Measuring definable sets in o-minimal fields

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May 31, 2014

Abstract

We introduce a non real-valued measure on the definable sets contained in the finite part of a cartesian power of an o-minimal field R. The measure takes values in an ordered semiring, the Dedekind completion of a quotient of R. We show that every measurable subset of R^n with non-empty interior has positive measure, and that the measure is preserved by definable C^1-diffeomorphisms with Jacobian determinant equal to ± 1.

1 Introduction

Let R be an o-minimal field, i.e. an o-minimal expansion of a real closed field. In [6], Hrushovski, Peterzil and Pillay ask, roughly, the following question: Let $B[n]$ be the lattice of all bounded R-definable subsets of R^n. Define an equivalence relation $\sim$ on $B[n]$ as follows: $X \sim Y$ if modulo a set of dimension $< n$ we have $\phi(X) = Y$ for some definable C^1-diffeomorphism ϕ with absolute value of the determinant of the Jacobian of ϕ at x equal to 1 for all $x \in X$. Suppose $X \in B[n]$ is of dimension n. Is there a finitely additive map $\mu\colon B[n] \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ which is $\sim$-invariant and such that $\mu X \in \mathbb{R}^{>0}$?

Note that for cardinality reasons it is impossible to find a real-valued measure that would assign a real non-zero value to every bounded definable set with non-empty interior in some big o-minimal field.

We remark that the answer to the question posed in [6] is yes if R is pseudo-real[1] in the sense of van den Dries ([2]): If there is an o-minimal field

[1] Let $\mathcal{L}$ be an expansion of the language of ordered rings, and let $T(\mathcal{L})$ be the collection of all $\mathcal{L}$-sentences true in all $\mathcal{L}$-expansions of the reals. A structure is called pseudo-real if it is a model of $T(\mathcal{L})$.

$\mathcal{S}$ (in the language $\mathcal{L}$) for which the answer to the question posed in [6] is no, then we can find definable bounded sets $X, Y \subseteq \mathcal{S}^n$ and a positive integer m so that $X \not\sim \emptyset$ and $(m+1)X\dot{\cup}Y \sim mX$, where $(m+1)X$ is the disjoint union of $m+1$ copies of X (see [6], Proposition 5.5, p. 576). But this fact is expressible by a parameter-free first-order sentence in $\mathcal{L}$, and this sentence is false in all $\mathcal{L}$-expansions of the reals, hence our structure is not pseudo-real.

While the framework of o-minimality was developed with a view towards structures on the reals (see Shiota [10] and van den Dries [3]), it is well-known that not all o-minimal structures are pseudo-real. More concretely, Lipshitz and Robinson show in [7] that the field of Puiseux series $\bigcup_n \mathbb{R}((t^{\frac{1}{n}}))$ in t over $\mathbb{R}$ expanded by functions given by overconvergent power-series (henceforth the L-R field) is o-minimal, and Hrushovski and Peterzil show in [5] that the L-R field is not pseudo-real.

Let V be the convex hull of $\mathbb{Q}$ in R. Then V is a convex subring of R, hence a valuation ring. Let $\pi\colon V \to \mathbf{k}$ be the corresponding residue/standard part map. The corresponding residue field $\mathbf{k}$ is the ordered real field $\mathbb{R}$ if R is at least ω-saturated. In [1], Berarducci and Otero define a measure on the lattice $SB[n]$ of all strongly bounded definable subsets of R^n, i.e. the definable subsets of V^n. Assuming that R is at least ω-saturated, one way to define the Berarducci-Otero measure is to assign to $X \in SB[n]$ the Lebesgue measure of πX. It was shown in [8] that the Berarducci-Otero measure is $\sim$-invariant, which yields a partial answer to the question posed in [6]: The answer is yes whenever the set $X \in B[n]$ in question is contained in V^n, and πX has non-empty interior. However, the Berarducci-Otero measure assigns zero to every set whose standard part has empty interior.

In this paper we drop the requirement of the measure being real-valued. We define a map $\mu\colon SB[n] \to \widetilde{V}$, where $\widetilde{V}$ is an ordered semiring, such that for all $X, Y \in SB[n]$, $\mu(X\dot{\cup}Y) = \mu X + \mu Y$, and $\mu X > 0$ iff the interior of X is nonempty (see Theorem 5.9). The underlying set of $\widetilde{V}$ is constructed as the Dedekind completion of a quotient of $V^{\geq 0}$. Taking a quotient of $V^{\geq 0}$ serves the purpose of identifying lower and upper measures. Having a definition in terms of both lower and upper measure yields Lemmas 5.2 and 5.3 - both crucial in proving invariance under a change of variables formula. On the non-negative part of the maximal ideal of V, the equivalence relation under consideration is, in general, strictly coarser than the one induced by the standard valuation (see Example 4.5 to see why this is necessary). Consequently, even though we could extend the equivalence relation on $V^{\geq 0}$ to an equivalence relation on $R^{\geq 0}$ while maintaining an ordered semiring structure, the semiring operations would not be compatible with the operations on $R^{\geq 0}$ anymore. This forces us to restrict the measurable sets to $SB[n]$ (see the last bullet point of Remark 3.9).

For measurable sets whose standard part has non-empty interior our measure agrees with the Berarducci-Otero measure. In fact, the minimal ring that embeds the maximal cancelative quotient of $\widetilde{V}$ is $\mathbb{R}$. On the collection of strongly bounded definable sets whose standard part has empty interior, μ resembles a dimension function: There, we have $\mu(X\dot{\cup}Y) = \max\{\mu X, \mu Y\}$, and if $\mu X < \mu Y$, then X can be isomorphically embedded (in the sense of [6]) into finitely many copies of Y (this follows from Lemmas 5.2 and 5.3). We do not know if the strict inequality above can be replaced by a nonstrict one. Corollary 5.7 shows that μ has the analogue of the $\sim$-invariance property defined in [6] (see Definition 5.1 for a precise statement).

In the case when the value group of the standard valuation of R embeds into the ordered additive group $\mathbb{R}$ (this case includes the L-R field), we can modify the definition of μ to obtain a finitely additive measure on all of $B[n]$. This measure takes values in the Dedekind completion of the value group of the standard valuation. It agrees with μ for sets $X \in SB[n]$ so that $\operatorname{int}(\pi X) = \emptyset$, but assigns the same value to all sets $X \in SB[n]$ with $\operatorname{int}(\pi X) \neq \emptyset$ (see Theorem 6.6).

We thank Michel Coste and Marcus Tressl for their advice. The first author whishes to thank the second author for his hospitality during a visit to Nagoya, Japan. We thank the anonymous referee for suggestions improving the exposition of this paper.

2 Notation and conventions

The letters k, l, m, n denote non-negative integers.

Let M be a structure. Then M-*definable* (or simply *definable*, if M is clear from the context) means definable in the language of M, with parameters from M. We denote by $\operatorname{Def}^n(M)$ the collection of all M-definable subsets of M^n.

We fix V to be the convex hull of $\mathbb{Q}$ in R. Then V is a convex subring of R, hence a valuation ring, with residue (standard part) map $\pi\colon V \to \mathbf{k}$, maximal ideal $\mathfrak{m}$, and (ordered) residue field $\mathbf{k}$. For $X \subseteq R^n$ we set $\pi X =$

$\pi(X \cap V^n)$. We denote by v the corresponding valuation $R \to \Gamma \cup \{\infty\}$, where $\Gamma = R^\times/(V \setminus \mathfrak{m})$ is the (divisible ordered abelian) value group.

Let M be an o-minimal structure. For $k < n$ we denote by p^n_k the projection map $M^n \to M^k$ given by $x \mapsto (x_1, \ldots, x_k)$. If $Y \subseteq M^n$ is definable and non-empty and $x \in M^n$, then

$$d(x, Y) := \inf\{d(x, y) : \ y \in Y\},$$

where $d(x, y)$ is the euclidean distance between x and y. For $X, Y \subseteq M^n$ we write $X \subseteq_0 Y$ if $\dim(X \setminus Y) < n$, and $X =_0 Y$ if $X \subseteq_0 Y$ and $Y \subseteq_0 X$. If $f \colon X \to M$, where $X \subseteq M^n$, is a function, then

$$\Gamma f := \{(x, y) : \ x \in X \text{ and } f(x) = y\}$$

is the graph of f.

For $X \subseteq R$ we set $X^{\geq r} := \{x \in X \colon \ x \geq r\}$. The sets $X^{\leq r}$, $X^{<r}$, and $X^{>r}$ are defined similarly. If Y is another subset of R, then $X^{>Y}$ is the set

$$\{x \in X : x > y \ \text{ for all } y \in Y\}.$$

The set $X^{<Y}$ is defined similarly.

A box in R^n is a set of the form $[a_1, b_1] \times \cdots \times [a_n, b_n]$, where $a_i < b_i$ and $a_i, b_i \in R^{>0}$.

If $X \subseteq M^n$, then $\mathrm{cl}(X)$ denotes the closure of X and $\mathrm{int}(X)$ denotes the interior of X with respect to the interval topology on M.

3 The set of values $\widetilde{V}$

In this section we define the set of values $\widetilde{V}$ of our measure, and we show that it can be equipped with the structure of an ordered semiring.

First, we define an equivalence relation $\sim$ on $V^{\geq 0}$.

Definition 3.1 *Let $x, y \in V^{\geq 0}$. Then $x \sim y$ if either*

- *both x and y are in $\mathfrak{m}^{\geq 0}$, and*

$$y^q \leq x \leq y^p \text{ for all } p, q \in \mathbb{Q}^{>0}, \ p < 1, \ q > 1, \ \text{or}$$

- *both x and y are $> \mathfrak{m}$, and $\pi x = \pi y$.*

Note that the ordering $\leq$ on R induces an ordering $\leq$ on $V^{\geq 0}/\sim$. For $x \in V^{\geq 0}$ we denote by $[x]$ the $\sim$-equivalence class of x.

In the next definition a *Dedekind cut in* $V^{\geq 0}/\sim$ is the union of a downward closed subset of $V^{\geq 0}/\sim$ without a greatest element with the set $V^{<0}/\sim$, where $\sim$ is extended to $V^{<0}$ by setting $x \sim y$ iff $-x \sim -y$, for $x, y \in V^{<0}$.

Definition 3.2 *We let $\widetilde{V}$ be the collection of all Dedekind cuts in $V^{\geq 0}/\sim$. We define an ordering $\leq$ and binary operations $+$ and $\cdot$ on $\widetilde{V}$ as follows. Let $X, Y \in \widetilde{V}$. Then*

a) $X \leq Y$ *iff* $\forall x \in \bigcup X \;\; \exists y \in \bigcup Y$ *with* $x \leq y$.

b) $X + Y := \{x + y : \, x \in \bigcup X \,\&\, y \in \bigcup Y\}/\sim$.

c) $X \cdot Y := \{x \cdot y : \, x \in \bigcup X^{\geq 0} \,\&\, y \in \bigcup Y^{\geq 0}\}/\sim \cup \, V^{<0}/\sim$.

