# The structure on the real field generated by the standard part map on an o-minimal expansion of a real closed field

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**Abstract**

Let $R$ be a sufficiently saturated o-minimal expansion of a real closed field, let $\mathcal{O}$ be the convex hull of $\mathbb{Q}$ in $R$, and let $\mathrm{st} : \mathcal{O}^n \to \mathbb{R}^n$ be the standard part map. For $X \subseteq R^n$ define $\mathrm{st}\, X := \mathrm{st}\,(X \cap \mathcal{O}^n)$. We let $\mathbb{R}_{\mathrm{ind}}$ be the structure with underlying set $\mathbb{R}$ and expanded by all sets of the form $\mathrm{st}\, X$, where $X \subseteq R^n$ is definable in $R$ and $n = 1, 2, \ldots$. We show that the subsets of $\mathbb{R}^n$ that are definable in $\mathbb{R}_{\mathrm{ind}}$ are exactly the finite unions of sets of the form $\mathrm{st}\, X \setminus \mathrm{st}\, Y$, where $X, Y \subseteq R^n$ are definable in $R$. A consequence of the proof is a partial answer to a question by Hrushovski, Peterzil and Pillay about the existence of measures with certain invariance properties on the lattice of bounded definable sets in $R^n$.

## 1 Introduction

Throughout, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $m, n$ range over $\mathbb{N}$.

Let $R$ be an o-minimal expansion of an ordered field (necessarily real closed), let $\mathcal{O} = \{a \in R : |a| \leq n \text{ for some } n\}$ be the convex hull of $\mathbb{Q} \subseteq R$ in $R$, and let $\mathfrak{m}$ be the maximal ideal of the valuation ring $\mathcal{O}$ of $R$, so $\mathfrak{m} = \{a \in R : |a| \leq 1/n \text{ for all } n > 0\}$. Let $\mathrm{st} : \mathcal{O} \to \mathbb{R}$ be the standard part map; it has kernel $\mathfrak{m}$ and induces for each $n$ a corresponding standard part map $\mathrm{st} : \mathcal{O}^n \to \mathbb{R}^n$. For $X \subseteq R^n$ we set $\mathrm{st}(X) := \mathrm{st}(X \cap \mathcal{O}^n)$.

From now on we assume that $R$ is $(2^{\aleph_0})^+$-saturated. In particular, the map $\text{st} : \mathcal{O} \to \mathbb{R}$ is surjective, and if $X \subseteq R^3$ is the graph of the addition operation of $R$, then $\text{st}(X) \subseteq \mathbb{R}^3$ is the graph of the addition operation of $\mathbb{R}$. The same is true for multiplication instead of addition.

By *definable* we shall mean *definable with parameters in the structure* $R$, unless specified otherwise. If another ambient structure is specified, then *definable* also means *definable with parameters* (in that structure).

Via the standard part map the definable sets of $R$ induce a structure on $\mathbb{R}$ as follows: let $\mathbb{R}_{\text{ind}}$ be the structure with underlying set $\mathbb{R}$ and with the sets $\text{st}(X)$ with definable $X \subseteq R^n$, $n = 0, 1, 2, \ldots$, as basic relations. Since the graphs of the addition and multiplication on $\mathbb{R}$ are among these basic relations, and the usual ordering of $\mathbb{R}$ is 0-definable from addition and multiplication, we may view $\mathbb{R}_{\text{ind}}$ as an expansion of the ordered field of real numbers, and we shall do so. It follows from a theorem by Baisalov and Poizat [1] that $\mathbb{R}_{\text{ind}}$ is o-minimal; this was observed by Hrushovski, Peterzil and Pillay [5], but their argument left open how logically complicated the definable relations of $\mathbb{R}_{\text{ind}}$ can be, compared to the basic relations. We answer this question here as follows:

**Theorem 1.1** *The subsets of* $\mathbb{R}^n$ *definable in* $\mathbb{R}_{\text{ind}}$ *are exactly the finite unions of differences* $\text{st}(X) \setminus \text{st}(Y)$ *with definable* $X, Y \subseteq R^n$.

This result is obtained without using the Baisalov-Poizat theorem, and thus gives another proof of the fact that $\mathbb{R}_{\text{ind}}$ is o-minimal. A previously known special case of Theorem 1.1 is when $R$ is an elementary extension of an o-minimal expansion $\mathbb{R}^{\#}$ of the ordered field of real numbers; see [4]. (The key fact in that case is that $\mathbb{R}_{\text{ind}}$ and $\mathbb{R}^{\#}$ have the same definable relations.)

The proof of the theorem goes as follows. We single out certain subsets of $\mathbb{R}^n$ as *good cells*; they have the form $\text{st}(X) \setminus \text{st}(Y)$ with definable $X, Y \subseteq R^n$, and for $n > 0$ the image of a good cell in $\mathbb{R}^n$ under the projection map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$ is a good cell in $\mathbb{R}^{n-1}$. The main step is to show by induction on $n$ that for any definable $X \subseteq R^n$ the set $\text{st}(X)$ is a finite union of good cells. More precisely, we have "good cell decomposition", Corollary 4.4. The theorem above then follows easily.

We also show that the closed subsets of $\mathbb{R}^n$ definable in $\mathbb{R}_{\text{ind}}$ are exactly the sets $\text{st}(X)$ with definable $X \subseteq R^n$.

As a consequence of a strengthening of good cell decomposition we obtain a partial answer to a question posed in [5], which is roughly as follows.

Let $B[n]$ be the lattice of all bounded definable subsets of $R^n$, and define $X, Y \in B[n]$ to be *isomorphic* iff, modulo a set of dimension $< n$, we have $\psi(X) = Y$ for some definable $C^1$-diffeomorphism $\psi$ with $|J\psi(x)| = 1$ for all $x \in X$. Let $X \in B[n]$ have nonempty interior. Is there a finitely additive $\mu : B[n] \to [0, \infty]$ with $0 < \mu(X) < \infty$ which is invariant under isomorphisms?

Our partial result is that there is such a $\mu$ if $X \subseteq \mathcal{O}^n$ and $\mathrm{st}(X)$ has nonempty interior. This follows by proving that the measure introduced by Berarducci and Otero in [2] on the lattice of definable sets contained in $\mathcal{O}^n$ is invariant under isomorphism. The main point here is that the standard part of a partial derivative of a definable function is almost everywhere equal to the corresponding partial derivative of the standard part of the function.

**Further notations and terminology.** An *interval* is always a *nonempty open interval* $(a, b)$, and intervals are in $R$ or in $\mathbb{R}$, as specified. For $m \leq n$ we let $p^n_m : R^n \to R^m$ and $\pi^n_m : \mathbb{R}^n \to \mathbb{R}^m$ be given by

$$p^n_m(x_1, \ldots, x_n) = (x_1, \ldots, x_m), \qquad \pi^n_m(x_1, \ldots, x_n) = (x_1, \ldots, x_m).$$

The *hull* of a set $C \subseteq \mathbb{R}^n$ is by definition the clopen set $C^h := \mathrm{st}^{-1}(C) \subseteq \mathcal{O}^n$. A point $x$ in $R^n$ or $\mathbb{R}^n$ has components $x_1, \ldots, x_n$, that is, $x = (x_1, \ldots, x_n)$.

Let $f : X \to R$, where $X \subseteq R^n$. Then the graph of $f$ as a subset of $R^{n+1}$ is denoted by $\Gamma f$, and we put

$$\begin{aligned}
(-\infty, f) &:= \{(x, y) \in R^{n+1} : x \in X \,\&\, y < f(x)\},\\
(-\infty, f] &:= \{(x, y) \in R^{n+1} : x \in X \,\&\, y \leq f(x)\},\\
(f, +\infty) &:= \{(x, y) \in R^{n+1} : x \in X \,\&\, f(x) < y\},\\
[f, +\infty) &:= \{(x, y) \in R^{n+1} : x \in X \,\&\, f(x) \leq y\}.
\end{aligned}$$

When also $g : X \to R$, then "$f < g$" abbreviates "$f(x) < g(x)$ for all $x \in X$" and if $f < g$ we put

$$(f, g) := \{(x, y) \in R^{n+1} : x \in X \,\&\, f(x) < y < g(x)\}.$$

Likewise, functions $X \to \mathbb{R}$ with $X \subseteq \mathbb{R}^n$ give rise to subsets of $\mathbb{R}^{n+1}$ that we denote in the same way. A $\mathbb{Q}$-box in $R^n$ is a cartesian product

$$I_1 \times \cdots \times I_n \subseteq R^n$$

of intervals $I_j$ in $R$ whose endpoints lie in $\mathbb{Q}$. Any unexplained terminology or notation is from [3].

