O-minimal fields with standard part map

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Abstract

Let R be an o-minimal field and V a proper convex subring with residue field $\boldsymbol{k}$ and standard part (residue) map $\mathrm{st}\colon V \to \boldsymbol{k}$. Let $\boldsymbol{k}_{\mathrm{ind}}$ be the expansion of $\boldsymbol{k}$ by the standard parts of the definable relations in R. We investigate the definable sets in $\boldsymbol{k}_{\mathrm{ind}}$ and conditions on (R, V) which imply o-minimality of $\boldsymbol{k}_{\mathrm{ind}}$. We also show that if R is ω-saturated and V is the convex hull of $\mathbb{Q}$ in R, then the sets definable in $\boldsymbol{k}_{\mathrm{ind}}$ are exactly the standard parts of the sets definable in (R, V).

1 Introduction

Throughout R is an o-minimal field, that is, an o-minimal expansion of a real closed field, and V is a proper convex subring with maximal ideal $\mathfrak{m}$, ordered residue field $\boldsymbol{k} = V/\mathfrak{m}$, and standard part (residue) map $\mathrm{st}\colon V \to \boldsymbol{k}$. This map induces a map $\mathrm{st}\colon V^n \to \boldsymbol{k}^n$ and for $X \subseteq R^n$ we put $\mathrm{st}\, X := \mathrm{st}(X \cap V^n)$. By $\boldsymbol{k}_{\mathrm{ind}}$ we denote the ordered field $\boldsymbol{k}$ expanded by the relations $\mathrm{st}\, X$ with $X \in \mathrm{Def}^{\mathrm{n}}(R)$, $n = 1, 2, \ldots$. Unless indicated otherwise, by "definable" we mean "definable with parameters in the structure R".

The most important case of a convex subring of R is the convex hull

$$\mathcal{O} := \{x \in R : \ |x| \leq q \text{ for some } q \in \mathbb{Q}^{>0}\}$$

of $\mathbb{Q}$ in R. If $V = \mathcal{O}$, then the ordered field $\boldsymbol{k}$ is archimedean and we identify $\boldsymbol{k}$ with its image in the ordered field $\mathbb{R}$ of real numbers via the unique ordered field embedding of $\boldsymbol{k}$ into $\mathbb{R}$. In particular, if R is ω-saturated and $V = \mathcal{O}$, then $\boldsymbol{k} = \mathbb{R}$.

We consider the following questions:

(1) Under what conditions on (R, V) is $\boldsymbol{k}_{\mathrm{ind}}$ o-minimal?

(2) How complicated are the definable relations of $\boldsymbol{k}_{\mathrm{ind}}$ in terms of the basic relations $\operatorname{st} X$ with definable $X \subseteq R^n$?

Here is a brief history of these problems. In 1983, Cherlin and Dickmann [4] proved quantifier elimination for real closed fields with a proper convex subring. In 1995 van den Dries and Lewenberg [8] identified the notion of *T-convex subring* of an o-minimal field as a suitable analogue of *convex subring of a real closed field* (here T is the theory of the given o-minimal field). A convex subring V of R is said to be $\mathrm{Th}(R)$-convex if $f(V) \subseteq V$ for every continuous $\emptyset$-definable function $f\colon R \to R$. The situation when V is a $\mathrm{Th}(R)$-convex subring of R is well-understood; see [8] and [6]. In particular, $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal in that case.

The structure $\boldsymbol{k}_{\mathrm{ind}}$ is not always o-minimal, as the example on page 17 shows. A theorem by Baisalov and Poizat [1] implies that $\boldsymbol{k}_{\mathrm{ind}}$ is always weakly o-minimal. Hrushovski, Peterzil and Pillay observe in [11] that if R is sufficiently saturated and $V = \mathcal{O}$, then it follows from [1] that $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal, because then $\boldsymbol{k} = \mathbb{R}$ and for expansions of the ordered field $\mathbb{R}$ weak o-minimality is the same as o-minimality. However, [11] gives no information about question (2) in that situation, which includes cases where $\mathcal{O}$ is not $\mathrm{Th}(R)$-convex; we say more about this in the remark on page 4.

Good cell decomposition. In [14] we answered (2) for the situation in [11] by means of *good cell decomposition*, which also gives the o-minimality of $\mathbb{R}_{\mathrm{ind}}$ without using [1]. In the present paper we obtain good cell decomposition (and thus o-minimality of $\boldsymbol{k}_{\mathrm{ind}}$) under more general *first-order* assumptions on the pair (R, V). More precisely, suppose $(R, V) \models \Sigma_{\mathrm{i}}$ where Σ_{i} is defined below. Theorem 2.21 says that then the subsets of $\boldsymbol{k}^n$ definable in $\boldsymbol{k}_{\mathrm{ind}}$ are the finite unions of differences $\operatorname{st} X \setminus \operatorname{st} Y$, where $X, Y \subseteq R^n$ are definable. It follows that $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal. Theorem 2.21 is proved in the same way as the corresponding theorem in [14], except that uses of saturation in [14] are replaced by uses of Σ_{i}. Also the proof of Lemma 4.1. in [14] does not generalize to our setting, and this is replaced here by a more elementary proof of Lemma 2.4 below.

The following conditions on (R, V) are related to good cell decomposition. To state these, let $I := \{x \in R : |x| \leq 1\}$, and for $X \subseteq R^{1+n}$ and $r \in R$, put

$$X(r) := \{x \in R^n : (r, x) \in X\}.$$

We let $\mathfrak{m}^{>r} := \{x \in \mathfrak{m} : x > r\}$ for $r \in \mathfrak{m}$. We define the conditions $\mathcal{I}$, Σ_{i}, Σ_{d}, Σ, and $\mathcal{C}$ on pairs (R, V) as follows:

($\mathcal{I}$) if $X, Y \subseteq I^n$ are definable, then there is a definable $Z \subseteq I^n$ such that $\operatorname{st} X \cap \operatorname{st} Y = \operatorname{st} Z$;

(Σ_{i}) if $X \subseteq I^{1+n}$ is definable and $X(r) \subseteq X(s)$ for all $r, s \in I$ with $r \leq s$, then there is $\epsilon_0 \in \mathfrak{m}^{>0}$ such that $\operatorname{st} X(\epsilon_0) = \operatorname{st} X(\epsilon)$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$;

(Σ_{d}) if $X \subseteq I^{1+n}$ is definable and $X(r) \supseteq X(s)$ for all $r, s \in I$ with $r \leq s$, then there is $\epsilon_0 \in \mathfrak{m}^{>0}$ such that $\operatorname{st} X(\epsilon_0) = \operatorname{st} X(\epsilon)$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$;

(Σ) if $X \subseteq I^{1+n}$ is definable, then there is $\epsilon_0 \in \mathfrak{m}^{>0}$ such that $\operatorname{st} X(\epsilon_0) = \operatorname{st} X(\epsilon)$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$;

($\mathcal{C}$) the $\boldsymbol{k}_{\mathrm{ind}}$-definable closed subsets of $\boldsymbol{k}^n$ are exactly the sets $\operatorname{st} X$ with definable $X \subseteq R^n$.

One should add here "for all n and X, Y" as initial clause to $\mathcal{I}$, and likewise with the other conditions. In Section 3 we prove that for all (R, V),

a) $\quad \mathcal{I} \iff \Sigma_{\mathrm{i}}$;

b) $\quad \Sigma_{\mathrm{i}} \implies \boldsymbol{k}_{\mathrm{ind}}$ is o-minimal;

c) $\quad \Sigma \implies \mathcal{C}$.

We do not know whether the converse of b) holds. In a subsequent paper with van den Dries [9] we shall prove the converse of c), and also $\Sigma_{\mathrm{i}} \implies \mathcal{C}$, yielding $\Sigma_{\mathrm{i}} \iff \Sigma$.

Our definition of $\mathcal{I}$ is not of first-order nature, but by a) it is equivalent to first-order conditions. Similarly $\mathcal{C}$ will turn out to be equivalent to first order conditions by c) and its converse in [9].

In Section 3 we also show that (R, V) satisfies Σ if any of the following holds:

(i) cofinality$(\mathfrak{m}) > 2^{|\boldsymbol{k}|}$;

(ii) V is T-convex, where $T := \operatorname{Th}(R)$;

(iii) R is ω-saturated and $V = \mathcal{O}$.

