Quantifier elimination for o-minimal structures expanded by a valuational cut

Clifton Ealy and Jana Maříková

Dept. of Mathematics and Philosophy, WIU

June 15, 2020

Abstract

Let R be an o-minimal expansion of a group in a language in which $\mathrm{Th}(R)$ eliminates quantifiers, and let C be a predicate for a valuational cut in R. We identify a condition that implies quantifier elimination for $\mathrm{Th}(R, C)$ in the language of R expanded by C and a small number of constants, and which, in turn, is implied by $\mathrm{Th}(R, C)$ having quantifier elimination and being universally axiomatizable. The condition applies for example in the case when C is a convex subring of an o-minimal field R and its residue field is o-minimal.

1 Introduction

Throughout, we let R be an o-minimal expansion of a group in a language $\mathcal{L}_0$ in which $\mathrm{Th}(R)$ eliminates quantifiers. We expand our language with a unary predicate C used to define a valuational cut in R. Such a structure, (R, C), is an instance of a valuational weakly o-minimal structure. Weakly o-minimal structures were first introduced by Dickmann [5]. In [10], Macpherson, Marker, Steinhorn proved a basic dichotomy: Every weakly o-minimal structure is either valuational or non-valuational. While the non-valuational ones are rather well-understood (see for example Bar-Yehuda, Hasson, Peterzil [3]), and have been shown to share many desirable properties with o-minimal structures, valuational weakly o-minimal structures have been, at

1

least at a certain level of generality, less well-explored.[1]

The main result of this paper says that if C is a cut in a model R, and C has the property $*$ (see Definition 2.3), then $Th(R, C)$ eliminates quantifiers in $\mathcal{L} := \mathcal{L}_0 \cup \{C\}$ expanded by a small number of constants. We note that C has property $*$ for example when R is an o-minimal expansion of a field and C is a convex subring of R such that its residue field with structure induced from R (equivalently, from (R, C)) is o-minimal (see Appendix A). However, it is not true, in the case when C is a convex subring of R, that C having property $*$ would be equivalent to the residue field of (R, C) being o-minimal (see Example 2.8). Neither is it the case that C has property $*$ whenever C is a convex subring of R (see Example 2.9).

Here is some context. Originally, our main interest was in the case when R expands a field and C is a convex subring. A particularly well-behaved class of such structures are the T-convex structures, as introduced by van den Dries and Lewenberg in [6]. But even when one relaxes the T-convexity condition to the strictly weaker condition of having an o-minimal residue field, one maintains some desirable properties such as first-order axiomatizability (see Maříková [12]), and model-completeness of (R, C) in an expansion of $\mathcal{L}$ by a small number of constants (see Ealy, Maříková [8]). Later, the second author noticed (in unpublished notes) that the proof in [8] can rather easily be extended to yield quantifier elimination. The question remained whether the assumption of o-minimality of the residue field is necessary. We show here that it can be replaced by the more general condition $*$.

In [8], to prove model completeness, one first establishes Theorem 3.3, p. 244, a criterion for elementary extensions of (R, C). While [8] assumes that R expands a field, C is a convex subring, in this paper we show that these hypotheses may be replaced by the hypotheses that R expands a group and C defines a valuational cut. The remainder of the proof in [8] relies heavily on the assumption of o-minimality of the residue field and is here replaced by an entirely different approach. We also provide a much simplified proof of (an analog of) Theorem 3.3, p.244 [8], and correct some errors from [8].

In particular, Fact 1.6, p.236 [8] is misstated: it holds for cuts, but not for non-cuts; in [8], we only apply it to cuts. More seriously, we claim to have proven that what we call property $*$ in this paper always holds (Theorem 2.7, p.239, [8]). It does hold where we use it to establish model completeness

---

[1]In [13], Laskowski, Shaw claim to prove the existence of definable Skolem functions for structures (R, C), but there is a gap in the proof.

(see Appendix), but it is not always true, as the examples in the next section show. In total, Corollary 2.9, Lemma 3.1, Lemma 3.2, and Theorem 3.3 of [8] require property $*$, and Corollary 3.4 requires the hypothesis of either property $*$ or power-boundedness (see Appendix).

Our notation and set-up are as follows. We let $\mathbb{N} = \{0, 1, \dots\}$. The variables $i, j, k, l, m, n, \dots$ range over $\mathbb{N}$, and, more generally, $\alpha, \beta, \gamma, \dots$ denote ordinals.

Definable shall mean definable with parameters, unless indicated otherwise. If M is a structure, then sometimes we write M-*definable* to mean definable in the structure M with parameters from M. Let M be an o-minimal structure. If $M \preceq \mathcal{M}$ and $a \in \mathcal{M} \setminus M$, then we denote by $M\langle a \rangle$ the structure generated by a in $\mathcal{M}$ over M (note that then $M \preceq M\langle a \rangle \preceq \mathcal{M}$).

We follow Marker, Steinhorn [11] (and many others) in saying that a type $q \in S_1(M)$ is a *cut* if there are nonempty disjoint subsets C^- and C^+ of M such that $C^- < C^+$, $C^- \cup C^+ = M$, C^- has no supremum in R, and q consists of the formulas $c < x$ for all $c \in C^-$ and $x < c$ for all $c \in C^+$. (Thus, cuts are precisely the nondefinable 1-types.) We call a cut in M *valuational* if there is $\epsilon \in M^{>0}$ such that C^- is closed under addition by ϵ.

As stated above, we let R be an o-minimal expansion of a group in a language $\mathcal{L}_0$. We assume that $\mathcal{L}_0$ is such that $T_0 = \mathrm{Th}(R)$ eliminates quantifiers in $\mathcal{L}_0$, and that $\mathcal{L}_0$ and T_0 have been expanded by definitions as follows: for each $\mathcal{L}_0$-formula $\phi(x_1, \dots, x_n, y)$ such that

$$T_0 \vdash \forall x_1 \dots \forall x_n \, \exists! y \, \phi(x_1, \dots, x_n, y)$$

we add a new function symbol f to $\mathcal{L}_0$ and the axiom

$$\phi(x_1, \dots, x_n, f(x_1, \dots, x_n))$$

to the theory T_0. Then T_0 is universally axiomatizable.

By C we shall denote a unary predicate for a convex, downward closed set such that the type

$$\{x < r : r \in C\} \cup \{x < r : r > C\}$$

is a cut. We write $C(M)$ for the realization of C in a structure M. We consider (R, C) in the language $\mathcal{L} = \mathcal{L}_0 \cup \{C\}$, and we use T to denote the theory of (R, C) (in the language $\mathcal{L}$). We work inside a monster model $(\mathcal{R}, C)$

3

of T. In particular, if $M \preceq \mathcal{R}$, then in (M, C) the predicate C is realized by the set $C(\mathcal{R}) \cap M$.

Finally, for $A \subseteq \mathcal{R}$, by $S_1(A)$ we mean the 1-types in the (o-minimal!) language $\mathcal{L}_0$ over A.

We will repeatedly use the following properties of nondividing which hold in all theories (see [1] Lemma 5.2 for a proof):

- (monotonicity) If $A \underset{C}{\overset{d}{\downarrow}} B$, $A_0 \subseteq A$ and $B_0 \subseteq B$, then $A_0 \underset{C}{\overset{d}{\downarrow}} B_0$.

- (base monotonicity) Suppose $D \subseteq C \subseteq B$. If $A \underset{D}{\overset{d}{\downarrow}} B$, then $A \underset{C}{\overset{d}{\downarrow}} B$.

- (left transitivity) Suppose $D \subseteq C \subseteq B$. If $B \underset{C}{\overset{d}{\downarrow}} A$ and $C \underset{D}{\overset{d}{\downarrow}} A$, then $B \underset{D}{\overset{d}{\downarrow}} A$.

Furthermore, we will use that forking and dividing are the same in o-minimal theories. (See the remarks preceding Proposition 2.8 together with Corollary 5.6 in [4] for a proof that forking is the same as dividing in weakly o-minimal theories). This allows us to also use the following:

- (extension) If $A \underset{C}{\downarrow} B$ and $\hat{B} \supseteq B$, then there is $A' \equiv_{BC} A$ such that $A' \underset{C}{\downarrow} \hat{B}$.

Finally, we note a useful, but sometimes overlooked, observation about o-minimal structures.

Fact 1.1 *Let $p \in S_1(R)$ be non-isolated. Then there is no $\mathcal{L}_0$-definable over R decreasing function mapping the set of realizations of p in $\mathcal{R}$ to itself.*

PROOF: A decreasing function with the above property would have a fixed point a realizing p. But then $a \in \mathrm{dcl}(R)$, a contradiction with p including $\{x \neq r | r \in R\}$. □

2 The (non)-symmetry of forking in Morley sequences in invariant one-types in o-minimal structures

In [8], it is claimed that, in o-minimal expansions of groups, forking in Morley sequences in invariant one-types is symmetric (p. 239, Theorem 2.7). This result is used in the proof of Theorem 3.3, p. 244, [8], whose analog we wish to establish here. We refer the reader to [8] for the necessary definitions.

Unfortunately, the proof of Theorem 2.7, [8] has a gap. In fact, the theorem is false in the stated generality, as follows from Proposition 2.5. (Note: In this paper, cuts for which Theorem 2.7 of [8] holds are said to satisfy $*$.) While this does not pose a problem for the model completeness result in [8] (Theorem 2.7 does hold in the context of [8], namely, when V is a convex subring of an o-minimal field R such that the corresponding residue field is o-minimal – see Appendix), here we need to bridge the gap in the proof of Theorem 2.7. We do this by additionally assuming that the cut C has property $*$ (see Definition 2.3).

Definition 2.1 *Given an A-invariant type p over $\mathcal{R}$, and $B \supseteq A$, a Morley sequence in p over B is any sequence $t_1, t_2, \ldots$ in $\mathcal{R}$ constructed as follows: let $t_1 \models p|_B$ and having defined $t_1, \ldots, t_n$, let $t_{n+1} \models p|_{Bt_1 \ldots t_n}$.*

Note that in a Morley sequence over B, $t_i \underset{B}{\cancel{\smile}} t_1 \ldots t_{i-1}$ and moreover the sequence is indiscernible over B.

Proposition 2.2 *Let $p \in S_1(\mathcal{R})$ be the global type $C(R) < x$ and $x < \mathcal{R}^{>C(R)}$, and let $t_1, \ldots, t_n$ form a finite Morley sequence in p over R. Let $q \in S_1(\mathcal{R})$ be the global type implied by $r < x$ for all $r \in \mathcal{R}$ with $r < R^{>C(R)}$, and $x < R^{>C(R)}$ (i.e. the R-invariant type at the opposite side of the cut from p). Then the following are equivalent:*

1. *There is no function f, $\mathcal{L}_0$-definable over $Rt_2t_3 \ldots t_{n-1}$, which maps a cofinal sequence in $C(R)$ onto a coinitial sequence in the convex hull of $R^{>C(R)}$ in $\mathcal{R}$.*

2. *$t_1 \underset{R}{\cancel{\smile}} t_2 \ldots t_n$.*

3. *$t_n, \ldots, t_1$ is a finite Morley sequence in $q \in S_1(\mathcal{R})$ over R.*

5

PROOF: First note that for $n = 2$, (1) is always true by Fact 1.1, as an $\mathcal{L}_0$-definable over R function mapping a cofinal segment of $C(R)$ to a coinitial segment of the convex hull of $R^{>C(R)}$ in $\mathcal{R}$ would be a decreasing $\mathcal{L}_0$-definable over R function mapping a complete type over R to itself. Also, it is always the case that $t_1 \underset{R}{\downarrow} t_2$ and $t_1 \models q|_{Rt_2}$, since if either of these failed to be true, there would be some $\mathcal{L}_0$-definable over R function f with $f(t_2) > t_1$. As already noted, such a function cannot be decreasing. But nor can it be increasing, as $f^{-1}(t_1)$ would be less than t_2, contradicting the choice of t_2.