## 2 Basic facts about standard part sets

It is easy to see that if $X \subseteq R^n$ is definable in $R$, then $\operatorname{st} X$ is closed in $\mathbb{R}^n$. Let $\mathrm{St}_n$ be the collection of all sets $\operatorname{st} X$ with definable $X \subseteq R^n$.

Note: if $X, Y \in \mathrm{St}_n$, then $X \cup Y \in \mathrm{St}_n$; if $X \in \mathrm{St}_m$ and $Y \in \mathrm{St}_n$, then $X \times Y \in \mathrm{St}_{m+n}$. The next lemma is almost obvious, with (1) a special case of (2). To state it we use the projection maps $\pi = \pi^{m+n}_m : \mathbb{R}^{m+n} \to \mathbb{R}^m$ and $p = p^{m+n}_m : R^{m+n} \to R^m$.

**Lemma 2.1** *Let $X \in \mathrm{St}_{m+n}$. Then*

(1) *if $X$ is bounded, then $\pi(X) \in \mathrm{St}_m$,*

(2) *if $X = \operatorname{st} X'$ where the set $X' \subseteq R^{m+n}$ is definable in $R$ and satisfies $X' \cap p^{-1}(\mathcal{O}^m) \subseteq \mathcal{O}^{m+n}$, then $\pi(X) \in \mathrm{St}_m$.*

**Lemma 2.2** *If $X, Y \in \mathrm{St}_n$, then $X \cap Y \in \mathrm{St}_n$.*

Proof: Let $X, Y \in \mathrm{St}_n$, and take definable $X', Y' \subseteq R^n$ such that $\operatorname{st}(X') = X$ and $\operatorname{st}(Y') = Y$. For each $a \in X \cap Y$ take $x_a \in X'$ and $y_a \in Y'$ such that $\operatorname{st}(x_a) = \operatorname{st}(y_a) = a$. By saturation (in a cardinal $> 2^{\aleph_0}$) we can take an infinitesimal $\varepsilon \in R^{>0}$ such that $d(x_a, y_a) < \varepsilon$ for all $a \in X \cap Y$. Hence, with

$$Z := \{(x, y) \in X' \times Y' : d(x, y) < \varepsilon\} \subseteq R^{2n},$$

$Z$ is definable and $X \cap Y$ is the image of $\operatorname{st}(Z) \subseteq \mathbb{R}^{2n}$ under the projection map $\pi^{2n}_n : \mathbb{R}^{2n} \to \mathbb{R}^n$. Now apply (2) of Lemma 2.1. □

**Lemma 2.3** *Let $X \subseteq R^n$ and $f : X \to R$ be definable, and put*

$$X^- := \{x \in X : \ f(x) < \mathbb{Q}\}, \quad X^+ := \{x \in X : \ f(x) > \mathbb{Q}\}.$$

*Their standard parts $\operatorname{st}(X^-)$ and $\operatorname{st}(X^+)$ belong to $\mathrm{St}_n$.*

Proof: To get $\operatorname{st}(X^-) \in \mathrm{St}_n$, use Lemma 2.2, the fact that

$$Y := \{(x, y) \in X \times R : \ f(x) < 0, \ f(x) \cdot y = 1\} \subseteq R^{n+1}$$

is definable, and

$$\operatorname{st}(X^-) \ = \ \pi^{n+1}_n\big((\operatorname{st} Y) \cap (\mathbb{R}^n \times \{0\})\big).$$

In the same way we see that $\operatorname{st}(X^+) \in \mathrm{St}_n$. □

**Lemma 2.4** *If $X \subseteq R$ is definable, then $\operatorname{st}(X)$ is a finite union of intervals and points in $\mathbb{R}$.*

Proof: This is immediate from the o-minimality of $R$. □

## 3 Good cells

The following notion turns out to be very useful.

**Definition 3.1** *Given functions $f : X \to R$ with $X \subseteq R^n$, and $g : C \to \mathbb{R}$ with $C \subseteq \mathbb{R}^n$, we say that $f$* induces *$g$ if $f$ is definable (so $X$ is definable), $C^h \subseteq X$, $f|C^h$ is continuous, $f(C^h) \subseteq \mathcal{O}$ and $\Gamma g = \mathrm{st}(\Gamma f) \cap (C \times \mathbb{R})$.*

**Lemma 3.2** *Let $C \subseteq \mathbb{R}^n$ and suppose $g : C \to \mathbb{R}$ is induced by the function $f : X \to R$ with $X \subseteq R^n$. Then $g$ is continuous.*

Proof: Let $x \in C$ and suppose towards a contradiction that $\epsilon \in \mathbb{Q}^{>0}$ is such that for every $\lambda \in \mathbb{Q}^{>0}$ we have $x_\lambda \in C$ with $|x_\lambda - x| < \lambda$ and $|g(x_\lambda) - g(x)| > \epsilon$. Pick $y \in \{x\}^h$ and for $\lambda \in \mathbb{Q}^{>0}$ pick $y_\lambda \in \{x_\lambda\}^h$. Then $|f(y) - f(y_\lambda)| \geq \epsilon$ for those $\lambda$, so by saturation we get a point $z \in \{x\}^h$ with $|f(y) - f(z)| \geq \epsilon$, contradicting that $g$ is a function. $\square$

For $C \subseteq \mathbb{R}^n$ we let $G(C)$ be the set of all $g : C \to \mathbb{R}$ that are induced by some definable $f : X \to R$ with $X \subseteq R^n$.

**Lemma 3.3** *Let $1 \leq j(1) < \cdots < j(m) \leq n$ and define*

$$\pi : \mathbb{R}^n \to \mathbb{R}^m, \quad \pi(x_1, \ldots, x_n) = (x_{j(1)}, \ldots, x_{j(m)}).$$

*Let $C \subseteq \mathbb{R}^n$ and suppose $g \in G(\pi C)$. Then $g \circ \pi|_C \in G(C)$.*

Proof: Take definable $f : Y \to R$ with $Y \subseteq R^m$ such that $f$ induces $g$, so $\Gamma g = \mathrm{st}(\Gamma f) \cap \big(\pi C \times \mathbb{R}\big)$. Let $p : R^n \to R^m$ be given by

$$p(x_1, \ldots, x_n) = (x_{j(1)}, \ldots, x_{j(m)}),$$

and put $X := p^{-1}(Y)$. Then $C^h \subseteq X$, and it is easy to check that

$$\Gamma(g \circ \pi|_C) = \mathrm{st}\left(\Gamma(f \circ p|_X)\right) \cap (C \times \mathbb{R}),$$

so $g \circ \pi|_C$ is induced by $f \circ p|_X$. $\square$

**Definition 3.4** *Let $i = (i_1, \ldots, i_n)$ be a sequence of zeros and ones. Good $i$-cells are subsets of $\mathbb{R}^n$ obtained by recursion on $n$ as follows:*

(i) *For* $n = 0$ *and* $i$ *the empty sequence, the set* $\mathbb{R}^0$ *is the only good* $i$*-cell, and for* $n = 1$*, a* good $(0)$-cell *is a singleton* $\{a\}$ *with* $a \in \mathbb{R}$*; a* good $(1)$-cell *is an interval in* $\mathbb{R}$*.*

(ii) *Let* $n > 0$ *and assume inductively that good* $i$*-cells are subsets of* $\mathbb{R}^n$*. A good* $(i, 0)$*-cell is a set* $\Gamma h \subseteq \mathbb{R}^{n+1}$ *where* $h \in G(C)$ *and* $C \subseteq \mathbb{R}^n$ *is a good* $i$*-cell. A good* $(i, 1)$*-cell is either a set* $C \times \mathbb{R}$*, or a set* $(-\infty, f) \subseteq \mathbb{R}^{n+1}$*, or a set* $(g, h) \subseteq \mathbb{R}^{n+1}$*, or a set* $(f, +\infty) \subseteq \mathbb{R}^{n+1}$*, where* $f, g, h \in G(C)$*,* $g < h$*, and* $C$ *is a good* $i$*-cell.*

One verifies easily that a good $i$-cell is open in $\mathbb{R}^n$ iff $i_1 = \cdots = i_n = 1$, and that if $i_1 = \cdots = i_n = 1$, then every good $i$-cell is homeomorphic to $\mathbb{R}^n$. A *good cell in* $\mathbb{R}^n$ is a good $i$-cell for some sequence $i = (i_1, \ldots, i_n)$ of zeros and ones.