Traces. Call a set $X \subseteq R^n$ a *trace* if $X = Y \cap R^n$ for some definable n-ary relation Y in some elementary extension of R, where we allow parameters from that elementary extension to define Y. In Section 4 we assume that R is ω-saturated and $V = \mathcal{O}$, and under these assumptions we characterize the definable sets in $\mathbb{R}_{\text{ind}}$ in terms of traces. As a corollary we obtain that if R is ω-saturated and $V = \mathcal{O}$, then

$$\text{Def}^n(\mathbb{R}_{\text{ind}}) \;=\; \{\text{st}\, X :\; X \in \text{Def}^n(R, \mathcal{O})\}.$$

We do not know if the analogue of this corollary holds under the more general first-order assumption Σ. We do know that if V is $\text{Th}(R)$-convex, then, for all n,

$$\text{Def}^n(\boldsymbol{k}_{\text{ind}}) = \{\text{st}\, X : X \in \text{Def}^n(R, V)\}.$$

Remark. In 1996 van den Dries [5] asked the following question: Let L be a language extending the language of ordered rings, and let $T(L, \mathbb{R})$ be the set of all sentences true in all L-expansions of the real field. Call R *pseudo-real* if $R \models T(L, \mathbb{R})$. Is every o-minimal field pseudo-real?

If R has an archimedean model, then R is pseudo-real, but the converse fails. Consider for example a proper elementary extension of the real field and extend its language by a name for an element $\lambda > \mathbb{R}$. Then the theory of R in the extended language does not have an archimedean model but R is of course pseudo-real as a structure for this extended language.

In 2006 Lipshitz and Robinson [12] considered the ordered Hahn field $\mathbb{R}((t^{\mathbb{Q}}))$ with operations given by overconvergent power series, and they proved its o-minimality. In 2007 Hrushovski and Peterzil [10] showed that this Lipshitz-Robinson field is not pseudo-real. It is easy to see that if R is a model of the theory T of the Lipshitz-Robinson field, then $\mathcal{O} \subseteq R$ is not T-convex.

Preliminaries. We assume familiarity with o-minimal structures and their basic properties; see for example [7]. Throughout we let m, n range over the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. Given a one-sorted structure $\mathcal{M} = (M; \cdots)$ we let $\text{Def}^n(\mathcal{M})$ be the boolean algebra of definable subsets of M^n. Let K be an ordered field. For $x \in K$ we put $|x| := \max\{x, -x\}$, for $a = (a_1, \dots, a_n) \in K^n$ we put

$$|a| := \max\{|a_i| :\; i = 1, \dots, n\} \;\text{ if } n > 0, \quad |a| := 0 \text{ if } n = 0,$$

and for $a, b \in K^n$ we put $d(a,b) := |a-b|$. A *box in* K^n is a cartesian product of open intervals

$$(a_1 - \delta, a_1 + \delta) \times \cdots \times (a_n - \delta, a_n + \delta),$$

where $a = (a_1, \ldots, a_n) \in K^n$ and $\delta \in K^{>0}$. A V*-box in* R^n is a box in R^n as above where $a \in V^n$ and $\delta \in V^{>\mathfrak{m}}$. So if $B \subseteq R^n$ is a V-box, then $B \subseteq V^n$ and $\operatorname{st} B$ contains a box in $\boldsymbol{k}^n$.

An *interval* is always a nonempty open interval (a, b) in R, or in $\mathbb{R}$, or in $\boldsymbol{k}$, as specified. We already defined $I := \{x \in R : \ |x| \leq 1\}$ and more generally, for each ordered field K we put $I(K) := \{x \in K : \ |x| \leq 1\}$. For $a \in R^n$ and definable nonempty $X \subseteq R^n$ we set

$$d(a, X) := \inf\{d(a, x) : \ x \in X\},$$

and likewise for $a \in \boldsymbol{k}^n$ and definable nonempty $X \subseteq \boldsymbol{k}^n$ when $\boldsymbol{k}_{\text{ind}}$ is o-minimal. A set $X \subseteq R^n$ is said to be V*-bounded* if there is $a \in V^{>0}$ such that $|x| \leq a$ for all $x \in X$. (For $V = \mathcal{O}$ this is the same as *strongly bounded*.) The *hull* of $X \subseteq \boldsymbol{k}^n$ is the set $X^h := \operatorname{st}^{-1}(X) \subseteq V^n$.

Given sets X, Y and $S \subseteq X \times Y$ we put

$$S(x) := \{y \in Y : \ (x, y) \in S\}.$$

If X is a subset of an ambient set M that is understood from the context, then

$$X^c := \{x \in M : \ x \notin X\}.$$

We often use the following projection maps for $m \leq n$:

$$\begin{aligned} p^n_m &: R^n \to R^m, \qquad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m) \\ \pi^n_m &: \boldsymbol{k}^n \to \boldsymbol{k}^m, \qquad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m). \end{aligned}$$

Given a map $f \colon X \to Y$ we let

$$\Gamma f := \{(x, y) \in X \times Y : \ f(x) = y\}$$

denote its graph.

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2 Good cell decomposition

2.1 General facts on standard part sets

Recall that R is an o-minimal field and V is a proper convex subring of R. We begin with some results requiring no extra assumption on (R, V). A very useful fact of this kind is the V-box Lemma (Corollary 2.5).

Lemma 2.1. *If $X \subseteq R^n$ is definable, then $\operatorname{st} X$ is closed.*

Proof. Let $X \subseteq R^n$ be definable and assume towards a contradiction that we have an $a \in \operatorname{cl}(\operatorname{st} X) \setminus \operatorname{st} X$. Take $a' \in R^n$ such that $\operatorname{st} a' = a$. Then, by o-minimality of R, $d(a', X)$ exists in R and $d(a', X) > \mathfrak{m}$. So there is a neighborhood $U \subseteq \boldsymbol{k}^n$ of a with $U \cap \operatorname{st} X = \emptyset$, a contradiction. □

Let St_n be the collection of all sets $\operatorname{st} X$ with definable $X \subseteq R^n$. Note that if $X, Y \in \mathrm{St}_n$, then $X \cup Y \in \mathrm{St}_n$; if $X \in \mathrm{St}_m$ and $Y \in \mathrm{St}_n$, then $X \times Y \in \mathrm{St}_{m+n}$. The next lemma is almost obvious. To state it we use the projection maps $\pi = \pi^{m+n}_m \colon \boldsymbol{k}^{m+n} \to \boldsymbol{k}^m$ and $p = p^{m+n}_m \colon R^{m+n} \to R^m$.

Lemma 2.2. *Let $X \in \mathrm{St}_{m+n}$. Then*

(1) *if X is bounded, then $\pi(X) \in \mathrm{St}_m$;*

(2) *if $X = \operatorname{st} X'$ where the set $X' \subseteq R^{m+n}$ is definable in R and satisfies $X' \cap p^{-1}(V^m) \subseteq V^{m+n}$, then $\pi(X) \in \mathrm{St}_m$.*

Lemma 2.3. *If $X \subseteq R$ is definable, then $\operatorname{st} X$ is a finite union of intervals and points in $\boldsymbol{k}$.*

Proof. This is immediate from the o-minimality of R. □

Recall the definition of a V-box from page 5. Below p is the projection map $R^{n+1} \to R^n$ given by $p(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$.

Lemma 2.4.

(A_n) *If $D \subseteq V^{n+1}$ is a V-box, and $f \colon Y \to R$, where $Y \subseteq V^n$, is definable and continuous with $f(Y) \subseteq V$, then there is a V-box $B \subseteq D$ with $B \cap \Gamma f = \emptyset$.*

(B_n) *If $D \subseteq V^n$ is a V-box, and $\mathcal{C}$ is a decomposition of D, then there is $C \in \mathcal{C}$ such that C contains a V-box.*

Proof. It is clear that (B_1) holds. We first show that (B_n) implies (A_n). Let $f\colon Y \to V$ be definable and continuous, with $Y \subseteq V^n$, and let

$$D = (a_1, b_1) \times \cdots \times (a_{n+1}, b_{n+1}) \subseteq V^{n+1}$$

be a V-box. Take $p, q \in V$ such that $a_{n+1} < p < q < b_{n+1}$ and

$$q - p, p - a_{n+1}, b_{n+1} - q > \mathfrak{m},$$

and pick $\delta > \mathfrak{m}$ with $\delta < \min\{p - a_{n+1}, \frac{q-p}{2}, b_{n+1} - q\}$. Define

$$\begin{aligned} X(p) &:= \{x \in p^{n+1}_n D \cap Y : \ f(x) \in (p - \delta, p + \delta)\} \\ X(q) &:= \{x \in p^{n+1}_n D \cap Y : \ f(x) \in (q - \delta, q + \delta)\}, \end{aligned}$$

and note that $X(p) \cap X(q) = \emptyset$. Take a decomposition $\mathcal{C}$ of R^n such that $\mathcal{C}$ partitions the sets $p^{n+1}_n D$, $X(p)$, and $X(q)$. By (B_n), there is $C \in \mathcal{C}$ such that $C \subseteq p^{n+1}_n D$ and C contains a V-box P. Then $P \times (p - \delta, p + \delta)$ or $P \times (q - \delta, q + \delta)$ yields the desired V-box B.

