 & & & \\
 & & 2\nearrow & & \nwarrow 4 & & 6 & & \\
 & 1\nearrow & & \color{red}{3\nearrow} & & 4\nearrow & & \nwarrow 7 \nearrow & \\
 \nwarrow 1 & & \color{red}{3\nearrow} & & \color{red}{\nwarrow 3 \nearrow} & & 7 & & \nwarrow 8
\end{array}$$

Generating function

We associate the following **weight** to a given arrowed Gelfand-Tsetlin pattern $A = (a_{i,j})_{1 \le j \le i \le n}$:

$$\mathrm{W}(A) = \mathrm{sgn}(A) \cdot t^{\#\emptyset} u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\!\nearrow} \prod_{i=1}^{n} X_i^{\sum_{j=1}^{i} a_{i,j} - \sum_{j=1}^{i-1} a_{i-1,j} + \#\nearrow \text{in row } i - \#\nwarrow \text{in row } i}$$

The weight of our example is

$$-t^3 u^5 v^3 w^3 X_1^3 X_2^4 X_3^4 X_4^6 X_5^6.$$

Compare to the Schur function weight for Gelfand-Tsetlin patterns!

Theorem (F., Schreier-Aigner). The generating function of arrowed Gelfand-Tsetlin patterns with bottom row $k_1, \ldots, k_n$ is

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1 \le i \le j \le n} (v + wX_i + tX_j + uX_iX_j) \prod_{i=1}^{n} X_i^{k_i - 1}\right]}{\prod_{1 \le i < j \le n} (X_j - X_i)}.$$

Application to our LHS

Our Littlewood-type identity, slightly rewritten:

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i\le j\le n}(1+wX_i+X_j+X_iX_j)\sum_{0\le k_1<k_2<\ldots<k_n}X_1^{k_1-1}X_2^{k_2-1}\cdots X_n^{k_n-1}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)}$$
$$=\prod_{i=1}^{n}\frac{X_i^{-1}+(1+w)+X_i}{1-X_i}\prod_{1\le i<j\le n}\frac{1+X_i+X_j+wX_iX_j}{1-X_iX_j}$$

The left-hand side is the generating function of all arrowed Gelfand-Tsetlin patterns with strictly increasing bottom row of non-negative integers when setting $t=u=v=1$.

IV. Bounded identities

Bounded classical Littlewood identity

Bounded? $\sum_{0 \le k_1 < k_2 < \ldots < k_n} \to \sum_{0 \le k_1 < k_2 < \ldots < k_n \le m}$

$$\sum_{\lambda \subseteq (m^n)} s_\lambda(X_1, \ldots, X_n) = \frac{\det_{1 \le i,j \le n} \left(X_i^{j-1} - X_i^{m+2n-j} \right)}{\prod_{i=1}^n (1 - X_i) \prod_{1 \le i < j \le n} (X_j - X_i)(1 - X_i X_j)}$$

Macdonald in his book.

Bounded Littlewood identity related to ASMs

$$\frac{1}{\prod_{1\le i<j\le n}(X_j-X_i)}\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i\le j\le n}(Q+(Q+r)X_i+X_j+X_iX_j)\right.$$
$$\left.\times\sum_{0\le k_1<k_2<\ldots<k_n\le \mathbf{m}}\left(\frac{X_1(1+X_1)}{Q+X_1}\right)^{k_1}\left(\frac{X_2(1+X_2)}{Q+X_2}\right)^{k_2}\cdots\left(\frac{X_n(1+X_n)}{Q+X_n}\right)^{k_n}\right]$$
$$=\frac{\det_{1\le i,j\le n}\left(a_{j,m,n}(Q,r;X_i)\right)}{\prod_{1\le i\le j\le n}(Q-X_iX_j)\prod_{1\le i<j\le n}(X_j-X_i)}$$

with

$$a_{j,m,n}(Q,r;X)=(1+QX^{-1})X^j(1+X)^{j-1}(Q+rX+QX)^{n-j}$$
$$-X^{2n}Q^{-n}\left(\frac{(1+X)X}{Q+X}\right)^m(1+X)\left(QX^{-1}\right)^j(1+QX^{-1})^{j-1}(Q+rQX^{-1}+Q^2X^{-1})^{n-j}.$$

The proof has more than 7 pages, but it is elementary.

