Geometric Properties of Semilinear and Semibounded Sets*

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Abstract

We calculate the universal Euler characteristic and universal dimension function on semilinear and semibounded sets and obtain some criteria for definable equivalence of semilinear and semibounded sets in terms of these invariants.

1 Introduction

In [9], [3], [4] the notions of weak and strong Euler characteristic and of abstract dimension are defined. These are functions assigning values in a ring (the Grothendieck ring in the case of the universal Euler characteristic), a semiring respectively, to definable sets in a given first order structure.

There are general ways how to construct the Grothendieck ring and the dimension semiring of a first order structure M from $\widetilde{\mathrm{Def}}(M)$, the family of all M-definable sets up to definable equivalence (see [9], [3], [4]). We say that two M-definable sets are definably equivalent if there is an M-definable bijection between them. More generally: if N is an expansion of M then two M-definable sets are said to be N-definably equivalent if there is an N-definable bijetion between them. Definability means always definability with parameters.

If M is an o-minimal expansion of a real closed field, then the usual definitions of Euler characteristic and dimension on o-minimal structures (see [1]) give already the universal Euler characteristic and universal dimension in the sense of [9], [3], [4], and, as proved in [1], two M-definable sets are

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definably equivalent iff they have the same Euler characteristic and dimension (as defined in [1]).

We denote by $R = (R, <, +, \cdot)$ a real closed field (hence the R-definable sets are exactly the semialgebraic sets), by $S = (R, <, +, \{\lambda_r : r \in R\})$, where λ_r denotes left scalar multiplication by r, the reduct of R in which exactly the semilinear sets are definable, and by $B = (R, <, +, \{\lambda_r : r \in R\}, \mathcal{B})$, where $\mathcal{B}$ is a predicate for a bounded R-definable set which is not S-definable, the reduct of R in which the definable sets are exactly the semibounded sets. By [6], [8], B is the unique structure which lies properly between R and S. (As above, we do not make a notational distinction between a structure and its underlying set, as it should always be clear from the context which of the two options applies.)

In the case of S and B, the standard definitions of Euler characteristic and dimension on o-minimal structures do not give the universal variants. To understand $\widetilde{\mathrm{Def}}(S)$, we introduce the notion of a basic cell (every basic cell is a cell in the standard sense, but not vice versa), prove a basic cell decomposition theorem for S, and obtain a criterion for definable equivalence in terms of basic cells: two semilinear sets are definably equivalent iff there are partitions of these sets into the same numbers of "same" basic cells. Two basic cells are called the same if there is an affine bijection between them. From this we obtain the Grothendieck ring and the universal dimension semiring of S, and the fact that two semilinear sets are definably equivalent iff they have the same universal Euler characteristic and universal dimension. Moreover, if M is an o-minimal expansion of a real closed field then two bounded semilinear sets are M-definably equivalent iff there is a semilinear bijection between them. Due to the results in [7], [2], and the Structure Theorem in particular (it is proved in [7] for the reals, [2] proves a general version), semibounded sets can be partitioned into sets that play the same role as basic cells in the semilinear case. We use this fact and our results on semilinear sets to prove that there is no semibounded bijection between two semilinear sets that are not already semilinearly equivalent. This statement is related to Shiota's Hauptvermutung Theorem in [11]. As a corollary we obtain the Grothendieck ring and the universal dimension on B.

Schanuel [9] defines Euler characteristic and the dimension function in the abstract context of distributive categories. He mentions several examples, and the categories of bounded and unbounded semilinear sets in particular, and he states how the two functions should look in these examples. No proofs are given in [9].

2 Preliminaries

We briefly recall some basic facts and notions concerning o-minimal structures, as a general reference see [1].

A linearly ordered first order structure M is called *o-minimal* if the only definable sets in M^1 are finite unions of points and intervals. Let M be an o-minimal structure, X a definable subset of M^k, and let $f, g : M^k \to M$ be functions such that $f(x) < g(x)$ for all $x \in X$, and $f|X$ is definable and continuous or constantly equal to $-\infty$, and $g|X$ is definable and continuous or constantly equal to ∞. Then $(f, g)_X := \{(x, r) : x \in X, f(x) < r < g(x)\}$. For any definable function $f : M^k \to M^l$, $\Gamma(f|X)$ denotes the graph of f restricted to X.