So we will assume $n > 2$, and set $\underline{t} = t_2, \ldots, t_{n-1}$.

3. $\implies$ 2. Clearly, $t_1 \models q|_{R\underline{t}t_n}$ implies $t_1 \underset{R}{\downarrow} \underline{t}t_n$.

2. $\implies$ 1. Assume that $f_{\underline{t}}$ maps a cofinal sequence in $C(R)$ to a coinitial sequence of the convex hull of $R^{>C(R)}$ in $\mathcal{R}$. Then $t_1 < f_{\underline{t}}(t_n) < R^{>C(R)}$, and thus t_1 is contained in the interval $(t_2, f_{\underline{t}}(t_n))$. This witnesses $t_1 \underset{R}{\not\downarrow} \underline{t}t_n$.

1. $\implies$ 3. Assume that $t_1 \not\models q|_{R\underline{t}t_n}$, and, without loss of generality, assume that n is the least such. Then there is an element $s \in R\langle \underline{t}t_n \rangle$ realizing $q|_R$ with $t_1 < s$. Write $s = f_{\underline{t}}(t_n)$ for some $\mathcal{L}_0$-definable over R function f. We claim that $f_{\underline{t}}$ maps a cofinal sequence in $C(R)$ onto a coinitial sequence in in the convex hull of $R^{>C(R)}$ in $\mathcal{R}$.

We first show that $f_{\underline{t}}$ is strictly decreasing on some interval I containing a cofinal sequence in $C(R)$ and that $f_{\underline{t}}(I \cap C(R))$ is contained in the convex hull of $R^{>C(R)}$ in $\mathcal{R}$: By o-minimality of $R\langle \underline{t} \rangle$, $f_{\underline{t}}$ is continuous and strictly monotone on some interval I with endpoints in $\mathrm{dcl}(R\underline{t})$ and containing t_n. Since $\underline{t}, t_n$ is a Morley sequence in p over R, the left endpoint of I is contained in the convex hull of $C(R)$ in $R\langle \underline{t} \rangle$. For sufficiently big $c \in C^-$ we have $t_1 < f_{\underline{t}}(c)$, since $t_n \in \{x : f_{\underline{t}}(x) > t_1\}$, and $\{x : f_{\underline{t}}(x) > t_1\}$ is $\mathcal{L}_0$-definable over $R\langle t_1 \underline{t} \rangle$. Moreover, $f_{\underline{t}}(c) < R^{>C(R)}$ would yield a contradiction with the minimality of n. So we have $f_{\underline{t}}(t_n) < f_{\underline{t}}(c)$, hence $f_{\underline{t}}$ is decreasing on I, and $f_{\underline{t}}(I \cap C(R))$ is contained in the convex hull of $R^{>C(R)}$ in $\mathcal{R}$.

It remains to show that $f_{\underline{t}}(C(R) \cap I)$ is coinitial in the convex hull of $R^{>C(R)}$ in $\mathcal{R}$. Suppose not. Then there is $c \in R^{>C(R)}$ with $c < f_{\underline{t}}(C(R) \cap I)$. Then $f_{\underline{t}}^{-1}(c) < t_n$ but realizes $p|_R$. This contradicts the definition of t_n.

$\square$

Definition 2.3 *Let $p \in S_1(\mathcal{R})$ be the global type $C(R) < x$ and $x < \mathcal{R}^{>C(R)}$. We say that the cut C has property $*$ if whenever $\underline{t} = t_1, \ldots, t_n$ form a finite Morley sequence in $p|_R$, then the equivalent conditions of Proposition 2.2 hold.*

6

We use C both for a predicate in the expanded language $\mathcal{L}$ and to denote a cut, i.e. a non-definable 1-type in the the o-minimal language $\mathcal{L}_0$. Property $*$ is better thought of as a property of an o-minimal 1-type rather than as a property of the predicate C.

Proposition 2.4 *Let $p \in S_1(\mathcal{R})$ be a global R-invariant type such that $p|_R$ is the type corresponding to C and assume further that C has property $*$. Let $t_1, \ldots, t_n$ be a finite Morley sequence in p over R. Then*

$$t_1 \ldots t_k \underset{R}{\downarrow} t_{k+1} \ldots t_n \text{ and } t_{k+1} \ldots t_n \underset{R}{\downarrow} t_1 \ldots t_k.$$

for all k with $1 \leq k < n$.

PROOF: Use 2. of Proposition 2.2 and the fact that for any a, b_1, b_2, c, one has $a \underset{c}{\downarrow} b_1 b_2 \implies a \underset{cb_1}{\downarrow} b_2$, and $a \underset{cb_1}{\downarrow} b_2$ and $b_1 \underset{c}{\downarrow} b_2 \implies ab_1 \underset{c}{\downarrow} b_2$. $\square$

Proposition 2.5 *C has property $*$ implies that C is a valuational cut.*

PROOF: Consider (R, C) with C non-valuational. We denote by $p \in S_1(\mathcal{R})$ the type implied by $C(R) < x$ and $x < \mathcal{R}^{>C(R)}$. Let t_1, t_2, t_3 be a finite Morley sequence in p over R. In particular, $t_3 < t_2 < t_1$.

Since C is non-valuational, $0 < t_1 - t_2 < R^{>0}$, so $t_2 - (t_1 - t_2) \models p|_R$ and thus $t_3 < t_2 - (t_1 - t_2)$. That is, $t_1 < t_2 + (t_2 - t_3)$. Likewise, $0 < t_2 - t_3 < R^{>0}$, so $t_2 + (t_2 - t_3) \models p|_R$. Hence the formula $t_2 < x < t_2 + (t_2 - t_3)$ witnesses $t_1 \underset{R}{\not\downarrow} t_2 t_3$. $\square$

Theorem 2.6 *Fix (R, C) and let $T = Th(R, C)$. The following are equivalent:*

1. *There is a model (R_1, C) of T where the cut C has property $*$.*

2. *There is a model (R_0, C) of T such that if $C(R_0) < t < R_0^{>C(R_0)}$, then $(R_0, C) \preceq (R_0\langle t \rangle, C)$ provided that either*

$$C(R_0\langle t \rangle) = \{x \in R_0\langle t \rangle : x < r \text{ for some } r \in C(R_0)\}$$

   or
$$C(R_0\langle t \rangle) = \{x \in R_0\langle t \rangle : x < r \text{ for all } r \in R_0^{>C(R_0)}\}.$$

3. $Th(R, C)$ *is universally axiomatizable and eliminates quantifiers after adding constants for a cofinal sequence in C^- and a coinitial sequence in C^+.*

PROOF:

1. $\implies$ 2. This follows from Theorem 3.1 in the next section.

2. $\implies$ 1. Let (R_0, C) be as in the assumption of 2. Then we can build an elementary extension (R_1, C) of (R_0, C) in which the cofinality of $C(R_1)$ differs from the coinitiality of $R_1^{>C(R_1)}$. Thus, by 1. of Proposition 2.2, $C(R_1)$ has property $*$.

1. $\implies$ 3. This is Theorem 4.11.

3. $\implies$ 2. Both interpretations of C in $R\langle t \rangle$ result in (R, C) being a substructure of $(R\langle t \rangle, C)$, and quantifier elimination together with universal axiomitization implies that all substructures are elementary.

$\square$

The remainder of this section is devoted to the construction of several examples. First an example of a structure (R, C) where the cut defined by C does not have property $*$.

Example 2.7 *Let R be the field of the real algebraic numbers and let $R' = R\langle \epsilon \rangle$, the structure generated over the real algebraic numbers by a positive infinitesimal ϵ. Let C be the cut of π. The cut of π is a non-valuational cut in R but a valuational cut in R'. However, the same argument as in Proposition 2.5 shows $t_1 \not\perp_{R'} t_2 t_3$.*

By Theorem 6.3, p. 5470 [10], there is a canonical way in which (R, C) defines a valuation ring. In Example 2.7, this valuation ring is the convex hull of $\mathbb{Q}$ in R', so its residue field is o-minimal by Corollary 2.4, p. 244 [8]. It is therefore not possible to weaken the assumption in Proposition 5.5, which says that if C is a cut corresponding to a convex subring of R such that the associated residue field is o-minimal, then C has property $*$, to: "the residue field of the canonical valuation associated to (R, C) is o-minimal".

On the other hand, if $C = V$ is a convex subring of R, then the o-minimality of the residue field is not a necessary condition for C to have property $*$, as the following example demonstrates.

8

Example 2.8 *Let R be a big elementary extension of the real exponential field, and let $a \in R^{>\mathbb{Q}}$. We set $V = \bigcup_n(-a^n, a^n)$. The residue field of (R, V) is not o-minimal, since the inverse function of the exponential induces a map $\ln\colon \mathbf{k}^{>0} \to \mathbf{k}$, where $\ln(\mathbf{k}^{>0}) \subseteq \mathbf{k}$ is bounded above but does not have a supremum in $\mathbf{k}$. However, letting $\mathcal{O}$ be the convex hull of $\mathbb{Q}$, we see that (R, V) and $(R, \mathcal{O})$ are interdefinable via the $\mathcal{L}_0$-definable function $f(x) = \frac{\ln x}{\ln a}$. The residue field of $(R, \mathcal{O})$ is $\mathbb{R}$ and is thus o-minimal. (In fact, $(R, \mathcal{O})$ is even T-convex.) Thus, the cut corresponding to $\mathcal{O}$ has property $*$, and therefore so does the cut corresponding to V.*

While Example 2.7 shows that (R, C) might not satisfy $*$, one might still hope that $*$ holds whenever C is a predicate for a valuation ring, but this is also not true. An example illustrating this is a bit more difficult to construct, and we shall devote the rest of the section doing so.

Example 2.9 *Set $R_0 = \mathbb{R}_{\exp}$, the real exponential field, and let $\mathcal{M}$ be a countably saturated elementary extension of R_0. We pick $a_1 \in \mathcal{M}^{>\mathcal{O}}$ and set $R_1 = R_0\langle a_1 \rangle$ and $V_1 = \bigcup_n(-a_1^n, a_1^n) \subseteq R_1$. We define (R_{k+1}, V_{k+1}) inductively by letting $a_{k+1} \in \mathcal{M}$ be such that $V_k < a_{k+1} < R_k^{>V_k}$, $R_{k+1} = R_k\langle a_{k+1} \rangle$, and $V_{k+1} = \bigcup_n(-a_{k+1}^n, a_{k+1}^n) \subseteq R_{k+1}$. Finally we set $R = \bigcup_k R_k$ and $V = \bigcup_k V_k$. Our aim is to show that V does not have property $*$.*

Lemma 2.10 *Let $f(x, y) = y^{\frac{\ln y}{\ln x}}$, and let $t_3 < t_2 < t_1$ be a finite Morley sequence in p over R, where $p \in S_1(\mathcal{R})$ is the type $V < x < \mathcal{R}^{>V}$. Then*

$$t_1 < f(t_3, t_2) < R^{>V}.$$

PROOF: *For each i, we let $g_i \in R_{i+1}$ be such that $a_i^{g_i} = a_{i+1}$, i.e. $g_i = \frac{\ln a_{i+1}}{\ln a_i}$.*

CLAIM 1 : *$\mathcal{O} < g_i < R_i^{>\mathcal{O}}$, where $\mathcal{O}$ is the convex hull of $\mathbb{Q}$ in R_i.*

By our choice of a_{i+1}, $g_i > k$ for all $k \in \mathbb{N}$. Moreover, if there was $r \in R_i^{>\mathcal{O}}$ with $r < g_i$, then $V_i < a_i^r < a_i^{g_i} = a_{i+1}$. But since $a_i^r \in R_i$, this would imply that $a_i^r \in V_i$, a contradiction.

