Type-definable and invariant groups in o-minimal structures

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Abstract

Let M be a big o-minimal structure and G a type-definable group in M^n. We show that G is a type-definable subset of a definable manifold in M^n that induces on G a group topology. If M is an o-minimal expansion of a real closed field, then G with this group topology is even definably isomorphic to a type-definable group in some M^k with the topology induced by M^k. Part of this result holds for the wider class of so-called invariant groups: each invariant group G in M^n has a unique topology making it a topological group and inducing the same topology on a large invariant subset of the group as M^n.

1 Introduction

Throughout k, m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$. Let M be a (one-sorted) structure. Recall that a *definable group in* M^n is a definable set $G \subseteq M^n$ with a group operation $G \times G \to G$ whose graph is a definable subset of M^{3n}. Likewise, if M is big[1], then a type-definable group in M^n is a type-definable set $G \subseteq M^n$ with a group operation $G \times G \to G$ whose graph is a type-definable subset of M^{3n}.

Let M be an o-minimal structure and X a set. A *definable atlas of dimension* k *on* X (tacitly: with respect to M) is a finite set $\mathcal{A} = \{h_i : i \in I\}$ of bijections $h_i : X_i \to h_i(X_i)$ between subsets X_i of X and definable open subsets $h_i(X_i)$ of M^k such that $X = \bigcup_{i \in I} X_i$, each $h_i(X_i \cap X_j)$ is a definable open subset of $h_i(X_i)$, and each transition map $h_{ij} : h_i(X_i \cap X_j) \to h_j(X_j \cap X_i)$ given by $h_{ij} = h_j \circ h_i^{-1}$ is definable and continuous (hence a homeomorphism).

[1]Here "big" means "κ-saturated and strongly κ-homogeneous for a certain infinite cardinal κ," and in this context "small" means "of cardinality $< \kappa$".

Given such an atlas $\mathcal{A}$, the $\mathcal{A}$-topology is the unique topology on X that makes each X_i an open subset of X and each h_i a homeomorphism. Two definable atlases of dimension k on X are said to be *equivalent* if their union is also a definable atlas of dimension k on X; this notion of equivalence is an equivalence relation on the set of definable atlases of dimension k on X. A *definable manifold of dimension* k (tacitly: with respect to M) is a set Y equipped with an equivalence class of definable atlases of dimension k on Y (and each member of this equivalence class is called an atlas of the manifold). Each definable manifold is given the $\mathcal{A}$-topology where $\mathcal{A}$ is an atlas of the manifold; this topology does not depend on the choice of $\mathcal{A}$. A *definable manifold of dimension* k *in* M^n is a definable manifold of dimension k with definable underlying set $Y \subseteq M^n$ and with an atlas of the manifold whose maps are all definable, that is, their graphs are definable subsets of M^{n+k}. (Note that then *every* atlas of the manifold has this definability property.)

Suppose now that M is an o-minimal structure and G is a definable group in M^n of dimension k. Then by [5] there is a unique definable manifold of dimension k in M^n with underlying set G such that the manifold topology makes G into a topological group. (In the case that M is an o-minimal expansion of a real closed field, this manifold is definably homeomorphic to a definable subset of some M^m. This follows from the proof of Lemma 10.4 in [1] and Theorem 1.8, Ch. 10 in [3].)

We prove here analogous results for type-definable groups in big o-minimal structures. In particular, if G is a type-definable group in M^n, where M is a big o-minimal expansion of a real closed field, then there is a unique topology on G making it a topological group with a type-definable homeomorphism onto a subspace of some M^k. In contrast to the ω-stable case, not every type-definable group in such an M^n is definable: consider for example the additive group of infinitesimals in a big real closed field.

Part of the above is true for the much wider class of *invariant groups*. A precise definition of this class in terms of *invariant sets* is in the next section. For example, let M be a big real closed field. Then the set $\mathbb{Z} \subseteq M$ as well as the convex hull of $\mathbb{Z}$ in M are invariant sets in M; as additive subgroups of M they are even invariant groups in M; these two sets are not type-definable in M. In fact, they are what is called $\bigvee$-definable in [4]. An example of an invariant group in a big real closed field M which is neither type- nor $\bigvee$-definable in M is the cartesian product of $\mathbb{Z}$ (as an additive subgroup of M) with the additive group of the infinitesimals. We assign to each invariant set a dimension and prove that each invariant group in M^n has a unique topology making it a topological group and inducing on some *large* invariant subset U of G the same topology as M^n.

Notation. From now on M is a big o-minimal structure, and A and A' denote small subsets of M (serving as sets of parameters). For $a = (a_1, \ldots, a_m) \in M^m$, $b = (b_1, \ldots, b_n) \in M^n$ we let $aA = Aa := A \cup \{a_1, \ldots, a_m\}$, and $\hat{ab} := (a_1, \ldots, a_m, b_1, \ldots, b_n) \in M^{m+n}$. We denote by $\mathrm{Aut}(M|A)$ the group of automorphisms of M that fix A point-wise. By a *box in* M^n we mean a cartesian product $(a_1, b_1) \times \cdots \times (a_n, b_n) \subseteq M^n$ where $a_i, b_i \in M$, $a_i < b_i$, for $i = 1, \ldots, n$. We let $f : X \rightharpoonup Y$ denote a *partial* function from the set X into the set Y, that is, a function $f : X' \to Y$ with $X' \subseteq X$. For a partial function $f : M^m \rightharpoonup M^n$ we let $\Gamma(f) \subseteq M^{m+n}$ denote the graph of f. If $X \subseteq M^n$ then $\partial X := \mathrm{cl}(X) - X$ denotes the *frontier* of X in M^n, and $\mathrm{int}(X)$ denotes the *interior* of X in M^n. Unless specified otherwise we use multiplicative notation for groups; in particular, the identity of a group is denoted by 1.

2 Invariant sets and invariant groups

Definition 2.1. *An* A-set in M^n *is a set* $X \subseteq M^n$ *such that* $\sigma(X) = X$ *for every* $\sigma \in Aut(M|A)$. *Likewise, an* A-group in M^n *is an* A*-set* $G \subseteq M^n$ *with a group operation* $G \times G \to G$ *whose graph is an* A*-set in* M^{3n}. *If* X *is an* A*-set in* M^n *for some (unspecified)* A*, then we call* X *an* invariant set in M^n. *Likewise, an* invariant group in M^n *is just an* A*-group in* M^n *for some unspecified* A.

Note that A-sets in M^n are unions of realizations of types over A in M^n, thus unions of intersections of A-definable cells in M^n. Also note that the A-sets in M^n form a boolean algebra of subsets of M^n, that if X and Y are A-sets in M^m and M^n, then $X \times Y$ is an A-set in M^{m+n}, and that the image of an A-set in M^{n+1} under the projection map

$$(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n) : M^{n+1} \to M^n$$

is an A-set in M^n. Also, every subset of M^n that is type-definable over A is an A-set, every type-definable group in M^n over A is an A-group, and every A-set is an A'-set when $A \subseteq A'$.