Definition 2.1. *Let $(i_1, \ldots, i_k)$ be a sequence of 0 and 1 of length $k \geq 1$. An $(i_1, \ldots, i_k)$-cell is a definable subset of M^k obtained by induction as follows:*

1. A (0)-cell is a singleton, and a (1)-cell is an open interval with endpoints in $M \cup \{\pm\infty\}$.

2. Suppose $(i_1, \ldots, i_k)$-cells are already defined. Then an $(i_1, \ldots, i_k, 0)$-cell is a set of the form $\Gamma(f|B_0)$, where B_0 is an $(i_1, \ldots, i_k)$-cell and $f : M^k \to M$ is a function that is definable and continuous on B_0. An $(i_1, \ldots, i_k, 1)$-cell is a set of the form $(f, g)_{B_0}$ where B_0 is an $(i_1, \ldots, i_k)$-cell.

Theorem 2.2. *(Cell Decomposition Theorem) If $X \subseteq M^n$ is definable then there is a partition of X into finitely many M-definable cells. Moreover, if $f : X \to M$, is a definable function then there is a partition of X into finitely many M-definable cells such that the restriction of f to each of them is continuous.*

Definition 2.3. *1. An* affine function *on R^m is a function $f : R^m \to R$ of the form*

$$f(x_1, \ldots, x_m) = \lambda_1 x_1 + \cdots + \lambda_m x_m + \lambda_0,$$

for some fixed $\lambda_i \in R$, $i = 0, \ldots, m$.

2. A basic semilinear set *in R^m is a set defined by formulas of the form*

$$f_1(x) = \cdots = f_p(x) = 0,\ g_1(x) > 0, \ldots, g_q(x) > 0,$$

where $x = (x_1, \ldots, x_m) \in R^m$ and f_i, g_j, for $i = 1, \ldots, p$, $q = 1, \ldots, q$, are affine functions on R^m.

3. A semilinear set *in R^m is a finite union of basic semilinear sets in R^m.*

For the rest of the paper we let $R = (R, <, +, \cdot)$ be a real closed field and $S = (R, <, +, \{\lambda_r\}_{r \in R})$, where $\lambda_r : R \to R : x \mapsto r \cdot x$ denotes left scalar multiplication by r, a reduct of R. Recall that the S-definable sets are exactly the semilinear sets. We shall make use of the following lemma (see [1], p.27, 7.6):

Lemma 2.4. *Each definable function in S is piecewise affine. More precisely: given an S-definable function $f : X \to R$, $X \subseteq R^m$, there is a partition of X into basic semilinear sets X_i, $i = 1, \ldots, k$, such that $f|X_i$ is the restriction of an affine function on R^m to X_i.*

3 Euler Characteristic, Grothendieck Rings and Abstract Dimension

Unless specified otherwise, M stands for an arbitrary first order structure. We denote by $\mathrm{Def}^\infty(M)$ the family of all definable sets in all M^n, $n \in \mathbb{N}$. If $X, Y \in \mathrm{Def}^\infty(M)$, then $X \sim Y$ stands for X, Y being definably equivalent. The set of all M-definable sets up to definable equivalence is denoted by $\widetilde{\mathrm{Def}}(M)$, and, for $X \in \mathrm{Def}^\infty(M)$, $[X]$ is the equivalence class of X in $\widetilde{\mathrm{Def}}(M)$. We regard $\widetilde{\mathrm{Def}}(M)$ also as a structure in the language $(+, \cdot, 0, 1)$ by defining:

- $0 := [\emptyset]$;
- $1 := [\{a\}]$, where $a \in M$;
- $[X] + [Y] := [X' \dot\cup Y']$, where $X' \in [X]$, $Y' \in [Y]$;
- $[X] \cdot [Y] := [X \times Y]$;

for $[X], [Y] \in \widetilde{\mathrm{Def}}(M)$. Note that $\widetilde{\mathrm{Def}}(M)$ is a semiring (in [9] also called *rig* - a "ring without negatives"). Formally: $(\widetilde{\mathrm{Def}}(M), +, 0)$ and $(\widetilde{\mathrm{Def}}(M), \cdot, 1)$ are commutative monoids related by $a \cdot 0 = 0$ and by distributivity.

We recall some notions and facts from [9], [3] and [4].