The case $Q = 1$

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1 \le i \le j \le n}(1 + wX_i + X_j + X_iX_j)\sum_{0 \le k_1 < k_2 < \ldots < k_n \le m} X_1^{k_1-1}X_2^{k_2-1}\cdots X_n^{k_n-1}\right]}{\prod_{1 \le i < j \le n}(X_j - X_i)}$$

$$= \prod_{i=1}^{n}(X_i^{-1} + 1 + w + X_i)$$

$$\times \frac{\det_{1 \le i,j \le n}\left(X_i^{j-1}(1+X_i)^{j-1}(1+wX_i)^{n-j} - X_i^{m+2n-j}(1+X_i^{-1})^{j-1}(1+wX_i^{-1})^{n-j}\right)}{\prod_{i=1}^{n}(1-X_i)\prod_{1 \le i < j \le n}(1-X_iX_j)(X_j - X_i)}.$$

LHS: Generating function of arrowed Gelfand-Tsetlin patterns with strictly increasing bottom that are bounded by m.

What about the RHS ?

V. Combinatorial interpretation of the RHS

The classical case

The classical bounded Littlewood identity:

$$\sum_{\lambda \subseteq (m^n)} s_\lambda(X_1, \ldots, X_n) = \frac{\det_{1 \le i,j \le n}\left(X_i^{j-1} - X_i^{m+2n-j}\right)}{\prod_{i=1}^n (1 - X_i) \prod_{1 \le i < j \le n} (X_j - X_i)(1 - X_i X_j)}$$

This identity is equivalent to

$$\sum_{\lambda \subseteq (m^n)} s_\lambda(X_1, \ldots, X_n) = \prod_{i=1}^n X_i^{m/2} so^{\text{odd}}_{(m/2, m/2, \ldots, m/2)}(X_1, \ldots, X_n),$$

where $so^{\text{odd}}_\lambda(X_1, \ldots, X_n)$ is the irreducible character of the special orthogonal group $SO_{2n+1}(\mathbb{C})$ associated with the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$.

Now: $so^{\text{odd}}_\lambda(X_1, \ldots, X_n)$ is the generating function of certain **halved** Gelfand-Tsetlin patterns.

Combinatorial interpretation of the RHS for $w = 0$ and m large

One needs to distinguish between the cases that m is odd or even.

Theorem (F., 2022). Assume that $w = 0$ and $m = 2l + 1$. In case $l \geq n - 2$, the RHS is the generating function of pairs of plane partitions (P, Q) of shape λ and μ, respectively, where

- μ is the complement of λ in the $n \times l$-rectangle,
- P is a column-strict plane partition such that the entries in the i-th row are bounded by $2n+2-2i$, and
- Q is a row-strict plane partition such that the entries in the i-th row are bounded by $n - i$.

The weight is

$$\prod_{i=1}^{n-1} X_i^l (X_i^{-1} + 1 + X_i)(1 + X_i) X_i^{\#\text{of } 2i-1 \text{ in P}} X_i^{-\#\text{of } 2i \text{ in P}}.$$