**Lemma 3.5** *Every good cell in* $\mathbb{R}^n$ *is of the form* $X \setminus Y$ *with* $X, Y \in \mathrm{St}_n$*.*

Proof: This is clear for $n = 1$. Suppose it holds for a certain $n \geq 1$, and consider first an $(i, 0)$-cell $\Gamma h \subseteq \mathbb{R}^{n+1}$ as in (ii) above, with $h \in G(C)$ induced by $f : X \to R$, where $X \subseteq R^n$. Then $\Gamma h = \mathrm{st}(\Gamma f) \cap (C \times \mathbb{R})$. Now $C = \mathrm{st}(P) \setminus \mathrm{st}(Q)$ with definable $P, Q \subseteq R^n$, so $C \times \mathbb{R} = \mathrm{st}(P \times R) \setminus \mathrm{st}(Q \times R)$, hence

$$\Gamma h = \big(\mathrm{st}(\Gamma f) \cap \mathrm{st}(P \times R)\big) \setminus \big(\mathrm{st}(\Gamma f) \cap \mathrm{st}(Q \times R)\big),$$

and we are done by Lemma 2.2. Next, consider an $(i, 1)$-cell $(g, h) \subseteq \mathbb{R}^{n+1}$ with $g, h \in G(C)$, $g < h$, with $g$ induced by $\phi : X \to R$ and $h$ induced by $\psi : Y \to R$ with $X, Y \subseteq R^n$. Then $\Gamma g = \mathrm{st}(\Gamma \phi) \cap (C \times \mathbb{R})$ and $\Gamma h = \mathrm{st}(\Gamma \psi) \cap (C \times \mathbb{R})$. It is easy to check that

$$\begin{aligned} (-\infty, g] &= \mathrm{st}((-\infty, \phi]) \cap (C \times \mathbb{R}), \\ [h, +\infty) &= \mathrm{st}([\psi, +\infty)) \cap (C \times \mathbb{R}), \text{ hence} \\ (g, h) &= (C \times \mathbb{R}) \setminus \mathrm{st}\,\big((-\infty, \phi] \cup [\psi, +\infty)\big). \end{aligned}$$

Now $C = \mathrm{st}(P) \setminus \mathrm{st}(Q)$ with definable $P, Q \subseteq R^n$, so

$$(g, h) = \mathrm{st}(P \times R) \setminus \mathrm{st}\,\big((Q \times R) \cup (-\infty, \phi] \cup [\psi, +\infty)\big),$$

and we are done. The other types of $(i, 1)$-cells are treated likewise. □

**Lemma 3.6** *Let $C \subseteq \mathbb{R}^n$ be a good $(i_1, \ldots, i_n)$-cell, and suppose $i_k = 0$ where $k \in \{1, \ldots, n\}$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be given by*

$$\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n).$$

*Then $\pi(C) \subseteq \mathbb{R}^{n-1}$ is a good cell, $\pi|C : C \to \pi(C)$ is a homeomorphism, and if $E \subseteq \pi(C)$ is a good cell, so is its inverse image $\pi^{-1}(E) \cap C$.*

PROOF: By induction on $n$. If $n = 1$, then $k = 1$ and $C$ is a singleton, and the lemma holds trivially in that case. Assume inductively that the lemma holds for a certain $n > 0$, let $C \subseteq \mathbb{R}^{n+1}$ be a good $(i_1, \ldots, i_{n+1})$-cell, let $k \in \{1, \ldots, n+1\}$ be such that $i_k = 0$, and let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be given by

$$\pi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}).$$

Our task is to establish the following.

*Claim.* $\pi(C) \subseteq \mathbb{R}^n$ is a good cell, $\pi|C : C \to \pi(C)$ is a homeomorphism, and if $E \subseteq \pi(C)$ is a good cell, then $\pi^{-1}(E) \cap C$ is a good cell in $\mathbb{R}^{n+1}$.

If $k = n + 1$, then $\pi = \pi^{n+1}_n$ and $C = \Gamma f$ with $f \in G(\pi(C))$, and then the claim follows easily. So we can assume $k \leq n$. Then we introduce the good cell $D := \pi^{n+1}_n(C)$ in $\mathbb{R}^n$ and the map $\pi_0 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ defined by

$$\pi_0(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n).$$

Then $\pi_0(D) \subseteq \mathbb{R}^{n-1}$ is a good cell, $\pi_0|_D : D \to \pi_0(D)$ is a homeomorphism, and for each good cell $F \subseteq \pi_0(D)$ its inverse image $\pi_0^{-1}(F) \cap D$ is a good cell in $\mathbb{R}^n$. Since $\pi(x, t) = (\pi_0(x), t)$ for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, it follows that $\pi|_{D \times \mathbb{R}} : D \times \mathbb{R} \to \pi_0(D) \times \mathbb{R}$ is a homeomorphism, so $\pi|C : C \to \pi(C)$ is a homeomorphism. We have $\pi^n_k D = \Gamma h$ where $h \in G\big(\pi^n_{k-1}(D)\big)$, and the map $(\pi_0|_D)^{-1} : \pi_0(D) \to D$ is given by

$$(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_{k-1}, h(x_1, \ldots, x_{k-1}), x_{k+1}, \ldots, x_n).$$

Let $h$ be induced by $\eta : Y \to R$, $Y \subseteq R^{k-1}$.

Consider first the case that $C = \Gamma f$ with $f \in G(D)$. It is routine to check that then $\pi(C) = \Gamma f_0$, where $f_0 := f \circ (\pi_0|_D)^{-1} : \pi_0(D) \to \mathbb{R}$ is given by

$$(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \mapsto f(x_1, \ldots, x_{k-1}, h(x_1, \ldots, x_{k-1}), x_{k+1}, \ldots, x_n).$$

Let $f$ be induced by $\phi : X \to R$, $X \subseteq R^n$, and let $Z$ be the set of all $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in R^{n-1}$ such that

$$(x_1, \ldots, x_{k-1}) \in Y, \ (x_1, \ldots, x_{k-1}, \eta(x_1, \ldots, x_{k-1}), x_{k+1}, \ldots, x_n) \in X.$$

One easily shows that then $f_0$ is induced by the function $Z \to R$ given by

$$(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \mapsto \phi(x_1, \ldots, x_{k-1}, \eta(x_1, \ldots, x_{k-1}), x_{k+1}, \ldots, x_n).$$

Thus $\pi(C) = \Gamma f_0$ is a good cell in $\mathbb{R}^n$. Let $E \subseteq \pi(C)$ be a good cell. Then $E = \Gamma(f_0|_F)$ where $F \subseteq \pi_0(D)$ is a good cell, so $B := \pi_0^{-1}(F) \cap D$ is a good cell in $\mathbb{R}^n$ by the inductive assumption. Then $\pi^{-1}(E) \cap C = \Gamma(f|_B)$, as is easy to check, so $\pi^{-1}(E) \cap C$ is indeed a good cell.

Next, consider the case $C = (f, g)$ where $f, g \in G(D)$, $f < g$. Then $\pi(C) = (f_0, g_0)$, where

$$\begin{aligned} f_0 &:= f \circ (\pi_0|_D)^{-1} \ : \ \pi_0(D) \to \mathbb{R}, \\ g_0 &:= g \circ (\pi_0|_D)^{-1} \ : \ \pi_0(D) \to \mathbb{R}, \end{aligned}$$

and as before one checks that $f_0, g_0 \in G(\pi_0(D))$, so $\pi(C)$ is a good cell. Let $E \subseteq \pi(C)$ be a good cell, and set $F := \pi^n_{n-1}(E)$. Then $F \subseteq \pi_0(D)$ is a good cell, so $B := \pi_0^{-1}(F) \cap D$ is a good cell in $\mathbb{R}^n$ by the inductive assumption. If $E = \Gamma s$ with $s \in G(F)$, then

$$\pi^{-1}(E) \cap C = \Gamma(s \circ \pi_0|_B),$$

as is easy to check, and $(s \circ \pi_0|_B) \in G(B)$ by Lemma 3.3, so $\pi^{-1}(E) \cap C$ is indeed a good cell. If $E = (s, t)$ with $s, t \in G(F)$, $s < t$, then

$$\pi^{-1}(E) \cap C = (s \circ \pi_0|_B, t \circ \pi_0|_B),$$

and $(s \circ \pi_0|_B), (t \circ \pi_0|_B) \in G(B)$ by Lemma 3.3, so $\pi^{-1}(E) \cap C$ is indeed a good cell.