Next, we show that (A_n) and (B_n) imply (B_{n+1}). Let $D \subseteq V^{n+1}$ be a V-box and let $\mathcal{C}$ be a decomposition of D. Then $p^{n+1}_n \mathcal{C}$ is a decomposition of $p^{n+1}_n D$ and by (B_n) we can take $C \in \mathcal{C}$ such that $p^{n+1}_n C$ contains a V-box P. Let $C_1, \ldots, C_k$ be the cells in $\mathcal{C}$ such that $p^{n+1}_n C = p^{n+1}_n C_i$ for $i = 1, \ldots, k$. After restricting the functions $p^{n+1}_n C \to R$ used to define $C_1, \ldots, C_k$ to P we see that it is enough to prove the following:

Let $f_1, \ldots, f_m\colon P \to V$ be definable and continuous and let $p, q \in V$ be such that $p < q$ and $|q - p| > \mathfrak{m}$. Then there is a V-box $B \subseteq P \times (p, q)$ with $B \cap \Gamma f_j = \emptyset$ for all j.

For $m = 1$ this statement follows from (A_n), and for $m > 1$ it follows by a straightforward induction on m using again (A_n). $\square$

Corollary 2.5. (V-Box Lemma) *Let $X \subseteq R^n$ be definable and let $D \subseteq \boldsymbol{k}^n$ be a box such that $D \subseteq \operatorname{st} X$. Then X contains a V-box B with $\operatorname{st} B \subseteq D$.*

Proof. We may assume that $X \subseteq V^n$, and that $\operatorname{cl}(D) \subseteq \operatorname{st} X$. Pick a V-box $D' \subseteq R^n$ such that $\operatorname{st} D' = \operatorname{cl}(D)$, and take a decomposition $\mathcal{C}$ of R^n which partitions both D' and X. By Lemma 2.4, we can take $C \in \mathcal{C}$ such that $C \subseteq D'$ and C contains a V-box B. It is clear that $B \cap X \neq \emptyset$, otherwise D would contain a box whose intersection with $\operatorname{st} X$ is empty. So $B \subseteq C \subseteq X$. $\square$

Corollary 2.6. *If $X \subseteq R^n$ is definable, then* $\mathrm{st}(X) \cap \mathrm{st}(X^c)$ *has empty interior in* $\boldsymbol{k}^n$.

By [1], $\boldsymbol{k}_{\mathrm{ind}}$ is weakly o-minimal. MacPherson, Marker and Steinhorn define in [13] a notion of dimension for weakly o-minimal structures:

Definition 2.7. *Let M be a weakly o-minimal structure, and let $X \subseteq M^n$ be definable in M. If $X \neq \emptyset$, then $\dim_w(X)$ is the largest integer $k \in \{0, \ldots, n\}$ for which there is a projection map*

$$p\colon M^n \to M^k, \qquad (x_1, \ldots, x_n) \mapsto (x_{\lambda(1)}, \ldots, x_{\lambda(k)}),$$

where $1 \leq \lambda(1) < \cdots < \lambda(k) \leq n$, such that $\mathrm{int}(pX) \neq \emptyset$. We set $\dim_w(\emptyset) = -\infty$.

Note that if M is o-minimal, then the above notion of dimension agrees with the usual dimension for o-minimal structures.

Corollary 2.8. $\dim_w(\mathrm{st}\, X) \leq \dim(X)$ *for V-bounded $X \in \mathrm{Def}^n(R)$.*

2.2 Good cells

We define good cells in analogy with [14], and we state some results needed in the proof of good cell decomposition. We omit proofs that are as in [14].

Definition 2.9. *Given functions $f\colon X \to R$ with $X \subseteq R^n$, and $g\colon C \to \boldsymbol{k}$ with $C \subseteq \boldsymbol{k}^n$, we say that f* induces *g if f is definable (so X is definable), $C^h \subseteq X$, $f|C^h$ is continuous, $f(C^h) \subseteq V$ and $\Gamma g = \mathrm{st}(\Gamma f) \cap (C \times \boldsymbol{k})$.*

Lemma 2.10. *Let $C \subseteq \boldsymbol{k}^n$ and suppose $g\colon C \to \boldsymbol{k}$ is induced by the function $f\colon X \to R$ with $X \subseteq R^n$. Then g is continuous.*

Proof. Assume towards a contradiction that g is not continuous at $c \in C$. Let $r \in \boldsymbol{k}^{>0}$ be such that for every neighborhood $B \subseteq \boldsymbol{k}^n$ of c there is $b \in B \cap C$ with $|g(c) - g(b)| \geq r$. Pick $c' \in R^n$ with $\mathrm{st}\, c' = c$ and define

$$Y := \{x \in X : \ |f(c') - f(x)| \geq \frac{r'}{2}\},$$

where $r' \in R^{>0}$ is such that $\mathrm{st}\, r' = r$. Then $d(c', Y)$ exists in R. If $d(c', Y)$ is infinitesimal then, since Y is closed, there is $y \in Y$ such that $\mathrm{st}\, y = \mathrm{st}\, c'$, a contradiction with f inducing a function. Hence $d(c', Y) > \mathfrak{m}$, but this yields a neighborhood $B \subseteq \boldsymbol{k}^n$ of c such that $g(B \cap C) \subseteq (g(c) - r, g(c) + r)$, a contradiction. $\square$

For $C \subseteq \boldsymbol{k}^n$ we let $G(C)$ be the set of all $g\colon C \to \boldsymbol{k}$ that are induced by some definable $f\colon X \to R$ with $X \subseteq R^n$.

Lemma 2.11. *Let $1 \le j(1) < \cdots < j(m) \le n$ and define $\pi\colon \boldsymbol{k}^n \to \boldsymbol{k}^m$ by*

$$\pi(x_1, \ldots, x_n) = (x_{j(1)}, \ldots, x_{j(m)}).$$

Let $C \subseteq \boldsymbol{k}^n$ and suppose $g \in G(\pi C)$. Then $g \circ \pi|_C \in G(C)$.

Definition 2.12. *Let $i = (i_1, \ldots, i_n)$ be a sequence of zeros and ones. Good i-cells are subsets of $\boldsymbol{k}^n$ obtained by recursion on n as follows:*

(i) *For $n = 0$ and i the empty sequence, the set $\boldsymbol{k}^0$ is the only good i-cell, and for $n = 1$,* a good (0)-cell *is a singleton $\{a\}$ with $a \in \boldsymbol{k}$;* a good (1)-cell *is an interval in $\boldsymbol{k}$.*

(ii) *Let $n > 0$ and assume inductively that good i-cells are subsets of $\boldsymbol{k}^n$. A good $(i, 0)$-cell is a set $\Gamma h \subseteq \boldsymbol{k}^{n+1}$ where $h \in G(C)$ and $C \subseteq \boldsymbol{k}^n$ is a good i-cell. A good $(i, 1)$-cell is either a set $C \times \boldsymbol{k}$, or a set $(-\infty, f) \subseteq \boldsymbol{k}^{n+1}$, or a set $(g, h) \subseteq \boldsymbol{k}^{n+1}$, or a set $(f, +\infty) \subseteq \boldsymbol{k}^{n+1}$, where $f, g, h \in G(C)$, $g < h$, and C is a good i-cell.*

One verifies easily that a good i-cell is open in $\boldsymbol{k}^n$ iff $i_1 = \cdots = i_n = 1$, and that if $i_1 = \cdots = i_n = 1$, then every good i-cell is homeomorphic to $\boldsymbol{k}^n$. A *good cell in $\boldsymbol{k}^n$* is a good i-cell for some sequence $i = (i_1, \ldots, i_n)$ of zeros and ones.