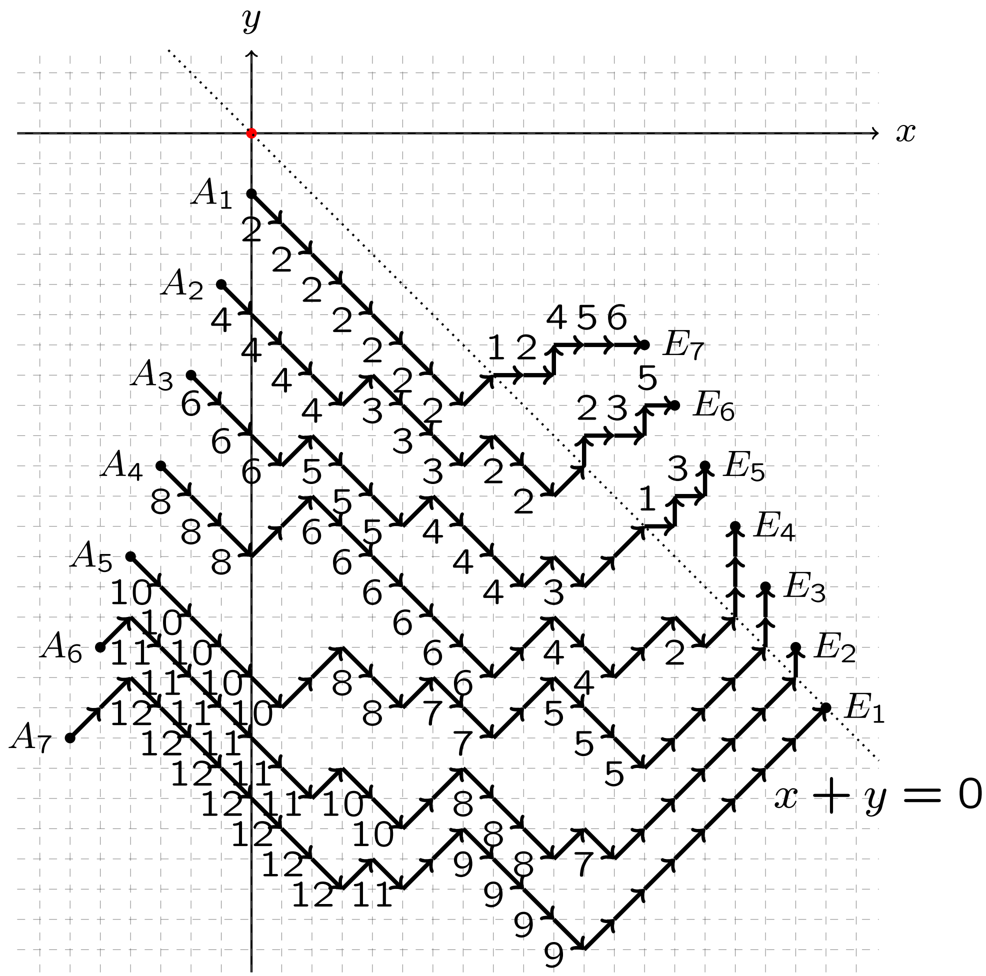
Remark.

- The Q's are in easy bijection with $2n \times 2n \times 2n$ TSSCPPs.
- The P's are in easy bijection with symplectic tableaux.

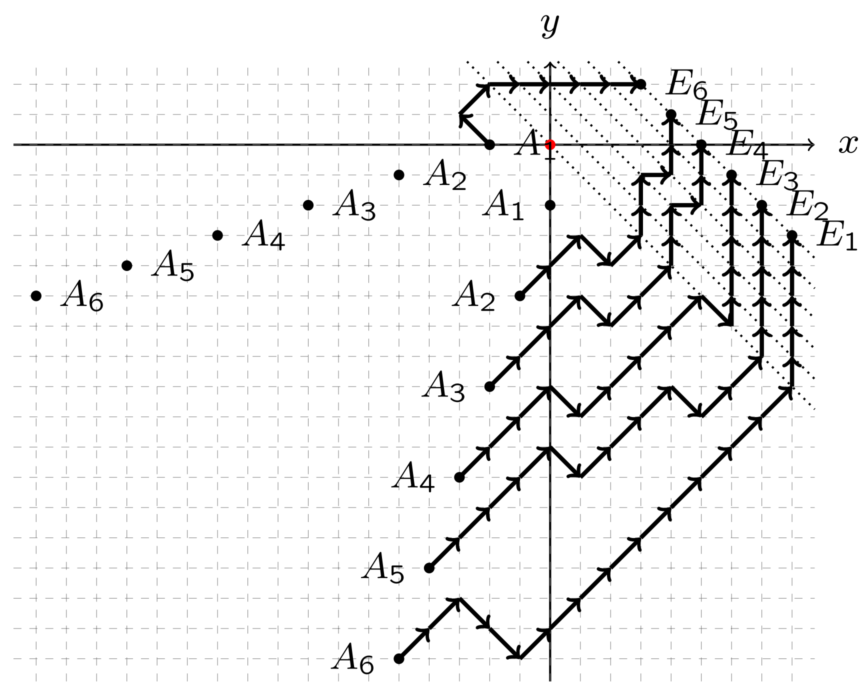
Example $n = 7$ and $l = 12$

$$\left(\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
12 & 12 & 12 & 12 & 12 & 12 & 12 & 11 & 9 & 9 & 9 & 9 \\
\hline
11 & 11 & 11 & 11 & 11 & 11 & 10 & 10 & 8 & 8 & 8 & 7 \\
\hline
10 & 10 & 10 & 10 & 10 & 8 & 8 & 7 & 7 & 5 & 5 & 5 \\
\hline
8 & 8 & 8 & 6 & 6 & 6 & 6 & 6 & 6 & 4 & 4 & 2 \\
\hline
6 & 6 & 6 & 5 & 5 & 5 & 4 & 4 & 4 & 3 \\
\cline{1-10}
4 & 4 & 4 & 4 & 3 & 3 & 3 & 2 & 2 \\
\cline{1-9}
2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\cline{1-7}
\end{array}\,,\;
\begin{array}{|c|c|c|c|c|}
\hline
6 & 5 & 4 & 2 & 1 \\
\hline
5 & 3 & 2 \\
\cline{1-3}
3 & 1 \\
\cline{1-2}
\end{array}\right)$$