The remaining cases, where $C = D \times \mathbb{R}$, or $C = (-\infty, f)$, or $C = (f, +\infty)$, with $f \in G(D)$, are treated in the same way. □

# 4 Good cell decomposition

A set $X \subseteq R^n$ is said to be *strongly bounded* if there is $q \in \mathbb{Q}^{>0}$ such that $|x| \leq q$ for all $x \in X$. The proof of good cell decomposition in this section works initially only for strongly bounded definable sets, because it uses part (1) of Lemma 2.1. Once we have good cell decomposition for that case we extend it to general definable sets using the homeomorphism $x \mapsto x/\sqrt{1+x^2} : \mathbb{R} \to (-1,1)$.

Berarducci and Otero [2] define a real-valued finitely additive measure $\mu = \mu^{(n)}$ on the lattice of strongly bounded definable subsets of $R^n$. The properties of this measure imply a fact that is useful for the inductive step in the proof of good cell decomposition:

**Lemma 4.1** *Syppose $Y \subseteq R^n$ is definable and $\operatorname{st} Y$ has nonempty interior in $\mathbb{R}^n$. Then $Y$ contains a $\mathbb{Q}$-box.*

Proof: We can assume $Y$ is strongly bounded. Then by Theorem 4.3 of [2] we have $\mu(Y) = \lambda(\operatorname{st} Y)$ where $\lambda$ is the usual Lebesgue measure on $\mathbb{R}^n$; in particular, $\mu(Y) > 0$. The way $\mu$ is defined in [2] guarantees that $Y$ contains a $\mathbb{Q}$-box. □

**Lemma 4.2** *Let $C \subseteq \mathbb{R}^n$ be a good i-cell, let $X \subseteq R^{n+1}$ be definable and suppose $k \in \{1, \dots, n\}$ is such that $i_k = 0$. Define $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ by*

$$\pi(x) = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}).$$

*Then $\pi\big(\operatorname{st}(X) \cap (C \times \mathbb{R})\big)$ is a difference of sets in $\mathrm{St}_n$.*

Proof: Let $\pi^n_k C = \Gamma g$, with $g : \pi^n_{k-1} C \to \mathbb{R}$ induced by $f : Y \to R$, $Y \subseteq R^{k-1}$. For infinitesimal $\varepsilon \in R^{>0}$, define $X_\varepsilon \subseteq X$ as follows:

$$X_\varepsilon := \{x \in X : (x_1, \dots, x_{k-1}) \in Y \text{ and } |f(x_1, \dots, x_{k-1}) - x_k| \leq \varepsilon\}$$

*Claim 1.* There is an infinitesimal $\varepsilon \in R^{>0}$ such that

$$\operatorname{st}(X) \cap (C \times \mathbb{R}) = \operatorname{st}(X_\epsilon) \cap (C \times \mathbb{R}).$$

To see this, pick for each $a \in \operatorname{st}(X) \cap (C \times \mathbb{R})$, an $x \in \operatorname{st}^{-1}(a)$. For such $x$,

$$\operatorname{st}(x_1, \dots, x_{k-1}) \in \pi^n_{k-1} C \quad\text{and}\quad |f(x_1, \dots, x_{k-1}) - x_k| \text{ is infinitesimal.}$$

Then saturation gives an infinitesimal $\varepsilon \in R^{>0}$ as claimed.

Define $p : R^{n+1} \to R^n$ by $p(x) = (x_1 \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1})$, and take an infinitesimal $\varepsilon \in R^{>0}$ with the property of Claim 1.

*Claim 2.* $\pi\big(\operatorname{st}(X_\varepsilon)\cap(C\times\mathbb{R})\big)=\operatorname{st}p(X_\varepsilon)\cap\pi(C\times\mathbb{R})$.

It is clear that $\pi\big(\operatorname{st}(X_\varepsilon)\cap(C\times\mathbb{R})\big)\subseteq\operatorname{st}p(X_\varepsilon)\cap\pi(C\times\mathbb{R})$. So take $x\in X_\varepsilon$ such that $(\operatorname{st}x_1,\dots,\operatorname{st}x_{k-1},\operatorname{st}x_{k+1},\dots,\operatorname{st}x_{n+1})\in\pi(C\times\mathbb{R})$. We claim that then

$$\operatorname{st}x\in\operatorname{st}(X_\varepsilon)\cap(C\times\mathbb{R}).$$

This follows from the definition of $X_\varepsilon$: clearly $\operatorname{st}(x_1,\dots x_{k-1})\in\pi^n_{k-1}C$ and $|x_k-f(x_1,\dots,x_{k-1})|$ is infinitesimal. Hence

$$\operatorname{st}x_k=\operatorname{st}f(x_1,\dots,x_{k-1})=g(\operatorname{st}(x_1,\dots,x_{k-1})),$$

and so $\operatorname{st}(x_1,\dots,x_k)\in\Gamma g$. □

We set $I:=[-1,1]\subseteq\mathbb{R}$ and $I(R):=\{x\in R: -1\leq x\leq 1\}$. A *good decomposition of* $I^n$ is a special kind of partition of $I^n$ into finitely many good cells. The definition is by recursion on $n$:

(i) a good decomposition of $I^1=I$ is a collection

$$\{(c_0,c_1),(c_2,c_3),\dots,(c_k,c_{k+1}),\{c_0\},\{c_1\},\dots,\{c_k\},\{c_{k+1}\}\}$$

of intervals and points in $\mathbb{R}$ where $c_0<c_1<\cdots<c_k<c_{k+1}$ are real numbers with $c_0=-1$ and $c_{k+1}=1$;

(ii) a good decomposition of $I^{n+1}$ is a finite partition $\mathcal{D}$ of $I^{n+1}$ into good cells such that $\{\pi^{n+1}_n C:\ C\in\mathcal{D}\}$ is a good decomposition of $I^n$.

**Theorem 4.3 (Good Cell Decomposition)**

($A_n$) *Given any definable $X_1,\dots,X_m\subseteq I(R)^n$, there is a good decomposition of $I^n$ partitioning each set $\operatorname{st}X_i$.*

($B_n$) *If $f:X\to I(R)$, with $X\subseteq I(R)^n$, is definable, then there is a good decomposition $\mathcal{D}$ of $I^n$ such that for every open $C\in\mathcal{D}$, either the set $\operatorname{st}(\Gamma f)\cap(C\times\mathbb{R})$ is empty, or $f$ induces a function $g:C\to I$.*

Proof: We proceed by induction on $n$. Item $(A_1)$ holds by Lemma 2.4. We now assume $(A_n)$, $n > 0$, and first prove $(B_n)$, and then $(A_{n+1})$.

Let $f : X \to I(R)$ be definable with $X \subseteq I(R)^n$. Take a decomposition $\mathcal{P}$ of $R^n$ that partitions $I(R)^n$ and $X$ such that if $P$ is an open cell of $\mathcal{P}$ contained in $X$, then $f$ is continuously differentiable on $P$ and $\partial f/\partial x_i$ has constant sign on $P$ for $i = 1, \ldots, n$. Let $P \in \mathcal{P}$ be an open cell contained in $X$, and let $i \in \{1, \ldots, n\}$.

Consider first the case that $(\partial f/\partial x_i) > 0$ on $P$, and put

$$P(i) := \{a \in P : (\partial f/\partial x_i)(a) > \mathbb{Q}\},$$

so $\operatorname{st} P(i) \in \mathrm{St}_n$ by Lemma 2.2. Then the set $\operatorname{st} P(i) \subseteq I^n$ has empty interior: otherwise, Lemma 4.1 gives a $\mathbb{Q}$-box $B \subseteq P(i)$, but then $f$ could not be $\mathbb{Q}$-bounded on $B$, a contradiction. In case $(\partial f/\partial x_i) \leq 0$ on $P$, put

$$P(i) := \{a \in P : (\partial f/\partial x_i)(a) < \mathbb{Q}\},$$

and then $\operatorname{st} P(i) \in \mathrm{St}_n$ and $\operatorname{st} P(i)$ has empty interior, by similar reasoning.