Definition 3.1. *A* (weak) Euler characteristic on M *with values in a commutative ring with unity K is a function*

$$\chi : Def^\infty(M) \to K,$$

such that

$$\chi = \chi' \circ [\ ],$$

where $[\ ] : \mathrm{Def}^\infty(M) \to \widetilde{\mathrm{Def}}(M) : A \mapsto [A]$ *is the quotient map, and* $\chi' : \widetilde{\mathrm{Def}}(M) \to K$ *is a $(+, \cdot, 0, 1)$-homomorphism.*

A strong Euler characteristic *on* M *is a weak Euler characteristic on* M *satisfying the fiber condition: if* $f : X \to Y$ *is a definable function, and there is* $c \in K$ *such that for all* $y \in Y$, $\chi(f^{-1}\{y\}) = c$, *then* $\chi(X) = c \cdot \chi(Y)$.

The fact that the values of χ are in K is sometimes denoted by χ/K.

Definition 3.2. *The* Grothendieck ring $K_0(M)$ *of* M *is the ring obtained by the following construction. We define an equivalence relation* $\sim_e$ *on* $\widetilde{\mathrm{Def}}(M)$*:* $[A] \sim_e [B]$ *if there is* $[C] \in \widetilde{\mathrm{Def}}(M)$ *such that* $[A] + [C] = [B] + [C]$. *Let* $M' = \widetilde{\mathrm{Def}}(M)/\sim_e$ *be the quotient semiring, then* $(M', +, 0)$ *is a cancellative monoid. The Grothendieck ring* $K_0(M)$ *is the unique minimal ring that embeds* M'. *The Euler characteristic* $\chi_0/K_0(M)$ *is called the* universal Euler characteristic*.*

The following theorem from [3] follows immediately from the definition of $K_0(M)$. We recall that the onto-pigeonhole principle (ontoPHP) holds in a structure M if there is no definable bijection between a definable set and the same set without one element.

Theorem 3.3. $K_0(M)$ *is nontrivial (i.e.* $0 \neq 1$*) iff* M *satisfies* ontoPHP. *If* $\chi : \mathrm{Def}^\infty(M) \to K$ *is an Euler characteristic on* M*, then* χ *factors through* χ_0.

Definition 3.4. *An* abstract dimension function *on* M *is a semiring homomorphism* $\mathrm{d} : \widetilde{\mathrm{Def}}(M) \to D$, *where* D *is a semiring satisfying* $d(1+1) = d(1)$.

Remark. *We may regard the semiring* D *in the definition above as an upper semi-lattice* $\langle D, e, \leq, \vee, 0, \oplus \rangle$, *where* $+$ *on* $\widetilde{\mathrm{Def}}(M)$ *becomes* $\vee$, $\cdot$ *becomes* $\oplus$, 0 *becomes* e, *and* 1 *maps to* 0.

Definition 3.5. *The* universal (abstract) dimension *on* M *is the map*

$$\dim_{univ} \circ [\ ] : \mathrm{Def}^\infty(M) \to \mathrm{D}(M),$$

where $[\ ] : \mathrm{Def}^\infty(M) \to \widetilde{\mathrm{Def}}(M) : X \mapsto [X]$ *is the quotient map and* $\dim_{univ}$ *is the universal map from* $\widetilde{\mathrm{Def}}(M)$ *to the semiring* $\mathrm{D}(M)$, *the quotient of* $\widetilde{\mathrm{Def}}(M)$ *by the minimal congruence enforced by* $1 + 1 = 1$.

The semiring $\mathrm{D}(M)$ is constructed as follows. We define a congruence $\sim_d$ on $\widetilde{\mathrm{Def}}(M)$ by $[X] \sim_d [Y]$ if $[X] \leq [Y]$ and $[Y] \leq [X]$, where $[X] \leq [Y]$ if there is $[Z] \in \widetilde{\mathrm{Def}}(M)$ and $n \in \mathbb{N}$ such that

$$[X] + [Z] = \underbrace{[Y] + \cdots + [Y]}_{\text{n times}}.$$

Then $\mathrm{D}(M)$ is the quotient of $\widetilde{\mathrm{Def}}(M)$ by $\sim_d$.

Note that $\mathrm{D}(M)$ is always nontrivial in the sense that $[\{a\}] \not\leq [\emptyset]$, and that every abstract dimension function on M factors through $\dim_{univ}$.

Example 3.6. *1. If M is a finite structure, then* $\mathrm{D}(M)$ *has cardinality 2. The two dimensions are e (for the empty set) and 0 (for nonempty finite sets).*

2. Let $\mathbb{Z}$ be the ring of integers. Then $\mathbb{Z}$-definable sets have three possible dimensions corresponding to the empty set, finite sets and infinite sets.

