TARSKI'S THEOREM FOR THE DEFINABLE SETTING

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Abstract. Working in any model theoretic structure, we single out a class of definable bipartite graphs that admit definable, close to perfect matchings. We use this result to prove a strengthening of a theorem by Tarski on paradoxical decompositions for the definable setting.

1. Introduction

A well-known theorem by Tarski from 1929 (see Tarski [10], and Tarski [11] for a proof) relates the non-existence of paradoxical decompositions with the existence of finitely additive measures. It can be stated as follows (see G. Tomkowicz, S. Wagon [12], p. 194, Theorem 11.1), where $\leq$ denotes the pre-order on the monoid M given by $\alpha \leq \beta$ if there is γ such that $\alpha + \gamma = \beta$.

Theorem 1.1 (Tarski). *Let $(M; 0, +, \alpha)$ be a commutative monoid with identity element 0 and a distinguished element α. Then the following are equivalent.*

(1) For every k, $(k+1)\alpha \not\leq k\alpha$.
(2) There is a homomorphism of monoids $\mu\colon M \to [0, \infty]$ such that $\mu\alpha = 1$.

From a model-theoretic perspective, a situation of interest is as follows. Given a structure $\mathcal{S}$, we let $\mathrm{Def}^{\mathrm{n}}(\mathcal{S})$ be the boolean algebra of definable subsets of S^n. We then set $M := \mathrm{Def}^{\mathrm{n}}(\mathcal{S})/_{\sim}$, the quotient of $\mathrm{Def}^{\mathrm{n}}(\mathcal{S})$ by the equivalence relation induced by isomorphisms, where isomorphisms are some singled-out definable bijections that we think of as being measure-preserving.[1] The binary operation $+$ is disjoint union, $0 = \emptyset/_{\sim}$ and $c = X/_{\sim}$, where $X \subseteq S^n$ is a definable set for which we would like to prove or disprove the existence of a finitely additive isomorphism-invariant measure on $\mathrm{Def}^{\mathrm{n}}(\mathcal{S})$ which assigns 1 to X. Such finitely additive measures on the boolean algebras of definable sets are called Keisler measures, and they have played a prominent role in model theory since the mid 2000's, especially in so-called NIP theories (see for instance Starchenko [9]).

We show that in the model-theoretic setting, (1) in Tarski's Theorem can be replaced, roughly, by "two copies of X almost embed isomorphically into

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[1] The particular notion of isomorphism will depend on the structure one is interested in. For instance, if $\mathcal{S}$ is an o-minimal expansion of a real closed field, then isomorphisms might be the definable C^1-diffeomorphisms $S^n \to S^n$ with Jacobian determinant equal to ± 1.

one copy of X". For the meaning of "Y almost isomorphically embeds into Z" (denoted by $Y \leq_0 Z$) see Definition 4.4. Intuitively, for every $m \in \mathbb{N}$, there is a definable $Y_0 \subseteq Y$ which is m times smaller than Y, and $Y \setminus Y_0$ isomorphically embeds into Z (see Definition 4.4). We obtain:

Theorem 1.2. *Let $\mathcal{S}$ be a model theoretic structure and $X \in \mathrm{Def}^{\mathrm{n}}(\mathcal{S})$. The following are equivalent.*

(1) $2X \not\leq_0 X$.
(2) There is a finitely additive invariant measure $\mu\colon \mathrm{Def}^{\mathrm{n}}(\mathcal{S}) \to [0, \infty]$ *such that* $\mu X = 1$.

Note that there is a finitely additive invariant measure $\mu\colon \mathrm{Def}^{\mathrm{n}}(\mathcal{S}) \to [0, \infty]$ assigning 1 to X iff there is a finitely additive invariant probability measure on the definable subsets of X. For the right to left implication, extend the probability measure by assigning ∞ to $Y \in \mathrm{Def}^{\mathrm{n}}(\mathcal{S})$ such that $Y/_\sim$ does not have a representative that is a definable subset of X.