By $(A_n)$ we have a good decomposition $\mathcal{D}$ of $I^n$ partitioning $\operatorname{st} P$ and $\operatorname{st} \partial P$ whenever $P \in \mathcal{P}$ is open and $P \subseteq I(R)^n$, and all $\operatorname{st} P(i)$, $1 \leq i \leq n$, for which $P \in \mathcal{P}$ is open and contained in $X$. We are going to show that $\mathcal{D}$ has the property required by $(B_n)$. Suppose $C \in \mathcal{D}$ is open. Take $P \in \mathcal{P}$ such that $C \subseteq \operatorname{st} P$. Then $P$ is an open cell contained in $I(R)^n$, so $C \cap \operatorname{st} \partial P = \emptyset$.

*Claim 1.* $C^{\mathrm{h}} \subseteq P$.

To see this, let $a \in C^{\mathrm{h}}$ and suppose $a \notin P$. Take $b \in P$ with $\operatorname{st} a = \operatorname{st} b$, and note that the straight line segment connecting $a$ to $b$ must contain a point $p \in \partial P$, but then $\operatorname{st} p = \operatorname{st} a \in C$, a contradiction.

Suppose now that $\operatorname{st}(\Gamma f) \cap (C \times \mathbb{R}) \neq \emptyset$. It remains to show that then $f$ induces a function $C \to I$. It follows from Claim 1 that $P \subseteq X$. Let $x \in C$ be given. Then there is $y$ in $I$ with $(x, y) \in \operatorname{st}(\Gamma f)$, and there is only one such $y$: if $(x, y_1), (x, y_2) \in \operatorname{st}(\Gamma f)$, with $y_1 \neq y_2$, take $a, b \in P$ with $\operatorname{st} a = x = \operatorname{st} b$ and $\operatorname{st} f(a) = y_1$ and $\operatorname{st} f(b) = y_2$. By Claim 1, the infinitesimal line segment connecting $a$ and $b$ is entirely contained in $P$, and by the Mean Value Theorem this line segment must contain a point $p \in P(i)$ with $i \in \{1, \ldots, n\}$, so $\operatorname{st} p = x \in \operatorname{st} P(i)$, contradicting $C \cap \operatorname{st} P(i) = \emptyset$. Thus $f$ induces a function $C \to I$. This finishes the proof of $(B_n)$.

Towards proving $(A_{n+1})$, we first establish the following.

*Claim 2.* Let $C_1, \dots, C_m \subseteq I^{n+1}$ be good cells; then there is a good decomposition of $I^{n+1}$ that partitions each $C_k$.

To prove this, take functions $\phi_1, \dots, \phi_M$, $(M \in \mathbb{N})$, where each $\phi_i \in G(D_i)$, $D_i$ a good cell in $I^n$, such that each $C_k$ is of the form $\Gamma\phi_i$ or $(\phi_i, \phi_j)$ (where in the latter case $D_i = D_j$). Let $1 \leq i < j \leq M$, and put

$$D_{ij} := \pi_n^{n+1}(\Gamma\phi_i \cap \Gamma\phi_j).$$

We show there are definable $P, Q \subseteq I(R)^n$ such that $D_{ij} = \mathrm{st}(P) \setminus \mathrm{st}(Q)$. To get such $P, Q$, take $f : X \to I(R)$ and $g : Y \to I(R)$ with $X, Y \subseteq I(R)^n$, such that $f$ induces $\phi_i$ and $g$ induces $\phi_j$. It is easy to check that then

$$D_{ij} = \pi_n^{n+1}\big(\mathrm{st}(\Gamma f) \cap \mathrm{st}(\Gamma g)\big) \cap D_i \cap D_j,$$

so $D_{ij}$ has the desired form, by part (1) of Lemma 2.1 and by Lemmas 2.2 and 3.5. By $(A_n)$ we can take a good decomposition $\mathcal{D}$ of $I^n$ that partitions all $D_i$ and all $D_{ij}$. It follows easily that there is a good decomposition $\mathcal{C}$ of $I^{n+1}$ that partitions all $C_k$ such that $\{\pi_n^{n+1}(C) : C \in \mathcal{C}\} = \mathcal{D}$. This finishes the proof of Claim 2.

To prove $(A_{n+1})$ we note that by cell decomposition in the structure $R$ and Claim 2 it suffices to establish the following special case:

*Claim 3.* Let $X \subseteq I(R)^{n+1}$ be a cell in $R^{n+1}$; then $\mathrm{st}(X)$ is a finite union of good cells in $\mathbb{R}^{n+1}$.

Assume first that $X = \Gamma f$, with $f : p_n^{n+1}X \to I(R)$. By $(A_n)$ and $(B_n)$, we have a finite partition $\mathcal{P}$ of $\mathrm{st}(p_n^{n+1}X)$ into good cells, such that if $C \in \mathcal{P}$ is open, then $f$ induces a function $C \to I$, so $\mathrm{st}(X) \cap (C \times I)$ is a good cell. Consider next a cell $C \in \mathcal{P}$ that is not open. Let $i = (i_1, \dots, i_n)$ be such that $C$ is a good $i$-cell, take $k \in \{1, \dots, n\}$ such that $i_k = 0$, and consider the map

$$\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n, \qquad \pi(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}).$$

It is easy to see that $\pi|_{C \times I} : C \times I \to \pi(C \times I)$ is a homeomorphism. By Lemma 4.2, the set $\pi\big(\mathrm{st}(X) \cap (C \times I)\big)$ is a difference of sets in $\mathrm{St}_n$. Thus by $(A_n)$,

$$\pi\big(\mathrm{st}(X) \cap (C \times I)\big) = \bigcup_{i=1}^{m} E_i,$$

where $E_1, \dots, E_m \subseteq I^n$ are good cells. Then

$$\operatorname{st}(X) \cap (C \times I) = \bigcup_{i=1}^{m} \pi^{-1}(E_i) \cap (C \times I),$$

and each $\pi^{-1}(E_i) \cap (C \times I)$ is a good cell by Lemma 3.6. It follows that Claim 3 holds for $X = \Gamma f$.

Next, assume that $X = (f, g)$ where $f, g : p_n^{n+1}X \to I(R)$, $f < g$. By $(B_n)$, we have a finite partition $\mathcal{P}$ of $\operatorname{st}(p_n^{n+1}X)$ into good cells such that if $C \in \mathcal{P}$ is open, then both $f$ and $g$ induce functions on $C$. By $(A_n)$, we can take a finite partition $\mathcal{P}'$ of $\operatorname{st}(p_n^{n+1}X)$ into good cells such that $\mathcal{P}'$ partitions each cell $C \in \mathcal{P}$ and for every open $C \in \mathcal{P}$ it partitions the set $\{\operatorname{st} x \in C : \operatorname{st} f(x) = \operatorname{st} g(x)\}$. So if $C \in \mathcal{P}'$ is open, then $\operatorname{st} X \cap (C \times I)$ is a good cell. If $C \in \mathcal{P}'$ is not open, then we show in the same way as in the case $X = \Gamma f$ that $\operatorname{st} X \cap (C \times I)$ is a finite union of good cells.

□

A *good decomposition of* $\mathbb{R}^n$ is a special kind of partition of $\mathbb{R}^n$ into finitely many good cells. The definition is by recursion on $n$:

(i) a good decomposition of $\mathbb{R}^1 = \mathbb{R}$ is a collection

$$\{(c_0, c_1), (c_2, c_3), \dots, (c_k, c_{k+1}), \{c_1\}, \dots, \{c_k\}\}$$

of intervals and points in $\mathbb{R}$ where $c_1 < \dots < c_k$ are real numbers and $c_0 = -\infty$, $c_{k+1} = \infty$;

(ii) a good decomposition of $\mathbb{R}^{n+1}$ is a finite partition $\mathcal{D}$ of $\mathbb{R}^{n+1}$ into good cells such that $\{\pi_n^{n+1}C : C \in \mathcal{D}\}$ is a good decomposition of $\mathbb{R}^n$.

We set $J := (-1, 1) \subseteq \mathbb{R}$ and $J(R) := (-1, 1) \subseteq R$. We shall use the definable homeomorphism

$$\tau_n : R^n \to J(R)^n : (x_1 \dots, x_n) \mapsto (\frac{x_1}{\sqrt{1 + x_1^2}}, \dots, \frac{x_n}{\sqrt{1 + x_n^2}}),$$

and we also let $\tau_n$ denote the homeomorphism

$$\tau_n : \mathbb{R}^n \to J^n : (x_1 \dots, x_n) \mapsto (\frac{x_1}{\sqrt{1 + x_1^2}}, \dots, \frac{x_n}{\sqrt{1 + x_n^2}}).$$

One easily checks that $\tau_1 : R \to J(R)$ induces $\tau_1 : \mathbb{R} \to J$.