CLAIM 2 : *$g_{i+1} < g_i^{\frac{1}{k}}$ for all $k \in \mathbb{N}$.*

9

By Claim 1, $g_{i+1} < g_i$. For any k, $f(x) = x^k$ is increasing and $f(\mathcal{O}) \subseteq \mathcal{O}$, so $g_{i+1}^k \geq g_i$ is impossible.

Note that $\{a_{i+1}^{g_i}\}_{i=1}^{\infty}$ is a coinitial sequence in $R^{>V}$: If not, then there would be $r \in R^{>V}$ such that $r < a_{i+1}^{g_i}$ for all i. But $r \in R_j^{>V_j}$ for some j, hence $a_{j+1}^{g_j} < r$, a contradiction. So Claim 3 will complete the proof.

CLAIM 3 : For $i \in \{1, 2\}$, we set $h_i := \frac{\ln t_i}{\ln t_{i+1}}$ (so $t_i = t_{i+1}^{h_i}$). Then, for all i,

$$t_1 < t_2^{h_2} < a_{i+1}^{g_i} = f(a_i, a_{i+1}).$$

To prove that $t_2^{h_2} < a_{i+1}^{g_i}$ for all i, assume towards a contradiction that i is such that $t_2^{h_2} \geq a_{i+1}^{g_i}$. Then

$$h_2 \geq \frac{g_i \ln a_{i+1}}{\ln t_2} \geq \frac{g_i \ln a_{i+1}}{g_{i+1} \ln a_{i+2}} = \frac{g_i}{(g_{i+1})^2}.$$

By Claim 2, we have $\frac{g_i}{(g_{i+1})^2} > g_i^{\frac{1}{2}}$, so $h_2 > g_i^{\frac{1}{2}}$. It follows that

$$t_2 = t_3^{h_2} > a_{i+1}^{(g_i^{\frac{1}{2}})} \in R_{i+1}^{>V},$$

a contradiction with t_2 realizing $p|_R$.

It is left to show that $t_1 < t_2^{h_2}$. If not, then $t_2^{h_1} > t_2^{h_2}$, so $h_1 > h_2$. We would then be able to find k such that $t_2^{\frac{1}{h_1}} < a_k$, which is equivalent to $t_2^{\frac{\ln t_2}{\ln a_k}} < t_1$. On the other hand, $\frac{\ln t_2}{\ln a_k} > \frac{\ln a_{k+1}}{\ln a_k} = g_k$. So

$$t_2^{\frac{\ln t_2}{\ln a_k}} > t_2^{g_k} > a_{k+1}^{g_k} > V,$$

a contradiction with $t_2^{\frac{\ln t_2}{\ln a_k}} < t_1$. $\qquad\square$

Lemma 2.10 shows that $t_1 \not\perp_R t_2 t_3$, and hence V does not have property $*$.

We conclude this section with a lemma showing that property $*$ is preserved by passing to certain superstructures.

Lemma 2.11 *Suppose C defines a cut in R which has property $*$, and let $p \in S(\mathcal{R})$ be the invariant type implied by*

$$\{x > c | c \in C(R)\} \cup \{x < r | r \in \mathcal{R} \text{ and } r > C(R)\}.$$

and $q \in S(\mathcal{R})$ the type implied by

$$\{x < r | r \in R^{>C(R)}\} \cup \{x > r | r \in \mathcal{R} \text{ and } r < R^{>C(R)}\}.$$

Then any of the following conditions implies C defines a cut in $R\langle (a_i)_{i<\kappa} \rangle$ which has property $*$:

1. *$(a_i)_{i<\kappa}$ forms a Morley sequence in p over R and*

$$\mathrm{dcl}((a_i)_{i<\kappa} \cap p|_R) > C(\mathcal{R}).$$

2. *$(a_i)_{i<\kappa}$ forms a Morley sequence in q over R and*

$$dcl_R((a_i)_{i<\kappa} \cap p|_R) \subseteq C(\mathcal{R}).$$

3. *$\mathrm{dcl}_R((a_i)_{i<\kappa} \cap p|_R) = \emptyset.$*

PROOF: Assume (1) holds. Take a Morley sequence $(t_i)_{i<\omega}$ in p over $R\langle (a_i)_{i<\kappa} \rangle$. Then $(a_i)_{i<\kappa}(t_i)_{i<\omega}$ is also a Morley sequence in p over R. Note that $t_1 (a_i)_{i<\kappa} \underset{R}{\downarrow} t_2 \ldots t_n$, because if not, there would be a finite subtuple $a_{i_0}, \ldots, a_{i_k}$ with

$$a_{i_0} \ldots a_{i_k} t_1 \underset{R}{\not\downarrow} t_2 \ldots t_n,$$

and, by indiscernibility of the Morley sequence, we would have

$$t_1 t_2 \ldots t_{k+2} \underset{R}{\not\downarrow} t_{k+3} \ldots t_{k+n+1},$$

a contradiction with C having property $*$. From $t_1 (a_i)_{i<\kappa} \underset{R}{\downarrow} t_2 \ldots t_n$, it follows that $t_1 \underset{R\langle (a_i)_{i<\kappa} \rangle}{\downarrow} t_2 \ldots t_n$.

If (2) holds, one carries out the analogous argument with the Morley sequence $(t_i)_{i<\omega}$ in q.

Now assume (3). Suppose for a contradiction that C does not define a cut in $R\langle (a_i)_{i<\kappa} \rangle$ that has property $*$, and that this is witnessed by a finite Morley sequence $t_k < t_{k-1} < \cdots < t_1$ in $p|_{R\langle (a_i)_{i<\kappa} \rangle}$ and an R-definable function f,

11

i.e. $t_1 < f_{\underline{a}}(\underline{t})$, where $\underline{a}$ is a finite subsequence of $(a_i)_{i<\kappa}$, $\underline{t} = t_2, \ldots, t_k$ and $f_{\underline{a}}(\underline{t}) \models p|_{R\langle(a_i)_{i<\kappa}\rangle}$. Then the cut defined by C in $R\langle\underline{a}\rangle$ does not have property $*$. Thus, if we could prove that C defining a cut in R with property $*$ implies that C also defines a cut with property $*$ in $R\langle a\rangle$, where a is a singleton and $\mathrm{dcl}_R(a) \cap p|_R = \emptyset$, then by applying this fact finitely many times, we could show C defines a cut with property $*$ in $R\langle\underline{a}\rangle$. So towards a contradiction, we assume C does not define a cut with property $*$ in $R\langle a\rangle$.

Let $t_k < t_{k-1} < \cdots < t_1$ be a finite Morley sequence in p over $R\langle a\rangle$ and f an R-definable function with $t_1 < f_a(\underline{t})$, where $\underline{t} = t_2, \ldots, t_k$, and $f_a(\underline{t}) \models p|_{R\langle a\rangle}$.

Set $t_1' = f_a(\underline{t})$ and let $t_k' < \cdots < t_2' < t_1'$ be a finite Morley sequence in p over $R\langle a\rangle$ so that $t_2' < t_k$. Since $\mathrm{tp}(\underline{t}/R\langle a\rangle) = \mathrm{tp}(\underline{t}'/R\langle a\rangle)$, we have

$$t_1' = f_a(\underline{t}) < f_a(\underline{t}') < R\langle a\rangle^{>C(R\langle a\rangle)}.$$

We now set $t_1'' = f_a(\underline{t}')$.

CLAIM   The sequence $t_1'', t_2', \ldots, t_k'$ is a finite Morley sequence in p over R.

PROOF: This is proved by induction on k. We clearly have $t_1'' \models p|_R$. Now suppose that $t_1'', t_2', \ldots, t_l'$ is a finite Morley sequence in p over R. If $t_1'', t_2', \ldots t_{l+1}'$ was not, then we would have $C < h(t_1'', t_2', \ldots t_l') < t_{l+1}'$. Let $H(x) = h(x, t_2', \ldots, t_l')$ and consider $H(t_1')$. The function H is continuous and strictly monotone on some interval, I, with left endpoint in $\mathrm{dcl}(t_2' \ldots t_l')$ and realizing the type $p|_R$, and right endpoint in the convex hull of $R^{>C(R)}$ in $\mathcal{R}$.

If H is increasing on that interval, then $H(t_1') < r$ for some $r \in C(R)$, since otherwise $t_1', \ldots, t_{l+1}'$ would not be a Morley sequence in p over R. But if $H(t_1') < r$, then any $r' > r$, $r' \in C(R)$ would have to be such that $t_1' < H^{-1}(r') < t_1''$, a contradicting that C defines a cut with property $*$ in R.

If H is decreasing, then a coinitial segment of $R^{>C(R)}$ would have to be mapped to a cofinal segment of $C(R)$, since if there were any $r \in R^{>C(R)}$ with $H(r) \models p|_R$ this would necessarily be less than t_{l+1}' (since $H(r) < H(t_1'')$), contradicting $t_1', \ldots, t_{l+1}'$ being a Morley sequence in p over R. If, on the other hand, $r \in C(R)$ is such that $H^{-1}(I \cap R^{>C(R)}) < r$, then $H^{-1}(r) > t_1''$, hence $H^{-1}(r)$ cannot satisfy $p|_R$ because the cut defined by C in R has property $*$. But H mapping a coinitial segment of $R^{>C}$ to a cofinal segment of $C(R)$ contradicts C defining a cut in R with property $*$. $\square$

Note that $a \in \mathrm{dcl}_R(t_1'', \underline{t}')$. Extend $t_1'', \underline{t}'$ to a finite Morley sequence $t_1'', \underline{t}', \underline{t}''$ in $p|_R$. Then $t_1'' < f_a(\underline{t}'')$ and $f_a(\underline{t}'') \models p|_{R\langle a \rangle}$. But $f_a(\underline{t}'') = f_{g(t_1'', \underline{t}')}(\underline{t}'')$ for some R-definable function g. Hence the function $f(g(\dots) \dots)$ and the finite Morley sequence $t_1'', \underline{t}', \underline{t}''$ in $p|_R$ witness the failure of $*$ in the cut defined by C in R, a contradiction.

$\square$

3 Elementary extensions

The theorem below is an analog of Theorem 3.3, p. 244 in [8]. The difference between the two statements is that in [8], R is assumed to expand a field and C (called "V" in [8]) is assumed to be a proper convex subring of R. Moreover, Theorem 3.3 in [8] needs an additional assumption such as V has property $*$ (though $(R, V) \models \Sigma$, i.e. the induced structure on the residue field of (R, V) is o-minimal, would also suffice) to be correct.

Note that while Theorem 3.3 in [8] is an implication, we state it here as an equivalence. The easy direction (which went unstated in [8]) being if $(R, C) \preceq (R\langle a \rangle, C)$, then we cannot have $C(R) < a, f(a) < R^{>C(R)}$, where f is an R-definable function and $a \in C(R\langle a \rangle)$ and $f(a) > C(R\langle a \rangle)$. Else,

$$(R\langle a \rangle, C) \models \exists x \in C \ f(x) > C,$$

hence

$$(R, C) \models \exists x \in C \ f(x) > C,$$

yielding a contradiction with f being increasing (see Fact 1.1).

Theorem 3.1 *Suppose C is a cut in R with property $*$, and let $a \in \mathcal{R} \setminus R$. Then (R, C) fails to be an elementary submodel of $(R\langle a \rangle, C)$ if and only if there is an element of $C(R\langle a \rangle)$ greater than any element of $C(R)$ (WLOG, we may assume this element is a) and an R-definable function, f, such that*

$$C(R\langle a \rangle) < f(a) < R^{>C(R)}.$$

PROOF: This may be proved in an almost identical fashion to the proof of Theorem 3.3 [8] (which encompasses paragraphs 2 and 3), and below, for the reader familiar with that proof, we outline the minor changes needed.

1. All instances of "V" need to be replaced by "C".

13

2. Replace Theorem 2.7, p. 239 [8] with Corollary 2.4.

3. Corollary 2.9, p. 241 [8] needs the additional assumption that $p|_R$ is the type corresponding to C and C has property $*$.