The proof of Theorem 1.2 goes via a definability result for "close to perfect" matchings in certain definable bipartite graphs. In particular, given any structure $\mathcal{S}$, we show that if a definable k-regular bipartite graph is built up from finitely many graphs of isomorphisms, then it admits, for every m, a definable matching, which covers the vertex V outside of a subset whose m copies embed isomorphically into V. The proof was inspired by results of Lyons, Nazarov [7] and Elek, Lippner [3] in the Borel setting. Namely, we first show, for every K, the existence of a definable matching that does not admit augumenting paths of length K. Then, given such a matching M, we use the regularity of the graph to compute a bound on the relative size of Y_0, the set of vertices not covered by M, in terms of K. The relative size of Y_0 is given by its the number of copies that embed isomorphically into the vertex set. We note that by an example of Laczkovich [6], our matching theorem cannot be improved to yield a perfect definable matching in the conclusion.

Our matching theorem yields approximate weak cancellation in $\mathrm{Def}^{\mathrm{n}}(\mathcal{S})/_\sim$ (Theorem 5.2), by essentially the same argument as in the non-definable setting. This, in turn, yields that Condition (1) from Tarski's Theorem is equivalent to having 2 copies of the set in question almost embed into 1 copy (Corollary 6.1), and hence the definable Tarski's Theorem 1.2.

We remark that Maříková [8] contains a very a special case of our matching theorem. Namely, it is shown that certain 2-regular measure-preserving graphs definable in o-minimal structures admit definable matchings covering all vertices outside of a set whose standard part has arbitrarily small positive Lebesgue measure. This result is of course only meaningful if the standard part of the vertex set has non-empty interior. The proofs in [8] rely heavily on nontrivial results about o-minimal structures, in particular, on a reduction to the reals, where certain definable colorings exist. Here, we replace these proofs by purely combinatorial ones. In particular, the use of colorings is entirely avoided by using that the edge relation is a union of finitely many graphs of bijections.

For a general reference on graph theory see for instance Diestel [1].

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2. Notation and definitions

Given a structure $\mathcal{S} = (S; \dots)$, definable shall mean "definable in $\mathcal{S}$", possibly with parameters. We let $\mathrm{Def}^{\mathrm{n}}(\mathcal{S})$ be the boolean algebra of definable subsets of S^n, and for $X \in \mathrm{Def}^{\mathrm{n}}(\mathcal{S})$, we set

$$\mathrm{Def}X := \{Y \in \mathrm{Def}^{\mathrm{n}}(\mathcal{S})\colon Y \subseteq X\}.$$

A function $f\colon A \to S^n$, $A \subseteq S^m$, is said to be definable, if its graph

$$\Gamma f = \{(x, y) \in S^{m+n}\colon f(x) = y\}$$

is definable. By $\pi^n_m\colon S^n \to S^m$ we denote the projection onto the first m coordinates, whenever $1 \leq m \leq n$.

A *graph* $\mathcal{G}$ is a pair of sets (V, E), where $V \neq \emptyset$ is called the set of vertices, and $E \subseteq V^2$ is a symmetric and antireflexive relation whose elements are called edges. Our graphs are thus not oriented, have no loops, and we shall think of edges, when convenient, as two-element subsets of V. In this sense, a vertex v and an edge e are *incident* if $v \in e$. Two vertices v, w are *adjacent* if $(v, w) \in E$. The degree of a vertex is the number of edges incident with it. If $v \in e$, then we also say that e *covers* v. If $Y \subseteq E$, then $Y(V)$ denotes the set of vertices covered by the edges in Y. A graph is *definable* if V and E are definable. A subset of $X \subseteq V$ is *independent* if no two distinct vertices in X are incident with the same edge. Similarly, a set $Y \subseteq E$ is independent if no two distinct edges in Y share a vertex. A graph is called k-regular, if each vertex is incident with exactly k edges.

A graph (V, E) is *bipartite* if $V = A \dot{\cup} B$ and $E \subseteq A \times B$. Such A, B is called a *bipartition*. A definable bipartite graph is a definable graph with a definable bipartition.