Lemma 2.13. *Let $C \subseteq \boldsymbol{k}^n$ be a good $(i_1, \ldots, i_n)$-cell, and let $k \in \{1, \ldots, n\}$ be such that $i_k = 0$. Let $\pi\colon \boldsymbol{k}^n \to \boldsymbol{k}^{n-1}$ be given by*

$$\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n).$$

Then $\pi(C) \subseteq \boldsymbol{k}^{n-1}$ is a good cell, $\pi|C\colon C \to \pi(C)$ is a homeomorphism, and if $E \subseteq \pi(C)$ is a good cell, so is its inverse image $\pi^{-1}(E) \cap C$.

2.3 More on good cells

Recall the conditions $\mathcal{I}$ and Σ_{i} on pairs (R, V) from page 3. We prove here that $(R, V) \models \mathcal{I}$ iff $(R, V) \models \Sigma_{\mathrm{i}}$. This yields that if $(R, V) \models \Sigma_{\mathrm{i}}$, then good cells in $\boldsymbol{k}^n$ are differences of standard parts of definable subsets of R^n.

It is not difficult to show that if $(R,V) \models \mathcal{I}$, then for all n and all definable $X, Y \subseteq R^n$ there is a definable $Z \subseteq R^n$ such that $\mathrm{st}(X) \cap \mathrm{st}(Y) = \mathrm{st}\, Z$: Set $J(\boldsymbol{k}) := (-1,1) \subseteq \boldsymbol{k}$ and $J := (-1,1) \subseteq R$. We shall use the definable homeomorphism

$$\tau_n \colon R^n \to J^n \colon (x_1 \dots, x_n) \mapsto (\frac{x_1}{\sqrt{1+x_1^2}}, \dots, \frac{x_n}{\sqrt{1+x_n^2}}),$$

and we also let τ_n denote the homeomorphism

$$\tau_n \colon \boldsymbol{k}^n \to J(\boldsymbol{k})^n \colon (x_1 \dots, x_n) \mapsto (\frac{x_1}{\sqrt{1+x_1^2}}, \dots, \frac{x_n}{\sqrt{1+x_n^2}}).$$

One easily checks that $\tau_1 \colon R \to J$ induces $\tau_1 \colon \boldsymbol{k} \to J(\boldsymbol{k})$, and that for $X \in \mathrm{Def}^n(R)$,

$$\tau_n(\mathrm{st}\, X) = \mathrm{st}(\tau_n X) \cap J(\boldsymbol{k})^n \quad \text{and} \quad \tau_n^{-1}(\mathrm{st}(X) \cap J(\boldsymbol{k})^n) = \mathrm{st}(\tau_n^{-1}(X)),$$

where $\tau_n^{-1} \colon J^n \to R^n$ and $\tau_n^{-1} \colon J(\boldsymbol{k})^n \to \boldsymbol{k}^n$ are the inverse functions of $\tau_n \colon R^n \to J^n$ and of $\tau_n \colon \boldsymbol{k}^n \to J(\boldsymbol{k})^n$ respectively.

Suppose (R,V) satisfies $\mathcal{I}$. Then for all n and all $X, Y \in \mathrm{Def}^n(R)$ there is $Z \in \mathrm{Def}^n(R)$ such that $\mathrm{st}(X) \cap \mathrm{st}(Y) = \mathrm{st}(Z)$. To see this, let $X, Y \in \mathrm{Def}^n(R)$. Then $\tau_n(X), \tau_n(Y) \subseteq J^n$, so we can take $Z \in \mathrm{Def}^n(R)$ such that

$$\mathrm{st}(\tau_n(X)) \cap \mathrm{st}(\tau_n(Y)) = \mathrm{st}\, Z.$$

We claim that

$$\mathrm{st}(X) \cap \mathrm{st}(Y) = \mathrm{st}(\tau_n^{-1}(Z \cap J^n)).$$

To prove this it is enough to show that

$$\tau(\mathrm{st}(X) \cap \mathrm{st}(Y)) = \tau(\mathrm{st}(\tau_n^{-1}(Z \cap J^n))). \tag{1}$$

Now the right-hand side of (1) is equal to

$$\mathrm{st}(Z \cap J^n) \cap J(\boldsymbol{k})^n = \mathrm{st}(Z) \cap J(\boldsymbol{k})^n,$$

and we have

$$\tau_n(\mathrm{st}(X) \cap \mathrm{st}(Y)) = \mathrm{st}(\tau_n X) \cap \mathrm{st}(\tau_n Y) \cap J(\boldsymbol{k})^n.$$

In view of $\mathrm{st}(\tau_n(X)) \cap \mathrm{st}(\tau_n(Y)) = \mathrm{st}\, Z$ this gives (1).

In a similar way the condition Σ_{i} implies its "unrestricted version", i.e. the variant obtained by substituting R for I. We shall often use these facts silently.

Lemma 2.14. *Suppose* (R, V) *satisfies* $\mathcal{I}$. *Then* $(R, V) \models \Sigma_{\mathrm{i}}$.

Proof. Let $X \subseteq I^{1+n}$ be definable and increasing in the first variable. Towards proving that X satisfies the conclusion of Σ_{i} we may assume that X is closed.

Claim 1. There is $\epsilon_0 \in \mathfrak{m}^{\geq 0}$ such that

$$\mathrm{st}(X) \cap (\{0\} \times I(\boldsymbol{k})^n) = \mathrm{st}(X \cap ([0, \epsilon_0] \times I^n)).$$

We set $Y := \{0\} \times I^n$ and take a definable $Z \subseteq I^{n+1}$ with $\mathrm{st}(X) \cap \mathrm{st}(Y) = \mathrm{st}(Z)$. We may assume that Z is closed and nonempty, and we set $\epsilon_1 := \sup\{d(z, X) : z \in Z\}$ and $\epsilon_2 := \sup\{d(z, Y) : z \in Z\}$. Then $\epsilon_1, \epsilon_2 \in \mathfrak{m}^{\geq 0}$, and we claim that $\epsilon_0 := \epsilon_1 + \epsilon_2$ works. Clearly,

$$\mathrm{st}(X \cap ([0, \epsilon_0] \times I^n)) \subseteq \mathrm{st}(X) \cap (\{0\} \times I(\boldsymbol{k})^n).$$

So let $a \in \mathrm{st}(X) \cap \mathrm{st}(Y)$. Then $a = \mathrm{st}\, z$ with $z \in Z$. We have $d(z, X) \leq \epsilon_1$ and $d(z, Y) \leq \epsilon_2$. Since Z is closed and V-bounded, we can take $x \in X$ and $y \in Y$ such that $d(x, z) \leq \epsilon_1$, $d(y, z) \leq \epsilon_2$. Then $d(x, y) \leq \epsilon_1 + \epsilon_2 = \epsilon_0$, and it follows that

$$a = \mathrm{st}\, x \in \mathrm{st}(X \cap ([0, \epsilon_0] \times I^n)).$$

This proves Claim 1. Let ϵ_0 be as in Claim 1.

Claim 2. $\mathrm{st}\, X(\epsilon) = \mathrm{st}\, X(\epsilon_0)$ for all $\epsilon \in \mathfrak{m}^{\geq \epsilon_0}$.

It is clear that $\mathrm{st}\, X(\epsilon_0) \subseteq \mathrm{st}\, X(\epsilon)$ for all $\epsilon \geq \epsilon_0$. To prove the other inclusion, let $a \in \mathrm{st}\, X(\epsilon)$ and take $x \in X(\epsilon)$ such that $\mathrm{st}\, x = a$. Then

$$(0, a) \in \mathrm{st}(X) \cap (\{0\} \times I(\boldsymbol{k})^n),$$

hence

$$(0, a) \in \mathrm{st}(X \cap ([0, \epsilon_0] \times I^n))$$

by Claim 1. Because X is increasing in the first variable, this implies $(0, a) \in \mathrm{st}\, X(\epsilon_0)$. $\square$

Lemma 2.15. $\Sigma_{\mathrm{i}} \implies \mathcal{I}$.