For an invariant set $X \subseteq M^n$ we define its dimension $\dim X \in \{0, \ldots, n\} \cup \{-\infty\}$ just as for definable sets in [3]: if X contains a cell of dimension d but no cell of dimension $d + 1$, then $\dim X := d$; also $\dim \emptyset = -\infty$.

This dimension is related to the rank function of the pregeometry of M in the same way as for definable sets. To explain this, recall that for $x = (x_1, \ldots, x_n) \in M^n$, $\mathrm{rk}(x|A)$ is the cardinality of a maximal subset of $\{x_1, \ldots, x_n\}$ that is algebraically independent over A. We shall consider $\mathrm{tp}(x|A)$ as the collection of all A-definable sets $X \subseteq M^n$ such that $x \in X$.

Let $\widehat{\mathrm{tp}}(x|A)$ denote the set of realizations of $\mathrm{tp}(x|A)$ in M^n, that is, the intersection of all sets $X \in \mathrm{tp}(x|A)$; equivalently, $\widehat{\mathrm{tp}}(x|A)$ is the orbit of x under the action of $\mathrm{Aut}(M|A)$ on M^n.

The connection between rank and dimension is based on the following:

Lemma 2.2. *If $x \in M^n$ and $\mathrm{rk}(x|A) = d$, then there is a cell C in M^n of dimension d, a box B in M^n, and a set $X \in \mathrm{tp}\big(x|A\big)$ such that*

$$x \in C \subseteq B \cap X \subseteq \widehat{\mathrm{tp}}(x|A).$$

Proof. By induction on n. The case $n = 0$ is trivial. Assume the lemma holds for a certain n. Let $(x_1, \ldots, x_n, x_{n+1}) \in M^{n+1}$ and $\mathrm{rk}((x_1, \ldots, x_n)|A) = d$. The inductive assumption gives a cell C in M^n of dimension d, a box B in M^n and a set $X \in \mathrm{tp}\big((x_1, \ldots, x_n)|A\big)$ such that

$$(x_1, \ldots, x_n) \in C \subseteq B \cap X \subseteq \widehat{\mathrm{tp}}\big((x_1, \ldots, x_n)|A\big).$$

Case 1. $x_{n+1} \in \mathrm{dcl}(A \cup \{x_1, \ldots, x_n\})$. Then $\mathrm{rk}((x_1, \ldots, x_n, x_{n+1})|A) = d$. Take an A-definable partial function $f : M^n \rightharpoonup M$ with $(x_1, \ldots, x_n)$ in its domain such that $f(x_1, \ldots, x_n) = x_{n+1}$. By cell decomposition we can assume that domain(f) is a cell, and that f is continuous, in particular,

$$C \subseteq \widehat{\mathrm{tp}}\big((x_1, \ldots, x_n)|A\big) \subseteq \mathrm{domain}(f).$$

Take $d, e \in M$ with $d < x_{n+1} < e$, and put

$$C' := \{(y, f(y)) : y \in C\}, \quad B' := B \times (d, e), \quad X' := (X \times M) \cap \Gamma(f).$$

Then $(x_1, \ldots, x_n, x_{n+1}) \in C' \subseteq B' \cap X' \subseteq \widehat{\mathrm{tp}}\big((x_1, \ldots, x_n, x_{n+1})|A\big)$.

Case 2. $x_{n+1} \notin \mathrm{dcl}(A \cup \{x_1, \ldots, x_n\})$. Then $\mathrm{rk}((x_1, \ldots, x_n, x_{n+1})|A) = d + 1$. For each A-definable continuous function $f : Y \to M$ with $Y \in \mathrm{tp}\big((x_1, \ldots, x_n)|A\big)$ and $f(x_1, \ldots, x_n) < x_{n+1}$ we take $d(f) \in M$ with $d(f) < x_{n+1}$ and a box $B(f)$ in M^n such that $(x_1, \ldots, x_n) \in B(f)$ and $f(y_1, \ldots, y_n) < d(f)$ for all $(y_1, \ldots, y_n) \in Y \cap B$. Likewise, for each A-definable continuous function $g : Y \to M$ with $Y \in \mathrm{tp}\big((x_1, \ldots, x_n)|A\big)$ and $g(x_1, \ldots, x_n) > x_{n+1}$ we take $e(g) \in M$ with $e(g) > x_{n+1}$ and a box $B(g)$ in M^n such that $(x_1, \ldots, x_n) \in B(g)$ and $g(y_1, \ldots, y_n) > e(g)$ for all $(y_1, \ldots, y_n) \in Y \cap B$. Next we take a box B_1 in M^n that is contained in B as well as in $B(f)$ and $B(g)$ for all f and g as above; we also take $d, e \in M$ such that $d(f) < d < x_{n+1} < e < e(g)$ for all f and g as above. (This is possible by saturation.) Let C_1 be a cell in M^n of dimension d such that $(x_1, \ldots, x_n) \in C_1 \subseteq B_1 \cap C$. Put

$$B' := B_1 \times (d, e), \quad X' := X \times M, \quad C' := C_1 \times (d, e).$$

Then $(x_1, \ldots, x_n, x_{n+1}) \in C' \subseteq B' \cap X' \subseteq \widehat{\mathrm{tp}}\big((x_1, \ldots, x_n, x_{n+1})|A\big)$. □

Corollary 2.3. *Let $X \subseteq M^n$ be a nonempty A-set. Then*

$$\dim X = \max \{\mathrm{rk}(x|A) : \ x \in X\}.$$

Proof. From the lemma above it is clear that $\dim X \geq \mathrm{rk}(x|A)$ for each $x \in X$. For the reverse inequality, take a projection map $p : M^n \to M^d$ with $d = \dim X$ such that pX contains a box B in M^d. It is easy to see that B contains a point of rank d over A, and thus X contains a point of rank $\geq d$ over A. $\square$

Definition 2.4. *Let $X \subseteq M^n$ be an A-set. A* generic of X over A *is an element $x \in X$ such that $\mathrm{rk}(x|A) = \dim X$.*

Lemma 2.5. *Let X be an A-set in M^n. Then*

1. *for each k the A-set $\{x \in M^n : \mathrm{rk}(x|A) \leq k\}$ is closed in M^n;*

2. *the set of generics of X over A is open in X;*

3. *if x is a generic of X over A, then $\widehat{\mathrm{tp}}(x|A)$ is open in X.*