A *matching* in a graph $\mathcal{G}$ is an independent set of edges. A matching is *perfect* if it covers V. Given a matching $M \subseteq E$, a path in $\mathcal{G}$ is *alternating* if its edges alternate belonging to M. An *augmenting path* is a simple alternating path such that the starting vertex and the final vertex are not covered by M. So an augmenting path is necessarily of odd length. A k-path is a path of length k, where the length of a path is its number of edges. A k-augmenting path is an augmenting path of length k. To *flip* an augmenting path p means to remove the edges of p that were in M from M, and to place the edges of p that were not in M into M. Note that, after flipping an augmenting path, M remains a matching.

The letters i, j, k, l, m, n, K, N are used to denote natural numbers, where $\mathbb{N} = \{0, 1, 2, \dots\}$.

3. Matchings without short augmenting paths

Let $\mathcal{S} = (S; \dots)$ be a first order-structure. In this section, we shall consider definable bipartite graphs built up from finitely many bijections.

Definition 3.1. Let $\mathcal{G}$ be an $\mathcal{S}$-definable bipartite graph with bipartition $A, B \subseteq S^n$ and edge relation $E \subseteq A \times B$. We call $\mathcal{G}$ *nice* if there is a definable finite partition $\{A_i\}$ of A, and for each i,

$$E \cap (A_i \times S^n) = \dot{\bigcup}_{j \in \mathcal{F}_i} \Gamma f_{ij},$$

where $\mathcal{F}_i$ is a finite set of definable bijections $f_{ij} \colon A_i \to B$.

Note that if $\mathcal{G}$ is nice, then there is N such that the degrees of the vertices of $\mathcal{G}$ are bounded by N. Also note that for every path $(x_0, x_1, \dots, x_k)$ starting in A there is a unique sequence $(f_1, \dots, f_k)$ of bijections from $\bigcup_i \mathcal{F}_i$ such that $x_{2i+1} = f_{2i+1}(x_{2i})$ and $x_{2i+2} = f_{2i+2}^{-1}(x_{2i+1})$. We shall call $(f_1, \dots, f_k)$ the *generating sequence* of the path $(x_0, \dots, x_k)$.

Proposition 3.2. *Let $\mathcal{G} = (V, E)$ be nice. Given a definable matching $M_0 \subseteq E$ and $K \in \mathbb{N}$ there is a definable matching $M \subseteq E$ which does not admit any augmenting paths of length $\leq 2K + 1$, and such that $M_0(V) \subseteq M(V)$.*

Proof. Let $\mathcal{G}$, $\{A_i\}$ and $\{\mathcal{F}_i\}$ be as in Definition 3.1. Note that we may assume that $\{A_i\} = \{A_1, \dots, A_l\}$ partitions $\pi_n^{2n} M_0$. This is because restrictions of definable maps to definable subsets of the original domain are again definable.

The proof proceeds by induction on K. At each stage we show that, after flipping a given set of augmenting paths, a) we obtain a matching, and b) we did not introduce any new augmenting paths of equal or shorter length than the flipped ones.

Let $i \in \{1, \dots, l\}$ be the first index such that there is a 1-augmenting path starting in A_i, say with generating sequence $f \in \mathcal{F}_i$. Then all vertices of A_i are starting vertices of an augmenting path of length 1 with generating sequence f, and we add Γf to M_0 to obtain M. Observe that M is a matching, because we only added edges between vertices not covered by M_0, and each vertex in $A_i \cup f(A_i)$ is incident with exactly one added edge, since f is a bijection. Also, we did not introduce any new 1-augmenting paths, $M_0(V) \subseteq M(V)$, and there are no more 1-augmenting paths starting in A_i.

Assume that k is maximal subject to $i \leq k \leq l$ and there are no 1-augmenting paths for M starting in $\bigcup_{j=1}^{k} A_j$. If $k = l$, then we are done. So suppose $k < l$ and A_{k+1} contains a starting vertex for a 1-augmenting path for M, say with generating sequence $f \in \mathcal{F}_{k+1}$. Then the set $X \subseteq A_{k+1}$ of starting vertices of 1-augmenting paths with generating sequence f is definable. We replace M by $M \cup \Gamma f|_X$ and observe that M is again a matching in which the previously covered vertices remain covered. No new 1-augmenting paths were introduced, and there are no more 1-augmenting paths starting in A_{k+1} with generating sequence f. One by one, we consider the remaining generating sequences $g \in \mathcal{F}_{k+1}$, and we proceed as before. Since $\mathcal{F}_{k+1}$ is finite, after finitely many steps we arrive at a matching M which does not admit any 1-augmenting paths starting in $\bigcup_{j=1}^{k+1} A_j$, and such that $M_0(V) \subseteq M(V)$.