**Corollary 4.4** *If $X_1, \dots, X_m \subseteq R^n$ are definable, then there is a good decomposition of $\mathbb{R}^n$ partitioning every* $\operatorname{st} X_i$.

PROOF: First note that by the remark right before this corollary, we have

$$\tau_n(\operatorname{st} X_i) = \operatorname{st} \tau_n(X_i) \cap J^n$$

for all $i$. Hence by Theorem 4.3 we have a good decomposition $\mathcal{D}$ of $I^n$ partitioning $J^n$ and every $\tau_n(\operatorname{st} X_i)$.

*Claim.* If $D \subseteq J^n$ is a good cell, then $\tau_n^{-1}(D) \subseteq \mathbb{R}^n$ is also a good cell.

We prove this by induction on $n$. The claim clearly holds for $n = 1$. Assume it holds for a certain $n \geq 1$, and let $D \subseteq J^{n+1}$ be a good cell. Put $C := \pi_n^{n+1} D$. We first consider the case $D = \Gamma g$, where $g : C \to J$ is induced by $f : X \to R$ with $X \subseteq R^n$. We can arrange that $X \subseteq J(R)^n$ and $f(X) \subseteq J(R)$. Then

$$\tau_{n+1}^{-1}(D) = \Gamma \tilde{g}, \quad \tilde{g} = \tau^{-1} \circ g \circ \tau_n|_{\tau_n^{-1} C} : \ \tau_n^{-1}(C) \to \mathbb{R}.$$

The set $\tau_n^{-1} C$ is a good cell by the inductive assumption and $\tilde{g}$ is induced by $\tau^{-1} \circ f \circ \tau_n|_{\tau_n^{-1} X}$. Thus $\tau_{n+1}^{-1}(D) = \Gamma \tilde{g}$ is a good cell in $\mathbb{R}^n$.

If $D = (-1, g)$ or $D = (g, 1)$, where $g$ is as above, then $\tau_{n+1}^{-1} D = (-\infty, \tilde{g})$ or $\tau_{n+1}^{-1} D = (\tilde{g}, \infty)$, with $\tilde{g}$ defined as above. We proceed likewise in the case $D = (g_1, g_2)$ with $g_1, g_2 : C \to J$. Finally, if $D = C \times (-1, 1)$, then we have $\tau_{n+1}^{-1} D = (\tau_n^{-1} C) \times \mathbb{R}$. This concludes the proof of the claim.

It follows that the collection of all $\tau_n^{-1} D$, where $D \in \mathcal{D}$ and $D \subseteq J^n$, is a good decomposition of $\mathbb{R}^n$ partitioning every $\operatorname{st} X_i$.

□

Theorem 1.1 from the Introduction is now obtained as follows. Let $n$ be given. By Corollary 4.4 and Lemma 3.5, the finite unions of sets $\operatorname{st}(X) \setminus \operatorname{st}(Y)$ with definable $X, Y \subseteq R^n$ are exactly the finite unions of good cells in $\mathbb{R}^n$, and these finite unions are also the elements of a boolean algebra of subsets of $\mathbb{R}^n$. Also, if $X$ is a finite union of good cells in $\mathbb{R}^n$, then $X \times \mathbb{R}$ and $\mathbb{R} \times X$ are finite unions of good cells in $\mathbb{R}^{n+1}$. Finally, the $\pi_n^{n+1}$-image of a finite union of good cells in $\mathbb{R}^{n+1}$ is clearly a finite union of good cells in $\mathbb{R}^n$.

# 5 Closed sets and connected sets

In this section $n \geq 1$. For $x \in R^n$, and definable $Y \subseteq R^n$, put

$$|x| := \max_i |x_i|, \quad d(x,Y) := \inf\{|x-y| :\ y \in X\} \in R \cup \{+\infty\}.$$

Likewise, for $x \in \mathbb{R}^n$ and any set $Y \subseteq \mathbb{R}^n$,

$$|x| := \max_i |x_i|, \quad d(x,Y) := \inf\{|x-y| :\ y \in X\} \in \mathbb{R} \cup \{+\infty\}.$$

**Proposition 5.1** *The closed subsets of* $\mathbb{R}^n$ *definable in* $\mathbb{R}_{\text{ind}}$ *are exactly the sets* $\operatorname{st} X$ *with definable* $X \subseteq R^n$.

Proof: The result will follow from Corollary 4.4 once we show that the closure of a good cell in $\mathbb{R}^n$ is of the form $\operatorname{st} X$ for some definable $X \subseteq R^n$. Let $\epsilon$ range over positive infinitesimals. Let $C \subseteq \mathbb{R}^n$ be a good cell.

*Claim 1.* There is an $r_0 \in \mathbb{Q}^{>0}$ and a definable family $\{X_r\}$ of subsets of $R^n$, indexed by the $r \in (0, r_0) \subseteq R$, such that

$$0 < r < r' < r_0 \Longrightarrow X_{r'} \subseteq X_r; \qquad \operatorname{st}\Big(\bigcap_\epsilon X_\epsilon\Big) = C.$$

The proof is by induction on $n$. If $C = \{c\} \subseteq \mathbb{R}$, then we take any positive rational $r_0$ and $a \in R$ with $\operatorname{st} a = c$ and define $X_r := \{a\}$ for every $r \in (0, r_0)$. If $C \subseteq \mathbb{R}$ is an open bounded interval, then take $a, b \in R$ such that $\operatorname{st} a < \operatorname{st} b$ are the endpoints of $C$ and let $X_r = (a+r, b-r)$ with $r \in (0, r_0)$ where $r_0$ is some positive rational $< \frac{b-a}{2}$. The family $\{X_r\}$ has the desired properties. The case that $C$ is an unbounded interval is left to the reader.

Assume the claim holds for certain $n \geq 1$, and let $D$ be a good cell in $\mathbb{R}^{n+1}$. For $C := \pi^{n+1}_n D$, let $\{X_r\}$ with $r \in (0, r_0)$ be a definable family as in the claim. We may assume that $r_0 < 1$.

Consider first the case $D = \Gamma g$ where $g : C \to \mathbb{R}$ is induced by a definable $f : X \to R$, $X \subseteq R^n$. After replacing $\{X_r\}$ by $\{X_r \cap X\}$ if necessary, we may assume that $X_r \subseteq X$ for every $r$. We define

$$Y_r := \{(x,y) \in R^{n+1} :\ x \in X_r \text{ and } f(x) = y\}.$$

It is easy to see that then $\operatorname{st}\big(\bigcap Y_\epsilon\big) = D$.

Next, assume $D = (\phi_1, \phi_2)$ with $\phi_1, \phi_2 : C \to \mathbb{R}$ induced by $f_1 : X_1 \to R$ and $f_2 : X_2 \to R$. Without loss of generality $X = X_1 = X_2$, $f_1 < f_2$ on $X$, and $X_r \subseteq X$ for all $r \in (0, r_0)$. For $r \in (0, r_0)$, define

$$\begin{aligned} Y_r \ &:= \ \{(x,y) \in R^{n+1} : x \in X_r \text{ and} \\ &\quad f_1(x) + \tfrac{f_2(x) - f_1(x)}{2}\, r < y < f_2(x) - \tfrac{f_2(x) - f_1(x)}{2}\, r\}. \end{aligned}$$

It is clear that if $0 < r < r' < r_0$, then $Y_{r'} \subseteq Y_r$. To get $D = \operatorname{st}\big(\bigcap_\epsilon Y_\epsilon\big)$, let $x \in C^h$. Then $f_2(x) - f_1(x) > q$ for some $q \in \mathbb{Q}^{>0}$, hence for $r \in (0, r_0)$ we have: $\frac{f_2(x)-f_1(x)}{2}\, r$ is infinitesimal iff $r$ is infinitesimal.

The cases $D = C \times \mathbb{R}$, $D = (-\infty, g)$, $D = (g, \infty)$ are left to the reader.

*Claim 2.* Let $\{X_r\}$, $r \in (0, r_0)$, be a definable family as in Claim 1. Then there is an $\epsilon$ such that $\operatorname{st} X_\epsilon = \operatorname{cl}(C)$.