Proof. Let $x = (x_1, \ldots, x_n) \in \mathrm{cl}(\{y \in X : \mathrm{rk}(y|A) \leq k\})$, with $\mathrm{rk}(x|A) = d$. After a suitable permutation of coordinates $x_1, \ldots, x_d$ are independent over A, and $f(x_1, \ldots, x_d) = (x_{d+1}, \ldots, x_n)$ where $f : M^d \rightharpoonup M^{n-d}$ is an A-definable map with $(x_1, \ldots, x_d) \in \mathrm{domain}(f)$. Lemma 2.2 gives a d-dimensional cell C in M^d such that

$$(x_1, \ldots, x_d) \in C \subseteq \widehat{\mathrm{tp}}\big((x_1, \ldots, x_d)|A\big) \subseteq \mathrm{domain}(f).$$

Note: C is open in M^d. There are $y = (y_1, \ldots, y_n) \in M^n$ arbitrarily close to x with $\mathrm{rk}(y|A) \leq k$, so there are y with with $\mathrm{rk}(y|A) \leq k$ and $(y_1, \ldots, y_d) \in C$, hence $(y_1, \ldots, y_d) \in \widehat{\mathrm{tp}}\big((x_1, \ldots, x_d)|A\big)$, so $\mathrm{rk}((y_1, \ldots, y_d)|A) = d$, so $k \geq d$. This proves item 1. Item 2 is an immediate consequence.

To obtain item 3, let $x = (x_1, \ldots, x_n)$ be a generic of X over A, say $\mathrm{rk}(x|A) = \dim X = d$. After a suitable permutation of coordinates, $x_1, \ldots, x_d$ are independent over A and we have the situation described in the proof of item 1; below we use the notation of that proof. Then

$$\widehat{\mathrm{tp}}(x|A) = \{(u, f(u)) : u \in \widehat{\mathrm{tp}}\big((x_1, \ldots, x_d)|A\big)\}.$$

Moreover, if $y = (y_1, \ldots, y_n) \in X$ is sufficiently close to x, then $(y_1, \ldots, y_d) \in C$ and $g(y_1, \ldots, y_d) = (y_{d+1}, \ldots, y_n)$ where $g : M^d \rightharpoonup M^{n-d}$ is an A-definable map with $(y_1, \ldots, y_d) \in \mathrm{domain}(g)$ (and thus $(x_1, \ldots, x_d) \in \mathrm{domain}(g)$).

Consider any A-definable map $g : M^d \rightharpoonup M^{n-d}$ with $(x_1, \ldots, x_d) \in \text{domain}(g)$. We claim that either $f(u) \neq g(u)$ for all $u \in \widehat{\text{tp}}\big((x_1, \ldots, x_d)|A\big)$, or $f(u) = g(u)$ for all $u \in \widehat{\text{tp}}\big((x_1, \ldots, x_d)|A\big)$. This claim follows by noting that if $f(u) = g(u)$ for some $u \in \widehat{\text{tp}}\big((x_1, \ldots, x_d)|A\big)$, then the action of $\text{Aut}(M|A)$ yields $f(u) = g(u)$ for all $u \in \widehat{\text{tp}}\big((x_1, \ldots, x_d)|A\big)$. By the continuity of the maps f and g on the open cell C and saturation there are boxes B_1 in M^d and B_2 in M^{n-d} such that

$$(x_1, \ldots, x_d) \in B_1 \subseteq C, \quad (x_{d+1}, \ldots, x_n) \in B_2,$$

and $g(u) \notin B_2$ for all $u \in B_1$ and all maps g as above for which there exists a $u \in \widehat{\text{tp}}\big((x_1, \ldots, x_d)|A\big)$ with $g(u) \neq f(u)$. We conclude that if $y \in X$ and $y \in B_1 \times B_2$, then $y \in \widehat{\text{tp}}(x|A)$. □

Recall that for $a \in M^m, b \in M^n$ we have

$$\text{rk}(a\hat{\ }b|A) = \text{rk}(a|bA) + \text{rk}(b|A).$$

Corollary 2.6. *If X and Y are A-sets in M^m and M^n, then* $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Corollary 2.7. *Let $X \subseteq M^n$ be an A-set, and $G \subseteq M^n$ an A-group.*

1. *If $a \in M^m$ and $b \in M^n$ are interdefinable over A, then* $\text{rk}(a|A) = \text{rk}(b|A)$.

2. *Let $X \subseteq M^n$ be an A-set. If a is a generic of X over A and b is a generic of X over aA, then a is a generic of X over bA.*

3. *If $b \in G$ and a is a generic of G over bA, then $a \cdot b$ is a generic of G over bA.*

4. *If $b \in G$ then there are $b_1, b_2 \in G$ such that b_1, b_2 are generics of G over bA and $b = b_1 \cdot b_2$.*

Lemma 2.8. *Let $f : M^n \rightharpoonup M^k$ be a partial map whose graph is an A-set in M^{n+k}. Then f is continuous at each generic of* domain(f) *over A.*

Proof. We can assume $k = 1$. By the definition of dimension we have $\dim(\Gamma(f)) \leq \dim(\text{domain}(f))$. Let x be a generic of domain(f) over A. By the inequality above, $(x, f(x))$ is a generic of $\Gamma(f)$ over A, so this inequality is actually an equality. In particular, $\text{rk}((x, f(x))|A) = \text{rk}(x|A)$, so $f(x) \in \text{dcl}(Ax)$. This gives an A-definable function $g : X \to M$ such that

$x \in X \subseteq M^n$ and $g(x) = f(x)$. By cell decomposition we can assume that g is continuous. For each $\sigma \in \mathrm{Aut}(M|A)$ we have

$$g(\sigma x) = \sigma(gx) = \sigma(fx) = f(\sigma x),$$

so f and g agree on $\widehat{\mathrm{tp}}(x|A)$, which is open in X by item 3 of Lemma 2.5. $\square$

Definition 2.9. *Let X, Y be invariant sets in M^n. We say that Y is* large *in X if* $\dim(X - Y) < \dim X$.

We show that on every invariant group G in M^n there is a unique topology making it a topological group and inducing the same topology on a large invariant subset of G as M^n.

In the rest of this section, $G \subseteq M^n$ is an invariant group of dimension d. For simplicity we assume it is an A-group for $A = \emptyset$. (This assumption is no loss of generality: if $A \neq \emptyset$ we just expand M by names for the elements of A.)