Since $\{A_i\}$ is finite, after repeating finitely many times the steps starting with "Assume that k is maximal subject to ...", we arrive at a definable matching M which covers all vertices that were covered by M_0, and which does not admit any 1-augmenting paths (as an augmenting path starting in B ends in A).

Now assume that $M \subseteq E$ is a definable matching which does not admit any augmenting paths of length $\leq 2K-1$, where $K \geq 1$. We wish to construct a matching which does not admit any augmenting paths of length $\leq 2K+1$ and that covers all vertices covered by M.

Let i be the first index in $\{1, \dots, l\}$ such that A_i contains a (definable) non-empty set X of starting vertices of $(2K+1)$-augmenting paths, say with generating sequence $(f) = (f_1, f_2, \dots, f_{2K+1})$. We shall denote the set of these augmenting paths by $P_{X,(f)} \subseteq V^{2K+2}$. Flip the paths in $P_{X,(f)}$ and call the resulting relation M'.

Claim M' is a matching in $\mathcal{G}$.

Proof of Claim:

The only way the claim could fail is if there were $p, q \in P_{X,(f)}$ such that for some $i, j \in \{0, \dots, 2K+1\}$, $p_i = q_j$. Then $i \neq j$ and both i and j are even, or both are odd. Say $i < j$. Then $p_1, \dots, p_i, q_{j+1}, \dots, q_{2K+2}$ is an augmenting path of length $\leq 2K+1$, contradicting the inductive assumption. □ (Claim)

Claim Suppose p is an augmenting path for M' of length $\leq 2K+1$ starting in A. Then p is an augmenting path for M.

Proof of Claim: We shall use the following convention. If $q = (q_0, \dots, q_n)$ is a path, then we denote by $\hat{q}$ the path q traversed in the opposite direction, i.e. $\hat{q} = (q_n, q_{n-1}, \dots, q_0)$. If q and q' are paths such that $q_{|q|} = q'_0$, then $qq' = q_0 \dots q_{|q|} q'_1 \dots q_{|q'|}$ is the concatenation of q and q'.

If p is a 1-augmenting path for M' and $|p| = 1$, then p is an augmenting path for M, since flipping augmenting paths cannot uncover vertices. So we may assume that $|p| \geq 3$. We assert towards a contradiction that p is an augmenting path for M' but not for M. Then p contains at least one edge that was flipped as part of a path in $\mathcal{P}_{X,(f)}$. Let $c^1, \dots, c^l$ be the connected components of p that were flipped as part of a path in $\mathcal{P}_{X.(f)}$, in the order of their occurrence in p, that is, the maximal subpaths of p consisting of flipped edges such that if $i < j$ then every edge in c^i occurs in p before every edge in c^j. Note that the first and last edge of each c^i are necessarily covered by M' and not covered by M, because M' is a matching by the previous claim. For each i, let q^i be the unique path in $\mathcal{P}_{X,(f)}$ such that $\hat{c}^i$ is a subpath of q^i. Then $|q^i| = 2K+1$ for each i. For simplicity, we shall proceed assuming $l = 3$, but the same proof works for any $l \geq 1$.