For each $\epsilon$ we have $C \subseteq \operatorname{st} X_\epsilon$, hence $\operatorname{cl}(C) \subseteq \operatorname{st} X_\epsilon$. Let $a \in \mathbb{R}^n \setminus \operatorname{cl}(C)$. Pick $q_a \in \mathbb{Q}^{>0}$ with $d(a, \operatorname{cl}(C)) > q_a$ and pick $b_a \in \mathcal{O}^n$ with $\operatorname{st}(b_a) = a$. Since $\operatorname{st} X_r \subseteq C$ for noninfinitesimal $r \in (0, r_0)$, this yields $d(b_a, X_r) > q_a$ for such $r$. By o-minimality of $R$ this gives $d(b_a, X_\epsilon) > q_a$ for all sufficiently large (positive infinitesimal) $\epsilon$. Then by saturation we obtain an $\epsilon$ such that $d(b_a, X_\epsilon) > q_a$ for *all* $a \in \mathbb{R}^n \setminus \operatorname{cl}(C)$. For this $\epsilon$ we have $a \notin \operatorname{st} X_\epsilon$ for all $a \in \mathbb{R}^n \setminus \operatorname{cl}(C)$, and thus $\operatorname{st} X_\epsilon = \operatorname{cl}(C)$. □

**Lemma 5.2** *Suppose $X \subseteq \mathbb{R}^n$ is closed. Then $X^h$ is the intersection of a type-definable set in $R^n$ with $\mathcal{O}^n$. In particular, if $X$ is bounded, then $X^h$ is type-definable.*

Proof: The complement of $X$ in $\mathbb{R}^n$ is a countable union of open boxes, so $X = \bigcap_{i \in \mathbb{N}} \operatorname{st} Y_i$ where each $Y_i \subseteq R^n$ is definable. Let

$$Y := \{x \in R^n : d(x, Y_i) < \frac{1}{n} \text{ for all } i \text{ and all } n > 0\}.$$

Then $Y$ is type-definable, and it is easy to check that $X^h = Y \cap \mathcal{O}^n$. The second part of the lemma follows immediately from the first part. □

**Proposition 5.3** *Let $X \subseteq R^n$ be definable, strongly bounded, and definably connected. Then $\operatorname{st} X \subseteq \mathbb{R}^n$ is connected.*

PROOF: Assume towards a contradiction that $\operatorname{st} X$ is not connected. Then $\operatorname{st} X$ is not definably connected with respect to the o-minimal structure $\mathbb{R}_{\text{ind}}$, [3], p. 59. So $\operatorname{st} X = Y_1 \dot\cup Y_2$ where $Y_1, Y_2$ are nonempty, definable in $\mathbb{R}_{\text{ind}}$, and closed in $\operatorname{st} X$, and thus closed in $\mathbb{R}^n$. Since

$$X = (X \cap Y_1^h) \dot\cup (X \cap Y_2^h),$$

and $Y_1^h$, $Y_2^h$ are type-definable and disjoint, the sets $X \cap Y_1^h, X \cap Y_2^h$ are definable, nonempty, and closed in $X$, a contradiction. □

# 6 Amenability

Note that the proof of Corollary 4.4 yields that if $f : X \to R$ is definable with $X \subseteq R^n$, then there is a good decomposition $\mathcal{D}$ of $\mathbb{R}^n$ such that if $D \in \mathcal{D}$ is open, then either $\operatorname{st} \Gamma f \cap (D \times \mathbb{R})$ is empty or $f$ induces a function $D \to \mathbb{R}$.

**Lemma 6.1** *Let both $C \subseteq \mathbb{R}^n$ and $X \subseteq R^n$ be open, and suppose $f : X \to R$ is definable, $C^1$, and $f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ induce functions $g, g_1, \ldots, g_n : C \to \mathbb{R}$. Then $g$ is $C^1$ and $g_i = \frac{\partial g}{\partial x_i}$ for all $i$.*

PROOF: Let $i \in \{1, \ldots, n\}$, let $e_i$ be the $i$-th unit vector in $R^n$ or in $\mathbb{R}^n$ (according to the context), that is, $e_{ij} = 1$ if $i = j$ and $e_{ij} = 0$ otherwise. Let $a \in C$, and take $b \in C^h$ with $\operatorname{st} b = a$. Take $q \in \mathbb{Q}^{>0}$ such that $a + te_i \in C$ for all $t \in \mathbb{R}$ with $|t| < q$, and also $b + te_i \in X$ for all $t \in R$ with $|t| < q$. By the Mean Value Theorem we have, for $t \in R$, $|t| < q$,

$$f(b + te_i) - f(b) = (\partial f/\partial x_i)(b + \theta e_i) \cdot t, \qquad (\theta \in R,\ |\theta| \le |t|),$$

and taking standard parts this gives for $t \in \mathbb{R}$, $|t| < q$,

$$g(a + te_i) - g(a) = g_i(a + \tau e_i) \cdot t, \qquad (\tau \in \mathbb{R},\ |\tau| \le |t|).$$

Letting $t \in \mathbb{R}$ go to 0 in this equality and using the continuity of $g_i$ shows that $\frac{\partial g}{\partial x_i}(a)$ exists and equals $g_i(a)$. Because this holds for all $i$ we conclude that $g$ is $C^1$. □

**Corollary 6.2** *Let $f : Y \to R$ with $Y \subseteq R^n$ be definable with strongly bounded graph. Then there is a good decomposition $\mathcal{D}$ of $\mathbb{R}^n$ that partitions $\operatorname{st} Y$ such that if $D \in \mathcal{D}$ is open and $D \subseteq \operatorname{st} Y$, then $f$ is continuously differentiable on an open definable $X \subseteq Y$ containing $D^h$, and $f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$, as functions on $X$, induce functions $g, g_1, \ldots, g_n : D \to \mathbb{R}$ such that $g$ is $C^1$ and $g_i = \frac{\partial g}{\partial x_i}$ for all $i$.*

PROOF: Since $\Gamma f$ is strongly bounded, we can reduce to the case that $\Gamma f \subseteq I(R)^{n+1}$. Then the proof of $(B_n)$ in Theorem 4.3 yields a good decomposition $\mathcal{D}$ of $I^n$ that partitions $\operatorname{st} Y$ such that if $C \in \mathcal{D}$ is open and $C \subseteq \operatorname{st} Y$, then there is an open $X \subseteq Y$ such that $f|_X$ and $C$ satisfy the assumptions of Lemma 6.1. $\square$

The following notions are from [5]. Let $X, Y \subseteq R^n$ be definable. Define

$$X \subseteq_0 Y :\iff \dim (X \setminus Y) < n,$$
$$X =_0 Y :\iff X \subseteq_0 Y \text{ and } Y \subseteq_0 X.$$

We say that a property holds for *almost all* elements of $X$ if it holds for all elements of $X$ outside a definable subset of dimension $< n$. We shall also use this notation and terminology when $X, Y \subseteq \mathbb{R}^n$ are definable in $\mathbb{R}_{\text{ind}}$, with $\mathbb{R}_{\text{ind}}$ replacing $R$.

Let $V[n]$ be the additive monoid of all definable $f : R^n \to R^{\geq 0}$ that are bounded with bounded support, with addition being pointwise addition of functions. If $f, g \in V[n]$, then by an *isomorphism* $\psi : f \to g$ we mean a definable $C^1$-diffeomorphism $\psi : U \to V$ with definable open $U, V \subseteq R^n$ such that $\operatorname{supp} f \subseteq_0 U$, $\operatorname{supp} g \subseteq_0 V$, and

$$f(x) = |J\psi(x)| g(\psi(x)) \ \text{ for almost all } x \in U,$$

where $|J\psi(x)|$ is the absolute value of the determinant of the Jacobian matrix of $\psi$ at $x \in U$. We call $f, g \in V[n]$ *isomorphic* if there is an isomorphism $f \to g$. Note that $f \in V[n]$ is isomorphic to 0 iff $f(x) = 0$ for almost all $x \in R^n$, and that isomorphism is an equivalence relation on $V[n]$.

**Definition 6.3** *An $n$-*volume *is a finitely additive $I : V[n] \to [0, \infty]$ such that $I(0) = 0$ and $I$ is invariant under isomorphisms.*[1]

Call a function $f \in V[n]$ *amenable* if there is an $n$-volume $I$ such that $0 < I(f) < \infty$. Note that then $f$ is not isomorphic to 0. Call $R$ *amenable for volumes* if for every $n$, every $f \in V[n]$ not isomorphic to 0 is amenable.