In [1] Euler characteristic and dimension on definable sets in arbitrary o-minimal structures are defined, based on the standard geometric intuition. In particular, the Euler characteristic assumes always integers as values and dimensions are natural numbers. In the case of an o-minimal expansion of a real closed field, these definitions give already the universal Euler characteristic and the universal dimension in the sense of the abstract definitions from this section.

Definition 3.7. *Let M be an o-minimal structure. If C is a $(i_1, \ldots, i_k)$-cell then its dimension is the natural number*

$$\dim C := \sum_{j=1}^{k} i_j.$$

The dimension of a nonempty set $X \in \mathrm{Def}^{\infty}(M)$ is defined to be

$$\dim X := \max\{\dim C : X \textit{ contains a cell } C\}.$$

To the empty set we assign the dimension $-\infty$.

Definition 3.8. *Let M be an o-minimal structure. If C is a cell of dimension d then its Euler characteristic is*

$$\mathrm{E}(C) := (-1)^{\dim C}.$$

To each $X \in \mathrm{Def}^{\infty}(M)$ we assign the Euler characteristic

$$\mathrm{E}(X) := \sum_{C \in P} \mathrm{E}(C),$$

where P is a finite partition of X into cells.

In [1] the following fact is proved:

Theorem 3.9. *Let M be an o-minimal expansion of a real closed field, and let $X, Y \in \mathrm{Def}^\infty(M)$. Then $X \sim Y$ iff $\dim X = \dim Y$ and $\mathrm{E}(X) = \mathrm{E}(Y)$.*

Corollary 3.10. *If M is an o-minimal expansion of a real closed field, then* $\dim$ *from Definition 3.7 is the universal dimension on M, and* E *from Definition 3.8 is the universal Euler characteristic on M.*

Proof. If $C_1, C_2 \in \mathrm{Def}^\infty(M)$ are cells, such that $\dim C_1 = \dim C_2$, then $C_1 \leq C_2$ and $C_2 \leq C_1$ by Theorem 3.9. So let $X, Y \in \mathrm{Def}^\infty(M)$ with $\dim X = \dim Y = d$. We partition X, Y into cells, and let X', Y' respectively, be exactly the union of the set of cells of dimension d in the given partition of X, Y respectively, into cells. Then $X' \leq Y'$ and $Y' \leq X'$, and obviously $X - X' \leq Y'$ and $Y - Y' \leq X'$, so $X \leq Y$ and $Y \leq X$.

To see that E is universal, let $X, Y \in \mathrm{Def}^\infty(M)$ with $\mathrm{E}(X) = \mathrm{E}(Y)$. Take $C \in \mathrm{Def}^\infty(M)$, disjoint from X, Y, such that $\max\{\dim X, \dim Y\} \leq \dim C$. Then $\dim(X \cup C) = \dim(Y \cup C)$ and $\mathrm{E}(X \cup C) = \mathrm{E}(Y \cup C)$. From Theorem 3.9 it follows that $X \cup C \sim Y \cup C$.

□

4 Cell Decomposition for S

We introduce here a slightly more restricted notion of cell for S than in Definition 2.1, which we call *basic cell*, and we prove that every S-definable set can be partitioned into a finite number of basic cells. We classify basic cells not only with regard to their geometric dimension but also with respect to what we call the *number of infinite directions*. For instance, there is no definable bijection between a bounded and an unbounded interval in S, even though they are both (1)-cells by Definition 2.1. Every basic cell is a cell as defined in 2.1 but not conversely. Roughly speaking, we take as basic cells only cells that are bounded by a minimal number of affine functions. For example the bounded two-dimensional cells will be triangles but not squares. The reason for this is that even though there is a definable bijection between a triangle and a square in the semilinear case, this bijection is not affine.

We use a notation of the form $\langle a_0, \ldots, a_{d-e}; \vec{u}_1, \ldots, \vec{u}_e \rangle$ for basic cells. There are points as well as vectors involved in this notation and we make this distinction for two reasons: firstly, a point indicates a vertex of a basic cell, whereas a vector indicates an infinite direction. Secondly, applying an affine function $f(x_1, \ldots, x_m) = \lambda_1 x_1 + \cdots + \lambda_m x_m + \lambda_0$ to a point means something different than applying it to a vector:

$$
\begin{aligned}
f(a) &= \lambda_1(a)_1 + \cdots + \lambda_m(a)_m + \lambda_0 \\
f(\vec{u}) &= \lambda_1(\vec{u})_1 + \cdots + \lambda_m(\vec{u})_m,
\end{aligned}
$$

since $f(\vec{u}) = f(u_1) - f(u_0)$, where u_0, u_1 are points for which $\vec{u} = u_1 - u_0$. By $(a)_i$ we denote the i^{th} coordinate of the point a, and $(\vec{u})_i$ denotes $(u_1)_i - (u_0)_i$.