Lemma 2.10. *Let $\phi : (x, y, z) \mapsto xy^{-1}z : G^3 \to G$ and let g be a generic of G. Then ϕ is continuous at (g, g, g).*

Proof. Let c be a generic of G over g. Consider the following maps

$$\begin{array}{lcl} \phi_1 & : & (x,y,z) \mapsto (cx, y^{-1}, z) : G^3 \to G^3, \\ \phi_2 & : & (x,y,z) \mapsto (xy, z) : G^3 \to G^2, \\ \phi_3 & : & (x,y) \mapsto (c^{-1}, xy) : G^2 \to G^2, \\ \phi_4 & : & (x,y) \mapsto xy : G^2 \to G. \end{array}$$

By Lemma 2.8, ϕ_1 is continuous at (g, g, g), ϕ_2 is continuous at (cg, g^{-1}, g), ϕ_3 is continuous at (c, g) and ϕ_4 is continuous at (c^{-1}, cg). Hence

$$\phi = \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$$

is continuous at (g, g, g). $\square$

We now pick some generic g of G, and use Lemma 2.2 to choose a box D in M^n and a $\emptyset$-definable set $Y \subseteq M^n$ such that $g \in D \cap Y \subseteq \widehat{\mathrm{tp}}(g)$ and $\dim Y = \mathrm{rk}(g)$. Take a definable set $X \subseteq M^k$ (for some k) and a definable set $B \subseteq X \times D$ such that $\{B(x) : x \in X\}$ is the set of all boxes in M^n that are contained in D and contain g as an element. So for each $x \in X$ the set $V(x) := B(x) \cap Y$ is definable and contained in G. By item 3 of Lemma 2.5, the collection $\{V(x) : x \in X\}$ is a basis of open neighborhoods of g in G for the topology on G induced by M^n. Let $U \subseteq X \times M^n$ be the set such that $U(x) = g^{-1}V(x)$ for each $x \in X$. To show that $\{U(x) : x \in X\}$ is a neighborhood basis of 1 for a group topology on G we use the following well-known fact (see [2]).

Fact 2.11. *Let H be a group and $\mathcal{U}$ a collection of subsets of H such that*

(a) $1 \in U$ for all $U \in \mathcal{U}$;

(b) for all $U_1, U_2 \in \mathcal{U}$ there is $U_3 \in \mathcal{U}$ with $U_3 \subseteq U_1 \cap U_2$;

(c) for all $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^{-1} \subseteq U$;

(d) for all $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$;

(e) for all $U \in \mathcal{U}$ and $a \in H$ there is $V \in \mathcal{U}$ such that $aVa^{-1} \subseteq U$.

Then there is a unique topology on H that makes H a topological group and has $\mathcal{U}$ as a neighborhood basis of 1.

Lemma 2.12. *The collection $\mathcal{U} := \{U(x) : x \in X\}$ satisfies the conditions (a)–(e) above for $H = G$.*

Proof. It is clear that conditions (a) and (b) are satisfied. To obtain (c) it is enough to show that for every $x \in X$ there is $y \in X$ such that $U(y) \subseteq U(x) \cap U(x)^{-1}$. Given any $x, y \in X$, we have: $U(y) \subseteq U(x) \cap U(x)^{-1}$ iff $V(y) \subseteq V(x) \cap gV(x)^{-1}g$. By Lemma 2.10, there is for each $x \in X$ an $y \in X$ such that $V(y) \subseteq gV(x)^{-1}g$. Thus (c) holds.

For (d) it suffices to show that for all $x \in X$ there is $y \in X$ such that $U(y)^2 \subseteq U(x)$. Given $x, y \in X$ we have:

$$U(y)^2 \subseteq U(x) \iff (g^{-1}V(y))^2 \subseteq g^{-1}V(x) \iff V(y)g^{-1}V(y) \subseteq V(x).$$

By Lemma 2.10 there is for each $x \in X$ a $y \in X$ with $V(y)g^{-1}V(y) \subseteq V(x)$.

To obtain (e) we show that for all $x \in X$ and $a \in G$ there is $y \in X$ such that $aU(y)a^{-1} \subseteq U(x)$. This amounts to showing that for all $x \in X$ and $a \in G$, there is $y \in X$ such that $gag^{-1}V(y)a^{-1} \subseteq V(x)$. Let $a \in G$; it suffices to show that then the map $\phi : u \mapsto gag^{-1}ua^{-1} : G \to G$ is continuous at g. To obtain this continuity, let b be a generic of G over $\{a, g\}$ and c a generic of G over $\{a, b, g\}$ and consider the following maps:

$$\begin{aligned}
\phi_1 &: u \mapsto (bgag^{-1}, u) : G \to G^2, \\
\phi_2 &: (u, v) \mapsto (uv, a^{-1}c) : G^2 \to G^2, \\
\phi_3 &: (u, v) \mapsto (b^{-1}, uv) : G^2 \to G^2, \\
\phi_4 &: (u, v) \mapsto (uv, c^{-1}) : G^2 \to G^2, \\
\phi_5 &: (u, v) \mapsto uv : G^2 \to G.
\end{aligned}$$

By Lemma 2.8, ϕ_1 is continuous at g, ϕ_2 is continuous at $(bgag^{-1}, g)$, ϕ_3 is continuous at $(bga, a^{-1}c)$, ϕ_4 is continuous at (b^{-1}, bgc), and ϕ_5 is continuous at (gc, c^{-1}). Hence

$$\phi = \phi_5 \circ \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$$

is continuous at g. □

The following is a well-known fact about topological groups (see [2]):

Fact 2.13. *Let E, F be topological groups and $f : E \to F$ a group homomorphism. If f is continuous at some point of E, then f is continuous.*

From now on t denotes the group topology on G for which $\mathcal{U}$ is a neighborhood basis of 1. Unless prefixed by t, topological terminology like "open", "neighborhood", "continuous", ... refers to the topology induced on G by M^n. We denote by Ω the set of all generics of G. Note that Ω is a large invariant subset of G which is open in G.

Theorem 2.14. *The set Ω is t-open and t induces on Ω the same topology as M^n. For any $a \in \Omega$, the set $\widehat{\mathrm{tp}}(a)$ is t-open, and t induces the same topology on $\widehat{\mathrm{tp}}(a)$ as M^n. The t-topology is the only group topology on G for which some generic of G has the same neighborhoods as the topology on G induced by M^n. In particular, the t-topology is the only group topology on G that induces the same topology on a large invariant subset of G as M^n.*

Proof. We call an element $a \in \Omega$ *good* if $\{t\text{-neighborhoods of } a \text{ in } G\} = \{\text{neighborhoods of } a \text{ in } G\}$. Note that g is good, since $\{V(x) : x \in X\}$ is both a neighborhood basis of g for the topology t and a neighborhood basis of g for the topology on G induced by M^n.

Suppose $a \in \Omega$ is good and b is a generic of G over a. Then we claim that b is good. To see this, note that a is generic over ba^{-1}, so the map $x \mapsto ba^{-1}x : G \to G$ is continuous at a, and maps a to b; likewise, its inverse $y \mapsto ab^{-1}y : G \to G$ is continuous at b. These two maps are also t-continuous, so the claim follows.

Next we claim: all elements of Ω are good. To see this, let $h \in \Omega$ and take a generic b of G over $\{g, h\}$. Then b is a generic of G over g, so b is good by the previous claim. Also, h is a generic of G over b, so h is good, again by the previous claim.