Question from [5]: *is $R$ amenable for volumes?* We give here a partial answer.

For $f \in V[n]$ we put $(0, f) := \{(x, y) \in R^{n+1} : 0 < y < f(x)\}$. Let $SV[n]$ be the submonoid of $V[n]$ of all $f \in V[n]$ such that $(0, f)$ is strongly bounded.

[1] Instead of isomorphism invariance, [5] requires that $I(f) = I(g)$ if $f = \sum_{i=1}^k f_i$ and $g = \sum_{i=1}^k g_i$ where $f_i, g_i \in V[n]$ are isomorphic for all $i$. This gives an equivalent definition.

**Lemma 6.4** *There is a finitely additive $I : SV[n] \to [0, \infty)$ with $I(0) = 0$, such that $I$ is invariant under isomorphism, and $I(f) > 0$ for all $f \in SV[n]$ for which $\mathrm{st}\,(0, f) \subseteq \mathbb{R}^{n+1}$ has nonempty interior.*

PROOF: Define $I : SV[n] \to [0, \infty)$ as follows. Let $f \in SV[n]$, and take a good decomposition $\mathcal{D}$ of $\mathbb{R}^n$ such that $f$ induces a function $f_D : D \to \mathbb{R}$ for every open $D \in \mathcal{D}$, and put $I(f) := \sum_D \int_D f_D$ where $\int$ is the Lebesgue integral and the sum is taken over all open $D \in \mathcal{D}$. It is easy to see that $I(f)$ is independent of the choice of such $\mathcal{D}$, and that $0 < I(f) < \infty$ if $\mathrm{st}\,(0, f)$ has nonempty interior in $\mathbb{R}^{n+1}$. It is also clear that $I$ is finitely additive and $I(0) = 0$. Thus it is left to show that $I(f) = I(g)$ whenever $f, g \in SV[n]$ are isomorphic.

So let $f, g \in SV[n]$ be isomorphic. Take a good decomposition $\mathcal{D}$ of $\mathbb{R}^n$ such that $f, g$ induce functions $f_D, g_D : D \to \mathbb{R}$ for every open $D \in \mathcal{D}$. We define functions $\hat{f}, \hat{g} : \mathbb{R}^n \to \mathbb{R}$ by

$$\hat{f}(x) = f_D(x) \text{ and } \hat{g}(x) = g_D(x) \text{ if } x \in D \text{ and } D \in \mathcal{D} \text{ is open,}$$
$$\hat{f}(x) = \hat{g}(x) = 0 \text{ if } x \notin D \text{ for all open } D \in \mathcal{D}.$$

Then $\hat{f}$ and $\hat{g}$ are $\mathbb{R}$-bounded with compact support and definable in $\mathbb{R}_{\mathrm{ind}}$. It is enough to show that $\int \hat{f} = \int \hat{g}$.

Take a definable $C^1$-diffeomorphism $\phi = (\phi_1, \ldots, \phi_n) : U \to V$ where $U, V$ are open subsets of $R^n$ with $\mathrm{supp}\, f \subseteq_0 U$, $\mathrm{supp}\, g \subseteq_0 V$ and

$$f(x) = |J\phi(x)|\, g(\phi(x)) \text{ for almost all } x \in U.$$

Note that $\phi(\mathrm{supp}\, f) =_0 \mathrm{supp}\, g$. So after replacing $\phi$ with $\phi|_{\mathrm{int}(\mathrm{supp}\, f \setminus Y)}$, where $Y$ is some definable subset of $\mathrm{supp}\, f$ of dimension $< n$, we may assume that the graph of $\phi$ is a strongly bounded subset of $R^{2n}$. Then, applying Corollary 6.2 to the components of $\phi$ and $\phi^{-1}$, we obtain open subsets $\hat{U}, \hat{V}$ of $\mathbb{R}^n$, definable in $\mathbb{R}_{\mathrm{ind}}$, such that each $\phi_i$ induces a $C^1$-function $\hat{\phi}_i : \hat{U} \to \mathbb{R}$, and $\hat{\phi} = (\hat{\phi}_1, \ldots, \hat{\phi}_n) : \hat{U} \to \mathbb{R}^n$ is a $C^1$-diffeomorphism onto its image $\hat{\phi}(\hat{U}) = \hat{V}$, each $\frac{\partial \phi_i}{\partial x_j}$ induces $\frac{\partial \hat{\phi}_i}{\partial x_j}$ and $\mathrm{supp}\, \hat{f} \subseteq_0 \hat{U}$, $\mathrm{supp}\, \hat{g} \subseteq_0 \hat{V}$. Then for almost all $x \in \hat{U}$ we have (taking $y \in \mathcal{O}^n$ such that $\mathrm{st}\, y = x$),

$$\hat{f}(x) = \mathrm{st}\, f(y) = \mathrm{st}\, |J\phi(y)|\, \mathrm{st}\, g(\phi(y)) = |J\hat{\phi}(x)| \hat{g}(\hat{\phi}(x)),$$

hence $\int \hat{f} = \int \hat{g}$. □

We let $B[n]$ be the collection of all bounded definable subsets of $R^n$. Let $X, Y \in B[n]$. Then an *isomorphism* $\psi : X \to Y$ is defined to be a definable $C^1$-diffeomorphism $\psi : U \to V$, where $U, V \subseteq R^n$ are open and definable, $X \subseteq_0 U$, $Y \subseteq_0 V$, $|J\psi(x)| = 1$ for almost all $x \in X \cap U$, and $\psi(X \cap U) =_0 Y$. Note that $\psi$ is an isomorphism $X \to Y$ iff it is an isomorphism $\chi_X \to \chi_Y$. (Here $\chi_X : R^n \to R$ is the characteristic function of $X$.) An *$n$-measure* is a finitely additive, isomorphism invariant $\mu : B[n] \to [0, \infty]$ with $\mu(\emptyset) = 0$.

It is straightforward that an $(n+1)$-measure $\mu$ yields an $n$-volume $I$ by $I(f) := \mu(0, f)$ for $f \in V[n]$, that an $n$-volume $I$ gives an $n$-measure $\mu$ by putting $\mu(X) := I(\chi_X)$, and that $R$ being amenable for volumes is equivalent to having for every $n$ and every $X \in B[n]$ with nonempty interior an $n$-measure $\mu$ with $0 < \mu(X) < \infty$.

Let $SB[n]$ be the collection of all strongly bounded definable subsets of $R^n$. The proof of Lemma 6.4 shows that the finitely additive measure $\mu = \mu^{(n)} : SB[n] \to [0, \infty)$ from [2] is invariant under isomorphism; it also has the property that $\mu(X) > 0$ for all $X \in SB[n]$ such that $\operatorname{st} X$ has nonempty interior.

**Theorem 6.5** *There is for each $n$ an $n$-volume $I$ such that $0 < I(f) < \infty$ for all $f \in SV[n]$ for which $\operatorname{st}(0, f)$ has nonempty interior in $\mathbb{R}^{n+1}$.*

PROOF: By the above remarks it suffices to show that for all $n$ the finitely additive $\mu = \mu^{(n)} : SB[n] \to [0, \infty)$ extends to an $n$-measure. We extend $\mu$ to $\mu^* : B[n] \to [0, \infty]$ as follows: if $X \in B[n]$ is isomorphic to $Y \in SB[n]$, then $\mu^*(X) := \mu(Y)$; if $X \in B[n]$ is not isomorphic to any $Y \in SB[n]$, then $\mu^*(X) := \infty$. Clearly, $\mu^*(\emptyset) = 0$ and $\mu^*$ is invariant under isomorphism. We claim that $\mu^*$ is finitely additive, and thus an $n$-measure. Let $X, Y \in B[n]$ be disjoint; we need to show that then $\mu^*(X) + \mu^*(Y) = \mu^*(X \cup Y)$. We can reduce to the case where $X \cup Y$ is isomorphic to $Z$ where $Z \in SB[n]$; it remains to show that then there are $X', Y' \in SB[n]$ isomorphic to $X$ and $Y$, respectively. Let $\psi : U \to V$ be an isomorphism $X \cup Y \to Z$; so $X \cup Y \subseteq_0 U$ and $Z \subseteq_0 V$. Then

$$\psi|_{\operatorname{int}(X \cap U)} : \; X \to X' := \psi(\operatorname{int}(X \cap U)) \cap Z$$

is an isomorphism and $X' \in SB[n]$, and likewise with $Y$. $\square$

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