4. The assumption that (R, V) has property $*$ must be added to the hypotheses of Lemma 3.1 and appeals to Corollary 2.9 are replaced by appeals to this hypothesis.

5. Note that the cut of type p in $R\langle a \rangle$ also has property $*$ by Lemma 2.11 of this paper.

6. In the proof of Lemma 3.1, p. 242, lines -18 to -15: The field assumption is used to reduce 4 cases to 2. But the two remaining cases can simply be proved in the same fashion as Case 2.

7. In the proof of Lemma 3.1, p. 243, lines 1 and 2: "Note that since V is a group, ..." should be replaced with "Note that since $(C, R^{>C})$ is a valuational cut, ..." and on page 243, lines 4,5 "But $\frac{1}{2}\beta$ is also greater than every element of V, ..." should be replaced with "But $\beta - \epsilon$, where $\epsilon > 0$ is such that $r - c > \epsilon$ for all $r \in R^{>C}$ and all $c \in C$, is also greater than every element of C, ...".

8. The hypothesis of satisfying property $*$ must be added to the hypotheses of Lemma 3.2 and Theorem 3.3 and appeals to Lemma 2.11 (of this paper) added as appropriate to their proofs.

$\square$

For readers not already familiar with the proofs in [8] this may be a significant amount of work, especially as the proofs in [8] themselves require one to go through the proofs in [2] to confirm results implicit, but not stated, therein. We have therefore included a new, simplified proof of Theorem 3.3 of [8] in the remainder of this section that includes an easy proof of the results implicit in [2] that we require. This new proof also contains a result that may be of independent interest, namely Proposition 3.6, a characterization of the divide between valuational and non-valuational cuts in neostability theoretic terms.

In order to prove Theorem 3.1, we will need a result on quantifier elimination of traces (Proposition 3.7) together with statement about the uniformity

of this quantifier elimination (Remark 3.8), both of which may be extracted from the proofs of of Baisalov and Poizat in [2].

We shall need the notion of separation which was introduced in [2]:

Definition 3.2 *Let $a \in R^m$, $b \in R^n$, and let $p \in S_1(A)$. Then a and b are separated in p over A if either*

$$\mathrm{dcl}_{\mathcal{L}_0}(aA) \cap p(R) < \mathrm{dcl}_{\mathcal{L}_0}(bA) \cap p(R) \text{ or } \mathrm{dcl}_{\mathcal{L}_0}(bA) \cap p(R) < \mathrm{dcl}_{\mathcal{L}_0}(aA) \cap p(R).$$

We say that a and b are A-separated if they are separated in all one-types over A.

We are able to provide short proofs by limiting ourselves to the case where the externally definable set is a trace of a formula whose parameters form a Morley sequence in a cut satisfying $*$, using our result (Corollary 2.4) on symmetry of dividing in Morley sequences, thus easily obtaining separated tuples. (This is also Corollary 2.9 of [8], although there it appeared (incorrectly) without the hypothesis $*$.)

Corollary 3.3 *Let $p \in S_1(\mathcal{R})$ be an R-invariant extension of the $\mathcal{L}_0$-type over R which states that x realizes the cut C, and assume that latter type has property $*$. If $t_1, \ldots, t_n$ is a finite Morley sequence in p over R and $1 \le k < n$, then $t_1, \ldots, t_k$ is R-separated from $t_{k+1}, \ldots, t_n$.*

PROOF: This is simply Corollary 2.4 together with the definition of separation. □

In addition, we shall use the following fact:

Fact 3.4 *Let $a_1, \ldots, a_k$ be such that each a_i realizes a cut in $R\langle a_1, \ldots a_{i-1}\rangle$. Then every element of $R\langle a_1, \ldots a_k\rangle \setminus R$ realizes a cut in R.*

PROOF: This is clear when $k = 1$. Suppose the statement is false, and consider the minimal k for which it fails. Let $b = f(a_1, \ldots, a_k)$ witness its failure. Replacing b with $1/b$, if necessary, we may assume that there is a closest element of R to b. Call this element r. Since b lies in a cut in $R\langle a_1, \ldots a_{k-1}\rangle$, there must be elements of $R\langle a_1, \ldots a_{k-1}\rangle$ between b and r. But these elements must therefore realize a definable type over R, contradicting the minimality of k. □

15

Finally, we will need to take a global type, p, invariant over a small model R where both the invariant type and its restriction to R are non-definable, build a Morley sequence $t_1, \ldots, t_n$ in p over R, and have that $p|_{Rt_1 \ldots t_n}$ is also nondefinable. The following example shows that this is not always the case:

Example 3.5 *Let A be the collection of rational numbers less than π and let p be the global type given by $\{x > r | r < a \in A\} \cup \{x < r | r > A\}$. Clearly p is invariant over the empty set and is valuational (as is any non-definable global type invariant over a small set). For most models R, $p|_R$ is valuational, but if R is archimedean, then $p|_R$ is either a non-valuational cut or a non-cut. Let R be an archimedean model. If $t_1, t_2, \ldots$ is a Morley sequence in $p|_R$ then the only realization of $p|_R$ in $R\langle t_1 \rangle$ is t_1 itself. Thus the type of t_2 over Rt_1 is a definable type.*

On the other hand, if $p|_A$ is a valuational type and $B \supseteq A$, then $p|_B$ is also valuational (witnessed by the same ϵ). In particular, $t_1, \ldots, t_k$ is a Morley sequence in $p|_A$, then $p|_{At_1 \ldots t_k}$ is also valuational, and we have the following characterization of valuational cuts:

Proposition 3.6 *If p is a global 1-type invariant over a small set A, and $B \supseteq A$, then the following are equivalent*

1. *$p|_B$ is a valuational type.*

2. *If $p|_B$ is a cut and $\mathrm{dcl}(Ba)$ contains $t \models p|_B$ then $\mathrm{dcl}(Ba)$ contains more than one realization of $p|_B$.*

3. *if $t_1, \ldots, t_k$ is a Morley sequence in $p|_B$ then $p|_{Bt_1 \ldots t_k}$ is nondefinable for all k*

4. *if t_1, t_2 is a Morley sequence in $p|_B$ then $\mathrm{tp}(t_1/B)$ and $\mathrm{tp}(t_2/Bt_1)$ are nondefinable.*

PROOF: Since p is a 1-type invariant over B, we may assume by replacing B with an appropriate subset of $\mathrm{dcl}(B)$ that either p is implied by $\{x > b \| b \in B\} \cup \{x < r | r > B\}$ or p is implied by $\{x < b \| b \in B\} \cup \{x > r | r < B\}$. For convenience, we assume the former. The proof in the latter case is identical.

1. $\implies$ 2. Let ϵ witness that $p|_B$ is valuational. Then $t - \epsilon \models p|_B$.

2. $\implies$ 1. Let $t, \tilde{t} \in \mathrm{dcl}(Ba)$ both realize $p|_B$. Note that $|t - \tilde{t}|$ cannot realize the right infinitessimal neighborhood of 0 in $\mathrm{dcl}(B)$, since $p|_B$ is a cut. So let ϵ be a positive element of $\mathrm{dcl}(B)$ less than $|t - \tilde{t}|$. This ϵ witnesses that $p|_B$ is valuational.

1. $\implies$ 3. Again, let ϵ witness that $p|_B$ is valuational. Consider any $a \in \mathrm{dcl}(Bt_1 \ldots t_k)$ with $a \models p$. Then $a - \epsilon \models p$. Thus there is no smallest element of $p|_B \cap \mathrm{dcl}(Bt_1 \ldots t_k)$. Since $p|_B$ is in particular a cut, there is no largest element of B. Thus, $p|_{Bt_1 \ldots t_k}$ is not definable.

3. $\implies$ 4. Clear.

4. $\implies$ 1. Assume that either $\mathrm{tp}(t_1/B)$ or $\mathrm{tp}(t_2/Bt_1)$ is definable. If the former is true, $p|_B$ is not even a cut. If $\mathrm{tp}(t_1/B)$ is a cut, B has no greatest element, so if $\mathrm{tp}(t_2/Bt_1)$ is not, it must be that $p|_{Bt_1}$ is the left infinitessimal neighborhood of $a \in \mathrm{dcl}(Bt_1)$ with $a \models p|_B$. That means there is no $\epsilon > 0$ in $\mathrm{dcl}(Bt_1)$ such that $a - \epsilon \models p|_B$. So in particular there is no such $\epsilon \in \mathrm{dcl}(B)$, and thus $p|_B$ is not valuational.

$\square$

Now we may turn to our proof of Proposition 3.7. We let p be a global 1-type which is invariant over the empty set and such that $p|_R$ has property $*$. Consider an $|R|$-saturated elementary extension $\widetilde{R}$ of R. Up to this point, $\mathcal{L}$ has been $\mathcal{L}_0$ expanded by a predicate for a cut. For the remainder of this section we work in more generality. Let $\tilde{S}$ be the realization of $\theta(x_1, \ldots, x_k, t_1, \ldots, t_n)$ in $\widetilde{R}$, where the parameters from R are suppressed and the $t_1, \ldots, t_n$ form a Morley sequence in p over R. Let $S = \tilde{S} \cap R$ and expand $\mathcal{L}_0$ to $\mathcal{L}$ by adding a a predicate S defining, in a slight abuse of notation, the set S in R.

Proposition 3.7 *Let φ be an $\mathcal{L}$-formula with parameters from R. Then there is an $\mathcal{L}_0$-formula, τ_φ, such that $\varphi(R) = \tau_\varphi(\widetilde{R}) \cap R^k$, where the parameters of τ_φ form a Morley sequence in $p|_R$.*

PROOF: Let $\underline{x} = (x_1, \ldots, x_k)$. It suffices to show that if $\varphi(x_0, \underline{x})$ is the trace of the $\mathcal{L}_0$-formula $\tau_\varphi(x_0, \underline{x}, t_1, \ldots, t_n)$, then $\exists x_0 \varphi(x_0, \underline{x})$ is the trace of

$$\tau_{\exists x_0 \varphi} = \exists x_0 (\tau_\varphi(x_0, \underline{x}, t_1, \ldots, t_n) \wedge \tau_\varphi(x_0, \underline{x}, t_{n+1}, \ldots, t_{2n})),$$

where $t_1, \ldots, t_{2n} \in \widetilde{R}$ form a Morley sequence in p over R. Suppose $r_1, \ldots, r_k \in R$ are such that there is $r_0 \in R$ with $\varphi(r_0, r_1, \ldots, r_k)$. Then

$$\widetilde{R} \models \tau_\varphi(r_0, r_1, \ldots, r_k, t_1, \ldots, t_n),$$

and since $t_1, \ldots, t_n \equiv_R t_{n+1}, \ldots, t_{2n}$, one has

$$\widetilde{R} \models \tau_\varphi(r_0, r_1, \ldots, r_k, t_{n+1}, \ldots, t_{2n})$$

as well.

Now suppose that there is no $r_0 \in R$ such that $\phi(r_0, r_1, \ldots, r_k)$ and assume for a contradiction that there is $\tilde{r}_0 \in \widetilde{R}$ with

$$\widetilde{R} \models \tau_\varphi(\tilde{r}_0, r_1, \ldots, r_k, t_1, \ldots, t_n) \wedge \tau_\varphi(\tilde{r}_0, r_1, \ldots, r_k, t_{n+1}, \ldots, t_{2n}).$$

We consider $\mathrm{tp}(\tilde{r}_0 / R t_1, \ldots, t_n)$. Either $\tilde{r}_0 \in \mathrm{dcl}(R t_1, \ldots, t_n)$ or there is an interval (a, b) containing $\tilde{r}_0$ with $a, b \in \mathrm{dcl}(R t_1, \ldots t_n)$ (possibly equal to $-\infty, \infty$) such that any $s \in (a, b)$ is such that $\tau_\varphi(s, r_1, \ldots, r_k, t_1, \ldots, t_n)$. Note that as we are supposing that there is no element of R in that interval, the entire interval realizes a single type $q \in S_1(R)$. Choose $\tilde{r}_1 \in (a, b)$ with $\tilde{r}_1 \in \mathrm{dcl}(R t_1, \ldots t_n)$. As $p|_R$ is valuational, $\mathrm{tp}(t_i / R t_1 \ldots t_{i-1})$ is a cut, by Proposition 3.6. This allows us to apply Proposition 3.4 to $\tilde{r}_1$, showing that q is not a definable type. Thus neither a nor b can be in R or equal to $-\infty, \infty$.