Translated into non-intersecting lattice paths



The general picture for m is odd



It is a signed enumeration!

VI. $\mathbf{1, 4, 60, 3328, 678912, \ldots}$

$\mathbf{1, 4, 60, 3328, 678912 \ldots}$

RHS of the new Littlewood-type identity for $Q = 1$:

$$\frac{\det_{1 \le i,j \le n} \left(X_i^{j-1} (1+X_i)^{j-1} (1+wX_i)^{n-j} - X_i^{m+2n-j} (1+X_i^{-1})^{j-1} (1+wX_i^{-1})^{n-j} \right)}{\prod_{i=1}^{n} (1-X_i) \prod_{1 \le i<j \le n} (1-X_i X_j)(X_j - X_i)}$$

Setting all $X_i = 1, w = -1$ and $m = n-1$, we obtain

$$1, 4, 60, 3328, 678912, \ldots = 2^{n(n-1)/2} \prod_{j=0}^{n-1} \frac{(4j+2)!}{(n+2j+1)!}.$$

- This is a consequence of our Theorems 1 and 2 below.

- In fact, these theorems involve the **additional parameter** m, and the special case $m = n-1$ is an unpublished conjecture of Florian Schreier-Aigner from 2018.

- Note that $m = n-1$ just means that we consider arrowed Gelfand-Tsetlin patterns with bottom row $(0, 1, \ldots, n-1)$.

These numbers also appear in recent work of Philippe Di Francesco related to the **twenty vertex model** and **domino tilings.** He conjectured the following theorem of which I saw a proof last week by Christoph Koutschan.

Theorem.

$$\det_{0\le i,j\le n-1}\left(2^i\binom{i+2j+1}{2j+1}-\binom{i-1}{2j+1}\right)=2^{n(n-1)/2}\prod_{j=0}^{n-1}\frac{(4j+2)!}{(n+2j+1)!}.$$

- Christian Krattenthaler has found a conjectural generalization.
- Michael Schlosser has found several variations.

Explicit product formulas in case $X_i = 1, w = -1$ and $m = 2l+1$

Theorem 1 (F., Schreier-Aigner, 2022). For $(X_1, \ldots, X_n) = (1, \ldots, 1), w = -1$ and $m = 2l+1$, we have that

$$\frac{\det_{1 \le i,j \le n}\left(X_i^{j-1}(1+X_i)^{j-1}(1+wX_i)^{n-j} - X_i^{m+2n-j}(1+X_i^{-1})^{j-1}(1+wX_i^{-1})^{n-j}\right)}{\prod_{i=1}^{n}(1-X_i)\prod_{1\le i<j\le n}(1-X_iX_j)(X_j-X_i)}$$

is, for n is odd, equal to

$$\frac{1}{3^{n-1}2^{\frac{(n-1)(n-2)}{2}}\prod_{i=1}^{n-1}(5/2)_{i-1}}$$
$$\times (2l+n+1)\prod_{i=0}^{(2n-3)/8}(2l-n+3+4i)_{2n-3-8i}\prod_{i=0}^{(2n-4)/8}(2l+n+5+4i)_{2n-4-8i}\prod_{i=0}^{(n-5)/4}(2l+n-3-4i),$$

and, for n is even, it is equal to

$$\frac{1}{3^{n-1}2^{\frac{(n-1)(n-2)}{2}}\prod_{i=1}^{n-1}(5/2)_{i-1}}$$
$$\times (2l+n+4)\prod_{i=0}^{(2n-2)/8}(2l-n+3+4i)_{2n-2-8i}\prod_{i=0}^{(2n-5)/8}(2l+n+6+4i)_{2n-5-8i}\prod_{i=0}^{(n-6)/4}(2l+n+8+4i)$$

with $(a)_n = 1$ if $n < 0$.