It follows from the second claim that each point of Ω has a t-neighborhood entirely contained in Ω, namely Ω itself. Thus Ω is t-open. It also follows that each point of Ω has the same t-neighborhoods in Ω as neighborhoods in Ω. Thus t induces the same topology on Ω as M^n.

If $a \in \Omega$, then $\widehat{\mathrm{tp}}(a)$ is an open subset of Ω by part 3 of Lemma 2.5, so the assertion about $\widehat{\mathrm{tp}}(a)$ follows.

Let t' be a group topology on G and let h be a generic of G such that $\{t'\text{-neighborhoods of } h \text{ in } G\} = \{\text{neighborhoods of } h \text{ in } G\}$. The map

$$x \mapsto x : (G, t) \to (G, t'),$$

as well as its inverse, is continuous at h. Since it is also a group isomorphism, it follows that it is a homeomorphism, so $t = t'$.

If a group topology on G induces the same topology on a large invariant subset of G as M^n, then some generic of G has the same neighborhoods in this group topology as in the topology on G induced by M^n. So t is the unique group topology having this property. □

We also refer to the t-topology as "the group topology of G".

Lemma 2.15. *Let E, F be invariant groups in M^m and M^n, and let $f : E \to F$ be a group homomorphism whose graph is an invariant set in M^{m+n}. Then f is continuous with respect to the group topologies of E and F.*

Proof. By Fact 2.13, it is enough to show that there is $a \in E$ such that f is continuous at a. But this is an immediate consequence of Lemma 2.8. □

Lemma 2.16. *If H is an invariant subgroup of G, then the group topology of G induces on H the group topology of H.*

Proof. Immediate by Theorem 2.14. □

Recall the following facts about topological groups (see [2]):

Fact 2.17. *A subgroup of a topological group is open iff it has an interior point. Every open subgroup of a topological group is closed.*

Fact 2.18. *A topological group is Hausdorff iff the set $\{1\}$ is closed in it.*

Lemma 2.19. *Let H be an invariant subgroup of G. Then H is t-closed.*

Proof. For simplicity we assume that H is an A-group for $A = \emptyset$. We denote by $\mathrm{cl_t}(H)$ the topological closure of H with respect to the group topology t of G. Note that $\mathrm{cl_t}(H)$ is an invariant subgroup of G and that $\dim(\mathrm{cl_t}(H)) = \dim(H)$. Then by item 3 of Lemma 2.5 and Theorem 2.14, H has t-interior in $\mathrm{cl_t}(H)$. So by Fact 2.17, H is closed in G. □

Corollary 2.20. *G is a Hausdorff topological group.*

In [4], a $\bigvee$-definable group over A in M^n is a group whose underlying set is a union $\bigcup_{i \in I} X_i$ of sets $X_i \subseteq M^n$ that are definable over A, such that for all $i, j \in I$ there is $k \in I$ with $X_i \cup X_j \subseteq X_k$, and the restriction of the group operation to $X_i \times X_j$ is definable over A as a function into M^n. Note that every $\bigvee$-definable group in M^n is an invariant group in M^n.

Corollary 2.21. *If H is a type-definable group in M^n then it has a $\bigvee$-definable subgroup K with* $\dim K = \dim H$.

Proof. First note that there is a definable map $f : M^{2n} \rightharpoonup M^n$ such that $\Gamma(f) \cap H^3$ is the graph of the group operation. Let $U(x)$ be as in the definition of the group topology in the discussion preceding Fact 2.11. Then $U(x)$ is a definable neighborhood of 1 with $\dim U(x) = \dim H$. Let K be the subgroup of H generated by $U(x)$. Then K is a $\bigvee$-definable subgroup of H of same dimension as K. $\square$

Corollary 2.21 does not hold for invariant groups in general, i.e. there are invariant groups which do not contain a $\bigvee$-definable subgroup of same dimension, as the following example shows.

Let $\{\lambda_n\}$ be a strictly increasing sequence in $\mathbb{Q} \cap (0,1)$ converging to 1 in $\mathbb{Q}$. We define a permutation τ of $\{\lambda_n : n \in \mathbb{N}\}$ by $\tau(\lambda_{2k}) = \lambda_{2k+1}$ and $\tau(\lambda_{2k+1}) = \lambda_{2k}$, where $k = 0, 1, 2, \dots$. Let M be a (big) o-minimal expansion of a divisible ordered abelian group, so M is in particular a vector space over $\mathbb{Q}$. We let τ induce a permutation τ_* of M^2 as follows. Let $x = (x_1, x_2) \in M^2$; if $x_2 \neq \lambda_n x_1$ for all n, put $\tau_*(x) = x$; if $x_2 = \lambda_n x_1$, put $\tau_*(x) = (x_1, \tau(\lambda_n)x_1)$. We define a $\emptyset$-group H in M^2 as follows: H has M^2 as underlying set, and its group operation $\oplus$ is defined by

$$\tau_*(x \oplus y) := \tau_*(x) + \tau_*(y).$$

Note that τ_* is an invariant isomorphism (with inverse τ_*) from the invariant group H onto the additive group of the vector space M^2. Hence H with its t-topology is homeomorphic to M^2 with its usual topology via τ_*. We shall prove that H does not contain a $\bigvee$-definable subgroup of the same dimension as H. Towards a contradiction, let K be a $\bigvee$-definable subgroup of H of dimension 2. Since K is of dimension 2, it contains an interior point, so by Fact 2.17, K is open, and hence K contains a t-neighborhood of 0. Thus K contains a set $\tau_*^{-1}(V) = \tau_*(V)$ with V a definable neighborhood of 0 in M^2. Note that $\tau_*(V)$ is also a neighborhood of 0 in M^2 in the usual topology (though not necessarily a definable one). Take A and a family $(X_i)_{i \in I}$ of A-definable sets contained in K such that $K = \bigcup_{i \in I} X_i$, and for all $i, j \in I$ there is $k \in I$ with $X_i \cup X_j \subseteq X_k$, and the group operation $\oplus$ restricted to $X_i \times X_j$ is A-definable. We arrange easily that $\dim X_i = 2$ for all i.

Claim 1. For some $i \in I$ the set

$$\{n : \text{ there is } (x_1, x_2) \in X_i \text{ such that } x_2 = \lambda_n x_1 \text{ and } x_1 \neq 0\}$$

is infinite.