For $a \in V^{\geq 0}$ we denote by $\widetilde{a}$ the cut

$$\{[x] : \; x \in V^{\geq 0} \text{ and } [x] < [a]\} \cup \; V^{<0}/\sim .$$

Next, we show that $+$ and $\cdot$ are well-defined, and that $\sim$ is a congruence. The lemma below is used throughout the paper without explicit reference.

Lemma 3.3 *Let $x, y \in \mathfrak{m}^{>0}$ and suppose $v(x) = v(y)$. Then $x \sim y$.*

Proof: First note that $x \sim nx$ for all n: If $p \in \mathbb{Q}^{>0}$, $p < 1$, then

$$v(x^p) = p \cdot v(x) < v(x) = v(nx),$$

hence $nx \leq x^p$.

Now assume $x < y$ (the other cases are similar). Since $v(x) = v(y)$, we have $\frac{y}{x} < n$ for some n. Hence $x < y < nx$, and so $x \sim y$. $\square$

Remark 3.4 We do not have $x \sim y$ iff $v(x) = v(y)$ on $\mathfrak{m}^{>0}$. To see this assume that R is ω-saturated, let $x \in \mathfrak{m}^{>0}$, and let y be any element realizing the type $p(z)$ consisting of all formulas $nx < z < x^p$, where $n = 1, 2, \ldots$ and p ranges over all positive rationals < 1. Then $x \sim y$ but $v(x) \neq v(y)$.

Lemma 3.5 *Let $X, Y \in \widetilde{V}$. Then $X + Y \in \widetilde{V}$.*

PROOF: It is clear that $X + Y$ is downward closed and contains $V^{<0}/\sim$. It is left to show that it does not have a greatest element. Let $x \in \bigcup X$ and $y \in \bigcup Y$. We may assume $x \leq y$.

If $y > \mathfrak{m}$, take $y' \in \bigcup Y$ so that $[y] < [y']$. Then $|y - y'| > \mathfrak{m}$, so $(x + y) - (x + y') > \mathfrak{m}$, hence $[x + y] < [x + y']$.

So suppose $y \in \mathfrak{m}^{>0}$. Let $y' \in \bigcup Y$ be such that $y < y^p < y'$ for some $p \in \mathbb{Q}^{>0}$ with $p < 1$. Then

$$v(x + y) = v(y) > p \cdot v(y) \geq v(y') = v(x + y'),$$

and so $[x + y] < [x + y']$ because $y \not\sim y^p$.

The case when $Y = \widetilde{0}$ is clear. □

Lemma 3.6 *Let $x, y \in V^{\geq 0}$. Then $\widetilde{x} + \widetilde{y} = \widetilde{x + y}$.*

PROOF: We may assume that $x \leq y$. It suffices to show that if $x' \sim x$ and $y' \sim y$, then $x' + y' \sim x + y$, and if $z \sim x + y$, then there are $x' \sim x$ and $y' \sim y$ so that $z = x' + y'$. The cases when $y = 0$ and when $y > \mathfrak{m}$ are clear.

So suppose $y \in \mathfrak{m}^{>0}$. If $x' \sim x$ and $y' \sim y$, then $v(x' + y') = v(y')$ and $v(x + y) = v(y)$, so

$$x' + y' \sim y' \sim y \sim x + y.$$

If $z \sim x + y$, then, since $v(x + y) = v(y)$, we have $z \sim y$, and so $x' = x$ and $y' = z - x$ work. □

Lemma 3.7 *Let $X, Y \in \widetilde{V}$. Then $X \cdot Y \in \widetilde{V}$.*

PROOF: It is clear that $X \cdot Y$ is a downward closed subset of $V/\sim$ containing $V^{<0}/\sim$. It is left to show that $X \cdot Y$ does not have a greatest element. The case when there is $x \in (\bigcup X)^{>\mathfrak{m}}$ and $y \in (\bigcup Y)^{>\mathfrak{m}}$ is clear, as is the case when $X = \widetilde{0}$ or $Y = \widetilde{0}$.

So suppose $x \in \bigcup X$ and $y \in \bigcup Y$ and assume $x \leq y$. If $x \in \mathfrak{m}^{>0}$ and $y > \mathfrak{m}$, then $[xy] < [x'y]$ for any $x' \in \bigcup X$ with $[x] < [x']$. If $x \in \mathfrak{m}^{>0}$ and $y \in \mathfrak{m}^{>0}$, then we can find $p \in \mathbb{Q}^{>0}$, $p < 1$ so that $x < x^p < x'$ and $y < y^p < y'$ for some $x' \in \bigcup X$ and $y' \in \bigcup Y$. Then $xy < x^p y^p < x'y'$, hence $[xy] < [x'y']$. □

Lemma 3.8 *Let $x, y \in V^{\geq 0}$. Then $\widetilde{x} \cdot \widetilde{y} = \widetilde{xy}$.*

Proof: We may assume that $x \leq y$. It suffices to show that if $x' \sim x$ and $y' \sim y$, then $x'y' \sim xy$, and if $z \sim xy$, then there are $x' \sim x$ and $y' \sim y$ so that $z \sim x'y'$. It is easy to check that the lemma holds if $x, y > \mathfrak{m}$ or if $x = 0$.

So suppose $x \in \mathfrak{m}^{>0}$, and let $x' \sim x$ and $y' \sim y$. If $y > \mathfrak{m}$, then $v(x'y') = v(x')$ and $v(xy) = v(x)$, hence $x'y' \sim x' \sim x \sim xy$. If $y \in \mathfrak{m}$, then $x'y' \sim xy$ is immediate from the definition of $\sim$.

Now let $z \sim xy$ and assume $xy < z$. It suffices to prove that $x \sim \frac{z}{y}$ (as then $z = \frac{z}{y} \cdot y \in \bigcup X \cdot Y$). Assume towards a contradiction that this is not the case. Then, as $x < \frac{z}{y}$, we would have $x^p < \frac{z}{y}$ for a positive rational $p < 1$. Moreover, since $xy \sim z$, we have $z \leq x^q y^q$ for all positive rationals $q < 1$. Thus $yx^p < x^q y^q$ for all $q < 1$, $q \in \mathbb{Q}^{>0}$. Then $x^{p-q} < y^{q-1}$ for all $q < 1$, $q \in \mathbb{Q}^{>0}$. For $q = \frac{p+1}{2} < 1$ we obtain $x^{\frac{p}{2}-\frac{1}{2}} < y^{\frac{p}{2}-\frac{1}{2}}$, where $\frac{p}{2} - \frac{1}{2} < 0$ (as $p < 1$), a contradiction with $x \leq y$.

The case when $z \sim xy$ and $z < xy$ is handled similarly and left to the reader. □

From now on we shall assume that R is ω-saturated, in order to have $\mathbf{k} = \mathbb{R}$. This is no loss of generality: By Theorem 3.3 in [4], for any elementary extension R' of R, the structure (R', V'), where V' is the convex hull of $\mathbb{Q}$ in R', is an elementary extension of (R, V).

Remark 3.9

- It is now easy to check that $(\widetilde{V}, \leq, +, \cdot, \widetilde{0}, \widetilde{1})$ is an ordered semiring.
- The Dedekind completion of $V^{>\mathfrak{m}}/\sim$ is $\mathbb{R}^{>0}$. We shall thus feel free to identify this part of $\widetilde{V}$ with $\mathbb{R}^{>0}$. For $a \in \mathbb{R}^{>0}$ we shall sometimes write $\widetilde{a}$ to indicate that a is viewed as an element of $\widetilde{V}$. Since R is ω-saturated, for any $a \in \mathbb{R}^{>0}$, $\widetilde{a} = \widetilde{r}$ for some $r \in V^{>\mathfrak{m}}$.
- Let $X, Y \in \widetilde{V}$.

  i) If $X \in \mathbb{R}^{>0}$ and $Y \notin \mathbb{R}^{>0}$, then $X + Y = X$.

  ii) If $X \notin \mathbb{R}^{>0}$ and $Y \notin \mathbb{R}^{>0}$, then $X + Y = \max\{X, Y\}$.
- We could extend Definition 3.1 to all of $R^{\geq 0}$ by setting $x \sim y$ iff $x^{-1} \sim y^{-1}$ for $x, y \in R^{>V}$, and the set of all Dedekind cuts in $R^{\geq 0}/\sim$ could be made into an ordered semiring similarly as in Definition 3.2. However, $\sim$ is not a congruence with respect to $\cdot$ when considered as

an equivalence relation on $R^{\geq 0}$. To see this, consider the product of ϵ and $\frac{1}{\epsilon}$ for $\epsilon \in \mathfrak{m}^{>0}$. We have $\widetilde{\epsilon \cdot \frac{1}{\epsilon}} = \widetilde{1}$, but $(n\epsilon) \sim \epsilon$, hence $n \in \bigcup \widetilde{\epsilon} \cdot \widetilde{\frac{1}{\epsilon}}$ for all $n = 1, 2, \ldots$. This would force us to assign to the box $[0, \epsilon] \times [0, \frac{1}{\epsilon}]$ measure $> \widetilde{n}$ for all n. In general, this problem cannot be fixed by identifying all of $\widetilde{R} \cap \mathbb{R}^{>0}$: Let $a, b \in \mathfrak{m}^{>0}$, $a < b$, be such that $a \sim b$ but $v(a) \neq v(b)$. Then there is $c \in R^{>V}$ with $\widetilde{c} < \widetilde{a} \cdot \widetilde{\frac{1}{b}} = \widetilde{1}$.

The special case when $v(a) = v(b)$ iff $a \sim b$ for all $a, b \in \mathfrak{m}^{\geq 0}$ will be dealt with in the last section of this paper.

4 Measuring definable subsets of $[0, 1]^n$

In this section, we define the lower and upper measures of definable sets contained in $[0, 1]^n$, and we show that they conincide. This yields a measure on the definable subsets of $[0, 1]^n$ which is then extended to a measure on the definable subsets of V^n in Section 5.

We shall consider the structure $\mathbb{R}_0$, which has as underlying set $\mathbb{R}$, and whose basic relations are the sets πX, where $X \in \mathrm{Def}^n R$ for some n. As a weakly o-minimal structure on the reals, $\mathbb{R}_0$ is necessarily o-minimal. We shall use the facts below; the first one is Proposition 5.1, p. 188, in [8], the second one is extracted from the proof of Lemma 2.15, p. 124, in [9], and the third is Corollary 2.5, p. 120 in [9].