Explicit product formulas in case $X_i = 1, w = -1$ and $m = 2l$

Theorem 2 (F., Schreier-Aigner, 2022). For $(X_1, \ldots, X_n) = (1, \ldots, 1), w = -1$ and $m = 2l$, we have that

$$\frac{\det_{1\le i,j\le n}\left(X_i^{j-1}(1+X_i)^{j-1}(1+wX_i)^{n-j} - X_i^{m+2n-j}(1+X_i^{-1})^{j-1}(1+wX_i^{-1})^{n-j}\right)}{\prod_{i=1}^{n}(1-X_i)\prod_{1\le i<j\le n}(1-X_iX_j)(X_j-X_i)}$$

is, for n is odd, equal to

$$\frac{1}{3^{n-1}2^{\frac{(n-1)(n-2)}{2}}\prod_{i=1}^{n-1}(5/2)_{i-1}}$$
$$\times (2l+n)\prod_{i=0}^{(2n-3)/8}(2l-n+2+4i)_{2n-3-8i}\prod_{i=0}^{(2n-4)/8}(2l+n+4+4i)_{2n-4-8i}\prod_{i=0}^{(n-5)/4}(2l+n-4-4i),$$

and, for n is even, it is equal to

$$\frac{1}{3^{n-1}2^{\frac{(n-1)(n-2)}{2}}\prod_{i=1}^{n-1}(5/2)_{i-1}}$$
$$\times (2l+n+3)\prod_{i=0}^{(2n-2)/8}(2l-n+2+4i)_{2n-2-8i}\prod_{i=0}^{(2n-5)/8}(2l+n+5+4i)_{2n-5-8i}\prod_{i=0}^{(n-6)/4}(2l+n+7+4i).$$

Thank you!

Sketch of proof of Theorem 1

After transforming the bialternant-type formula into a Jacobi-Trudi-type formula (in which we can then easily set $X_i = 1$), it turns out that we need to compute

$$\det_{1\le i,j\le n}\left(\sum_p \binom{n-j}{p}\binom{l-p+i}{2i-j}\right).$$

We could guess the LU-decomposition.

Let

$$x_{i,j} = \begin{cases} (-1)^{i+1}\dfrac{(j)_j}{(2l-n+3j+2)_{j-1}(2l-n+i+2)_j} \\ \quad\times \displaystyle\sum_t 2^{2i-4t-n}\frac{(i-j-2t+1)_{2t}(i-2j+1)_{j-1-t}(l-n/2+j/2+t+3/2)_{i-2t-1}}{(1)_t(1)_{i-2t-1}} & i\le j \\ 0 & \text{otherwise} \end{cases}.$$

Setting $a_{i,j} = \sum_p \binom{n-j}{p}\binom{l-p+i}{2i-j}$, we could prove that $(a_{i,j})_{1\le i,j\le n}\cdot(x_{i,j})_{1\le i,j\le n}$ is a lower triangular matrix with 1's on the main diagonal.

A triple sum

It suffices to show that

$$\sum_{\substack{s,t\\k\le j}}(-1)^{1+k+s+t}2^{2-k-s}$$

$$\times\frac{(k-n)_s(-r)_{k-1+2t}(-1/2-i+2k-n/2-r/2+s)_{2i-k-s}(2-2j+r)_{j-1-2t}(j-t)_{j-t-1}}{(1)_{2i-k-s}(1)_s(1)_{k-1-2t}(1)_t}$$

$$=\begin{cases}\frac{(-r)_{2j-1}(-1+3j-r)_{j-1}}{(j)_j}, & i=j,\\ 0, & i<j\end{cases}.$$

We use Sister Celine's algorithm, which is implemented in the MultiSum package from RISC. It turned out that the case $j\ge 2i$ is easier to deal with and this serves as our induction hypothesis.

Now let $f(n,r,i,j,k,s,t)$ denote the summand of the tripe sum. Then MultiSum found the following recursion.

$$4(3j-r-2)(r+1)_4 f(n,r,i,j,k,s,t)=$$
$$2(2j+1)(2j-r-3)_2 f(n+2,r+4,i+1,j+1,k+2,s,t+1)+(j-r-3)f(n+2,r+4,i+1,j+2,k+2,s,t+1)$$

Given its simplicity, it is not difficult to prove it without computer algebra. This is the main ingredient in the proof of the tripe sum.