Definition 4.1. *A (d,e)-basic cell, where d, e are positive integers with $e \leq d$, is an S-definable set defined by induction as follows:*

1. A $(0,0)$-basic cell is a singleton $\{a_0\}$, $a_0 \in R$; we denote it by $\langle a_0; \rangle$. A $(1,0)$-basic cell is a bounded interval (a_0, a_1) with endpoints in R, denoted by $\langle a_0, a_1; \rangle$. A $(1,1)$-basic cell is any of the intervals $(-\infty, a_0)$, $(a_0, +\infty)$, where $a_0 \in R$, and it is denoted by $\langle a_0; \vec{u}_1 \rangle$, where $\vec{u}_1 = u_1 - a_0$, $u_1 \in R$ and $u_1 < a_0$, $a_0 < u_1$ respectively.

2. Let $B_0 = \langle a_0, \ldots, a_{d-e}; \vec{u}_1, \ldots, \vec{u}_e \rangle \subseteq R^n$ be a (d,e)-basic cell.

2.1 Then $B = \Gamma(f|B_0) \subseteq R^{n+1}$, where f is an affine function on R^n, is also a (d,e)-basic cell; B is denoted by

$$
\langle (a_0, f(a_0)), \ldots, (a_{d-e}, f(a_{d-e})); (\vec{u}_1, f(\vec{u}_1)), \ldots, (\vec{u}_e, f(\vec{u}_e)) \rangle,
$$

where $f(\vec{u}_i) = f(u) - f(a_0)$ for $\vec{u}_i = u - a_0$.

2.2 Let f, g be affine functions on R^n, and let

$$
\begin{aligned}
\Gamma(f|B_0) &= \langle b_0, \ldots, b_{d-e}; \vec{v}_1, \ldots, \vec{v}_e \rangle, \\
\Gamma(g|B_0) &= \langle c_0, \ldots, c_{d-e}; \vec{w}_1, \ldots, \vec{w}_e \rangle.
\end{aligned}
$$

2.2.1 Then $B = (f,g)_{B_0}$ is a $(d+1,e)$-basic cell if the set of points

$$
M = \{b_0, \ldots, b_{d-e}, b_0 + \vec{v}_1, \ldots, b_0 + \vec{v}_e, r\}
$$

is affine independent for exactly one

$$
r \in \{c_0, \ldots, c_{d-e}, b_0 + \vec{w}_1, \ldots, b_0 + \vec{w}_e\}
$$

and this r is of the form c_i for some $i = 0, \ldots, d-e$; we shall write $B = \langle b_0, \ldots, b_{d-e}, c_i; \vec{v}_1, \ldots, \vec{v}_e \rangle$.

2.2.2 $(f,g)_{B_0}$ is a $(d+1,e+1)$-basic cell if M is affine independent for exactly one $r \in \{c_0, \ldots, c_{d-e}, b_0 + \vec{w}_1, \ldots, b_0 + \vec{w}_e\}$ and this r is of the form $b_0 + \vec{w}_j$ for some $j = 1, \ldots, e$, or exactly one of the functions f, g is constantly $-\infty$, $+\infty$ respectively; B is denoted by $\langle b_0, \ldots, b_{d-e}; \vec{v}_1, \ldots, \vec{v}_e, \vec{s} \rangle$, where $\vec{s}$ is either $\vec{w}_j$ or the vector $(b_0, f(b_0) - 1) - (b_0, f(b_0))$, $(b_0, f(b_0) + 1) - (b_0, f(b_0))$ respectively.

We write shortly (d,e)-cell instead of (d,e)-basic cell. Here are some examples of basic cells in R^2: the triangle $\langle (0,0),(1,0),(1,1);\rangle$ is a $(2,0)$-basic cell (note however that not every triangle is a basic cell), the open first quadrant is a $(2,2)$-basic cell and the set $\{(x_1,x_2) : 2 < x_1 < 3 \ \& \ x_2 > 0\}$ is a $(2,1)$-basic cell.