Proof. Suppose (R, V) satisfies Σ_{i}. Let $X, Y \subseteq I^n$ be definable and nonempty. For $\epsilon \in R^{\geq 0}$ define

$$Y^\epsilon := \{x \in R^n : d(x, Y) \leq \epsilon\}.$$

We claim that

$$\bigcup_{\epsilon} \mathrm{st}\,(X \cap Y^{\epsilon}) = \mathrm{st}\, X \cap \mathrm{st}\, Y,$$

where ϵ ranges over all positive infinitesimals. If $a \in \mathrm{st}\,(X \cap Y^{\epsilon})$, then clearly $a \in \mathrm{st}\, X$ and $a \in \mathrm{st}\, Y$. If $a \in \mathrm{st}\, X \cap \mathrm{st}\, Y$, then we can take $a' \in X$ and $a'' \in Y$ such that $\mathrm{st}\, a' = \mathrm{st}\, a'' = a$ and $d(a', a'') < \epsilon$ for some $\epsilon \in \mathfrak{m}^{>0}$. Hence $a' \in X \cap Y^{\epsilon}$.

Now by Σ_{i}, there is a positive infinitesimal ϵ_0 such that

$$\mathrm{st}\,(X \cap Y^{\epsilon_0}) = \bigcup_{\epsilon} \mathrm{st}\,(X \cap Y^{\epsilon}).$$

$\square$

The proofs of the following two lemmas are similar to the proofs of their counterparts in [14].

Lemma 2.16. *Suppose (R, V) satisfies $\mathcal{I}$, and let $X \subseteq R^n$ and $f\colon X \to R$ be definable, and put*

$$X^{-} := \{x \in X :\ f(x) < V\}, \quad X^{+} := \{x \in X :\ f(x) > V\}.$$

Then $\mathrm{st}(X^{-})$ and $\mathrm{st}(X^{+})$ belong to St_n.

Corollary 2.17. *If (R, V) satisfies $\mathcal{I}$, and $X \subseteq R^n$ and $g\colon X \to R$ are definable, then $\mathrm{st}(\{x \in X : g(x) \in \mathfrak{m}\}) \in \mathrm{St}_n$.*

Conversely, if the conclusion of this corollary holds for all n and definable $g\colon X \to R$ with $X \subseteq R^n$, then (R, V) satisfies $\mathcal{I}$. To see this, let $X, Y \subseteq V^n$ be definable with $Y \neq \emptyset$. Assume the conclusion of the corollary holds for the function $x \mapsto d(x, Y)\colon X \to R$. Then we have a definable $Z \subseteq V^n$ such that $\mathrm{st}(Z) = \mathrm{st}(\{x \in X :\ d(x, Y) \in \mathfrak{m}\})$. This gives $\mathrm{st}(X) \cap \mathrm{st}(Y) = \mathrm{st}(Z)$.

From now on until the end of Section 2 we assume $(R, V) \models \Sigma_{\mathrm{i}}$.

The following lemma is now proved as in [14].

Lemma 2.18. *Every good cell in $\boldsymbol{k}^n$ is of the form $X \setminus Y$ with $X, Y \in \mathrm{St}_n$.*

2.4 Good cell decomposition

We obtain good cell decomposition, namely, if $X_1, \dots, X_m \subseteq R^n$ are definable, then there is a finite partition of $\boldsymbol{k}^n$ into good cells that partitions every $\mathrm{st}(X_i)$. A consequence of this is that the $\boldsymbol{k}_{\mathrm{ind}}$-definable subsets of $\boldsymbol{k}^n$ are finite unions of differences $\mathrm{st}(X) \setminus \mathrm{st}(Y)$, where $X, Y \in \mathrm{Def}^n(R)$.

Lemma 2.19. *Let $C \subseteq \boldsymbol{k}^n$ be a good i-cell, let $X \subseteq R^{n+1}$ be definable and suppose $k \in \{1, \dots, n\}$ is such that $i_k = 0$. Define $\pi\colon \boldsymbol{k}^{n+1} \to \boldsymbol{k}^n$ by*

$$\pi(x) = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}).$$

Then $\pi\big(\mathrm{st}(X) \cap (C \times \boldsymbol{k})\big)$ is a difference of sets in St_n.

A *good decomposition of $I(\boldsymbol{k})^n$* is a special kind of partition of $I(\boldsymbol{k})^n$ into finitely many good cells. The definition is by recursion on n:

(i) a good decomposition of $I(\boldsymbol{k})$ is a collection

$$\{(c_0, c_1), (c_2, c_3), \dots, (c_k, c_{k+1}), \{c_0\}, \{c_1\}, \dots, \{c_k\}, \{c_{k+1}\}\}$$

of intervals and points in $\boldsymbol{k}$ where $c_0 < c_1 < \cdots < c_k < c_{k+1}$ are real numbers with $c_0 = -1$ and $c_{k+1} = 1$;

(ii) a good decomposition of $I(\boldsymbol{k})^{n+1}$ is a finite partition $\mathcal{D}$ of $I(\boldsymbol{k})^{n+1}$ into good cells such that $\{\pi^{n+1}_n C : C \in \mathcal{D}\}$ is a good decomposition of $I(\boldsymbol{k})^n$.

Theorem 2.20. (Good Cell Decomposition)

(A_n) *Given any definable $X_1, \dots, X_m \subseteq I^n$, there is a good decomposition of $I(\boldsymbol{k})^n$ partitioning each set $\mathrm{st}\, X_i$.*

(B_n) *If $f\colon X \to I$, with $X \subseteq I^n$, is definable, then there is a good decomposition $\mathcal{D}$ of $I(\boldsymbol{k})^n$ such that for every open $C \in \mathcal{D}$, either the set $\mathrm{st}(\Gamma f) \cap (C \times \boldsymbol{k})$ is empty, or f induces a function $g\colon C \to I(\boldsymbol{k})$.*

Using the lemmas above the proof is very similar to that of Theorem 4.3 in [14].

A *good decomposition of $\boldsymbol{k}^n$* is a special kind of partition of $\boldsymbol{k}^n$ into finitely many good cells. The definition is by recursion on n:

(i) a good decomposition of $\boldsymbol{k}^1 = \boldsymbol{k}$ is a collection

$$\{(c_0, c_1), (c_2, c_3), \ldots, (c_k, c_{k+1}), \{c_1\}, \ldots, \{c_k\}\}$$

of intervals and points in $\boldsymbol{k}$, where $c_1 < \cdots < c_k \in \boldsymbol{k}$ and $c_0 = -\infty$, $c_{k+1} = \infty$;

(ii) a good decomposition of $\boldsymbol{k}^{n+1}$ is a finite partition $\mathcal{D}$ of $\boldsymbol{k}^{n+1}$ into good cells such that $\{\pi^{n+1}_n C : \ C \in \mathcal{D}\}$ is a good decomposition of $\boldsymbol{k}^n$.

Corollary 2.21. *If $X_1, \ldots, X_m \subseteq R^n$ are definable, then there is a good decomposition of $\boldsymbol{k}^n$ partitioning every $\operatorname{st} X_i$.*

Theorem 2.22. *The $\boldsymbol{k}_{\text{ind}}$-definable subsets of $\boldsymbol{k}^n$ are exactly the sets of the form $\operatorname{st}(X) \setminus \operatorname{st}(Y)$ with $X, Y \in \operatorname{Def}^n(R)$.*

As in [14] we obtain that the standard part of a partial derivative of a definable function is almost everywhere equal to the corresponding partial derivative of the standard part of the function:

Theorem 2.23. *Let $f\colon Y \to R$ with $Y \subseteq R^n$ be definable with V-bounded graph. Then there is a good decomposition $\mathcal{D}$ of $\boldsymbol{k}^n$ that partitions $\operatorname{st} Y$ such that if $D \in \mathcal{D}$ is open and $D \subseteq \operatorname{st} Y$, then f is continuously differentiable on an open definable $X \subseteq Y$ containing D^h, and $f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$, as functions on X, induce functions $g, g_1, \ldots, g_n \colon D \to \boldsymbol{k}$ such that g is C^1 and $g_i = \frac{\partial g}{\partial x_i}$ for all i.*

3 The conditions $\mathcal{C}$, Σ_{i}, Σ_{d} and Σ

In this section we show that $\big(\Sigma_{\text{i}}\ \&\ \Sigma_{\text{d}}\big)$ implies $\mathcal{C}$, we prove that various conditions imply Σ, and we give an example to the effect that $\boldsymbol{k}_{\text{ind}}$ is not always o-minimal.

3.1 Closed and definably connected sets

The conditions Σ_{d} and $\mathcal{C}$ on pairs (R, V) are stated on page 3. Note that if (R, V) satisfies $\mathcal{C}$, then $\boldsymbol{k}_{\text{ind}}$ is o-minimal by Lemma 2.3. For (R, V) to satisfy $\mathcal{C}$ it suffices that for each n the closed $\boldsymbol{k}_{\text{ind}}$-definable subsets of $I(\boldsymbol{k})^n$ are exactly the sets $\operatorname{st} X$ with definable $X \subseteq I^n$. (This follows by means of the homeomorphisms τ_n.)