Fact 4.1 *Let $X \in \mathrm{Def}^n(\mathbb{R}_0)$. Then there is $Y \in \mathrm{Def}^n(R)$ so that $\pi Y = \mathrm{cl}(X)$.*

Fact 4.2 *Let $X, Y \in \mathrm{Def}^n(R)$ be non-empty. Then there is $\epsilon \in \mathfrak{m}^{>0}$ so that $\pi(X \cap Y^\epsilon) = \pi X \cap \pi Y$, where $Y^\epsilon = \{x \in R^n : d(x, Y) \leq \epsilon\}$.*

Fact 4.3 *Let $X \in \mathrm{Def}^n(R)$, and suppose $\mathrm{int}(\pi X) \neq \emptyset$. Then there is a box $B \subseteq X$ with $\mathrm{int}(\pi B) \neq \emptyset$.*

Definition 4.4 *1. Let $X \subseteq [0, 1]^n$ be an $(i_1, \ldots, i_n)$-cell. We define the* lower measure $\underline{\mu}$ *and* upper measure $\overline{\mu}$ *of X by induction on n.*

(a) If X is a (0)-cell, then $\underline{\mu}X = \overline{\mu}X = 0$. If $X = (a, b)$ where $a < b$, then

$$\underline{\mu}X = \overline{\mu}X = \widetilde{b - a} \in \widetilde{V}.$$

(b) *Suppose* $\underline{\mu}X$ *and* $\overline{\mu}X$ *have been defined for* $(i_1, \dots, i_n)$*-cells. If* X *is an* $(i_1, \dots, i_{n+1})$*-cell so that* $i_j = 0$ *for some* $j \in \{1, \dots, n+1\}$*, then* $\underline{\mu}X = \overline{\mu}X = 0$*. If* $X = (f, g)$ *is an* $(i_1, \dots, i_{n+1})$*-cell so that* $i_j = 1$ *for all* $j \in \{1, \dots, n+1\}$*, then set* $h = g - f$ *and define* $\underline{\mu}X$ *to be the supremum of*

$$\sum_{i=1}^{k} \widetilde{z}_{i-1} \cdot \underline{\mu}(h^{-1}[z_{i-1}, z_i])$$

as $k \to \infty$ *and* $z_0, \dots, z_k$ *range over all elements of* $[0,1]_R$ *with*

$$0 = z_0 < \cdots < z_k = 1.$$

The upper measure $\overline{\mu}X$ *is defined to be the infimum of*

$$\sum_{i=1}^{k} \widetilde{z}_{i} \cdot \overline{\mu}(h^{-1}[z_{i-1}, z_i])$$

as $k \to \infty$ *and* $z_0, \dots, z_k$ *range over all elements of* $[0,1]_R$ *with*

$$0 = z_0 < \cdots < z_k = 1.$$

2. *Let* $X \subseteq [0,1]^n$ *be definable, and let* $\mathcal{D}$ *be a decomposition of* R^n *into cells that partitions* X*. Suppose* $X = D_1 \cup D_2 \cup \cdots \cup D_k$*, where all* $D_i \in \mathcal{D}$*. Then* $\underline{\mu}_{\mathcal{D}}X = \sum_{i=1}^{k} \underline{\mu}D_i$ *and* $\overline{\mu}_{\mathcal{D}}X = \sum_{i=1}^{k} \overline{\mu}D_i$*.*

We shall also refer to the sum

$$\sum_{i=1}^{k} \widetilde{z}_{i-1} \cdot \underline{\mu}(h^{-1}[z_{i-1}, z_i])$$

in the definition above as the *lower sum* of f corresponding to the partition $\{z_0, \dots, z_k\}$, and to the sum

$$\sum_{i=1}^{k} \widetilde{z}_{i} \cdot \overline{\mu}(h^{-1}[z_{i-1}, z_i])$$

as the *upper sum* of f corresponding to the partition $\{z_0, \dots, z_k\}$.

Example 4.5 In general, there is no hope of proving that the lower and upper measures of definable subsets of $[0,1]^n$ coincide if we replace the definition of $\sim$ on $\mathfrak{m}^{\geq 0}$ by $x \sim y$ iff $v(x) = v(y)$. To see this, consider the function $f\colon [\epsilon^2, \epsilon] \to [0,1]$ given by $f(x) = \frac{\epsilon^2}{x}$, where $\epsilon \in \mathfrak{m}^{>0}$. Let $\delta \in \mathfrak{m}^{>0}$ be such that

$$v(\epsilon^p) < v(\delta) < v(\epsilon^2),$$

for all $p \in \mathbb{Q}^{<2}$. It is easy to see that then $\underline{\mu}(0, f) = \widetilde{\epsilon}^2$, but there is no finite partition of $[0,1]$ so that the corresponding upper sum U of f would be such that $U \leq \widetilde{\delta}$.

Until Theorem 4.8 has been proven, we shall write $\underline{\mu}C$ and $\overline{\mu}C$ for the lower and upper measures of a cell $C \subseteq [0,1]^n$ computed as in part 1 of Definition 4.4 (this is in contrast to $\underline{\mu}_{\mathcal{D}}C$ and $\overline{\mu}_{\mathcal{D}}C$ which are computed as in part 2.).

Lemma 4.6 *Let $X \subseteq [0,1]^n$ be definable with $\mathrm{int}(\pi X) = \emptyset$, and let $\mathcal{D}$ be a decomposition of R^n into cells that partitions X. Then there is no $x \in \bigcup \underline{\mu}_{\mathcal{D}} X$ with $x > \mathfrak{m}$.*

Proof: The proof is by induction on n. The case $n = 1$ is clear, so suppose the lemma holds for $1, \ldots, n$, and let $X \subseteq [0,1]^{n+1}$. Suppose $X = D_1 \cup \cdots \cup D_m$, where $D_i \in \mathcal{D}$. Assume towards a contradiction that $x \in \bigcup \underline{\mu} X$ is so that $x > \mathfrak{m}$. Then there is $i \in \{1, \ldots, m\}$ such that $\bigcup \underline{\mu} D_i$ contains some $x > \mathfrak{m}$. Then $\mathrm{int}(D_i) \neq \emptyset$, so suppose $D_i = (f, g)$ and set $h = g - f$. There are

$$0 = y_0 < y_1 < \cdots < y_k = 1$$

so that $\bigcup \sum_{i=0}^{k-1} \widetilde{y_i} \cdot \underline{\mu} h^{-1}[y_i, y_{i+1}]$ contains an element $> \mathfrak{m}$, hence

$$\widetilde{y_i} \cdot \underline{\mu} h^{-1}[y_i, y_{i+1}] = \widetilde{a}$$

for some $a \in V^{>\mathfrak{m}}$ and $i \in \{0, \ldots, k-1\}$. It follows that $y_i > \mathfrak{m}$, and there is $x \in \bigcup \underline{\mu} h^{-1}[y_i, 1]$ with $x > \mathfrak{m}$. But then, by the inductive assumption, $\mathrm{int}(\pi h^{-1}[y_i, y_{i+1}]) \neq \emptyset$, hence

$$\mathrm{int}(\pi\big(h^{-1}[y_i, y_{i+1}] \times [0, y_i]\big)) \neq \emptyset,$$

a contradiction. □

Lemma 4.7 *If $X = (f,g) \subseteq [0,1]^n$ is an open cell with* $\mathrm{int}(\pi X) = \emptyset$*, then for each $a \in V^{\geq 0}$ with $\widetilde{a} < \underline{\mu} X$ there is $y \in [0,1]$ so that*

$$\widetilde{a} < \widetilde{y} \cdot \underline{\mu} h^{-1}[y,1],$$

where $h = g - f$.

Proof: Immediate from Lemma 4.6 and ii) in the third bullet point of Remark 3.9. $\square$

Theorem 4.8 *Let $X \subseteq [0,1]^n$ be definable. Then*

$$\underline{\mu}_{\mathcal{E}} X = \underline{\mu}_{\mathcal{F}} X = \overline{\mu}_{\mathcal{F}} X = \overline{\mu}_{\mathcal{E}} X,$$

for all decompositions $\mathcal{E}$ and $\mathcal{F}$ of R^n into cells that partition X.

We shall refer to the common value of the upper and lower measures of X as the measure of X *and denote it by μX.*

Proof: We may as well assume $\mathrm{int}(X) \neq \emptyset$. The proof is by induction on n. The case when $n = 1$ holds by Lemma 3.6, so assume inductively that the theorem holds for $1, \ldots, n$, and let $X \subseteq [0,1]^{n+1}$.

Case 1. Suppose $\mathrm{int}(\pi X) = \emptyset$.

Claim 1. Let $X = (f,g)$ be an open cell. Then $\underline{\mu} X = \overline{\mu} X$.

Proof of Claim 1. We set $h = g - f$, and we define

$$A := \sup_{y \in [0,1]} \{\widetilde{y} \cdot \mu(h^{-1}[y,1])\} \in \widetilde{V},$$

where the expression $\mu h^{-1}[y,1]$ makes sense by the inductive assumption. We shall say that *property $*$ holds for h* if there is $x \in \mathfrak{m}^{>0}$ such that

$$\widetilde{y} \cdot \mu(h^{-1}[y,1]) < \widetilde{x}$$

for all $y \in [0,1]$, and there is $y \in [0,1]$ and $q \in \mathbb{Q}^{>1}$ so that

$$\widetilde{x}^q < \widetilde{y} \cdot \mu(h^{-1}[y,1]).$$

We distinguish two cases.

1. First, assume that property $*$ holds for h.

   Let $x \in \mathfrak{m}^{>0}$ witness that $*$ holds for h. We set

$$\mathcal{S} := \{q \in \mathbb{Q}^{>1} \colon \exists y \in [0,1]\ \widetilde{x}^q < \widetilde{y} \cdot \mu h^{-1}[y,1]\}.$$

   Then $\mathcal{S}$ is a nonempty subset of $\mathbb{R}$ that is bounded below, hence the infimum of $\mathcal{S}$ exists in $\mathbb{R}$. We set $c := \inf \mathcal{S}$.

   SUBCLAIM: Let $q_1, q_2 \in \mathbb{Q}^{>0}$ be so that $q_1 < c < q_2$. Then

$$\widetilde{x}^{q_2} < \underline{\mu}(0,h) \leq \overline{\mu}(0,h) < \widetilde{x}^{q_1}.$$

   PROOF OF SUBCLAIM: We first show that $\widetilde{x}^{q_2} < \underline{\mu}(0,h)$. By the definition of c, we can find $q \in \mathcal{S}$ so that $c < q < q_2$, and we let $y \in [0,1]$ satisfy

$$\widetilde{x}^q < \widetilde{y} \cdot \mu h^{-1}[y,1].$$

   Then

$$\widetilde{x}^{q_2} < \widetilde{y} \cdot \mu h^{-1}[y,1] \leq \underline{\mu}(0,h).$$

   To prove $\overline{\mu}(0,h) < x^{q_1}$, let $q_3 \in \mathbb{Q}^{>0}$ and a positive integer l be such that $q_1 + 2q_3 < c$ and $q_1 + q_3 < lq_3$. Then the upper sum of h corresponding to the partition $\{0, x^{lq_3}, x^{(l-1)q_3} \dots, x^{q_3}, 1\}$ of $[0,1]$ is

$$\mu h^{-1}[x^{q_3}, 1] + \sum_{i=1}^{l-1} \widetilde{x}^{iq_3} \mu h^{-1}[x^{(i+1)q_3}, x^{iq_3}] + \widetilde{x}^{lq_3} \mu h^{-1}[0, x^{lq_3}].$$

   Now $\mu h^{-1}[x^{q_3}, 1] < \widetilde{x}^{q_1+q_3}$, because else $\mu h^{-1}[x^{q_3}, 1] \geq \widetilde{x}^{q_1+q_3}$ would imply $\widetilde{x}^{q_3} \cdot \mu h^{-1}[x^{q_3}, 1] \geq \widetilde{x}^{q_1+2q_3}$, a contradiction with $\widetilde{x}^c < \widetilde{x}^{q_1+2q_3}$.