To prove the claim, increase A if necessary, and take a $q > 0$ in M such that $[-2q, 2q]^2 \subseteq \tau_*(V)$ and $q \in \mathrm{dcl} A$. By saturation we can take $p_0 \in M$

such that $p_0 < \mu q$ for all rational $\mu > 1$, and $p_0 > a$ for all $a \in \mathrm{dcl}A$ satisfying $q \leq a < \mu q$ for all rational $\mu > 1$. Take $i \in I$ such that $(p_0, q) \in X_i$ (this is possible because $[-2q, 2q]^2 \subseteq \tau_*(V)$). Since $p_0 \notin \mathrm{dcl}A$ and X_i is A-definable, there are $c, d \in \mathrm{dcl}A$ such that $c < p_0 < d$ and for all x_1 with $c < x_1 < d$, $(x_1, q) \in X_i$. By the choice of p_0 we can take $p_1 \in M$ and a positive rational $\lambda < 1$ such that $p_0 < p_1 < d$ and $q = \lambda p_1$. Hence, since $\{\lambda_n\}$ converges to 1, there is $p_2 \in M$ with $p_0 < p_2 < p_1$ (and so $(p_2, q) \in X_i$) and $q = \lambda_n p_2$ for some n. It follows that there are infinitely many $n \in \mathbb{N}$ such that there is $x_1 \in M$ with $q = \lambda_n x_1$ and $(x_1, q) \in X_i$.

We now take an i as in the claim.

Claim 2. There is an A-definable $S \subseteq X_i$ such that $\dim S = 1$ and the set

$$\{n : \text{ there is } (x_1, x_2) \in S \text{ such that } x_2 = \lambda_n x_1 \text{ and } x_1 \neq 0\}$$

is infinite.

To see this, let $\mathcal{D}$ be a decomposition of X_i into A-cells. Take a cell $C \in \mathcal{D}$ such that the set

$$\{n : \text{ there is } (x_1, x_2) \in C \text{ such that } x_2 = \lambda_n x_1 \text{ and } x_1 \neq 0\}$$

is infinite. Then either $\dim C = 1$ or $\dim C = 2$. In the first case we can let S be C. For the second case note that after possibly taking a refinement of $\mathcal{D}$ we may assume that C is bounded and that $(0,0) \notin \mathrm{cl}(C)$. Now infinitely many lines through the origin with slope λ_n have nonempty intersection with C, and hence infinitely many such lines have nonempty intersection with ∂C, which is an A-definable set of dimension 1. After possibly shrinking C we may assume that $\mathrm{cl}(C) \subseteq X_i$, hence ∂C does the job.

In what follows we fix a set S as in Claim 2.

Claim 3. There is $(a, b) \in X_i$ with $\tau_*(a, b) = (a, b)$ and whenever $x \in S$ is such that $\tau_*(x) \neq x$ then $\tau_*((a, b) \oplus x) = (a, b) \oplus x$.

First let $B \subseteq X_i$ be an A-definable box such that for all $x = (x_1, x_2) \in B$, $\tau_*(x) = x$ and $x_2 < x_1$. Since there are only finitely many n with the set $\{(x_1, x_2) \in S : x_2 = \lambda_n x_1 \text{ and } x_1 \neq 0\}$ being infinite, after possibly deleting these infinite sets, we may assume that for all n, the set $\{(x_1, x_2) \in S : x_2 = \lambda_n x_1 \text{ and } x_1 \neq 0\}$ is finite. By saturation and continuity of $+$ on M^2, there is $(a, b) \in B$ such that for all $x \in S$ with $\tau_*(x) \neq x$, $\tau_*((a, b) \oplus x) = (a, b) \oplus x$.

Fixing an element $(a, b) \in X_i$ as in Claim 3, it is easy to see that the map $S \to M^2 : (x_1, x_2) \mapsto (a, b) \oplus (x_1, x_2)$ is not continuous at (x_1, x_2) for every $(x_1, x_2) \in S$ with $x_2 = \lambda_n x_1$ for some n. Definability of $\oplus$ restricted to

$X_i \times X_i$ implies definability of the above map, which yields, when considering a suitable projection of S, a contradiction with o-minimality.

The following lemma is essential in the next section.

Lemma 2.22. *Let $H \subseteq M^n$ be an A-group, and $X \subseteq H$ an A-set in M^n that is large in H. If $\dim H = d$ and $(h_0, \ldots, h_d) \in H^{d+1}$ is any generic of H^{d+1} over A, then $H = \cup_{i=0}^d h_i X$.*

Proof. Let $\dim H = d$, let $(h_0, \ldots, h_d)$ be a generic of H^{d+1} over A, and let $j \in \{0, \ldots, d-1\}$ be such that $\bigcup_{i=0}^{j} h_i X$ is a proper subset of H. We claim that then $\dim (H - \bigcup_{i=0}^{j+1} h_i X) < \dim (H - \bigcup_{i=0}^{j} h_i X)$.

Let g be a generic of $H - \bigcup_{i=0}^{j} h_i X$ over $h_{j+1}A'$, where $A' := Ah_0 \ldots h_j$. The claim follows if we can show that $g \in h_{j+1}X$, because then $g \notin H - (\bigcup_{i=0}^{j+1} h_i X)$. We have $g = h_{j+1}(h_{j+1}^{-1} g)$, so it remains to show that $h_{j+1}^{-1} g$ is a generic of H over A.

Let $d = \dim H = \mathrm{rk}(h_{j+1}|A')$, and $e = \dim (H - \bigcup_{i=0}^{j} h_i X) = \mathrm{rk}(g|h_{j+1}A')$. We have

$$\begin{aligned} \mathrm{rk}(\hat{g h}_{j+1}|A') &= \mathrm{rk}(g|h_{j+1}A') + \mathrm{rk}(h_{j+1}|A') = && (1) \\ &= \mathrm{rk}(h_{j+1}|gA') + \mathrm{rk}(g|A'). && (2) \end{aligned}$$

By (1), $\mathrm{rk}(\hat{g h}_{j+1}|A') = d + e$. Since $\mathrm{rk}(g|A') = \mathrm{rk}(g|h_{j+1}A') = e$, we obtain from (2) that $\mathrm{rk}(h_{j+1}|gA') = d$. So h_{j+1} is a generic of H over gA', and hence by point 1 in Corollary 2.7, h_{j+1}^{-1} is a generic of H over gA'. Thus by point 3 of Corollary 2.7, $h_{j+1}^{-1} g$ is a generic of H over A.

□

3 Type-definable groups

In this section we prove that if M is a big o-minimal expansion of a real closed field and G is a type-definable group in M^n, then G with its group topology is definably isomorphic to a type-definable group in some M^k with the topology induced by M^k. The idea of the proof is to construct a definable Hausdorff manifold in M^n containing G as a subset whose topology induces the group topology on G, and then to apply the results from [1] and [3] to this manifold. (NB: it is easy to construct a definable manifold in M^2 of dimension 1 that is not Hausdorff.)

From now on, let $G \subseteq M^n$ be a type-definable group, for simplicity over $\emptyset$, of dimension d. To obtain a manifold having the desired properties we first construct a definable set $V \subseteq M^n$ such that V is large in G and gives rise to a chart of the desired manifold.