Proposition 3.1. *Suppose* $(R, V) \models \Sigma_{\mathrm{i}}$ *and* $(R, V) \models \Sigma_{\mathrm{d}}$. *Then* (R, V) *satisfies* $\mathcal{C}$. *(In particular,* $\Sigma \Rightarrow \mathcal{C}$.*)*

Proof. The result will follow from Corollary 2.21 once we show that the closure of a good cell in $\boldsymbol{k}^n$ is of the form $\operatorname{st} X$ for some definable $X \subseteq R^n$. Let ϵ range over all positive infinitesimals, and let $C \subseteq \boldsymbol{k}^n$ be a good cell.

Claim. There is $r_0 \in R^{>\mathfrak{m}}$ and a definable $X \subseteq (0, r_0) \times R^n$ such that

$$0 < r < r' < r_0 \Longrightarrow X(r') \subseteq X(r); \qquad \operatorname{st}\Big(\bigcap_{\epsilon} X(\epsilon)\Big) = C.$$

This claim follows by the same argument as the corresponding claim in the proof of Proposition 5.1 in [14]. Let $X \subseteq (0, r_0) \times R^n$ be as in the Claim. Then, since $(R, V) \models \Sigma_{\mathrm{d}}$, we can take $\epsilon \in \mathfrak{m}^{>0}$ such that $\operatorname{st} X(\epsilon) = \operatorname{cl}(C)$. $\square$

For $Z \subseteq V^n$ we let $Z^h := \operatorname{st}^{-1}(\operatorname{st}(Z))$.

Proposition 3.2. *Suppose* (R, V) *satisfies* $\mathcal{C}$, *and let* $X \subseteq V^n$ *be definable and definably connected in* R. *Then* $\operatorname{st} X$ *is definably connected.*

Proof. Assume to the contrary that $\operatorname{st} X$ is not definably connected. Then $\operatorname{st} X = \operatorname{st} Y_1 \dot{\cup} \operatorname{st} Y_2$ for some definable, nonempty $Y_1, Y_2 \subseteq R^n$. We may assume that Y_1, Y_2 are closed. Let

$$q := \inf\{d(y, \operatorname{st} Y_2) : \ y \in \operatorname{st} Y_1\}.$$

Since $\operatorname{st} Y_1, \operatorname{st} Y_2$ are closed and bounded, $q \in \boldsymbol{k}^{>0}$. Define

$$X_1 := \{x \in R^n : \ d(x, Y_1) \leq \frac{q}{4}\} \text{ and } X_2 := \{x \in R^n : \ d(x, Y_2) \leq \frac{q}{4}\}.$$

Then X_1, X_2 are closed and disjoint, and $Y_1^h \subseteq X_1$, $Y_2^h \subseteq X_2$. Since $X^h = Y_1^h \cup Y_2^h$, we have $X = (X \cap X_1) \cup (X \cap X_2)$, where $X \cap X_1$, $X \cap X_2$ are nonempty, disjoint, and closed in X, a contradiction with X being definably connected. $\square$

3.2 Conditions implying Σ

In the next lemma we use the following convention. Let $C \subseteq R^n$ be an $(i_1, \ldots, i_n)$-cell of dimension k. Let

$$\lambda \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$$

be such that

$$1 \leq \lambda(1) < \cdots < \lambda(k) \leq n$$

and $i_{\lambda(1)} = \cdots = i_{\lambda(k)} = 1$. We define

$$C_0 := \{a \in R^k : \quad \text{there is } x \in C \text{ such that } x_{\lambda(1)} = a_1 \,\&\, \ldots \,\&\, x_{\lambda(k)} = a_k\}.$$

Then C_0 is the homeomorphic image of C under a coordinate projection $p\colon R^n \to R^k$. For a definable C^1-function $f\colon C \to R$ we let $\hat{f}\colon C_0 \to R$ be defined by $\hat{f}(p(x)) = f(x)$ where $x \in C$. We denote by $\frac{\partial f}{\partial x_j}(a)$, where $a \in C$ and $j \in \{1, \ldots, k\}$, the j-th partial derivative of $\hat{f}$ at $p(a)$.

Lemma 3.3. *Suppose* $\operatorname{cofinality}(\mathfrak{m}) > 2^{|\boldsymbol{k}|}$. *Then* (R, V) *satisfies* Σ.

Proof. Let $X \in \operatorname{Def}^{1+n}(R)$. By cell decomposition we may assume that X is an $(i_1, \ldots, i_{n+1})$-cell satisfying for every $k = 1, \ldots, n+1$ the following: If $p_k^{n+1}X = (f, g)$, then all $\frac{\partial f}{\partial x_i}$, $\frac{\partial g}{\partial x_i}$ have constant sign on $p_{k-1}^{n+1}X$. If $p_k^{n+1}X = \Gamma f$, then all $\frac{\partial f}{\partial x_i}$ have constant sign on $p_{k-1}^{n+1}X$.

Now there are $2^{|\boldsymbol{k}|}$ many distinct subsets of $\boldsymbol{k}^n$. Let $f\colon \mathfrak{m}^{>0} \to \mathcal{P}(\boldsymbol{k}^n)$, where $\mathcal{P}(\boldsymbol{k}^n)$ is the power set of $\boldsymbol{k}^n$, be given by $\epsilon \mapsto \operatorname{st} X(\epsilon)$. Assume to the contrary that for every $\epsilon_1 \in \mathfrak{m}^{>0}$ we can find $\epsilon_2 \in \mathfrak{m}^{>\epsilon_1}$ such that $\operatorname{st} X(\epsilon_1) \neq \operatorname{st} X(\epsilon_2)$. Then the above assumption on X yields a cofinal subset of $\mathfrak{m}$ such that f is injective on this subset, a contradiction.

□

Note that, together with 5.3 and 6.4 in [6], this lemma implies that if V is a T-convex subring of R, then $(R, V) \models \Sigma$.

Lemma 3.4. *Let R be ω-saturated. Then* $(R, \mathcal{O}) \models \Sigma$.

Proof. Let $X \subseteq R^{1+n}$ be defined over $a \in R^k$. Since R is ω-saturated, we can take $\epsilon \in \mathfrak{m}$ such that $\epsilon > \delta$ for every $\delta \in \operatorname{dcl}(a)$ with $\delta < \mathbb{Q}^{>0}$. Then for every $\epsilon' \in \mathfrak{m}^{>\epsilon}$, $\operatorname{tp}(\epsilon'|a) = \operatorname{tp}(\epsilon|a)$, and, in particular, $\operatorname{st} X(\epsilon') = \operatorname{st} X(\epsilon)$. Otherwise we could find $x \in \operatorname{st} X(\epsilon') \triangle \operatorname{st} X(\epsilon)$ and a box $B = (p_1, q_1) \times \cdots \times (p_n, q_n) \subseteq \mathbb{R}^n$ with $p_i, q_i \in \mathbb{Q}$ such that $x \in B$ and either $\operatorname{cl}(B) \cap \operatorname{st} X(\epsilon) = \emptyset$ or $\operatorname{cl}(B) \cap \operatorname{st} X(\epsilon') = \emptyset$. Then $B' = (p_1, q_1) \times \cdots \times (p_n, q_n) \subseteq R^n$ is such that $B' \cap X(\epsilon) = \emptyset$ and $B' \cap X(\epsilon') \neq \emptyset$, or vice versa, a contradiction. □

We saw in Section 2 that if $(R, V) \models \Sigma_{\mathrm{i}}$, then $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal. However, the following example shows that $\boldsymbol{k}_{\mathrm{ind}}$ is not always o-minimal.