   For $i = 1, \dots, l-1$, we have

$$\widetilde{x}^{iq_3} \mu h^{-1}[x^{(i+1)q_3}, x^{iq_3}] < \widetilde{x}^{q_1+q_3},$$

   because else

$$\widetilde{x}^{iq_3} \mu h^{-1}[x^{(i+1)q_3}, x^{iq_3}] \geq \widetilde{x}^{q_1+q_3}$$

   would imply

$$\widetilde{x}^{(i+1)q_3} \mu h^{-1}[x^{(i+1)q_3}, x^{iq_3}] \geq \widetilde{x}^{q_1+2q_3},$$

   again a contradiction with $\widetilde{x}^c < \widetilde{x}^{q_1+2q_3}$.

Also,

$$\widetilde{x}^{lq_3}\mu h^{-1}[0,x^{lq_3}] \le \widetilde{x}^{lq_3} < \widetilde{x}^{q_1+q_3}.$$

So the upper sum of h corresponding to $\{0, x^{lq_3}, x^{(l-1)q_3} \ldots, x^{q_3}, 1\}$ is smaller than $(l+1)\cdot \widetilde{x}^{q_1+q_3} = \widetilde{x}^{q_1+q_3} < \widetilde{x}^{q_1}$. □ (SUBCLAIM)

It now follows that $\underline{\mu}(0,h) = \overline{\mu}(0,h)$: If not, then we can find $y, z \in V^{>0}$ so that $x^{q_2} < y < z < x^{q_1}$ for all $q_1, q_2 \in \mathbb{Q}^{>0}$ with $q_1 < c < q_2$, and $y \not\sim z$. Hence $y < z^q$ for some $q \in \mathbb{Q}^{>1}$. Then

$$x^{q_2} < y < z^q < x^{qq_1}$$

for all $q_1, q_2 \in \mathbb{Q}^{>0}$ with $q_1 < c < q_2$. But picking q_1 so that $qq_1 > c$ yields a contradiction with $\widetilde{x}^{q_2} < y$ for all $q_2 \in \mathbb{Q}^{>c}$.

2. Suppose $*$ does not hold for h.

   In this case, if $x \in \mathfrak{m}^{>0}$, then either $A < \widetilde{x}^p$ for all $p \in \mathbb{Q}^{>0}$, or $\widetilde{x}^p < A$, for all $p \in \mathbb{Q}^{>0}$. We shall show that $\overline{\mu}(0,h) \le A \le \underline{\mu}(0,h)$.

   To prove that $A \le \underline{\mu}(0,h)$, let $a \in V^{>0}$ be such that $\widetilde{a} < A$. Then we can find $y \in [0,1]$ so that $\widetilde{a} < \widetilde{y}\cdot \mu h^{-1}[y,1] \le \underline{\mu}(0,h)$.

   To see that $\overline{\mu}(0,h) \le A$, let $y \in V^{>0}$ be such that $A < \widetilde{y}$.

   First, suppose $\mathfrak{m} < y < 1$. Then $\mu h^{-1}[\frac{y}{2},1] < \widetilde{(\frac{y}{2})}$, because else

$$\widetilde{(\frac{y}{2})}\cdot \overline{\mu} h^{-1}[\frac{y}{2},1] \ge \widetilde{(\frac{y}{2})}^2 > A,$$

   would yield a contradiction with the definition of A. So

$$\overline{\mu}(0,h) \le \mu h^{-1}[\frac{y}{2},1] + \frac{\widetilde{y}}{2}\cdot \overline{\mu} h^{-1}[0,\frac{y}{2}] < \widetilde{(\frac{y}{2})} + \widetilde{(\frac{y}{2})} = \widetilde{y}.$$

   So assume that $y \in \mathfrak{m}^{>0}$. Then $A < \widetilde{y}^2$, because $*$ fails for h. Hence $\mu h^{-1}[y,1] < \widetilde{y}$, else $\widetilde{y}\cdot \mu h^{-1}[y,1] \ge \widetilde{y}^2 > A$, a contradiction. So

$$\overline{\mu}(0,h) \le \mu h^{-1}[y,1] + \widetilde{y}\cdot \mu h^{-1}[0,y] < \widetilde{y} + \widetilde{y} = \widetilde{y}.$$

   It follows that $\underline{\mu}(0,h) = \overline{\mu}(0,h) = \mu(0,h)$.

   This finishes the proof of Claim 1.

Claim 2. Let $X = (f, g)$ be an open cell, and let $\mathcal{D}$ be a decomposition of R^{n+1} into cells that partitions X. Then $\mu X = \mu_{\mathcal{D}} X$.

Proof of Claim 2. Let $D_1, \dots, D_k \in \mathcal{D}$ be open with $X =_0 D_1 \cup \dots \cup D_k$. To see that $\mu X \leq \sum_{i=1}^{k} \mu D_i$, let $a \in V^{\geq 0}$ be so that $\widetilde{a} < \mu X$. By Lemma 4.6, $a \in \mathfrak{m}^{\geq 0}$. We need to show that $\widetilde{a} < \sum_{i=1}^{k} \mu D_i$. By Lemma 4.7, we can find $y \in [0, 1]$ such that $\widetilde{a} < \widetilde{y} \cdot \mu h^{-1}[y, 1]$, where $h = g - f$.

- If there is no $x \in \bigcup \mu h^{-1}[y, 1]$ with $x > \mathfrak{m}$, then, using the inductive assumption, we can find $D \in \{D_1, \dots, D_k\}$ so that

  $$\mu h^{-1}[y, 1] = \mu(h^{-1}[y, 1] \cap p_n^{n+1} D).$$

  Let $\{E_1, \dots, E_m\}$ be the subset of $\{D_1, \dots, D_k\}$ consisting of all D_i's with $p_n^{n+1} D_i = p_n^{n+1} D$. For each $i \in \{1, \dots, m\}$, let $E_i = (f_i, g_i)$, set $h_i = g_i - f_i$, and define

  $$F_i := \{x \in h^{-1}[y, 1] \cap p_n^{n+1} D : \ h_i(x) \geq h_j(x) \text{ for } j = 1, \dots, m\}.$$

  Then $h^{-1}[y, 1] \cap p_n^{n+1} D = \bigcup_{i=1}^{m} F_i$, and hence we can take $j \in \{1, \dots, m\}$ so that

  $$\mu F_j = \mu(h^{-1}[y, 1] \cap p_n^{n+1} D).$$

  We claim that $\widetilde{a} < \mu E_j$. This is because if $y \in \mathfrak{m}^{>0}$, then $\widetilde{y} \leq \widetilde{h_j(x)}$ for each $x \in F_j$. And if $y > \mathfrak{m}$, then $\widetilde{y} \cdot \mu h^{-1}[y, 1] = \mu h^{-1}[y, 1]$ and $(g_j - f_j)(x) > \mathfrak{m}$ for each $x \in F_j$.

- Now suppose there is $x \in \bigcup \mu h^{-1}[y, 1]$ with $x > \mathfrak{m}$. Let $D \in \{D_1, \dots, D_k\}$ be such that $\bigcup \mu(h^{-1}[y, 1] \cap p_n^{n+1} D)$ contains some $x > \mathfrak{m}$. Then

  $$\widetilde{y} \cdot \mu h^{-1}[y, 1] = \widetilde{y} \cdot \mu(h^{-1}[y, 1] \cap p_n^{n+1} D) = \widetilde{y}.$$

  Define $\{E_1, \dots, E_m\}$ and the sets F_i for D as in the previous case. Then for some $i \in \{1, \dots, m\}$, there is $x \in \bigcup \mu F_i$ so that $x > \mathfrak{m}$. Hence $\mu E_i > \widetilde{a}$.

To see that $\sum_{i=1}^{k} \mu D_i \leq \mu X$, let $a \in V^{\geq 0}$ be such that $\widetilde{a} < \sum_{i=1}^{k} D_i$. By Lemma 4.6, $a \in \mathfrak{m}^{\geq 0}$. Then $\sum_{i=1}^{k} \mu D_i = \mu D_j$ for some $j \in \{1, \dots, k\}$. Let $D_j = (f_j, g_j)$ and set $h_j = g_j - f_j$. Then there is $y \in [0, 1]$ with

$$\widetilde{y} \cdot \mu(h_j^{-1}[y, 1]) > \widetilde{a},$$

and

$$\widetilde{y} \cdot \mu\big(h_j^{-1}[y,1]\big) \leq \widetilde{y} \cdot h^{-1}[y,1] \leq \mu X.$$

This finishes the proof of Claim 2.

Claim 3. Let X be a definable set, and let $\mathcal{C}$ and $\mathcal{D}$ be decompositions of R^{n+1} into cells that partition X. Then $\mu_{\mathcal{C}} X = \mu_{\mathcal{D}} X$.

Proof of Claim 3. Let $\mathcal{E}$ be a decomposition of R^{n+1} into cells which is a common refinement of $\mathcal{C}$ and $\mathcal{D}$. Then

$$\mu_{\mathcal{D}} X = \sum_{D_i \subseteq X} \mu D_i = \sum_{D_i \subseteq X} \sum_{E_{ij} \subseteq D_i} \mu E_{ij} = \sum_{C_k \subseteq X} \sum_{E_{kl} \subseteq C_k} \mu E_{kl} = \sum_{C_k \subseteq X} \mu C_k = \mu_{\mathcal{C}} X,$$

where $D_i \in \mathcal{D}$, $E_{ij}, E_{kl} \in \mathcal{E}$ and $C_k \in \mathcal{C}$.

This finishes the proof of Claim 3, and we have thus proven Case 1.

Case 2. $\mathrm{int}(\pi X) \neq \emptyset$.