Let Γ be the graph of the group operation. An easy saturation argument shows that there is a $\emptyset$-definable function $f : Z^2 \to M^n$ such that $G \subseteq Z \subseteq M^n$, $\dim G = \dim Z = d$, and $\Gamma(f) \cap G^3 = \Gamma$. Moreover, we may assume (also by saturation) that f has the following properties:

(a) $f(f(x,y),z) = f(x,f(y,z))$ for $x,y,z \in Z$ if both sides are defined.

(b) For all $x \in Z$, $f(x,1) = f(1,x) = x$.

(c) For every $x \in Z$ there is $y \in Z$ such that $f(x,y) = f(y,x) = 1$. (By (a) and (b), such a y is unique for a given x.)

From now on we shall write xy instead of $f(x,y)$ when $x,y \in Z$, and for $x \in Z$ we let x^{-1} denote the unique $y \in Z$ such that $xy = yx = 1$.

Let g, $\{U(x) : x \in X\}$ be as in the discussion preceding Fact 2.11. Define

$$V_0 := \{z \in Z : \{zU(x) : x \in X\} \text{ is a neighborhood basis of } z \text{ in } Z\}$$

Take A so that X and g are definable over A; then V_0 is definable over A. Below we use the set Ω of generics of G, which is large in G.

Lemma 3.1. *The set Ω is contained in V_0, and $V_0 \cap G$ is open in V_0.*

Proof. If $a \in \Omega$, then a is *good* as defined in the proof of Theorem 2.14, that is, $\{aU(x) : x \in X\}$ is a neighborhood basis of a in G for the topology induced on G by M^n, and hence a neighborhood basis of a in Z, since Ω is both open in G and in Z.

Given any $z \in V_0 \cap G$, take some $x \in X$, and note that $zU(x) \cap V_0$ is a neighborhood of z in V_0 contained in $V_0 \cap G$. Thus $V_0 \cap G$ is open in Z. □

Lemma 3.2. *The group topology and the topology on M^n induce the same topology on $V_0 \cap G$. In particular, given any $a \in G$, the map $x \mapsto ax : T \to aT$, where $T = \{x \in V_0 \cap G : ax \in V_0 \cap G\}$, is a homeomorphism for the topologies on T and aT induced by M^n.*

Note that the set T in the lemma is large in $V_0 \cap G$.

Proof. The first part of the lemma is immediate from the definition of V_0. The second part follows from the first part. □

According to [3], p. 68, a *stratification* $\mathcal{S}$ of a closed definable set $S \subseteq M^n$ is a partition of S into finitely many cells, such that for each cell $C \in \mathcal{S}$, $\text{cl}(C) \setminus C$ is a union of (necessarily lower-dimensional) cells in $\mathcal{S}$. (Note that then each cell in $\mathcal{S}$ of dimension $\dim S$ is open in S.) It is shown there that if $S \subseteq M^n$ is closed and definable and $T \subseteq S$ is definable, then there is a stratification $\mathcal{S}$ of S that partitions T. The proof shows that if S and T are definable over a certain parameter set, then we can take the cells in $\mathcal{S}$ to be definable over the same parameter set.

Below we fix a generic $(a_0, \ldots, a_d)$ of G^{d+1} over A. Then by Lemma 2.22, $G \subseteq \bigcup_{j=0}^{d} a_j V_0$. We shall denote by Ω_A the set of generics of G over A.

We define the A-set V_1 as follows: Let $\mathcal{S}$ be a stratification of $\mathrm{cl}(V_0)$ whose cells are definable over A such that $\mathcal{S}$ partitions V_0, and let V_1 be the union of the cells of dimension d in $\mathcal{S}$ which are contained in V_0. Thus $\Omega_A \subseteq V_1$ and V_1 is open in V_0. Next we put $A' := Aa_0a_1 \ldots a_d$ and introduce A'-definable subsets V_2, V_3, and V of V_1:

$$\begin{aligned} V_2 &:= \{x \in V_1 : \text{ for } i,j = 0,\ldots,d,\ a_i x \in Z \,\&\, (a_i^{-1}a_j)x \in Z\}, \\ V_3 &:= \{x \in V_2 : \text{ for } i,j = 0,\ldots,d,\ \text{if } (a_i^{-1}a_j)x \in V_0, \text{ then the map} \\ &\qquad y \mapsto (a_i^{-1}a_j)y : V_2 \rightharpoonup V_0 \ \text{ is continuous at } x\}. \end{aligned}$$

Let V be the interior of V_3 in V_1.

Lemma 3.3. *The set V is open in V_0, and $G \subseteq \bigcup_{i=0}^{d} a_i V$.*

Proof. That V is open in V_0 is because V_1 is open in V_0 and V is open in V_1. To obtain $G \subseteq \bigcup_{i=0}^{d} a_i V$ it is enough to show that $V \cap G$ is an A-set that is large in G. It is clear that $V_1 \cap G = V_2 \cap G$; also $V_2 \cap G = V_3 \cap G$ by Lemmas 3.1 and 3.2. Since $V_0 \cap G$ is open in V_0, the set $V_1 \cap G$ is open in V_1, hence $V_3 \cap G = V \cap G$. Thus $V \cap G$ is an A-set and is large in G. □

Let $\{C \in \mathcal{S} : C \subseteq V_1\} = \{C_i : i \in I\}$ with $C_i \neq C_{i'}$ for $i \neq i'$. Then each C_i is a cell of dimension d, open in $\mathrm{cl}(V_0)$, and $V_1 = \bigcup_{i \in I} C_i$. For every $i \in I$, let ρ_i' be an A-definable homeomorphism of C_i onto an open subset of M^d, and let ρ_i be ρ_i' restricted to $C_i \cap V$. Note that $V = \bigcup_{i \in I} C_i \cap V$ and that $C_i \cap V$ is open in C_i, since V is open in V_0. So for every $i \in I$, $\rho_i : C_i \cap V \to \rho_i(C_i \cap V)$ is an A'-definable homeomorphism of $C_i \cap V$ onto an open subset of M^d, and every $x \in V$ is in the domain of some ρ_i.

For $x \in V$ and $j, l \in \{0, \ldots, d\}$, both $(a_j^{-1}a_l)x$ and $a_j^{-1}(a_l x)$ are defined, so they are equal (and in Z), and will accordingly be written as $a_j^{-1}a_l x$.