Example. Let $\mathbb{R}_{\exp}$ be the real exponential field and let R be a proper elementary extension. Take $\lambda \in R$ such that $\lambda > \mathbb{R}$, and let V be the smallest convex subring of R containing λ, i.e.

$$V := \{y : |y| < \lambda^n \text{ for some } n\},$$

and let $\boldsymbol{k}$ be the corresponding residue field. We define $\log\colon R^{>0} \to R$ to be the inverse function of $\exp\colon R \to R^{>0}$. Then $\log(V^{>0}) = V$ and it induces an increasing and injective function $\boldsymbol{k}^{>0} \to \boldsymbol{k}$, which, for simplicity, we shall also denote by log. Now the set $\{\mathrm{st}(\lambda)^n : n \in \mathbb{N}\}$ is cofinal in $\boldsymbol{k}^{>0}$, hence $\{\log \mathrm{st}(\lambda)^n : n \in \mathbb{N}\}$ is cofinal in $\log \boldsymbol{k}^{>0}$. So the set $\log \boldsymbol{k}^{>0}$ is definable in $\boldsymbol{k}_{\mathrm{ind}}$, but, because $\log \mathrm{st}(\lambda)^n = n \log \mathrm{st}(\lambda)$, it is not cofinal in $\boldsymbol{k}^{>0}$, nor does it have a supremum. It follows that $\boldsymbol{k}_{\mathrm{ind}}$ cannot be o-minimal, nor does (R, V) satisfy Σ_{i}.

4 Traces

Recall from the Introduction that a set $X \subseteq R^n$ is a *trace* if $X = Y \cap R^n$ for some n-ary relation Y defined in some elementary extension of R using parameters from that extension. Note that every $X \in \mathrm{Def}^n(R)$ is a trace, and that if $X, Y \subseteq R^n$ are traces, then so are $X \cup Y$, $X \cap Y$ and X^c. An example of a trace is $V \subseteq R$: take an element λ in an elementary extension of R such that $V < \lambda < R^{>V}$. Then $V = (-\lambda, \lambda) \cap R$ where the interval $(-\lambda, \lambda)$ is taken in the extension.

We let R^* be the expansion of R by all traces $X \subseteq R^n$, for all n. By the main result of [1] every subset of R^n definable in R^* is a trace. It follows that every subset of R^n definable in (R, V) is a trace.

Lemma 4.1. *Let $\boldsymbol{k}^*$ be the expansion of the ordered field $\boldsymbol{k}$ by the sets $\mathrm{st}(X) \subseteq \boldsymbol{k}^n$ for all traces $X \subseteq R^n$ and all n. Then, for all n,*

$$\mathrm{Def}^n(\boldsymbol{k}^*) = \{\mathrm{st}(X) : X \subseteq R^n \textit{ is a trace }\}.$$

Proof. We first show that for every n, the collection

$$\mathcal{C}_n := \{\mathrm{st}(X) : X \subseteq R^n \text{ is a trace }\}$$

is a boolean algebra on $\boldsymbol{k}^n$. It is clear that

$$\mathrm{st}(X_1) \cup \mathrm{st}(X_2) = \mathrm{st}(X_1 \cup X_2)$$

for all traces $X_1, X_2 \subseteq R^n$. To see that $\mathcal{C}_n$ is closed under complements, let $X \subseteq R^n$ be a trace, and note that

$$(\operatorname{st} X)^c = \operatorname{st}\{y \in R^n : d(y,x) > \mathfrak{m} \text{ for every } x \in X\}.$$

Since $\mathfrak{m}$ is a trace, the set $\{y \in R^n : d(y,x) > \mathfrak{m} \text{ for all } x \in X\}$ is definable in R^*, hence, by [1], it is itself a trace. We conclude that the sets $\operatorname{st}(X)$, where $X \subseteq R^n$ is a trace, are the elements of a boolean algebra on $\boldsymbol{k}^n$.

Now let $X \subseteq R^n$ be a trace, and let $0 \le m \le n$. We may assume that $X \subseteq V^n$ (since V is a trace). Then $\pi^n_m(\operatorname{st}(X)) = \operatorname{st}(p^n_m(X))$, and by [1], $p^n_m(X)$ is a trace. $\square$

It follows from Lemma 4.1 that $\boldsymbol{k}^*$ is weakly o-minimal.

Lemma 4.2. *Let S_1 be a weakly o-minimal structure and S_2 an o-minimal structure on the same underlying ordered set S. Suppose for every n and for every $X_1 \in \operatorname{Def}^n(S_1)$ there is $X_2 \in \operatorname{Def}^n(S_2)$ such that $X_1 \triangle X_2$ has empty interior in S^n. Then $\operatorname{Def}^n(S_1) \subseteq \operatorname{Def}^n(S_2)$, for all n.*

Proof. We proceed by induction on n. Let $n = 1$. If $X \subseteq S$ is a finite union of convex sets, and $Y \subseteq S$ is a finite union of points and intervals, then either $X \triangle Y$ is finite, or $X \triangle Y$ has nonempty interior. It follows that $\operatorname{Def}^1(S_1) \subseteq \operatorname{Def}^1(S_2)$ and, in particular, S_1 is o-minimal.

So assume $\operatorname{Def}^k(S_1) \subseteq \operatorname{Def}^k(S_2)$ holds for $k = 1, \dots, n$. Since S_1 and S_2 are o-minimal, it suffices to show that every S_1-cell in S^{n+1} is definable in S_2. It is even enough to prove this for S_1-cells Γg; here $g\colon C \to S$ is a continuous and S_1-definable function on an S_1-cell $C \subseteq S^n$. Let Γg be such an S_1-cell.

First, suppose C is an open cell. By the inductive assumption $C \in \operatorname{Def}^n(S_2)$ and we can take $X \in \operatorname{Def}^{n+1}(S_2)$ with $X \subseteq C \times S$ such that $(-\infty, g) \triangle X$ does not contain a box. Let $p\colon S^{n+1} \to S^n$ be given by $p(x_1, \dots, x_{n+1}) = (x_1, ..., x_n)$. For $X, Y \subseteq S^{n+1}$ we say that $X < Y$ if for all $a \in S^n$ and $(a,x) \in X$, $(a,y) \in Y$ we have $x < y$. Now take an S_2-decomposition $\mathcal{D}$ of S^{n+1} which partitions X, and let $C_1, \dots, C_k$ be the open cells in $p\mathcal{D}$ with $C_i \subseteq pX$. We claim that $\Gamma(g|C_i) \in \operatorname{Def}^{n+1}(S_2)$ for every i.

So let $i \in \{1, \dots, k\}$, and let $D_1, \dots, D_l$ be the open cells in $\mathcal{D}$ with $D_j \subseteq X$ and $pD_j = C_i$ for all j. If $D_j = (f_j, g_j)$ and $D_j \cap \Gamma(g|C_i) \neq \emptyset$ for some $j \in \{1, \dots, l\}$, then there is $x \in C_i$ with $g(x) < g_j(x)$. Then, by continuity of g and g_j, we obtain a box $B \subseteq X \setminus (-\infty, g)$, a contradiction. So $D_j \cap \Gamma g = \emptyset$, and, in particular, $D_j < \Gamma(g|C_i)$ for every j.

Let $d \in \{1, \ldots, l\}$ be such that $D_j < D_d = (f_d, g_d)$ for all $j \neq d$. If $g_d < g|C_i$ on a subset of C_i with nonempty interior, then, again by continuity of g and g_d, we find a box $B \subseteq (-\infty, g)$ with $\Gamma(g_d|pB) < B$. Since B intersects X in only at most finitely many cells of the form Γh, where $h\colon C_i \to S$ is continuous, we can find a box $B' \subseteq (-\infty, g) \setminus X$, a contradiction. So $g_d = g|C_i$ outside a subset of C_i with empty interior, hence $g_d = g|C_i$ by continuity of g and g_d.

We have shown that $\Gamma(g|C_i)$ is S_2-definable for all $i = 1, \ldots, k$. It is easy to check that then

$$\Gamma g = \mathrm{cl}(\bigcup_{i=1}^{k} \Gamma(g|C_i)) \cap (C_i \times S),$$

hence $\Gamma g \in \mathrm{Def}^{n+1}(S_2)$.

So let $\Gamma g \in \mathrm{Def}^{n+1}(S_2)$ be an $(i_1, \ldots, i_n, 0)$-cell with $i_k = 0$ where $1 \leq k \leq n$, and let

$$q\colon S^{n+1} \to S^n\colon (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}).$$

By the inductive assumption, $q(\Gamma g) \in \mathrm{Def}^n(S_2)$. We define Γg in S_2 as

$$\{(x, y) :\ x \in C \text{ and } (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n, y) \in q(\Gamma g)\}.$$

$\square$

The main result of this section is Theorem 4.4, where we assume that R is ω-saturated and $V = \mathcal{O}$. This assumption is essential in that Theorem: Suppose $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal but $\boldsymbol{k}$ is not isomorphic to $\mathbb{R}$. Then $\boldsymbol{k}$ has a nonempty bounded convex subset X without a least upper bound in $\boldsymbol{k}$, so X is not definable in $\boldsymbol{k}_{\mathrm{ind}}$. However, $X^h \subseteq R$ is a trace, and so $X = \mathrm{st}\, Y$ for some trace set $Y \subseteq R^n$.