Since πX is definable in the o-minimal structure $\mathbb{R}_0$, it is Lebesgue measurable, and $\underline{\mu}_{\mathcal{P}} \pi X = \overline{\mu}_{\mathcal{P}} \pi X = \widetilde{a}$, where $a \in \mathbb{R}^{>0}$ is the Lebesgue measure of πX, and $\mathcal{P}$ is any decomposition of $\mathbb{R}^{n+1}$ into cells that partitions πX. We shall thus write μY instead of $\overline{\mu}_{\mathcal{P}} Y$ and $\underline{\mu}_{\mathcal{P}} Y$ if Y is an $\mathbb{R}_0$-definable subset of $[0,1]^m \subseteq \mathbb{R}^m$.

Our aim is to show that $\underline{\mu}_{\mathcal{D}} X = \overline{\mu}_{\mathcal{D}} X = \widetilde{a}$. Since this is clearly satisfied when $X \subseteq [0,1]$, we may assume that the inductive assumption holds in this a priori stronger form.

Claim 1. Suppose $X = (f,g) \subseteq [0,1]^{n+1}$ is a cell. Then $\underline{\mu} X = \overline{\mu} X = \widetilde{a}$.

Proof of Claim 1. We set $h = g - f$. By o-minimality of $\mathbb{R}_0$, there are continuous $\mathbb{R}_0$-definable functions f_0, g_0, and h_0 with

$$\mathrm{domain}(f_0) = \mathrm{domain}(g_0) = \mathrm{domain}(h_0) =_0 \pi p_n^{n+1} X,$$

and such that for all $x \in \mathrm{domain}(f_0)$ up to a definable set of dimension $< n$,

$$f_0(x) = \pi f(x'),\ g_0(x) = \pi g(x') \text{ and } h_0(x) = \pi h(x')$$

for all $x' \in p_n^{n+1} X$ with $\pi(x') = x$.

Let $\mathcal{C}_0$ be a decomposition of $\mathbb{R}^n$ into cells that partitions the domain of h_0 and is such that whenever $C \in \mathcal{C}_0$ is open and $C \subseteq \mathrm{domain}(h_0)$, then h_0 is differentiable on C and each $\frac{\partial h_0}{\partial x_i}$ has constant sign.

By Fact 4.1, we can find for each $C \in \mathcal{C}_0$ an R-definable set X_C so that $\pi X_C = \mathrm{cl}(C)$. Let $\mathcal{D}_0$ be a decomposition of R^n partitioning $p^{n+1}_n X$ and X_C for each $C \in \mathcal{C}_0$ with $C \subseteq \mathrm{domain}(h_0)$.

Subclaim: Let $D \in \mathcal{D}_0$ be such that $D \subseteq p^{n+1}_n X$. Set $X_D := (0,h) \cap (D \times R)$ and suppose $\mathrm{int}(\pi X_D) \neq \emptyset$. Then $\underline{\mu} X_D = \overline{\mu} X_D = \widetilde{d}$, where d is the Lebesgue measure of πX_D.

Proof of subclaim: We replace for the moment h with $h|_D$, and h_0 with $h_0|_{\mathrm{int}(\pi D)}$. We shall show $\overline{\mu}(0,h) \leq \widetilde{d}$ and $\widetilde{d} \leq \underline{\mu}(0,h)$. To prove the first inequality, let $d' \in \mathbb{R}$ be such that $\widetilde{d} < \widetilde{d'}$. We wish to show that $\overline{\mu}(0,h) < \widetilde{d'}$. Let $0 = a_0 < \cdots < a_k = 1$ be real numbers so that

$$\sum_{i=0}^{k-1} \widetilde{a_{i+1}} \cdot \mu h_0^{-1}[a_i, a_{i+1}] < \widetilde{d'}.$$

By Fact 4.2, for each i, we can find $\epsilon_i \in \mathfrak{m}^{\geq 0}$ so that

$$\pi\big(\Gamma h \cap (R^n \times [b_i - \epsilon_i, b_{i+1} + \epsilon_i])\big) = \Gamma h_0 \cap (\mathbb{R}^n \times [a_i, a_{i+1}])$$

up to a set $Y \subseteq \mathbb{R}^{n+1}$ with $\dim p^{n+1}_n Y < n$, and where $b_i, b_{i+1} \in R$ are such that $\pi b_i = a_i$ and $\pi b_{i+1} = a_{i+1}$. Inductively,

$$\mu h^{-1}[b_i - \epsilon_i, b_{i+1} + \epsilon_i] = \mu \pi h^{-1}[b_i - \epsilon_i, b_{i+1} + \epsilon_i],$$

hence

$$\mu h^{-1}[b_i - \epsilon_i, b_{i+1} + \epsilon_i] = \mu h_0^{-1}[a_i, a_{i+1}].$$

So

$$\begin{aligned}&\textstyle\sum_{i=0}^{k-1} \widetilde{b_{i+1} \cdot \mu h^{-1}}[b_i, b_{i+1}] \leq \\ &\textstyle\sum_{i=0}^{k-1} (b_{i+1} + \epsilon_i) \cdot \mu h^{-1}[b_i - \epsilon_i, b_{i+1} + \epsilon_i] = \\ &\textstyle\sum_{i=0}^{k-1} \widetilde{a_i} \cdot \mu h_0^{-1}[a_i, a_{i+1}] < \widetilde{d'}.\end{aligned}$$

Next, we need to show that $\widetilde{d} \leq \underline{\mu}(0,h)$. There are two cases to be considered.

1. Suppose $\frac{\partial h_0}{\partial x_j} = 0$ for all j.

   Then $\widetilde{d} = \mu p^{n+1}_n(0,h_0) \cdot \widetilde{h_0(x)}$ for all $x \in p^{n+1}_n(0,h_0)$. Let $b \in V^{>\mathfrak{m}}$ be such that $\pi(b) = h_0(x)$. By Fact 4.2, we can find $\epsilon \in \mathfrak{m}^{\geq 0}$ so that

$$\pi(\Gamma h \cap (R^n \times [b - \epsilon, b + \epsilon])) = \Gamma h_0$$

up to a set $Y \subseteq \mathbb{R}^{n+1}$ such that $\dim p_n^{n+1} Y < n$. Then

$$\widetilde{d} = \widetilde{h_0(x)} \cdot \mu p_n^{n+1}(0, h_0) \leq \widetilde{(b-\epsilon)} \cdot \mu h^{-1}[b-\epsilon, b+\epsilon] + \widetilde{(b+\epsilon)} \cdot \mu h^{-1}[b+\epsilon, 1]$$

by the inductive assumption.

2. Suppose $\frac{\partial h_0}{\partial x_j} \neq 0$ for some j. Let $d' \in V^{>\mathfrak{m}}$ be such that $\widetilde{d'} < \widetilde{d}$. We wish to show that $\widetilde{d'} < \underline{\mu}(0, h)$.

   Let
   $$0 = a_0 < \cdots < a_k = 1$$
   be elements of $\mathbb{R}$ so that $\widetilde{d'} < \sum_{i=0}^{k} \widetilde{a_i} \cdot \mu h_0^{-1}[a_i, a_{i+1}]$, and let
   $$0 = b_0 < \cdots < b_k = 1$$
   be elements of R such that $\pi b_i = a_i$ for each i. Then, for each i, $\mu \pi h^{-1}[b_i, b_{i+1}] = \mu h_0^{-1}[a_i, a_{i+1}]$: The inequality
   $$\mu \pi h^{-1}[b_i, b_{i+1}] \leq \mu h_0^{-1}[a_i, a_{i+1}]$$
   is clear by the inductive assumption. To prove the other inequality, let $\epsilon \in \mathfrak{m}^{>0}$ be such that
   $$\pi h^{-1}[b_i - \epsilon, b_{i+1} + \epsilon] =_0 h_0^{-1}[a_i, a_{i+1}].$$
   Then
   $$\pi h^{-1}[b_i - \epsilon, b_{i+1} + \epsilon] = \pi h^{-1}[b_i - \epsilon, b_i] \cup \pi h^{-1}[b_i, b_{i+1}] \cup \pi h^{-1}[b_{i+1}, b_{i+1} + \epsilon],$$
   where the sets on the right-hand side are disjoint apart from a set of dimension $< n$. Hence
   $$\mu h_0^{-1}[a_i, a_{i+1}] =_0 \mu \pi h^{-1}[b_i - \epsilon, b_i] + \mu \pi h^{-1}[b_i, b_{i+1}] + \mu \pi h^{-1}[b_{i+1}, b_{i+1} + \epsilon]$$
   by the inductive assumption. But
   $$\mu \pi h^{-1}[b_i - \epsilon, b_i] = \mu \pi h^{-1}[b_{i+1}, b_{i+1} + \epsilon] = 0,$$
   because $\mu h_0^{-1}(a_i) = \mu h_0^{-1}(a_{i+1}) = 0$.

   It follows that
   $$\widetilde{d'} < \sum_{i=0}^{k-1} \widetilde{a_i} \cdot \mu h_0^{-1}[a_i, a_{i+1}] = \sum_{i=0}^{k-1} \widetilde{b_i} \cdot \mu h^{-1}[b_i, b_{i+1}].$$

This proves $\widetilde{d} < \underline{\mu}(0,h)$, and hence $\underline{\mu}X_D = \overline{\mu}X_D = \widetilde{d}$. □ (SUBCLAIM)

Now let $p_n^{n+1}X = \bigcup_{i=1}^k D_i$, where each $D_i \in \mathcal{D}_0$. Then each set $X \cap (D_i \times R)$ is a cell, and

$$X = \big((D_1 \times R) \cap X\big)\dot{\cup}\dots\dot{\cup}\big((D_k \times R) \cap X\big).$$

Let $\mathcal{D}$ be a decomposition of R^{n+1} into cells such that $(D_i \times R) \cap X \in \mathcal{D}$ for each $i \in \{1,\dots,k\}$, and let $I \subseteq \{1,\dots,k\}$ consist of all the i with

$$\mathrm{int}(\pi((D_i \times R) \cap X)) \neq \emptyset.$$

By the subclaim, if $i \in I$, then we can find $a_i \in V^{>\mathfrak{m}}$ so that πa_i is the Lebesgue measure of $\pi\big((D_i \times R) \cap X\big)$ and

$$\mu\big((D_i \times R) \cap X\big) = \underline{\mu}\big((D_i \times R) \cap X\big) = \overline{\mu}\big((D_i \times R) \cap X\big) = \widetilde{a}_i.$$

For $i \in \{1,\dots,k\} \setminus I$, we set $a_i = 0$. Note that $\sum_{i=1}^k \pi a_i = \pi a$. To prove $\underline{\mu}X = \overline{\mu}X = \widetilde{a}$, let $a' \in R^{>\mathfrak{m}}$ be such that $\widetilde{a} < \widetilde{a'}$. We need to show $\overline{\mu}X < \widetilde{a'}$. Let for each $i \in \{1,\dots,k\}$,

$$0 = b_{i_0} < b_{i_1} < \dots < b_{i_{k_i}} = 1$$

be a partition of $[0,1]$ so that the corresponding upper sum of $h|_{D_i}$ has measure at most $\widetilde{a}_i + \widetilde{\frac{a'-a}{k}}$. Such a partition exists for $i \in I$ by the subclaim, and for $i \in \{1,\dots,k\} \setminus I$ by Case 1.