We write p as a concatenation of paths $a^1 c^1 a^2 c^2 a^3 c^3 a^4$, where the a^j's are uniquely determined by the c^i's. Furthermore, we write $q^i = q^{i2} \hat{c}^i q^{i1}$ for $i = 1, 2, 3$. Then

$$\sum_{i=1}^{4} |a^i| + \sum_{i=1}^{3} |c^i| \leq 2K+1 \tag{3.1}$$

and

$$|q^{i1}| + |q^{i2}| + |c^i| = 2K + 1 \text{ for each } i \in \{1, 2, 3\}. \tag{3.2}$$

We aim to show that then at least one of

$$a^1 q^{11},\ q^{21}\hat{a}^2 q^{12},\ q^{31}\hat{a}^3 q^{22},\ \hat{a}^4 q^{32}$$

is an augmenting path for M of length $\leq 2K - 1$, which will yield a contradiction with our assumption on M. Clearly, each of these paths is an augmenting path for M. Suppose to the contrary that the four augmenting paths above are all of length $> 2K - 1$. Since they are augmenting paths, they are of odd length, hence

$$\begin{array}{rcl} |a^1| + |q^{11}| & \geq & 2K + 1 \\ |a^4| + |q^{32}| & \geq & 2K + 1 \\ |q^{21}| + |a^2| + |q^{12}| & \geq & 2K + 1 \\ |q^{31}| + |a^3| + |q^{22}| & \geq & 2K + 1. \end{array}$$

Then

$$\sum_{i=1}^{4} |a^i| + \sum_{i=1}^{3} \sum_{j=1}^{2} |q^{ij}| \geq 8K + 4,$$

and so by equations 3.1 and 3.2,

$$8K + 4 - 2\sum_{i=1}^{3} |c_i| \geq 8K + 4,$$

thus $0 \geq \sum_{i=1}^{3} c_i$, contradicting the assumption that at least one edge in p was flipped while flipping $\mathcal{P}_{X,(f)}$.

□ (Claim)

Given that there are only finitely many possible generating sequences for $(2K + 1)$-augmenting paths starting in A_i, using the above two claims, we can eliminate all $(2K+1)$-augmenting paths starting in A_i in finitely many steps while maintaining that M' is a matching and keeping covered vertices covered. Each subsequent A_j is handled similarly. After finitely many steps we thus arrive at a matching M'' which does not admit any augmenting paths of length $\leq 2K+1$ starting in A, and for which $M_0(V) \subseteq M''(V)$.Then M'' also does not admit any augmenting paths of length $\leq 2K + 1$ starting in B, since such a path would end in A.

□

4. Matchings in k-regular bipartite graphs

The matching result in this section applies to all nice k-regular graphs whose bijections are isomorphisms, i.e. they belong to some specified subcollection of the set of definable bijections which is closed under composition and inverses. While one could take for the collection of isomorphisms the collection of all definable bijections, some choices will be more meaningful than others, since isomorphisms are used to compare "sizes" of sets. For instance, if $\mathcal{S}$ is an o-minimal expansion of a real closed field, and we chose to have the collection of isomorphisms be equal to the collection of all definable bijections (rather than, for instance, the maps from the footnote on the first page), then, the conclusion of Theorem 4.5 would simply be that we can find a definable matching covering the vertex set up to a subset that could potentially be the entire vertex set. This is because there is a definable bijection between two $\mathcal{S}$-definable sets iff the sets have the same Euler characteristic and dimension (see van den Dries [2], p. 132, 2.11).

Definition 4.1. Given n, a set of *isomorphisms* $\mathcal{I}_n$ is any specified collection of definable bijections $S^n \to S^n$ that is closed under composition and inverses (so $\mathcal{I}_n$ forms a group under composition of functions and with identity element $x \mapsto x$).

Remark 4.2. Our proofs would still go through if in the above definition, we would only require that $\mathcal{I}_n$ is a pseudogroup of partial definable bijections $S^n \rightharpoonup S^n$.

Definition 4.3. Given a collection of isomorphisms $\mathcal{I}_n$, a *measure-preserving graph* $\mathcal{G} = (A\dot{\cup}B, E)$, with $A, B \subseteq S^n$, is a nice graph whose bijections are restricted isomorphisms, i.e. there are $\{A_i\}$ and $\{\mathcal{F}_i\}$ as in Definition 3.1, such that for each i and each $f \in \mathcal{F}_i$, there is $\widetilde{f} \in \mathcal{I}_n$ such that $f = \widetilde{f}|_{A_i}$.

Definition 4.4. Let $X, Y \in \mathrm{Def}^{\mathrm{n}}(\mathcal{S})$.