Likewise when considering $\mathrm{tp}(\tilde{r}_0 / R t_{n+1}, \ldots t_{2n})$ we obtain either $\tilde{r}_0 \in \mathrm{dcl}(R t_{n+1}, \ldots t_{2n})$ or the existence of an interval (c, d) analogous to (a, b) above.

Now one notes that we have contradicted the separation of $t_1, \ldots, t_n$ from $t_{n+1}, \ldots, t_{2n}$ (which follows from Corollary 3.3). For either there is an element not in R common to their definable closures, or else there is an element of the definable closure of one contained in an interval defined over the other. $\square$

Remark 3.8 *It is easy to see, inspecting the proof of Proposition 3.7, that the construction of τ_φ is uniform in the sense that it is independent of p, R, and $\widetilde{R}$. To be precise, let $R_1 \preceq \widetilde{R}_1$ (in the sense of $\mathcal{L}_0$) with $\widetilde{R}_1$ sufficiently saturated, and let $S_1 = \tilde{S}_1 \cap R_1$ where $\tilde{S}_1$ is defined by $\theta(x_1, \ldots, x_k, s_1, \ldots s_n)$ with $s_1, \ldots, s_n$ a Morley sequence in $p_1|_{R_1}$, with p_1 a global 1-type invariant over $\emptyset$ such that $p_1|_{R_1}$ has property $*$. Then given any φ, the proof above creates the same τ_φ for R_1 and p_1 as it does for R and p*

Note that the uniformity observed in Remark 3.8 translates easily into a fact about elementary substructures. Namely, we have the following lemma.

Corollary 3.9 *Consider R and R_1 as in Proposition 3.7 where in addition $R \subseteq R_1$, $\widetilde{R} = \widetilde{R}_1$. Consider $p = p_1$ invariant over R such that $p|R$ and $p|_{R_1}$ have property $*$, and moreover $s_1, \ldots, s_n = t_1, \ldots, t_n$. Then $R \preceq R_1$ as $\mathcal{L}$-structures.*

PROOF: We may add constants for R to $\mathcal{L}$ to make p invariant over $\emptyset$ so that we may apply Proposition 3.7. Suppose that $R_1 \models \exists x \varphi(x, \vec{r})$ where $\vec{r} \in R^m$. This happens if and only if $\vec{r_0} \in \tau_{\exists x \varphi}(\widetilde{R}) \cap R_1^m$, where $\tau_{\exists x \varphi}$ is the $\widetilde{R}$-definable set whose trace $\exists x \varphi(x, \vec{R}_1^m)$ is. But the set $\exists x \varphi(x, R^m)$ is also the trace of $\tau_{\exists x \varphi}$. Obviously $\vec{r} \in \tau_{\exists x \varphi}(\widetilde{R}) \cap R^m$. Thus $R \models \exists x \varphi(x, \vec{r})$, and we are done by the Tarski-Vaught test. $\square$

We may now use this corollary to prove a slightly strengthened version (in that we only need to assume the cut is valuational, rather than an actual valuation ring) of Theorem 3.3 of [8]. One may check that the proof in [8] goes through with minimal changes, but given the above corollary, one may also give a far shorter proof, which we do below.

First note that if $R\langle a \rangle$ is a superstructure of R (as an $\mathcal{L}_C$-structure) and there is an element (which we may assume to be a) in $C^{>R}$ with $f(a)$ in the same cut in R as a but with $f(a)$ greater than C, then we cannot hope to satisfy the hypotheses of Corollary 3.9. In particular, the requirement that $s_1 \ldots, s_n = t_1, \ldots, t_n$ can not be satisfied. After all, C is the trace of $x < t$ and t must lie between a and $f(a)$, but an R-invariant global type whose restriction to R is the cut of a in R would produce a Morley sequence either less than a or greater than $f(a)$. Thus this Morley sequence could not include t.

However, this is the only obstruction:

Lemma 3.10 *[Lemma 3.2 [8]] There is an R-invariant global type q with $q|_R$ being the cut in R given by C and $q|_{R\langle a \rangle}$ being the cut in $R\langle a \rangle$ given by C, and there is a finite Morley sequence $t_1, \ldots, t_n$ in q over $R\langle a \rangle$ (and hence over R) in each of the following cases:*

   a) a realizes the cut in R given by C and $C(R\langle a \rangle)$ is the convex hull of $C(R)$ in $R\langle a \rangle$.

b) a realizes the cut given by C and $C(R\langle a\rangle) = \{x \in R\langle a\rangle : x < R^{>C(R)}\}$.

c) a is such that $\mathrm{dcl}_R(a)$ does not realize the cut given by C (hence $C(R\langle a\rangle)$ is the convex hull of C in $R\langle a\rangle$ and $C(R\langle a\rangle) = \{x \in R\langle a\rangle : x < R^{>C(R)}\}$).

PROOF: In a), we may take q to be the global type implied by

$$\{x > c | c \in C(R)\} \cup \{x < r | r \in \mathcal{R} \text{ and } r > C(R)\}.$$

In b), we let q be the global type implied by

$$\{x < r | r \in R^{>C}\} \cup \{x > r | r \in \mathcal{R} \text{ and } r < R^{>C}\}.$$

In c), of course, we may do either.

$\square$

Now we obtain our criterion for elementary extensions:

Theorem 3.1 *[Theorem 3.3 [8]] Let $a \in \mathcal{R}$ and assume that (R, C) has property $*$. Then $(R\langle a\rangle, C)$ fails to be an elementary extension of (R, C) if and only if there are R-definable one-variable functions f and g with $C(R) < f(a), g(a) < R^{>C(R)}$ and $f(a) \in C$ and $g(a) > C$.*

PROOF: First assume there are no functions f, g as in the statement of the theorem. Then we may apply Lemma 2.11 to obtain that $(R\langle a\rangle, C)$ has property $*$. The lack of such f, g also allows us to apply Lemma 3.10 to find an R-invariant q with $q|_R$ being the cut in R given by C and $q|_{R\langle a\rangle}$ being the cut in $R\langle a\rangle$ given by C. Now we have satisfied the hypotheses of Corollary 3.9, and conclude that $(R, C) \preceq (R\langle a\rangle, C)$.

Now assume that there are such functions f and g, and define h to be $g \circ f^{-1}$. Note that since h maps $f(a)$ to $g(a)$, h maps the cut in R corresponding to C to itself. Thus h is increasing (see Fact 1.1) on an interval containing this cut. Thus

$$(R\langle a\rangle, C) \models \text{ for all sufficiently large } x \in C, \ h(x) > C,$$

hence

$$(R, C) \models \text{ for all sufficiently large } x \in C, \ h(x) > C$$

Take $r \in R$ with $h(r) > C$. But $r < f(a)$ while $h(r) > h(f(a))$, a contradiction to h being increasing.

$\square$

20

4 Substructures of models of T and quantifier elimination

In this section, we obtain our quantifier elimination, universal axiomitization, and definable Skolem function results. The key result is that for a suitable $(R_0, C) \preceq (R, C)$, any single element $a \in R$ results in $(R_0, C) \preceq (R_0 \langle a \rangle) \preceq (R, C)$. We will obtain rather easily that $(R_0, C) \preceq (R_0 \langle a \rangle)$, but to continue to inductively build up to (R, C) we will also need that $(R_0 \langle a \rangle) \preceq (R, C)$, and this will take more work, in particular in the case where $(R_0 \langle a \rangle)$ contains no elements realizing the cut in R_0 corresponding to C.

Lemma 4.1 *Suppose $(R_0, C) \preceq (R, C)$, C defines a cut in R_0 with property $*$, and $a \in R \setminus R_0$. Then $(R_0, C) \preceq (R_0 \langle a \rangle, C)$.*

PROOF: If not, then, by Theorem 3.1, we may assume that a realizes the type
$$C(R_0) < x < R_0^{>C(R_0)},$$
and there is an $\mathcal{L}_0$-definable over R_0 function f such that $f(a)$ also realizes the same type, and moreover $a \in C(R_0 \langle a \rangle)$ and $f(a) > C(R_0 \langle a \rangle)$. Hence
$$(R, C) \models \exists x \in C \ f(x) > C.$$

By Fact 1.1, f is increasing, and so $f(C(R_0)) \subseteq C(R_0)$ and $f(R_0^{>C(R_0)}) \subseteq R_0^{>C(R_0)}$, contradicting $(R_0, C) \preceq (R, C)$. $\square$

Lemma 4.2 *Suppose $(R_0, C) \preceq (R, C)$, and suppose C defines a cut in R_0 with property $*$. Let $a_1, \ldots, a_n \in R$ and set $R_n = R_0 \langle a_1, \ldots, a_n \rangle$ and $C_n = C(\mathcal{R}) \cap R_n$. Then $(R_0, C) \preceq (R_n, C)$. Moreover, suppose $a_1, \ldots, a_n$ is an independent tuple such that, for some $m \leq n$, $a_{i+1} \in C_{i+1}^{>C_i}$ for each $i \leq m$, and C_m is cofinal in C_n. Then for all i, $(R_i, C) \preceq (R_{i+1}, C)$.*

PROOF: We may as well assume that $a_1, \ldots, a_n$ are as in the "Moreover, ..." part of the statement of the lemma. Note that once we have shown $(R_i, C) \preceq (R_{i+1}, C)$, then, by Lemma 3.1 and Lemma 2.11, we have that C defines a cut in R_{i+1} which has property $*$, and hence we are at liberty to use Lemma 3.1 at this stage, too.

We start by showing that $(R_i, C) \preceq (R_{i+1}, C)$ for $i < m$, which we do by induction.

i = 0: This is Lemma 4.1.

Inductive Step: Towards a contradiction, assume there is an R_0-definable function f such that $f(a_1, \ldots, a_i, x_{i+1})$ maps the set of realizations of the type

$$C_i < x < R_i^{>C_i}$$

to itself and $f(a_1, \ldots, a_i, a_{i+1})$ is greater than C_{i+1}. Since each $a_{j+1} > C_j$ for $j < i$, we have, by Lemma 3.1, that $R_0^{>C_0}$ is coinitial in $R_i^{>C_i}$.