Now let $\{b_0,\dots,b_m\}$ be a partition of $[0,1]$ which is a common refinement of all $\{b_{i_0},\dots,b_{i_{k_i}}\}$ where $i = 1,\dots,k$. Then the upper sum of $h|_{D_i}$ corresponding to this new partition is again at most $\widetilde{a}_i + \widetilde{\frac{a'-a}{k}}$. Furthermore,

$$\sum_{i=1}^m \widetilde{b}_i \cdot \mu h^{-1}[b_{i-1}, b_i] = \sum_{i=1}^m (\sum_{j=1}^k \widetilde{b}_i \cdot \mu(h^{-1}[b_{i-1}, b_i] \cap D_j)) < \widetilde{a'},$$

where the first equality follows from the inductive assumption. The inequality $\widetilde{a} \leq \underline{\mu}X$ is proved similarly. This finishes the proof of Claim 1.

Claim 2. $\underline{\mu}X_{\mathcal{E}} = \overline{\mu}_{\mathcal{E}}X = \widetilde{a}$.

Proof of Claim 2. Let $E_1, \dots, E_k \in \mathcal{E}$ be open such that $X =_0 \bigcup_{i=1}^k E_i$. Since $\mathrm{int}(\pi E_i) \neq \emptyset$ for at least one i, we may as well assume (by Lemma 4.6) that $\mathrm{int}(\pi E_i) \neq \emptyset$ for each i. Now, by the above, $\underline{\mu} E_i = \overline{\mu} E_i = \widetilde{b}_i$, where πb_i is the Lebesgue measure of πD_i. Hence

$$\underline{\mu} X = \sum_{i=1}^{k} \underline{\mu} D_i = \sum_{i=1}^{k} \overline{\mu} D_i = \overline{\mu} X.$$

This finishes the proof of Claim 2, thus the proof of Case 2, and hence the proof of the theorem. □

5 Measuring definable subsets of V^n and invariance of μ under isomorphisms

The following definition is from [6]. By $J\phi(x)$ we denote the determinant of the Jacobian of a diffeomorphism ϕ at x.

Definition 5.1 *Let $SB[n]$ be the lattice of all R-definable subsets of V^n, and let $X, Y \in SB[n]$. An* isomorphism *$\phi\colon X \to Y$ is defined to be a definable C^1-diffeomorphism $\phi\colon U \to V$, where U and V are open definable subsets of R^n, $X \subseteq_0 U$, $Y \subseteq_0 V$, $|J\phi(x)| = 1$ for all $x \in U \cap X$ up to a set of dimension $< n$, and $\phi(X) =_0 Y$.*

Let $C \subseteq V^n$ be an open cell with $C = (f_n, g_n)$ and $p^n_k C = (f_k, g_k)$ for $k = 1, \dots, n-1$. Suppose that f_i and g_i are continuously differentiable for $i = 2, \dots, n$. We define a map

$$\tau_C = (\tau_1, \dots, \tau_n)\colon C \to \tau C$$

by setting $\tau_k(x) = x_k - f_k(x_1, \dots, x_{k-1})$ for $x = (x_1, \dots, x_n) \in C$ and $k = 1, \dots, n$. It is routine to check that τ is an isomorphism $C \to \tau C$.

Lemma 5.2 *Let $X \subseteq [0,1]^n$ be definable and such that* $\mathrm{int}(\pi X) = \emptyset$*. Then for each $a \in V^{>0}$ with $\widetilde{a} < \mu X$, there is a cell $C \subseteq X$ and a box $B \subseteq_0 \tau_C C$ with $\mu B > \widetilde{a}$.*

Proof: Let $a \in V^{\geq 0}$ be such that $\widetilde{a} < \mu X$. Let $\mathcal{D}$ be a decomposition of R^n into cells that partitions X. Suppose $X = D_1 \cup \dots \cup D_m$, where each

$D_i \in \mathcal{D}$. Since $\operatorname{int}(\pi X) = \emptyset$, by Lemma 4.6 we can find $D \in \{D_1, \dots, D_m\}$ so that $\mu D = \mu X$. We shall find a box $B \subseteq_0 \tau_D(D)$ with $\mu D > \widetilde{a}$.

If $n = 1$, then τD is the required box. So assume the lemma holds for $1, \dots, n$, and let $X \subseteq [0,1]^{n+1}$. Suppose $\tau_D D = (0, h)$. Then we can find a partition $0 = y_0 < y_1 < \dots < y_l = 1$ of $[0,1]$ so that

$$\widetilde{a} < \sum_{i=1}^{l} \widetilde{y}_{i-1} \cdot \mu h^{-1}[y_{i-1}, y_i],$$

and $\sum_{i=1}^{l} \widetilde{y}_{i-1} \cdot \mu h^{-1}[y_{i-1}, y_i] = \widetilde{y}_{j-1} \cdot \mu h^{-1}[y_{j-1}, y_j]$ for some $j \in \{1, \dots, l\}$.

If $\operatorname{int}(\pi h^{-1}[y_{j-1}, y_j]) = \emptyset$, then $h^{-1}[y_{j-1}, y_j]$ contains a cell C of measure $\mu h^{-1}[y_{j-1}, y_j]$, and

$$\tau_C C \subseteq \tau_{p_n^{n+1} D} h^{-1}[y_{j-1}, y_j] \subseteq \tau_{p_n^{n+1} D} p_n^{n+1} D = p_n^{n+1} \tau_D D.$$

Let $c \in V^{>0}$ be such that $\widetilde{c} < \mu h^{-1}[y_{j-1}, y_j]$ and $\widetilde{a} < \widetilde{y}_{j-1} \cdot \widetilde{c}$. By the inductive assumption, $\tau_C C$ contains a box B_0 with $\widetilde{c} < \mu B_0$. Then $B_0 \times [0, y_{j-1}] \subseteq \tau_D D$ and $\mu(B_0 \times [0, y_{j-1}]) > \widetilde{a}$.

If $\operatorname{int}(\pi h^{-1}[y_{j-1}, y_j]) \neq \emptyset$, then $h^{-1}[y_{j-1}, y_j]$ contains a cell C such that $\operatorname{int}(\pi C) \neq \emptyset$. Then $\operatorname{int}(\pi \tau_C C) \neq \emptyset$ and, by Fact 4.3, $\tau_C C$ contains a box B_0 of measure $> \mathfrak{m}$. Then $B_0 \times [0, y_{j-1}]$ is as required. □

Lemma 5.3 *Let $X \subseteq [0,1]^n$ be definable with non-empty interior, and let $a \in V^{>0}$ be such that $\mu X < \widetilde{a}$. Then there are open cells $C_1, \dots, C_k \subseteq [0,1]^n$ so that $X =_0 C_1 \dot\cup \dots \dot\cup C_k$, and for each $i \in \{1, \dots, k\}$ there are boxes $B_{i1}, \dots, B_{ik_i} \subseteq [0,1]^n$ with $\tau C_i \subseteq \bigcup_{j=1}^{k_i} B_{ij}$ and $\sum_{i=1}^{k} \sum_{j=1}^{k_i} \mu B_{ij} < \widetilde{a}$.*

Proof: First assume that $X = (f,g)$ is a cell and set $h = g - f$. The proof is by induction on n. If $n = 1$, then $X = (c,d)$ for some $c, d \in V^{\geq 0}$. Then $\tau X_{(f,g)} \subseteq [0, d-c]$ and $\mu[0, d-c] = \mu X < \widetilde{a}$.

So suppose the lemma holds for $1, \dots, n$, and let $X \subseteq [0,1]^{n+1}$. Let

$$0 = y_0 < y_1 < \dots < y_k = 1$$

be such that $\sum_{i=1}^{k} \widetilde{y}_i \cdot \mu h^{-1}[y_{i-1}, y_i] < \widetilde{a}$.

Case 1. There is no $c \in \mathfrak{m}^{\geq 0}$ with $\mu X < \widetilde{c}$.

In this case $a > \mathfrak{m}$, and we fix $b \in V^{>\mathfrak{m}}$ so that $\mu X < \widetilde{b} < \widetilde{a}$. It suffices to prove the conclusion of the lemma for each

$$X' := X \cap (h^{-1}[y_{i-1}, y_i] \times [0,1])$$

instead of X and $\mu X' + \widetilde{\frac{a-b}{k}}$ in place of $\widetilde{a}$.

1. If $y_i \in \mathfrak{m}^{\geq 0}$, then let $\mathcal{D}$ be any decomposition of R^{n+1} into cells partitioning X'. For each open $D \in \mathcal{D}$ with $D \subseteq X'$ we have $\tau_D D \subseteq [0,1]^n \times [0, y_i]$, which is a box of measure $\widetilde{y_i}$, and $l \cdot \widetilde{y_i} < \mu X' + \widetilde{\frac{a-b}{k}}$ for any non-negative integer l.

2. If $\mu h^{-1}[y_{i-1}, y_i] < \widetilde{d}$ for some $d \in \mathfrak{m}^{\geq 0}$, then we use the inductive assumption to find open cells $C_1, \ldots, C_k$ so that

$$\mu h^{-1}[y_{i-1}, y_i] =_0 C_1 \cup \cdots \cup C_k,$$

and for each $i \in \{1, \ldots, k\}$ a family of boxes $\{B_{ij} \colon j = 1, \ldots, k_i\}$ covering $\tau_{C_i} C_i$ so that $\sum_{j=1}^{k_i} \mu B_{ij} < \widetilde{d}$. Then the cells $X' \cap (C_i \times [0,1])$ and the families of boxes

$$\{B_{ij} \times [0, y_i] \colon \ j = 1, \ldots, k_i\},$$

where $i = 1, \ldots, k$, are as in the conclusion of the lemma.

3. So suppose $y_i > \mathfrak{m}$, and there is no $d \in \mathfrak{m}^{\geq 0}$ with $\mu h^{-1}[y_{i-1}, y_i] < \widetilde{d}$. Then $\mu h^{-1}[y_{i-1}, y_i] < \frac{\widetilde{b}}{y_i}$. By the inductive assumption, we can find open cells $C_1, \ldots, C_k \subseteq [0,1]^n$ so that

$$h^{-1}[y_{i-1}, y_i] =_0 C_1 \cup \cdots \cup C_k,$$

and for each $i \in \{1, \ldots, k\}$ a family of boxes $\{B_{ij} \colon \ j = 1, \ldots, k_i\}$ covering $\tau_{C_i} C_i$ with

$$\sum_{i=1}^{k} \sum_{j=1}^{k_i} \mu B_{ij} < \frac{\widetilde{b}}{y_i}.$$

Then the cells $X' \cap (C_i \times [0,1])$ and the families of boxes

$$\{B_{ij} \times [0, y_i] \colon j = 1, \ldots, k_i\},$$

where $i = 1, \ldots, k$, are as required.