(1) If there is a finite definable partition $\{X_i\}$ of X and there are isomorphisms $f_i \colon S^n \to S^n$, such that $f_i(X_i) \cap f_j(X_j) = \emptyset$ whenever $i \neq j$, and $\bigcup_i f_i(X_i) \subseteq Y$, then we say that X *isomorphically embeds* into Y, and we write $X \leq Y$. If moreover $\bigcup f_i(X_i) = Y$, then we write $X \simeq Y$.

(2) We write nX for "union of n disjoint copies of X", and $\frac{p}{q}X \leq \frac{s}{t}Y$ is a short-hand for $ptX \leq qsY$.

(3) For $m > 0$, we write $X \leq_m Y$ if there are $p, q \in \mathbb{N}^{>0}$ such that $m \leq \frac{p}{q}$ and $pX \leq qY$.

(4) We write $X \leq_0 Y$ if for every $m > 0$ there is $X_0 \subseteq X$ such that $X_0 \leq_m X$ and $(X \setminus X_0) \leq Y$.

Note that if there is a Keisler measure ν on $\mathcal{S}$ which is invariant under isomorphisms, then $X \leq_m Y$ implies $m \cdot \nu X \leq \nu Y$.

In the proof of the theorem below, for $X \subseteq V$ and $M \subseteq E$, we set

$$N_M(X) := \{y \in V \colon \exists x \in X \, \exists e \in M \; e = (x, y)\}.$$

Theorem 4.5. *Let $\mathcal{G} = (A\dot{\cup}B, E)$ be a k-regular measure-preserving graph, where $k \geq 2$. Then for every positive integer m, there is a definable matching $M \subseteq G$ covering the vertex set outside of a subset Y_0 satisfying $Y_0 \leq_m A\dot{\cup}B$.*

Proof. Let $m \geq 1$ and K odd such that $m \leq \frac{k+1}{k}(1 + \frac{K-1}{2k})$. By Prop 3.2, we can find a definable matching $M \subseteq G$ which does not admit any augmenting paths of length $\leq K$. We shall show that M has the required property. For $1 \leq i \leq K$, we define Y_i as follows. If i is odd, then $Y_i = N_G(Y_{i-1})$. If i is even, then $Y_i = N_M(Y_{i-1})$.

Note that every vertex $v \in Y_1$ is incident with at most $k-1$ many vertices of Y_0, since Y_1 is covered by M. To obtain a lower bound on Y_1 in the sense of Definition 4.4, consider that

$$Y_1 \,\dot\cup\, (k-2)D_{k-1} \,\dot\cup\, (k-3)D_{k-2} \,\dot\cup\, \ldots \,\dot\cup\, D_2 \simeq kY_0,$$

where D_j consists of the vertices of Y_1 that are incident with exactly j vertices from Y_0. Thus

$$Y_1 \,\dot\cup\, (k-2)Y_1 \geq kY_0,$$

and hence $Y_1 \geq \frac{k}{k-1}Y_0$.

For i odd, $1 \leq i \leq K-2$, we have $Y_i \simeq Y_{i+1}$, since each vertex of Y_i is covered by M - otherwise there would be an augmenting path of length i.

Now let i be odd such that $1 < i \leq K$. Note that then Y_i contains Y_1, and $Y_i = Y_1 \dot\cup (Y_i \setminus Y_1)$. If $v \in Y_1$, then v is adjacent to some element from Y_0, hence can be adjacent to at most $(k-1)$ many vertices from Y_{i-1}. It follows that

$$Y_i \dot\cup (k-2) Y_1 \dot\cup (k-1)(Y_i \setminus Y_1) \geq kY_{i-1},$$

hence

$$(k-1)Y_i \dot\cup (Y_i \setminus Y_1) \geq kY_{i-1},$$

and so $kY_i \geq kY_{i-2} \dot\cup Y_1$. Inductively, we obtain $kY_i \geq (k + \frac{i-1}{2})Y_1$. Taking K for i yields

$$Y_K \geq \big(1 + \frac{K-1}{2k}\big)\frac{k+1}{k}Y_0.$$

So $Y_0 \leq_m A \dot\cup B$ by our choice of K.