In the rest of this section we assume that R is ω-saturated and $V = \mathcal{O}$. In particular, $\boldsymbol{k} = \mathbb{R}$.

Lemma 4.3. *Let $Y \subseteq R^n$ be a trace. Then there is a definable $Z \subseteq R^n$ such that $\mathrm{st}(Y) \triangle \mathrm{st}(Z)$ has empty interior in $\mathbb{R}^n$.*

Proof. Take an elementary extension R' of R with a definable set $Y' \subseteq R'^n$ such that $Y = Y' \cap R^n$. Then Y' is defined in R' by a formula $\phi(a, y)$ where

$a \in R'^m$ and $\phi(x, y)$ is a formula in the language of R, $x = (x_1, \dots, x_m), y = (y_1, \dots, y_n)$. By ω-saturation of R we can take $b \in R^m$ such that $\mathrm{tp}(b|\emptyset) = \mathrm{tp}(a|\emptyset)$. Let $Z \subseteq R^n$ be defined in R by $\phi(b, y)$. Then $Y \cap \mathcal{O}^n \subseteq \bigcup_\epsilon Z^\epsilon$, where ϵ ranges over all positive infinitesimals and

$$Z^\epsilon := \{y \in R^n : \ d(y, Z) \leq \epsilon\}.$$

Otherwise there would be $y \in (Y \cap \mathcal{O}^n)$ such that $d(y, Z) > \mathfrak{m}$, so for some $\mathcal{O}$-box $P \subseteq R^n$, we would have $P \cap Y \neq \emptyset$ and $P \cap Z = \emptyset$, a contradiction with $\mathrm{tp}(b|\emptyset) = \mathrm{tp}(a|\emptyset)$.

It follows that $\mathrm{st}(Y) \subseteq \mathrm{st}(Z)$. We claim that $\mathrm{int}(\mathrm{st}(Y) \triangle \mathrm{st}(Z)) = \emptyset$. Otherwise, we can take a box $B \subseteq \mathbb{R}^n$ such that $B \subseteq \mathrm{st}(Z) \setminus \mathrm{st}(Y)$, so the V-box lemma yields an $\mathcal{O}$-box $P \subseteq Z$ such that $P \cap Y = \emptyset$, contradicting $\mathrm{tp}(b|\emptyset) = \mathrm{tp}(a|\emptyset)$. □

Theorem 4.4. *For all n,*

$$\mathrm{Def}^n(\mathbb{R}_{\mathrm{ind}}) = \{\mathrm{st}(X) : \ X \subseteq R^n \textit{ is a trace}\}.$$

Proof. By Lemma 4.1,

$$\{\mathrm{st}(X) : \ X \subseteq R^n \text{ is a trace}\} = \mathrm{Def}^n(\mathbb{R}^*),$$

for all n, and it is clear that $\mathrm{Def}^n(\mathbb{R}_{\mathrm{ind}}) \subseteq \mathrm{Def}^n(\mathbb{R}^*)$. So let $X \subseteq R^n$ be a trace. By Lemma 4.3, we can take $Y \in \mathrm{Def}^n(R)$ such that $\mathrm{int}(\mathrm{st}(X) \triangle \mathrm{st}\, Y) = \emptyset$, hence, by Lemma 4.2, $\mathrm{Def}^n(\mathbb{R}^*) \subseteq \mathrm{Def}^n(\mathbb{R}_{\mathrm{ind}})$. □

Corollary 4.5. $\mathrm{Def}^n(\mathbb{R}_{\mathrm{ind}}) = \{\mathrm{st}(X) : \ X \in \mathrm{Def}^n(R, \mathcal{O})\}$*, for all n.*

Proof. It is clear that $\{\mathrm{st}(X) : \ X \in \mathrm{Def}^n(R, \mathcal{O})\} \subseteq \mathrm{Def}^n(\mathbb{R}^*)$, so by Theorem 4.4, $\{\mathrm{st}(X) : \ X \in \mathrm{Def}^n(R, \mathcal{O})\} \subseteq \mathrm{Def}^n(\mathbb{R}_{\mathrm{ind}})$. To see that

$$\mathrm{Def}^n(\mathbb{R}_{\mathrm{ind}}) \subseteq \{\mathrm{st}(X) : \ X \in \mathrm{Def}^n(R, \mathcal{O})\},$$

recall that the $\mathbb{R}_{\mathrm{ind}}$-definable subsets of $\mathbb{R}^n$ are finite unions of sets $\mathrm{st}\, Y \setminus \mathrm{st}\, Z$, where $Y, Z \in \mathrm{Def}^n(R)$, and observe that

$$\mathrm{st}\, Y \setminus \mathrm{st}\, Z = \mathrm{st}\{x \in Y : \ d(x, Z) > \mathfrak{m}\},$$

and that $\mathfrak{m}$ is definable in the structure $(R, \mathcal{O})$. □

5 Open problems

1. We proved that Σ_{i} implies o-minimality of $\boldsymbol{k}_{\mathrm{ind}}$. Is the converse true?

2. We showed that if cofinality$(\mathfrak{m}) > 2^{|\boldsymbol{k}|}$, then $(R, V) \models \Sigma$. Conversely, if $(R, V) \models \Sigma$, is there an elementary extension of (R, V) satisfying this inequality?

3. Does an analogue of Corollary 4.5 hold under more general conditions, for example $(R, V) \models \Sigma$?

4. Let R be an ω-saturated elementary extension of the Lipshitz-Robinson structure. Are the definable sets of $\mathbb{R}_{\mathrm{ind}}$ just the semialgebraic sets?

5. The following question was posed by Lou van den Dries and Jonathan Kirby:

   $(*)$ Let R be ω-saturated and $V = \mathcal{O}$; is $\mathbb{R}_{\mathrm{ind}}$ elementarily equivalent to a definable reduct of R?

   To state this question precisely we assign to each $X \in \mathrm{Def}^n(\mathbb{R}_{\mathrm{ind}})$ an n-ary relation symbol P_X, we let L_{ind} be the language $L = \{<, 0, 1, -, +, \cdot\}$ of ordered rings augmented by these new relation symbols P_X, and we construe $\mathbb{R}_{\mathrm{ind}}$ as a structure for the language L_{ind} in the obvious way, by interpreting each P_X as X. The formal statement of question $(*)$ is as follows: does there exist an L_{ind}-structure R' such that

   (i) L-reduct of R' = L-reduct of R,

   (ii) each n-ary symbol P_X is interpreted in R' as a set $X' \in \mathrm{Def}^n(R)$,

   (iii) $\mathbb{R}_{\mathrm{ind}} \equiv R'$?

A positive solution might be hard to come by. To explain this, let $L_{\exp}$ be the language of ordered rings augmented by a unary function symbol exp, and consider the $L_{\exp}$-theory $T_{\exp}$ of the ordered exponential field $\mathbb{R}_{\exp}$. Peterzil pointed out that by an argument as in Berarducci and Servi [3] we have:

Proposition 5.1. *Suppose* $(*)$ *has a positive answer. Then* $T_{\exp}$ *is decidable.*

Proof. By [3] we have a recursive set Σ_{o} of $L_{\exp}$-sentences such that $T_{\exp} \models \sigma$ for all $\sigma \in \Sigma_{\mathrm{o}}$ and all $L_{\exp}$-models of Σ_{o} are o-minimal. We can of course assume that Σ_{o} includes the usual axioms for real closed fields, as well as an axiom expressing that exp is a C^1-function with $\exp(0) = 1$ and $\exp' = \exp$.

Claim. Σ_o axiomatizes the (complete) theory T_{exp}.

To prove this claim, let R be an ω-saturated model of Σ_o. Then the exponential function $\exp_R$ of R induces the standard exponential function on $\mathbb{R}$. Since we assume that $(*)$ has a positive answer for R, this gives a definable function $e\colon R \to R$ such that $\mathbb{R}_{\text{exp}} \equiv (R, e)$ (with the last R denoting its underlying ordered field). But this function e must be the exponential function $\exp_R$ by a uniqueness result for solutions of differential equations in o-minimal fields; see Otero, Peterzil and Pillay [15]. Thus $\mathbb{R}_{\text{exp}} \equiv R$. $\square$

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