Case 2. There is $c \in \mathfrak{m}^{>0}$ with $\mu X < \widetilde{c}$.

In this case we may assume that $a \in \mathfrak{m}^{>0}$. We fix $b \in \mathfrak{m}^{>0}$ with

$$\mu X < \widetilde{b} < \widetilde{a}.$$

It suffices to prove the conclusion of the lemma for each set

$$X' := X \cap (h^{-1}[y_{i-1}, y_i] \times [0,1])$$

in place of X.

1. Suppose $y_i \in \mathfrak{m}^{\geq 0}$ and $\mu h^{-1}[y_{i-1}, y_i] < \widetilde{c}$, where $c \in \mathfrak{m}^{\geq 0}$.

   If $\widetilde{y_i} < \widetilde{a}$, then we find open cells $C_1, \ldots, C_k$ so that

   $$h^{-1}[y_{i-1}, y_i] =_0 C_1 \cup \cdots \cup C_k,$$

   and for each i a family of boxes $\{B_{ij}\colon\ j = 1, \ldots, k_i\}$ covering $\tau_{C_i} C_i$ such that $\sum_{i=1}^{k} \sum_{j=1}^{k_i} \mu B_{ij} < \widetilde{c}$. Then the cells $X' \cap (C_i \times [0,1])$ and the families of boxes $\{B_{ij} \times [0, y_i]\colon j = 1, \ldots, k_i\}$ are as required.

   If $\widetilde{a} \leq \widetilde{y_i}$, then $\widetilde{b} < \widetilde{y_i}$, hence $z = \frac{b}{y_i} \in \mathfrak{m}^{>0}$. Note that $\mu h^{-1}[y_{i-1}, y_i] < \widetilde{z}$. We proceed exactly as above, except that we require

   $$\sum_{i=1}^{k} \sum_{j=1}^{k_i} \mu B_{ij} < \widetilde{z}.$$

2. It is obvious how to handle the case when $y_i > \mathfrak{m}$.

3. Suppose $y_i \in \mathfrak{m}^{\geq 0}$, and there is no $c \in \mathfrak{m}^{>0}$ so that $\mu h^{-1}[y_{i-1}, y_i] < \widetilde{c}$.

   Then $\widetilde{y_i} < \widetilde{a}$: If $\widetilde{a} \leq \widetilde{y_i}$, then $\widetilde{b} < \widetilde{y_i}$, hence $\widetilde{b} < \widetilde{y_i}^{\frac{m}{n}} < \widetilde{y_i}$ for some $\frac{m}{n} \in \mathbb{Q}^{>1}$. But $y_i^{\frac{m-n}{n}} \in \mathfrak{m}^{>0}$ and $\widetilde{y_i} \cdot \widetilde{y_i}^{\frac{m-n}{n}} = \widetilde{y_i}^{\frac{m}{n}}$, a contradiction with $\widetilde{y_i} \cdot \widetilde{\epsilon} < \widetilde{b}$ for all $\epsilon \in \mathfrak{m}^{>0}$.

   It is now obvious how to handle this case as well.

We established the lemma for $X \subseteq [0,1]^n$ a cell. Now suppose that $X \subseteq [0,1]^n$ is a definable set. Let $\mathcal{D}$ be a decomposition of R^n into cells partitioning X, and let $X =_0 D_1 \cup \cdots \cup D_m$, where each $D_i \in \mathcal{D}$ is open.

The case when $a \in \mathfrak{m}^{\geq 0}$ follows immediately from Case 2 above. So suppose there is no $c \in \mathfrak{m}^{>0}$ so that $\mu X < \widetilde{c}$. Let $b \in V^{>0}$ be such that $\mu X < \widetilde{b} < \widetilde{a}$. By Case 1, each τD_i can be covered by finitely many boxes B_{ij} of total measure $< \mu D_i + \widetilde{\frac{a-b}{m}}$. Then the sum of the measures of all the boxes is $< \widetilde{a}$. □

Theorem 5.4 *Let $X, Y \subseteq [0,1]^n$ be definable and isomorphic. Then $\mu X = \mu Y$.*

Proof: Let ϕ be an isomorphism $X \to Y$. It suffices to show that $\mu X \leq \mu Y$, since ϕ^{-1} is an isomorphism $Y \to X$. If $\mathrm{int}(\pi X) \neq \emptyset$, then the theorem is obvious from the proof of Theorem 6.5, p. 194 in [8].

So suppose $\mathrm{int}(\pi X) = \emptyset$. Assume towards a contradiction that $\mu Y < \mu X$, and let $a \in \mathfrak{m}^{>0}$ be such that $\mu Y < \widetilde{a} < \mu X$. By Lemma 5.3, we can find open cells $C_1, \dots, C_k \subseteq [0,1]^n$ so that

$$Y =_0 C_1 \dot{\cup} C_2 \dot{\cup} \dots \dot{\cup} C_k,$$

ϕ is defined on each C_i, and so that for each i, we can find a family of boxes $\{B_{ij} : j = 1, \dots, k_i\}$ with $\tau_{C_i} C_i \subseteq_0 \bigcup_{j=1}^{k_i} B_{ij}$ and $\sum_{i=1}^{k} \sum_{j=1}^{k_i} \mu B_{ij} < \widetilde{a}$. Then

$$X =_0 \phi^{-1} C_1 \dot{\cup} \dots \dot{\cup} \phi^{-1} C_k.$$

We set $C := C_l$, where $l \in \{1, \dots, k\}$ is such that $\mu X = \mu \phi^{-1}(C_l)$, and we replace X by $\phi^{-1}(C)$ and ϕ by $\phi|_{\phi^{-1}C}$. Then $\tau_C \circ \phi$ is an isomorphism $X \to \tau_C C$. Let $\mathcal{D}$ be a decomposition of R^n into cells partitioning each

$$X \cap \phi^{-1}(\tau_C^{-1}(B_{ij} \cap \tau_C C)).$$

Then

$$X =_0 D_1 \dot{\cup} \dots \dot{\cup} D_m,$$

where each D_i is an open cell from $\mathcal{D}$. Let $D := D_l$ for $l \in \{1, \dots, m\}$ so that $\mu X = \mu D_l$, and let $B := B_{ij}$ so that $\tau_C \circ \phi(D) \subseteq B_{ij}$. By Lemma 5.2, we can find a box $P \subseteq \tau_D(D)$ with $\mu P > \widetilde{a}$. Then

$$\tau_P P = [0, \epsilon_1] \times [0, \epsilon_2] \times \dots \times [0, \epsilon_n],$$

where each $\epsilon_i \in V^{>0}$ and $\mu P = \Pi_{i=1}^{n} \widetilde{\epsilon_i}$. Let $\theta \colon [0,1]^n \to R^n$ be given by $\theta(x) = (\epsilon_1 x_1, \dots, \epsilon_n x_n)$. Then $\theta([0,1]^n) = \tau_P P$.

We define another map $\hat{\theta} \colon \tau_B B \to R^n$ by $\hat{\theta}(x) = (\delta_1 x_1, \dots, \delta_n x_n)$, where $\delta_1, \dots, \delta_n \in R^{>0}$ are chosen in such a way that $\det(\hat{\theta}) = \frac{1}{\det \theta}$, and $\hat{\theta}(\tau_B B) \subseteq V^n$ (this is possible since $\mu B < \mu P$). Then $\pi \hat{\theta}(\tau_B B)$ has empty interior. However, the map

$$\hat{\theta} \circ \tau_B \circ \tau_C \circ \phi \circ \tau_D^{-1} \circ \tau_P^{-1} \circ \theta$$

is an isomorphism $[0,1]^n \to \hat{\theta}(\tau_B B)$, a contradiction with the theorem being true in the case when $\mathrm{int}(\pi X) \neq \emptyset$. $\square$

Definition 5.5 *For a definable set $X \subseteq V^n$ we set*

$$\mu X := \widetilde{\frac{1}{\det A}} \cdot \mu(TX),$$

where

$$T : R^n \to R^n : x \mapsto Ax + b$$

is an affine map with affine transformation matrix $A = (a_{ij})$ *such that* $a_{ij} = \lambda \in V^{>\mathfrak{m}}$ *whenever* $i = j$*, and* $a_{ij} = 0$ *whenever* $i \neq j$*,* $b \in V^n$*, and* $AX \subseteq [0,1]^n$*.*

The next Lemma shows that μX is well-defined on $SB[n]$.

Lemma 5.6 *Let* $X \subseteq V^n$ *be definable, and let*

$$T : R^n \to R^n : x \mapsto Ax + b \text{ and } T' : R^n \to R^n : x \mapsto A'x + b'$$

be affine transformations for X *as in Definition 5.5. Then*

$$\widetilde{\frac{1}{\det A}} \cdot \mu(TX) = \widetilde{\frac{1}{\det A'}} \cdot \mu(T'X).$$

PROOF: Note that $\operatorname{int}(\pi X) = \emptyset$ iff $\operatorname{int}(\pi(TX)) = \emptyset$, and the lemma holds whenever $\operatorname{int}(\pi X) \neq \emptyset$, since it holds in $\mathbb{R}_0$. So we may assume $\operatorname{int}(\pi X) = \emptyset$, in which case

$$\widetilde{\frac{1}{\det A}} \cdot \mu TX = \mu TX \text{ and } \widetilde{\frac{1}{\det A'}} \cdot \mu T'X = \mu T'X,$$

so it suffices to show that $\mu(TX) = \mu(T'X)$. We set $Y := TX$ and $S := T' \circ T^{-1}$. Then $Y, SY \subseteq [0,1]^n$, and S is an affine transformation with diagonal affine transformation matrix so that each entry on the diagonal is a fixed $\lambda \in V^{>\mathfrak{m}}$.

To see that $\mu Y \leq \mu SY$, let $a \in \mathfrak{m}^{>0}$ be such that $\widetilde{a} < \mu Y$. We can find a cell $C \subseteq Y$ and a box $B \subseteq \tau_C C$ with $\widetilde{a} < \mu B$. But then $SC \subseteq SY$ is a also a cell, and $SB \subseteq \tau_{SC} SC$ is a box such that $\widetilde{a} < \mu SB$.

The inequality $\mu SY \leq \mu Y$ follows by a similar argument when considering $S^{-1}\colon SY \to Y$ instead of S. □

Corollary 5.7 *Let* $X, Y \subseteq V^n$ *be definable and let* $\phi\colon X \to Y$ *be an isomorphism. Then* $\mu X = \mu Y$*.*

PROOF: Since μ is invariant under translations, we may assume that $X, Y \subseteq [0,m]^n$. Let $\theta\colon R^n \to R^n$ be given by $\theta(x) = (\frac{1}{m}x_1, \ldots, \frac{1}{m}x_n)$. Then

$$\theta|_Y \circ \phi \circ \theta^{-1}|_{\theta X} : \theta X \to \theta Y$$

is an isomorphism between subsets of $[0,1]^n$, hence by Theorem 5.4, $\mu\theta X = \mu\theta Y$, and so $\mu X = \mu Y$ by the definition of μ. □

Lemma 5.8 *Let $X \subseteq [0,1]^m$ and $Y \subseteq [0,1]^n$ be definable. Then $\mu(X \times Y) = \mu Y \cdot \mu X$.*

Proof: If $\mathrm{int}(X) = \emptyset$ or $\mathrm{int}(Y) = \emptyset$, then the lemma holds trivially, so assume that $\mathrm{int}(X)$ and $\mathrm{int}(Y)$ are nonempty.