□

The following is an immediate consequence of Theorem 4.5.

Corollary 4.6. *Let $\mathcal{G} = (A \dot\cup B, E)$ be a k-regular measure-preserving graph with $k \geq 2$. Suppose μ is a finitely additive invariant probability measure on* $\mathrm{Def}(A \dot\cup B)$. *Then for every $\epsilon \in \mathbb{R}^{>0}$ there is a definable matching $M \subseteq G$ such that M covers $A \dot\cup B$ apart from a definable subset of μ-measure $< \epsilon$.*

Here is a straight-forward corollary of the proof of Theorem 4.5 for the case when we are interested in matchings covering only the first part of the bipartition:

Corollary 4.7. *Let $\mathcal{G} = (A \dot\cup B, E)$ be a measure-preserving graph which is k-regular in A, and of maximal degree k in B. Then for every positive integer m there is a definable matching M covering A outside of a definable $Y_0 \subseteq A$ with $Y_0 \leq_m A$.*

It is easy to see that the above also holds for definable measure-preserving bipartite *multigraphs*, which will appear in the proof of the Weak Cancellation Law. Those are defined in the same way as definable measure-preserving bipartite graphs, except that one is additionally given a definable symmetric map $I \colon E \to \{1, \ldots, N\}$. For $e \in E$, the value $I(e)$ is called the *multiplicity* of e. The degree of a vertex is then the number of edges incident with it, counting multiplicities. So an edge (v, w) with $I(v, w) = i$ contributes i towards the degree of v.

Corollary 4.8. *If $\mathcal{G} = (A\dot{\cup}B, E)$ is a definable measure-preserving bipartite multigraph which is k-regular in A, and of maximal degree k in B, then for every $m > 0$ there is a definable matching covering A up to a definable $Y_0 \subseteq A$ such that $Y_0 \leq_m A$.*

5. Cancellation

We let K_{semi} be the Grothendieck semigroup of $\mathcal{S}$. That is, K_{semi} is the commutative monoid

$$K_{semi}(\mathcal{S}) = (\mathrm{Def}^{\mathrm{n}}(\mathcal{S})/_{\simeq}\,; 0, +),$$

where $0 = \emptyset/_{\simeq}$ and $+$ is disjoint union (recall that $\simeq$ was introduced in Definition 4.4). We define a pre-ordering on K_{semi} by $\alpha \leq \beta$ if there is γ such that $\alpha + \gamma = \beta$. Note that $A \leq B$ iff $A/_{\simeq} \leq B/_{\simeq}$ in K_{semi}. The Grothendieck semigroup of a structure has appeared in numerous model theoretic settings; see e.g. Krajíček, Scanlon [5], or Hrushovski, Peterzil, Pillay [4].

Definition 5.1. Let $\alpha, \beta \in K_{semi}$.

1. For $m > 0$, we write $\alpha \leq_m \beta$ to mean there are positive integers p, q with $m \leq \frac{p}{q}$ such that $p\alpha \leq q\beta$.
2. We write $\alpha \leq_0 \beta$ if for every $m > 0$ there are $\alpha', \alpha'' \in K_{semi}$ such that $\alpha' + \alpha'' = \alpha$, $\alpha' \leq_m \alpha$ and $\alpha'' \leq \beta$.

In [12], p. 177, a proof of the Weak Cancellation Law in type semigroups is presented in the general, non-definable setting. It uses the Hall-Rado-Hall infinite Marriage Theorem and is a variation on König's method. Essentially the same proof goes through here, except that we use Corollary 4.8 instead of the infinite Marriage theorem. We recount the proof for the reader's convenience.