Note that we cannot repeat the argument of Lemma 4.1 exactly, for while $(R_0, C) \preceq (R_i, C)$, we do not know $(R_i, C) \preceq (R, C)$. But as $(R_0, C) \preceq (R, C)$, we have that for any $r_{01}, \ldots, r_{0i} \in C_0$,

$$
\begin{aligned}
(R, C) \quad \models \quad & \exists x_1 \in C^{>r_{01}} \ldots \exists x_i \in C^{>r_{0i}} \; x_1 < \cdots < x_i \\
& \exists y \in C^{>x_i} \forall x_{i+1} \in C^{>y} f(x_1, \ldots, x_i, x_{i+1}) > C \\
& \text{and } f \text{ is increasing as a function of } x_{i+1} \text{ on } C^{>y}.
\end{aligned}
$$

Since $(R_0, C) \preceq (R, C)$, (R_0, C) satisfies the same $\mathcal{L}_{R_0}$-formulas. Thus it also satisfies

$$
\begin{aligned}
(R_0, C) \quad \models \quad & \forall z_1 \in C \ldots \forall z_i \in C \, \exists x_1 \in C^{>z_1} \ldots \exists x_i \in C^{>z_i} x_1 < \cdots < x_i \\
& \exists y \in C^{>x_i} \forall x_{i+1} \in C^{>y} f(x_1, \ldots, x_i, x_{i+1}) > C \\
& \text{and } f \text{ is increasing as a function of } x_{i+1} \text{ on } C^{>y}.
\end{aligned}
$$

Now since $(R_0, C) \preceq (R_1, C)$, (R_1, C) satisfies the same sentence. So we may choose $a_1' \in C_1^{\geq a_1}$ (and thus $\mathrm{tp}_{\mathcal{L}_0}(a_1'/R_0) = \mathrm{tp}_{\mathcal{L}_0}(a_1/R_0)$) such that

$$
\begin{aligned}
(R_1, C) \quad \models \quad & \forall z_2 \in C \ldots \forall z_i \in C \, \exists x_2 \in C^{>z_2} \ldots \exists x_i \in C^{>z_i} a_1' < x_2 < \cdots < x_i \\
& \exists y \in C^{>x_i} \forall x_{i+1} \in C^{>y} f(a_1', x_2, \ldots, x_i, x_{i+1}) > C \\
& \text{and } f \text{ is increasing as a function of } x_{i+1} \text{ on } C^{>y}.
\end{aligned}
$$

Similarly, we may find $a_2' \in C_2^{\geq a_2}$ (and thus $\mathrm{tp}_{\mathcal{L}_0}(a_1' a_2'/R_0) = \mathrm{tp}_{\mathcal{L}_0}(a_1 a_2/R_0)$) such that

$$
\begin{aligned}
(R_2, C) \quad \models \quad & \forall z_3 \in C \ldots \forall z_i \in C \, \exists x_3 \in C^{>z_3} \ldots \exists x_i \in C^{>z_i} a_1' < a_2' < \cdots < x_i \\
& \exists y \in C^{>x_i} \forall x_{i+1} \in C^{>y} f(a_1', a_2', x_3, \ldots, x_i, x_{i+1}) > C \\
& \text{and } f \text{ is increasing as a function of } x_{i+1} \text{ on } C^{>y}.
\end{aligned}
$$

Continuing in this fashion, we find that

$$(R_i, C) \models \exists y \in C^{>a_i'} \, \forall x_{i+1} \in C^{>y} \, f(a_1', \ldots, a_i', x_{i+1}) > C$$
and f is increasing as a function of x_{i+1} on $C^{>y}$.

Note that $\mathrm{tp}_{\mathcal{L}_0}(\underline{a}'/R_0) = \mathrm{tp}_{\mathcal{L}_0}(\underline{a}/R_0)$, where $\underline{a}' = (a_1', \ldots, a_i')$ and $\underline{a} = (a_1, \ldots, a_i)$. Furthermore, the above sentence is witnessed by $y = h(\underline{a}')$ for some R_0-definable function h.

Let g be another R_0-definable function such that $h(\underline{a}') < g(\underline{a}')$ and $g(\underline{a}') \in C_i$ (so f is increasing at $g(\underline{a}')$). Since $f(\underline{a}', g(\underline{a}')) > C_i$, $f(\underline{a}', g(\underline{a}')) > r_0$ for some $r_0 \in R_0^{>C_0}$ by coinitiality of $R_0^{>C_0}$ in $R_i^{>C_i}$. Then $g(\underline{a}) < a_{i+1}$ (again by coinitiality of $R_0^{>C_0}$ in $R_i^{>C_i}$) but $f(\underline{a}, a_{i+1}) < r_0 < f(\underline{a}, g(\underline{a}))$, a contradiction with f being increasing in x_{i+1} on an interval containing $g(\underline{a})$ and a_{i+1}. This finishes the inductive argument.

Finally $(R_m, C) \preceq (R_n, C)$, by Theorem 3.1. $\qquad\qquad\square$

Remark 4.3 *Note that instead of choosing $a_1, \ldots, a_m \in C_n$ successively larger realizations of the type $C_0 < x < R_0^{>C_0}$ until C_m was cofinal in C_n, we could just as easily have chosen successively smaller realizations $a_1, \ldots, a_k$ of $C_0 < x < R_0^{>C_0}$, until $R_k^{>C_k}$ was coinitial in $R_n^{>C_n}$. Then the inductive argument above is easily modified to yield $(R_i, C) \preceq (R_{i+1}, C)$ for $i < k$, and Lemma 3.1 shows $(R_k, C) \preceq (R_n, C)$.*

What the above lemma shows is that if one builds up from (R_0, C) to (R, C) by taking successively larger elements of $C(R)$, each realizing the cut determined by C in the previous model, until one has a cofinal subset of $C(R)$ and then builds the rest of the way to R in any fashion, then each step from R_0 to R is an elementary extension. More precisely:

Lemma 4.4 *Suppose $(R_0, C) \preceq (R, C)$ and suppose C defines a cut in R_0 with property $*$. Let $(a_\alpha)_{\alpha < \kappa}$ be such that $R = R_0 \langle (a_\alpha)_{\alpha < \kappa} \rangle$, and set $R_\alpha = R_0 \langle (a_\beta)_{\beta < \alpha} \rangle$ and $C_\alpha = R_\alpha \cap C(\mathcal{R})$. Assume further that*

1. *there is $\gamma \leq \kappa$ such that $a_\alpha \in C_{\alpha+1}^{>C_\alpha}$ for all $\alpha < \gamma$, and C_γ is cofinal in $C(R)$.*

2. *there is $\delta \leq \kappa$ such that $R_\delta^{>C_\delta}$ is coinitial in $R^{>C(R)}$ and for all α with $\gamma \leq \alpha < \delta$, $a_\alpha \in R_{\alpha+1}^{>C_{\alpha+1}}$ but $a_\alpha < R_\alpha^{>C_\alpha}$.*

Then
$$(R_\alpha, C) \preceq (R_{\alpha+1}, C) \preceq (R, C)$$

for each $\alpha < \kappa$.

PROOF: Let γ be as indicated in the lemma. Note that $(a_\alpha)_{\alpha<\gamma}$ is a Morley sequence in q (in the sense of Lemma 2.11 with R_0 plays the role of R).

By Lemma 4.1, $(R_0, C) \preceq (R_1, C)$. Then by Lemma 2.11, C defines a cut in R_1 with property $*$. Suppose $\alpha_0 < \gamma$ is the first ordinal such that $(R_{\alpha_0}, C) \not\preceq (R_{\alpha_0+1}, C)$.

By applying Lemma 4.1 and then Theorem 3.1 at successor ordinals less than α_0, we see that $(a_\alpha)_{\alpha<\alpha_0}$ satisfies condition (2) of Lemma 2.11. Thus in (R_{α_0}, C), one has that C_{α_0} satisfies property $*$ and we may apply Theorem 3.1 to obtain $f(a_{\alpha_0})$ not in C_{α_0+1} but smaller than $R_{\alpha_0}^{>C_{\alpha_0}}$. Suppose the parameters in f are $a_{\beta_0}, \ldots, a_{\beta_n}$, where each $\beta_i < \alpha_0$. Then

$$R_0\langle a_{\beta_0}, \ldots a_{\beta_n}\rangle \not\preceq R_0\langle a_{\beta_0}, \ldots a_{\beta_n}, a_{\alpha_0}\rangle,$$

contradicting Lemma 4.2.

Note $(a_\alpha)_{\gamma\leq\alpha<\delta}$ is a Morley sequence in p (in the sense of Lemma 2.11) over R_γ, showing that the cut C_α in R_α has property $*$ for each $\gamma < \alpha < \delta$, by condition (1) of Lemma 2.11. Also note that $(a_\alpha)_{\delta\leq\alpha}$ satisfies condition (3) of Lemma 2.11 (where R_δ plays the role of R in Lemma 2.11), showing that property $*$ holds for the cut C_α in R_α for each $\delta < \alpha$.

For any $\alpha \geq \gamma$, the fact that C_γ is cofinal in $C(R)$ means that Theorem 3.1 implies $(R_\alpha, C) \preceq (R_{\alpha+1}, C)$. Taking unions at limit ordinals yields $(R_\beta, C) \preceq (R, C)$ for any $\beta < \kappa$. $\qquad\square$

Remark 4.5 *Suppose instead of having the sequence $(a_\alpha)_{\alpha<\kappa}$ in the order described in the hypotheses of Lemma 4.4, one has*

1. *$\gamma \leq \kappa$ such that $R_\gamma^{>C_\gamma}$ is coinitial in $R^{>C(R)}$ and for all α with $\alpha < \gamma$, $a_\alpha \in R_{\alpha+1}^{>C_{\alpha+1}}$ but $a_\alpha < R_\alpha^{>C_\alpha}$, and*

2. *$\delta \leq \kappa$ such that $a_\alpha \in C_{\alpha+1}^{>C_\alpha}$ for all $\gamma \leq \alpha < \delta$, and C_γ is cofinal in $C(R)$.*

then one can see that the conclusion of Lemma 4.4 still holds using the same proof except flipping the use of condition (1) and condition (2) of Lemma 2.11 and obtaining a contradiction with Remark 4.3 instead of Lemma 4.2.

Lemma 4.6 *Suppose $(R_1, C) \preceq (R_2, C)$ and suppose that (R_1, C) has property $*$. Let $C_1 = C(R_1)$. Suppose that $a \in R_2$ with $C_1 < a < R^{C_1}$ and $C(a)$. Suppose that $b \in R_2 \setminus R_1$ and there is no $\tilde{b} \in \mathrm{dcl}_{R_1}(b)$ with $C_1 < \tilde{b} < R^{>C_1}$. Then $(R_1\langle b\rangle, C) \preceq (R_1\langle ab\rangle)$.*

PROOF: First note that by Lemma 4.1, $(R_1, C) \preceq (R_1\langle a \rangle, C)$ and $(R_1, C) \preceq (R_1\langle b \rangle, C)$, and applying Lemma 2.11, we see that $(R_1\langle a \rangle, C)$ and $(R_1\langle b \rangle, C)$ have property $*$. Also note that a can be taken to be the beginning of a sequence as in the hypotheses of Lemma 4.4, and so, applying that lemma, we see that $(R_1\langle a \rangle, C) \preceq (R_2, C)$. Thus applying Lemma 4.1, we see that $(R_1\langle a \rangle, C) \preceq (R_1\langle ab \rangle, C)$.

Applying Theorem 3.1 to $(R_1\langle a \rangle, C) \preceq (R_1\langle ab \rangle, C)$, we see that, *a priori*, there are three possibilities:

1. $\mathrm{dcl}_{R_1\langle a \rangle}(b)$ contains no elements greater than $C(R_1\langle a \rangle)$ and less than $R_1\langle a \rangle^{>C(R_1\langle a \rangle)}$,

2. The elements of $\mathrm{dcl}_{R_1\langle a \rangle}(b)$ that are greater than $C(R_1\langle a \rangle)$ and less than $R_1\langle a \rangle^{>C(R_1\langle a \rangle)}$ are all in C,

3. The elements of $\mathrm{dcl}_{R_1\langle a \rangle}(b)$ that are greater than $C(R_1\langle a \rangle)$ and less than $R_1\langle a \rangle^{>C(R_1\langle a \rangle)}$ are all greater than C.

In the first two cases, Theorem 3.1 implies that $(R_1\langle b \rangle, C) \preceq (R_1\langle ab \rangle)$, while in the third case it implies that $(R_1\langle b \rangle, C) \not\preceq (R_1\langle ab \rangle)$.