Note that in the case when $\mu X, \mu Y \in \mathbb{R}^{>0}$, the lemma holds, since then μX and μY are just the Lebesgue measures of πX and πY respectively. So suppose $\mu X \notin \mathbb{R}^{>0}$ or $\mu Y \notin \mathbb{R}^{>0}$ (and hence $\mu(X \times Y) \notin \mathbb{R}^{>0}$).

Let $\mathcal{C}$ be a decomposition of R^{m+n} into cells that partitions $X \times Y$. To see that $\mu(X \times Y) \leq \mu X \cdot \mu Y$, let $C \in \mathcal{C}$ be such that $C \subseteq X \times Y$ and $\mu C = \mu(X \times Y)$. Let further $a \in V^{>0}$ be so that $\widetilde{a} < \mu(X \times Y)$. By Lemma 5.2, we can find a box $B \subseteq \tau_C C$ with $\widetilde{a} < \mu B$. Then $B = pB \times qB$, where $p\colon R^{m+n} \to R^m$ denotes the projection onto the first m coordinates and $q\colon R^{m+n} \to R^n$ is the projection onto the last n coordinates. By Lemma 3.8, $\mu B = \mu pB \cdot \mu qB$. Now $\tau_{pC}^{-1} pB \subseteq X$ and $\mu \tau_{pC}^{-1} pB = \mu pB$, since $\tau_{pC}^{-1}|_{pB}$ is an isomorphism $pB \to \tau_{pC}^{-1}(pB)$. Hence $\mu pB \leq \mu X$. We now define a map $\hat{\tau}$ on qB. Suppose $p_{m+k}^{m+n} C = (f_k, g_k)$ for $1 \leq k \leq n$. Fix $c \in C$, and let $\hat{f}_k = f_k(p_{m+k-1}^{m+n} c)$. Set $\hat{\tau} := (\tau_1, \ldots, \tau_n)$, where $\tau_k(x) = x_k + \hat{f}_k$ for $x \in qB$. Then $\hat{\tau} qB \subseteq Y$ and, since $\hat{\tau}$ is an isomorphism, $\mu qB = \mu \hat{\tau} qB$, so $\mu qB \leq \mu Y$. It follows that $\widetilde{a} < \mu X \cdot \mu Y$, hence $\mu(X \times Y) \leq \mu X \cdot \mu Y$.

To see that $\mu X \cdot \mu Y \leq \mu(X \times Y)$, let $a \in V^{>0}$ be such that $\widetilde{a} < \mu X \cdot \mu Y$. Then we can find $b, c \in V^{>0}$ with $\widetilde{a} \leq \widetilde{b} \cdot \widetilde{c}$ and $\widetilde{b} < \mu X$ and $\widetilde{c} < \mu Y$. First, suppose $\mu X \notin \mathbb{R}^{>0}$ and $\mu Y \notin \mathbb{R}^{>0}$. Then we can find cells $C \subseteq X$ and $D \subseteq Y$ such that $\mu C = \mu X$ and $\mu D = \mu Y$. By Lemma 5.2, there are boxes $B \subseteq \tau_C C$ and $P \subseteq \tau_D D$ so that $\widetilde{b} < \mu B$ and $\widetilde{c} < \mu P$. Note that $C \times D \subseteq X \times Y$ is a cell. We have $P \times Q \subseteq \tau_{C \times D}(C \times D)$, and hence $\widetilde{a} < \mu(X \times Y)$ because $\tau_{C \times D}$ is an isomorphism.

Finally, suppose that $\mu X \notin \mathbb{R}^{>0}$ and $\mu Y \in \mathbb{R}^{>0}$ (the case when $\mu X \in \mathbb{R}^{>0}$ and $\mu Y \notin \mathbb{R}^{>0}$ is similar). Proceed as in the previous case, but let $D \subseteq Y$ be any cell so that $\mathrm{int}(\pi D) \neq \emptyset$, and let $P \subseteq \tau_D D$ be a box so that $\mathrm{int}(\pi P) \neq \emptyset$. $\square$

We now have the following theorem:

Theorem 5.9 *For each n, there is a map $\mu_n\colon SB[n] \to \widetilde{V}$ such that for all $X, Y \in SB[n]$, $\mu_n(X \dot{\cup} Y) = \mu_n X + \mu_n Y$, and $\mu_n X > 0$ iff $\mathrm{int}(X) \neq \emptyset$. Furthermore, if $X \in SB[m]$ and $Y \in SB[n]$, then $\mu_{m+n}(X \times Y) = \mu_m X \cdot \mu_n Y$, and $\mu_n Y = \mu_n \phi(Y)$ whenever ϕ is an isomorphism $Y \to \phi(Y)$.*

PROOF: For a given n, let $\mu_n : SB[n] \to \widetilde{V}$ be as in Definition 5.5. Finite additivity of μ_n follows from Theorem 4.8. It follows from Lemma 5.2 and Theorem 5.4 that for $X \in SB[n]$, $\mu_n(X) > 0$ implies $\mathrm{int}(X) \neq \emptyset$. The reverse implication is immediate from the definition of μ_n. For $X \in SB[m]$ and $Y \in SB[n]$, $\mu_{m+n}(X \times Y) = \mu_m X \cdot \mu_n Y$ is implied by Lemma 5.8. Finally, invariance under isomorphisms is Corollary 5.7.

□

6 A special case

In this section, we assume that R is such that for all $x, y \in \mathfrak{m}^{\geq 0}$, $x \sim y$ iff $v(x) = v(y)$. Note that this condition is equivalent to Γ being archimedean, and hence to Γ being embeddable into the ordered additive group of $\mathbb{R}$. We shall modify the definition of μ to obtain a finitely additive measure ν on all of $B[n]$, which takes values in the Dedekind completion of Γ, and is such that $\nu X > 0$ iff $\mathrm{int}(X) \neq \emptyset$. The price we pay for extending the collection of measurable sets to $B[n]$, is that we need to identify all sets of "finite, non-infinitesimal size". For example, $\nu X = \nu Y$ whenever $X, Y \in SB[n]$ are such that πX and πY have non-empty interior.

Note that the results of this section apply in particular when the underlying set of R is the field of Puiseux series $\bigcup_n \mathbb{R}((t^{\frac{1}{n}}))$ in t over $\mathbb{R}$. The results of this section thus apply to the L-R field (see [7], and Introduction).

Definition 6.1 *Let $x, y \in R^{\geq 0}$. Then $x \approx y$ iff $v(x) = v(y)$.*

We define Dedekind cuts in $R^{\geq 0}/\approx$ analogously to Dedekind cuts in $V^{\geq 0}/\sim$ (see the paragraph above Definition 3.2), and we let $\widetilde{R}$ be the collection of all Dedekind cuts in $R^{\geq 0}/\approx$. We define $\leq$ and $+$ and $\cdot$ on $\widetilde{R}$ as in Definition 3.2, with $\widetilde{R}$ in place of $\widetilde{V}$.

The proof of the next lemma is straight-forward and left to the reader.

Lemma 6.2 *The operations $+$ and $\cdot$ are well-defined and make $\widetilde{R}$ into an ordered semiring.*

For $x \in V^{\geq 0}$ we shall abuse notation by identifying the element $\widetilde{x} \in \widetilde{V}$ with its image in $\widetilde{R}$ under the $(+, \cdot, 0, 1)$-homomorphism induced by the map

$$V^{\geq 0}/\sim \;\to R^{\geq 0}/\approx : [x]_\sim \mapsto [x]_\approx .$$

Lemma 6.3 *For all $x, y \in R^{\geq 0}$, $\widetilde{x} + \widetilde{y} = \widetilde{x+y}$ and $\widetilde{x} \cdot \widetilde{y} = \widetilde{x \cdot y}$.*

PROOF: Straight-forward and left to the reader. □

Definition 6.4 *For $X \in B[n]$ we set*

$$\nu X := \widetilde{\frac{1}{\det A}} \cdot \mu(TX),$$

where

$$T : R^n \to R^n : x \mapsto Ax + b$$

is an affine map with a diagonal affine transformation matrix $A = (a_{ij})$ such that $a_{ii} = \lambda \in (0, 1]$ for $i = 1, \ldots, n$, $b \in R^n$, and $TX \subseteq [0, 1]^n$.

The next Lemma shows that νX is well-defined.

Lemma 6.5 *Let $X \in B[n]$ be definable, and let*

$$T : R^n \to R^n : x \mapsto Ax + b \text{ and } T' : R^n \to R^n : x \mapsto A'x + b'$$

be affine transformations for X as in Definition 6.4. Then

$$\widetilde{\frac{1}{\det A}} \cdot \mu(TX) = \widetilde{\frac{1}{\det A'}} \cdot \mu(T'X).$$

PROOF: We set $Y = TX$ and

$$S = T' \circ T^{-1} \colon [0, 1]^n \to [0, 1]^n.$$

Then S is an affine map with diagonal transformation matrix (a_{ij}), where $a_{ii} = \alpha \in R^{>0}$ for $i = 1, \ldots, n$. It suffices to show that $\mu Y = \widetilde{\frac{1}{\alpha^n}} \cdot \mu SY$. This is clearly satisfied if $\mathrm{int}(\pi Y) \neq \emptyset$, since then $\widetilde{\alpha} = \widetilde{1}$. It also holds in the case when Y is a box. So if $\mu Y < \widetilde{1}$, then the lemma is implied by Lemma 5.2 and Lemma 5.3. □

It is now clear that we have an analog of Theorem 5.9 (where an isomorphism between sets in $B[n]$ is defined as in Definition 5.1 after replacing $SB[n]$ by $B[n]$):

Theorem 6.6 *Suppose Γ is archimedean. Then, for each n, there is a map $\mu_n \colon B[n] \to \widetilde{R}$ such that for all $X, Y \in B[n]$, $\mu_n(X \dot{\cup} Y) = \mu_n X + \mu_n Y$, and $\mu_n X > 0$ iff $\mathrm{int}(X) \neq \emptyset$. Furthermore, if $X \in B[m]$ and $Y \in B[n]$, then*

$$\mu_{m+n}(X \times Y) = \mu_m X \cdot \mu_n Y,$$

and $\mu_n Y = \mu_n \phi(Y)$ whenever ϕ is an isomorphism $Y \to \phi(Y)$.

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