Theorem 5.2 (Approximate Weak Cancellation Law)**.** *Let $\alpha, \beta \in K_{semi}$. If $k\alpha \leq k\beta$, then $\alpha \leq_0 \beta$.*

Proof. Suppose $k\alpha \leq k\beta$ is witnessed by $\theta\colon kA \to kB$, where $A/_{\simeq} = \alpha$ and $B/_{\simeq} = \beta$, and

$$kA = A_1\dot{\cup}A_2 \ldots \dot{\cup}A_k,$$

where $A_1 = A$ and ϕ_i witnesses $A \simeq A_i$. We set $\phi_1 = \mathrm{id}_A$, and $\underline{a} = (\phi_1(a), \ldots, \phi_k(a))$ for $a \in A$. Similarly,

$$kB = B_1\dot{\cup}B_2 \ldots \dot{\cup}B_k,$$

where $B_1 = B$, and ψ_i witnesses $B \simeq B_i$, with $\psi_1 = \mathrm{id}_B$, and $\underline{b} = (\psi_1(b), \ldots, \psi_k(b))$.

Let $\mathcal{H}$ be the bipartite multigraph with bipartition

$$\{\underline{a}\colon a \in A \text{ and } a_i = \phi_i(a)\} \dot{\cup} \{\underline{b}\colon b \in B \text{ and } b_i = \psi_i(b)\}$$

and edge $(\underline{a}, \underline{b})$ iff there are $1 \leq i, j \leq k$ such that $\theta(a_i) = b_j$. Then $\mathcal{H}$ is k-regular in the first part and of maximal degree k in the second part, so by Corollary 4.8, for $m > 0$ there is a definable matching M covering the first part of the bipartition outside of a definable set Y_0 with

$$mY_0 \leq \{\underline{a} \in R^k \colon a \in A \text{ and } a_i = \phi_i(a_1)\}.$$

Let

$$C_{ij} = \{a \in A \colon \exists b\big(b \in B \,\&\, (\underline{a}, \underline{b}) \in M \,\&\, \theta \circ \phi_i(a) = \psi_j(b)\big)\}$$

and

$$D_{ij} = \{b \in B \colon \exists a\big(a \in A \,\&\, (\underline{a}, \underline{b}) \in M \,\&\, \theta \circ \phi_i(a) = \psi_j(b)\big)\}.$$

Then the conclusion of the theorem is witnessed by the partition $\{C_{ij}\}$ of $A \setminus Y_0$, the collection $\{D_{ij}\}$ of pairwise disjoint subsets of B, and the maps $\psi_j^{-1} \circ \theta \circ \phi_i \colon C_{ij} \to D_{ij}$.

□

6. Paradoxical decompositions

We shall now use Theorem 5.2 to show that, roughly, if $(k+1)X \leq kX$ for some $k \geq 1$, then this is already witnessed by $k = 1$ and the same definable set X.

The proof of the below corollary of Theorem 5.2 is standard and can be found for instance in [12]. Again, we include it here for the reader's convenience.

Corollary 6.1. *Let $X \in \mathrm{Def^n}(\mathcal{S})$ be such that $(n+1)X \leq nX$. Then $2X \leq_0 X$.*

Proof. By substituting $(n+1)X \leq X$ into itself, we obtain

$$nX \geq (n+1)X = nX + X \leq (n+1)X + X = nX + 2X,$$

and after repeating this finitely many times

$$nX \geq nX + nX = 2(nX).$$

We now apply Theorem 5.2 to $nX \geq n(2X)$ to obtain $2X \leq_0 X$. □

Proof of Theorem 1.2. By Tarski's Theorem, it suffices to show that if $(k+1)X \leq kX$ for some k, then $2X \leq_0 X$, and that $2X \leq_0 X$ is an obstruction to the existence of an invariant finitely additive measure on $\mathrm{Def^n}(\mathcal{S})$ that assigns 1 to X.

The first implication follows from Corollary 6.1. For the latter, suppose $2X \leq_0 X$. Then there is $Y \in \mathrm{Def}(2X)$ such that $(2X \setminus Y) \leq X$ and $3Y \leq 2X$. If μ was an invariant finitely additive measure on $\mathrm{Def^n}(\mathcal{S})$ such that $\mu X = 1$, then $\mu(2X \setminus Y) \leq 1$ because $(2X \setminus Y) \leq X$. On the other hand, $\mu Y \leq \frac{2}{3}$, so $\mu(2X \setminus Y) \geq 2 - \frac{2}{3} > 1$, a contradiction.

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