Thus, for a contradiction, we assume that there is a R_1-definable function $f(x, y)$ with $C(R_1\langle a \rangle) < f(a, b) < R_1\langle a \rangle^{>C(R_1\langle a \rangle)}$. Let $B_x = \{y | f(x, y) > C\}$. Suppose first that $f(a, y)$ is increasing as a function of y. Then B_a, realized as a subset of $R_1\langle a \rangle$, includes everything to the right of the cut of b in $R_1\langle a \rangle$, since $f(a, b) > C(R_1\langle a \rangle)$. (And note that $\mathrm{tp}(b/R_1\langle a \rangle)$ is a cut since $f(a, y)$ maps this type to the cut corresponding to C in $R_1\langle a \rangle$.) B_a cannot include any $d \in R_1$ to the left of the cut of b, since then $C(R_1\langle a \rangle) < f(a, d) < f(a, b) < R_1\langle a \rangle^{>C(R_1\langle a \rangle)}$ contradicting $(R_1, C) \preceq (R_1\langle a \rangle, C)$. And since $R_1\langle b \rangle$ contains nothing realizing $\mathrm{tp}(a/R_1)$, $R_1\langle a \rangle$ can contain no points realizing $\mathrm{tp}(b/R_1)$. Thus if B_a included anything to the right of the cut of b in $R_1\langle a \rangle$, it would contain a element of R_1. Similarly if $f(a, y)$ is decreasing, B_a contains everything to the left of the cut of b and contains nothing to its right.

Note that $f(x, b)$ maps $\mathrm{tp}(a/R_1)$ to itself. Thus $f(x, b)$ is increasing on a neighborhood of a. So there is an interval I including b so that for all $y \in I$, $f(x, y)$ is increasing in x. We restrict ourselves to such an I.

Since $f(x, y)$ is increasing as a function of x, $x_1 < x_2$ implies that $B_{x_1} \subseteq B_{x_2}$. Choose $\tilde{a} \in C(R_1\langle a \rangle)$ with $\tilde{a} > a$. $B_a \subseteq B_{\tilde{a}}$ so $B_{\tilde{a}}$ also includes everything on one side of the cut of b, and we claim that $B_{\tilde{a}}$ also cannot contain any d on the other side of the cut of b. If it did contain a d on the

25

other side of b, then we could choose such a d in R_1. Then $R_1\langle a\rangle$ would satisfy the sentence "there is x in C such that B_x contains d", and R_1, being an elementary submodel, would satisfy the same sentence. Thus R_1 would contain an r in C such that $d \in B_r$. Thus $f(r,d) > C$, but $f(r,d) < f(a,d) < f(a,b)$, so $f(r,d) \in R_1$ would realize the cut of a in R_1.

So for any $\tilde{a} > a$ in $C(R_1\langle a\rangle)$, $B_{\tilde{a}} = B_a$. Thus $(R_1\langle a\rangle, C) \models \exists x_0 \bigcup_{x\in C} B_x = B_{x_0}$. Since $(R_1, C) \preceq (R_1\langle a\rangle, C)$, we see that $(R_1, C) \models \exists x_0 \bigcup_{x\in C} B_x = B_{x_0}$.

Say $a_0 \in R_1$ is such an x_0. (R_1, C) satisfies $\bigcup_{x\in C} B_x = B_{a_0}$, so $(R_1\langle a\rangle, C)$ does as well. Thus $(R_1\langle a\rangle, C) \models B_a = B_{a_0}$. Since $(R_1\langle a\rangle, C) \preceq (R_1\langle ab\rangle, C)$, $B_a = B_{a_0}$ holds in $(R_1\langle ab\rangle, C)$ as well. In particular, $f(a_0, b) > C$. Since $\neg C(f(a_0, b)$ is quantifier free, it holds in $(R_1\langle b\rangle, C)$ as well. As we also have $f(a_0, b) < f(a, b)$, we see that $C_1 < f(a_0, b) < R^{>C_1}$, contradicting our choice of b.

$\square$

Lemma 4.7 *Suppose $(R_0, C) \preceq (R, C)$, C defines a cut in R_0 with property $*$, and $b \in R \setminus R_0$. Then $(R_0\langle b\rangle, C) \preceq (R, C)$.*

PROOF:

Case 1: There is an element of $\operatorname{dcl}_{R_0}(b)$ in $C(R)^{>C(R_0)}$: We may assume that this element is b. We may find $(a_\alpha)_{\alpha<\kappa}$ as in the hypotheses of Lemma 4.4 with $a_0 = b$ to obtain $(R_0\langle b\rangle, C) \preceq (R, C)$.

Case 2: There is an element of $\operatorname{dcl}_{R_0}(b)$ greater than $C(R_0\langle b\rangle)$ but less than $R_0^{>C(R_0)}$: As in Case 1, but using Remark 4.5.

Case 3. No element of $\operatorname{dcl}_{R_0}(b)$ is greater than $C(R_0)$ but less than $R_0^{>C(R_0)}$: Choose $(a_\alpha)_{\alpha<\kappa}$ as in the hypotheses of Lemma 4.4 (and let R_α and γ be as in the proof of Lemma 4.4 as well) with the additional requirement that if at any stage α there is an element of $\operatorname{dcl}_{R_0}(b(a_\beta)_{\beta<\alpha})$ that could be chosen as a_α then one chooses such an element. If this happens at some stage prior to γ, let β denote that stage. Otherwise, let $\beta = \gamma$. Thus one has:

$$(R_0\langle b\rangle, C) \xrightarrow{\rho_1} (R_1\langle b\rangle, C) \xrightarrow{\rho_2} \ldots \xrightarrow{\rho_\beta} (R_\beta\langle b\rangle, C) \xrightarrow{\rho} (R, C)$$
$$\sigma_0 \Big\uparrow \qquad\qquad \sigma_1 \Big\uparrow \qquad\qquad\qquad \sigma_\beta \Big\uparrow$$
$$(R_0, C) \xrightarrow{\tau_1} (R_1, C) \xrightarrow{\tau_2} \ldots \xrightarrow{\tau_\beta} (R_\beta, C)$$

where each τ_i is an elementary embedding by Lemma 4.4, each σ_i is an elementary embedding by Lemma 4.1, and ρ is an elementary embedding

26

since either $\beta < \gamma$ and ρ continues the chain of elementary embeddings $(R_\alpha, C) \preceq (R_{\alpha+1}, C)$, or $\beta = \gamma$ and $C(R_\beta \langle b \rangle)$ is cofinal in $C(R)$.

Note that, except for possibly $i = \beta$, $R_i \langle b \rangle$ contains no x with $C(R_i) < x < R_i^{C(R_i)}$, so Lemma 4.6 shows that each ρ_i is an elementary embedding. Thus composing the ρ_i and ρ, one see that that $(R_0 \langle b \rangle, C) \preceq (R, C)$.

$\square$

Proposition 4.8 *Suppose (R_0, C) is substructure of (R, C) such that (R, C) can be embedded over R_0 into an elementary extension of (R_0, C_0), and suppose C defines a cut in R_0 with property $*$. Then $(R_0, C_0) \preceq (R, C)$.*

PROOF: This follows from Lemmas 4.1 and 4.7 by transfinite induction. $\square$

The above shows that, after adding constants for an elementary submodel of (R, C) where the cut C has property $*$, any substructure of (R, C) is an elementary submodel. This easily leads to quantifier elimination in the language with the new constants.

We now give a more concrete description of the constants needed. We use the following fact, found for example in Hodges [9] (p. 294, Theorem 6.5.1) (where his B and $\bar{a}$ are both A below).

Fact 4.9 *Let A be a substructure of C, and assume that every existential formula with parameters from A that is true in C is also true in A. Then there is an embedding of C into an elementary extension, D, of C, and this embedding can be chosen to be the identity on A.*

Lemma 4.10 *Fix a model (R, C) where the cut C has property $*$. Choose a sequence $(c_i)_{i \in I}$ cofinal in C and a sequence $(d_j)_{j \in J}$ coinitial in $R^{>C}$. Let R_0 be $\mathrm{dcl}_{\mathcal{L}}((c_i)_{i \in I}(d_j)_{j \in J})$. Then (R_0, C) has property $*$, and (R, C) can be embedded over R_0 into an elementary extension $(\widetilde{R}, C)$ of (R_0, C).*

PROOF: Note that by Proposition 2.2 (1), (R_0, C) clearly satisfies $*$. By Fact 4.9, it suffices to show that any existential statement with parameters from R_0 true in (R, C) is also true in (R_0, C). Note that any quantifier free formula in $\mathcal{L}_C$ is a finite disjunction of formulas of the form

$$\theta(x) \wedge C(f_1(x)) \wedge \cdots \wedge C(f_n(x)) \wedge \cdots \wedge \neg C(g_1(x)) \wedge \cdots \wedge \neg C(g_m(x))$$

where θ, each f_i, and each g_j are $\mathcal{L}_0$-definable over R_0 (and where x may be a tuple). By setting

$$f(x) = max(f_1(x), \ldots, f_n(x)) \text{ and } g(x) = min(g_1(x), \ldots, g_m(x)),$$

we see that every quantifier free formula is a finite disjunction of formulas

$$\varphi(x) = \theta(x) \wedge C(f(x)) \wedge \neg C(g(x)).$$

Note that in (R, C), $C(x)$ iff $\bigvee_{i \in I} x < c_i$, and $\neg C(x)$ iff $\bigvee_{j \in J} x > d_j$. Thus,

$$\varphi(R) = \theta(R) \wedge \bigvee_{i \in I} f^{-1}(R^{<c_i}) \wedge \bigvee_{j \in J} g^{-1}(R^{>d_j}).$$

Then the structure (R, C) eliminates quantifiers Now assume that $(R, C) \models \exists x \varphi(x)$ is witnessed by r. In particular, there are i, j such that r satisfies the formula $\theta(x) \wedge f(x) < c_i \wedge g(x) > d_j$. As this set is $\mathcal{L}_0$-definable over R_0, and R_0 considered as an $\mathcal{L}_0$-structure has definable Skolem functions, there is $r_0 \in \mathrm{dcl}_{\mathcal{L}_0}(R_0(c_i)_{i \in I}(d_j)_{j \in J})$ also contained in this set, and thus $(R_0, C) \models \exists x \varphi(x)$. $\square$

Theorem 4.11 *Fix a model (R, C) where the cut C has property $*$. Choose a sequence $(c_i)_{i \in I}$ cofinal in C and a sequence $(d_j)_{j \in J}$ coinitial in $R^{>C}$. Let R_0 be $\mathrm{dcl}_{\mathcal{L}}((c_i)_{i \in I}(d_j)_{j \in J})$ and let $\mathcal{L}_{R_0}$ be the language $\mathcal{L}$ with constant symbols for the c_i and d_j. We let T_{R_0} be the theory of (R, C) in $\mathcal{L}_{R_0}$. Then T_{R_0} is universally axiomatizable, has quantifier elimination and definable Skolem functions.*

PROOF: By Lemma 4.10, (R_0, C) has property $*$.

So we may apply Lemma 4.10 to embed a copy of (R, C) into an elementary extension $(\widetilde{R}, C)$ of (R_0, C). Then we apply Proposition 4.8 to see that (R_0, C) is an elementary substructure of the copy of (R, C), showing that the $\mathcal{L}_{R_0}$-theory of (R_0, C) is T_{R_0}.

By choosing a sufficiently saturated $(\widetilde{R}, C)$, universal axiomatizability follows from showing that every substructure of $(\widetilde{R}, C)$ is a submodel, and model completeness follows from showing every submodel of $(\widetilde{R}, C)$ is an elementary submodel. Since any $\mathcal{L}_{R_0}$-substructure contains R_0, both of these follow from Proposition 4.8.