Lemma 3.4. *For $i \in I$, $j = 0, \ldots, d$, let $h_{ij} : a_j(C_i \cap V) \to \rho_i(C_i \cap V)$ be defined by $h_{ij}(x) = \rho_i(a_j^{-1}x)$. Then $\{h_{ij} : i \in I,\ j = 0, \ldots, d\}$ is a definable atlas of dimension d on the set $\bigcup_{i,j} a_j(C_i \cap V)$, $\quad (i \in I, j = 0, \ldots, d)$.*

Proof. Let $i \in I$ and $j \in \{0, \dots, d\}$. By the remark just before this lemma we have a bijection

$$x \mapsto a_j^{-1}x : a_j(C_i \cap V) \to C_i \cap V,$$

hence h_{ij} is a definable bijection.

Let also $k \in I$ and $l \in \{0, \dots, d\}$. To see that $h_{ij}\big(a_j(C_i \cap V) \cap a_l(C_k \cap V)\big)$ is open in $h_{ij}\big(a_j(C_i \cap V)\big) = \rho_i(C_i \cap V)$, first note that

$$\begin{aligned} h_{ij}\big(a_j(C_i \cap V) \cap a_l(C_k \cap V)\big) &= \rho_i\big(a_j^{-1}\big(a_j(C_i \cap V) \cap a_l(C_k \cap V)\big)\big) \\ &= \rho_i\big((C_i \cap V) \cap a_j^{-1}a_l(C_k \cap V)\big). \end{aligned}$$

Now $\rho_i\big((C_i \cap V) \cap a_j^{-1}a_l(C_k \cap V)\big)$ is open in $\rho_i(C_i \cap V)$ iff $(C_i \cap V) \cap a_j^{-1}a_l(C_k \cap V)$ is open in $C_i \cap V$. Since

$$(C_i \cap V) \cap a_j^{-1}a_l(C_k \cap V) = \{x \in C_i \cap V : \ a_l^{-1}a_jx \in C_k \cap V\}$$

the set on the left is indeed open in $C_i \cap V$. The transition map

$$h_{(ij)(kl)} : h_{ij}\big(a_j(C_i \cap V) \cap a_l(C_k \cap V)\big) \to h_{kl}\big(a_j(C_i \cap V) \cap a_l(C_k \cap V)\big)$$

is given by $h_{(ij)(kl)}(x) = \rho_k\big(a_l^{-1}a_j\rho_i^{-1}(x)\big)$, so $h_{(ij)(kl)}$ is continuous by the definition of V_3. $\square$

Lemma 3.5. *For all $i, k \in I$, $j, l \in \{0, \dots, d\}$, $x \in C_i \cap V \cap G$, $y \in C_k \cap V \cap G$ with $a_jx \neq a_ly$, there are boxes $B_i \subseteq \rho_i(C_i \cap V)$ and $B_k \subseteq \rho_k(C_k \cap V)$ in M^d such that $\rho_i(x) \in B_i$, $\rho_k(y) \in B_k$, and $h_{ij}^{-1}(B_i) \cap h_{kl}^{-1}(B_k) = \emptyset$.*

Proof. Let i range over I, j over $\{0, \dots, d\}$, and let $Y \subseteq G$. We claim that Y is open in the group topology iff $a_j^{-1}Y \cap (C_i \cap V)$ is open in $C_i \cap V$ for all i, j. Indeed, if Y is open in the group topology, then $a_j^{-1}Y$ is open in the group topology, hence $a_j^{-1}Y \cap (C_i \cap V)$ is open in $C_i \cap V$ by Lemma 3.2.

For the converse, recall that Ω_A is the set of generics of G over A, and note that by Theorem 2.14, Y is open in the group topology iff $a_j^{-1}Y \cap \Omega_A$ is open in Ω_A for all j. But if $a_j^{-1}Y \cap (C_i \cap V)$ is open in $C_i \cap V$ for all i, j, then $a_j^{-1}Y \cap V$ is open in V for all j, and hence $a_j^{-1}Y \cap \Omega_A$ is open in Ω_A for all j, (since $\Omega_A \subseteq V$ by the proof of Lemma 3.3), so Y is open in the group topology. This finishes the proof of the claim about Y.

Since $h_{ij}\big(Y \cap a_j(C_i \cap V)\big) = \rho_i\big(a_j^{-1}Y \cap (C_i \cap V)\big)$, Y is open in the group topology iff $h_{ij}(Y \cap a_j(C_i \cap V))$ is open in $\rho_i(C_i \cap V)$ for all i, j.

The lemma follows from Corollary 2.20. $\square$

We have $G = \bigcap_{\lambda \in \Lambda} X_\lambda$, where each X_λ is $\emptyset$-definable, and for all $\lambda, \mu \in \Lambda$ there is $\nu \in \Lambda$ such that $X_\nu \subseteq X_\lambda \cap X_\mu$. By Lemma 3.5, there is $\alpha \in \Lambda$ such that for all $i, k \in I$, $j, l \in \{0, \ldots, d\}$, $x \in C_i \cap V \cap X_\alpha$, $y \in C_k \cap V \cap X_\alpha$ with $a_j x \neq a_l y$, there are boxes $B_i \subseteq \rho_i(C_i \cap V)$ and $B_k \subseteq \rho_k(C_k \cap V)$ in M^d such that $\rho_i(x) \in B_i$, $\rho_k(y) \in B_k$, and $h_{ij}^{-1}(B_i) \cap h_{kl}^{-1}(B_k) = \emptyset$.

Let W be the interior of $V \cap X_\alpha$ in V.

Lemma 3.6. *For $i \in I$, $j \in \{0, \ldots, d\}$, let h'_{ij} be the restriction of h_{ij} to $a_j(C_i \cap W)$. The collection $\mathcal{A} = \{h'_{ij} : i \in I,\ j = 0, \ldots, d\}$ is a definable atlas of dimension d on $\bigcup_{i,j} a_j(C_i \cap W)$, and $\mathcal{A}$ makes $\bigcup_{i,j} a_j(C_i \cap W)$ a definable Hausdorff manifold in M^n. Moreover, $G \subseteq \bigcup_{i,j} a_j(C_i \cap W)$ and the manifold topology induces the group topology on G.*

Proof. The claim about $\mathcal{A}$ follows from Lemmas 3.4 and 3.5 using that W is open in V. Since $W \cap G$ is an A-set that is large in G, $G \subseteq \bigcup_{i,j} a_j(C_i \cap W)$. The group topology and the manifold topology agree on G by an argument as in the proof of Lemma 3.5. □

If M is an o-minimal expansion of an abelian group, then by [1], every definable Hausdorff manifold in M^n is regular as a topological space. Since the definable manifold $\bigcup_{i,j} a_j(C_i \cap W)$ is a definable space in the sense of [3], Theorem 1.8. on p.159 of [3] yields:

Theorem 3.7. *Let M be an o-minimal expansion of a real closed field. Then every type-definable group in M^n with its group topology is, for some m, definably isomorphic to a type-definable group in M^m whose group topology is induced by M^m.*[2]

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[2] Here "definably isomorphic" means that the topological group isomorphism is given by restricting a definable map $M^n \to M^m$.

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