Then the structure (R, C) eliminates quantifiers A model complete theory eliminates quantifiers precisely when $T^\forall$ has the amalgamation property. Thus, model completeness together with universal axiomatizability implies quantifier elimination. Furthermore, quantifier elimination together with uniform axiomatizability implies the existence of definable Skolem functions. $\square$

Corollary 4.12 *If $f \colon R \to R$ is (R, C)-definable, then there are R-definable functions $f_1, \ldots, f_k \colon R \to R$ such that for each $a \in R$, $f(a) = f_i(a)$ for some $i \in \{1, \ldots, k\}$.*

PROOF: This is a consequence of T_A having definable Skolem functions and a universal axiomatization. $\square$

5 Appendix – convex valuations with o-minimal residue field

Here we show that if C is a predicate for a convex subring (hence valuation ring) of R with o-minimal residue field, then the cut defined by C has property $*$.

In this section, we assume that $C = V$ is realized as a proper convex subring of its ambient o-minimal field. For an o-minimal field S we denote by $\overline{x}$ the residue of $x \in V(S)$, and we let

$$I_S = \{x \in S \colon 0 \le x \le 1\}.$$

We assume that $p \in S_1(\mathcal{R})$ is the global R-invariant type

$$\{r < x \colon r \in V(R)\} \cup \{x < r \colon r \in \mathcal{R}^{>V(R)}\}.$$

For $X \subseteq S^{1+n}$, we set

$$X(a) := \{(\underline{x}) \in S^n \colon (a, \underline{x}) \in X\}.$$

Definition 5.1 *For an o-minimal field S, we say that $(S, V) \models \Sigma(n)$ if there is $\epsilon_0 \in \mathfrak{m}^{>0}$ such that for each $\epsilon \in \mathfrak{m}^{>\epsilon_0}$,*

$$\overline{X(\epsilon_0)} = \overline{X(\epsilon)}$$

where $X \subseteq I_S^{1+n}$ is definable in the language of S with parameters from S, and $\mathfrak{m}$ is the maximal ideal of $V(S)$.

We shall use the following two facts.

Fact 5.2 ([12], Theorem 1.2) *The residue field with structure induced from R (equivalently, from (R, V), by [8], Theorem 4.6) is o-minimal iff $(R, V) \models \Sigma(1)$.*

Fact 5.3 (M., van den Dries [7], Theorem 1.2) *For any n,*

$$(R, V) \models (\Sigma(1) \Rightarrow \Sigma(n)).$$

Lemma 5.4 *Suppose $(R, V) \models \Sigma(1)$, t realizes $p|_R$, and let $R_t = R\langle t \rangle$. Suppose that for each $a \in R_t$ with $V(R) < a < R^{>V(R)}$ one has $V(a)$. Then $(R_t, V) \models \Sigma(1)$.*

PROOF: We shall denote by $\mathfrak{m}$ and $\mathfrak{m}_t$ the maximal ideal of $V(R)$ and of $V(R_t)$ respectively, and note that $\mathfrak{m}_t$ is the convex hull of $\mathfrak{m}$ in R_t. Assume to the contrary $(R_t, V) \models \neg\Sigma(1)$. Then there is an R-definable function $f \colon I_S^2 \to I_S$ such that for all $\epsilon \in \mathfrak{m}_t$ there is $\delta_\epsilon \in \mathfrak{m}_t^{>\epsilon}$ with

$$\overline{f(\frac{1}{t}, \epsilon)} \neq \overline{f(\frac{1}{t}, \delta_\epsilon)}.$$

By Fact 5.3, we can find $\epsilon_0 \in \mathfrak{m}^{>0}$ such that

$$\overline{\left(\Gamma f \cap (I_R \times \{\epsilon\} \times I_R)\right)} = \overline{\left(\Gamma f \cap (I_R \times \{\epsilon_0\} \times I_R)\right)} \in V(R)/\mathfrak{m} \qquad (1)$$

for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$. We shall show that then $\overline{f(\frac{1}{t}, \epsilon_0)} = \overline{f(\frac{1}{t}, \epsilon)} \in V(R_t)/\mathfrak{m}_t$ for all $\epsilon \in \mathfrak{m}_t^{>\epsilon_0}$.

Fix $\epsilon \in \mathfrak{m}_t^{>\epsilon_0}$ such that $\overline{f(\frac{1}{t}, \epsilon_0)} \neq \overline{f(\frac{1}{t}, \epsilon)}$ (where both residues are elements of $V(R_t)/\mathfrak{m}_t$). Since $\mathfrak{m}_t$ is the convex hull of $\mathfrak{m}$ in R_t, we may assume that $\epsilon \in \mathfrak{m}$. Since $(R, V) \models \Sigma(1)$, there are $a \in \mathfrak{m}^{>0}$ and $b \in I_R^{\geq \mathfrak{m}}$ such that

$$\overline{\Gamma f \cap ([a, b] \times \{\epsilon\} \times I_R)} \text{ and } \overline{\Gamma f \cap ([a, b] \times \{\epsilon_0\} \times I_R)} \qquad (2)$$
$$\text{are graphs of functions } [0, \bar{b}] \to V(R)/\mathfrak{m}.$$

Define $g(x) = |f(x, \epsilon) - f(x, \epsilon_0)|$ for $x \in [a, b] \subseteq R_t$. By 1 and 2, $g(x) \in \mathfrak{m}$ for all $x \in [a, b] \subseteq R$. After possibly shrinking $[a, b] \subseteq R$ subject to the condition that $a < \frac{1}{t} < b$, we may moreover assume that g is continuous and monotone on (a, b), contradicting $g(\frac{1}{t}) > \mathfrak{m}$. $\square$

30

Proposition 5.5 *If $(R, V) \models \Sigma(1)$, then the cut corresponding to V has property $*$.*

PROOF: Assume to the contrary that $\underline{t} = t_1, \ldots, t_n$ is a finite Morley sequence in $p|_R$ and $f \colon R^{n+1} \to R$ is $\mathcal{L}_0$-definable over R such that $f_{\underline{t}}$ takes a cofinal segment of $V(R)$ to a coinitial segment of the convex hull of $R^{>V(R)}$ in $\mathcal{R}$. Note that we may replace $\underline{t}$ with any other tuple satisfying the same o-minimal type over R, and thus we may assume that, letting $R_{\underline{t}} = R\langle \underline{t} \rangle$, one has $V(a)$ for each $a \in R_{\underline{t}}$ with $V(R) < a < R^{>V(R)}$. Thus by Lemma 5.4, $(R_{\underline{t}}, V) \models \Sigma(1)$.

After replacing $f_{\underline{t}}$ (restricted to a suitable interval) with $\frac{1}{f_{\underline{t}}^{-1}} \circ \frac{1}{x}$, we obtain a function that maps a cofinal segment of $\mathfrak{m}$ to a coinitial segment of $V(R)^{>\mathfrak{m}}$, where $\mathfrak{m}$ is the maximal ideal of $V(R)$. The function $f_{\underline{t}}$ is definable over R_t, decreasing, and maps a cofinal segment of $\mathfrak{m}_t$ (the maximal ideal of $V(R_t)$) to a proper convex subset of $V(R_t)^{>\mathfrak{m}_t}$ and, after composition with the residue map, we obtain a function definable in R_t whose image is not eventually constant on $\mathfrak{m}_t$, contradicting $(R_t, V) \models \Sigma(1)$.

$\square$

We now turn to Corollary 3.4 [8] in which we claim that $(R, V) \models \Sigma(1)$ whenever V is the convex hull of $\mathbb{Q}$. Our proof of this now requires the additional assumption of power-boundedness.

Theorem 5.6 *[Corollary 3.4 [8]] Consider (R, V) where V is the convex hull of $\mathbb{Q}$. If (R, V) has property $*$ or if R is power-bounded, then $(R, V) \models \Sigma(1)$.*

PROOF: First consider the case where R is power-bounded. Build extensions of R until one has R' in which the residue field associated to the convex hull of $\mathbb{Q}$ is $\mathbb{R}$. By power-boundedness, we may use Wilkie's inequality build up to R' with only residual extensions. Since the residue field is $\mathbb{R}$, it is o-minimal and (R', V) has property $*$. Now note that $V(R)$ is cofinal in $V(R')$ (since $V(R)$ contains $\mathbb{Q}$) and $R^{>V}$ is co-initial in $R'^{>V}$ (since the value groups of (R,V) and (R, V') are the same). Theorem 4.11 shows that (R, V) is an elementary submodel of (R', V) and hence $(R, V) \models \Sigma(1)$.

Now assume that (R, V) has property $*$. Work in a sufficiently saturated elementary extension $(\mathcal{R}, V)$. Note that each cut in $\mathbb{Q}$ is realized in $\mathcal{R}$. Take an element $a \in \mathcal{R}$ realizing such a cut. If $R\langle a \rangle$ contains no element $\tilde{a}$ with $V(R) < \tilde{a} < R^{>V(R)}$ then note that in $(R\langle a \rangle, V)$, V is again the convex hull of

$\mathbb{Q}$. If $R\langle a\rangle$ contains $\tilde{a}$ with $V(R) < \tilde{a} < R^{>V(R)}$ then there is an R-definable function, f, mapping the cut of a to the cut in R corresponding to V. If $f(a) > V$, then note that $V(R\langle a\rangle)$ is again the convex hull of $\mathbb{Q}$ (by Theorem 3.1 and Lemma 4.1). If $V(f(a))$, choose $b \in \mathcal{R}$ with $V(\mathcal{R}) < b < R^{>V(R)}$, and replace a with $f^{-1}(b)$. Again, $V(R\langle a\rangle)$ is the convex hull of $\mathbb{Q}$.

By Lemma 4.7, $(R\langle a\rangle, V) \preceq (\mathcal{R}, V)$. Thus we may repeatedly add realizations of cuts in $\mathbb{Q}$ while keeping V the convex hull of $\mathbb{Q}$ until we have built a model (R_1, V) which is an elementary extension of (R, V) and which has $\mathbb{R}$ as a residue field. This residue field is necessarily o-minimal, and hence $(R_1, V) \models \Sigma(1)$, as does (R, V). $\qquad\square$

References

[1] H. Adler *A geometric introduction to forking and thorn forking*, J. Math. Logic 9 (2009), no. 1, 1–20.

[2] Y. Baisalov, B. Poizat, *Paires de Structures O-Minimales.* J. Symb. Log. 63 (2): 570-578 (1998).

[3] E. Bar-Yehuda, A. Hasson, Y. Peterzil, *A theory of pairs for non-valuational structures.* J. Symb. Log. 84 (2), pp. 664-683 (2019).

[4] S. Cotter, S. Starchenko, *Forking in vc-minimal theories*, J. Symb. Log. 77 (4), pp. 1257–1271 (2012).

[5] M. A. Dickmann, *Elimination of quantifiers for ordered valuation rings.* J. Symb. Log. 52 (1): 116-128 (1987).

[6] L. van den Dries, A. H. Lewenberg, *T-convexity and tame extensions.* J. Symb. Log. 60: 74-102 (1995).

[7] L. van den Dries, J. Maříková, *Triangulation in o-minimal fields with standard part map*, Fund. Math. no. 2, 133-155.

[8] C. F. Ealy, J. Maříková, *Model-completeness of o-minimal fields with convex valuations*, J. Symb. Log. 80 (2015), no. 1, 234-250.

[9] W. Hodges, *Model Theory*, Encyclopedia of Mathematics and its Applications, Volume 42, Cambridge University Press, 1993.

[10] D. Macpherson, D. Marker, Charles Steinhorn, *Weakly o-minimal structures and real closed fields*, Trans. Amer. Math. Soc. 352 (2000), 5435-5483.

[11] D. Marker, C. I. Steinhorn, *Definable Types in O-Minimal Theories*, J. Symb. Log., Vol. 59, No. 1 (Mar., 1994), pp. 185-198.

[12] J. Maříková, *O-minimal residue fields of o-minimal fields* Ann. Pure Appl. Logic 162 (2011), no. 6, 457-464.

[13] M. C. Laskowski, C. Shaw, *Definable choice for a class of weakly o-minimal theories*, Archive for Math Logic 55 (2016), no. 5-6, 735–748.