If $B_0 = \langle a_0,\ldots,a_{d-e};\vec{u}_1,\ldots,\vec{u}_e\rangle$ is a (d,e)-cell, then the set of points $\{a_0,\ldots,a_{d-e},a_0+\vec{u}_1,\ldots,a_0+\vec{u}_e\}$ is affine independent, and $\dim B = d$. We call $a_0,\ldots,a_{d-e}$ the *vertices* of B, and e the *number of infinite directions* of B, shortened nid(B).

Lemma 4.2. *Let $B = \langle a_0,\ldots,a_{d-e};\vec{u}_1,\ldots,\vec{u}_e\rangle$ be a basic cell, $B \subseteq R^n$. Then*

$$B = \{\sum_{i=0}^{d-e} t_i a_i + \sum_{j=1}^{e} t'_j \vec{u}_j : \ \sum_{i=0}^{d-e} t_i = 1, \ t_i, t'_j > 0, \ t_i, t'_j \in R\}.$$

Proof. First note that if the lemma holds for a (d,e)-cell $B_0 \subseteq R^n$, then it also holds for a (d,e)-cell of the form $\Gamma(f|B_0) \subseteq R^{n+1}$: let

$$B_0 = \langle a_0,\ldots,a_{d-e};\vec{u}_1,\ldots,\vec{u}_e\rangle,$$

let $f(x) = \lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_0$ be an affine function on R^n, and suppose that $x = (x_1,\ldots,x_n) \in B_0$ iff $x = \sum_{i=0}^{d-e} t_i a_i + \sum_{j=1}^{e} t'_j \vec{u}_j$ for some t_i, $t'_j > 0$ with $\sum_{i=0}^{d-e} t_i = 1$. Then $x_{n+1} = \lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_0$ iff $x_{n+1} = \sum_{i=0}^{d-e} t_i f(a_i) + \sum_{j=1}^{e} t'_j f(\vec{u}_j)$. Hence $(x_1,\ldots,x_{n+1}) \in B$ iff $(x_1,\ldots,x_{n+1}) = \sum_{i=0}^{d-e} t_i (a_i, f(a_i)) + \sum_{j=1}^{e} t'_j (\vec{u}_j, f(\vec{u}_j))$, for some $t_i, t'_j > 0$ with $\sum_{i=0}^{d-e} t_i = 1$.

We proceed by induction on d, e:

1. For $(0,0)$- , $(1,0)$- , and $(1,1)$-cells the lemma clearly holds.

2.1. We assume that the lemma holds for (d,d)-cells, and want to derive it for $(d+1,d+1)$-cells. Let $B_0 = \langle a_0;\vec{u}_1,\ldots,\vec{u}_d\rangle$ be a (d,d)-cell, and let $B = (f,g)_{B_0}$ be a $(d+1,d+1)$-cell. The cases when f is $-\infty$ or g is $+\infty$, are easy to check and left to the reader. So let $\Gamma(f|B_0) = \langle b_0;\vec{v}_1,\ldots,\vec{v}_d\rangle$, $\Gamma(g|B_0) = \langle b_0;\vec{w}_1,\ldots,\vec{w}_d\rangle$, and let for exactly one $l \in \{1,\ldots,d\}$ the set

$$\{b_0, b_0+\vec{w}_1,\ldots,b_0+\vec{w}_d, b_0+\vec{v}_l\}$$

be affine independent. To see that

$$\begin{array}{ll} & \{t(b_0+\sum_{j=1}^{d} t'_j \vec{v}_j) + (1-t)(b_0+\sum_{j=1}^{d} t'_j \vec{w}_j) : \ t'_j > 0, 0 < t < 1\} \\ \subseteq & \{b_0 + \sum_{j=1}^{d} s_j \vec{w}_j + s\vec{v}_l : \ s_j, s > 0\} \end{array}$$

use

$$\begin{aligned} x &= t(b_0 + \textstyle\sum_{j=1}^{d} t'_j\vec{v}_j) + (1-t)(b_0 + \sum_{j=1}^{d} t'_j\vec{w}_j) \\ &= b_0 + \textstyle\sum_{j=1}^{l-1} t'_j\vec{w}_j + \sum_{j=l+1}^{d} t'_j\vec{w}_j + tt_l\vec{v}_l + (1-t)t'_l\vec{w}_l. \end{aligned}$$

We put $s_j := t'_j$ for $j \in \{1, \ldots, l-1, l+1, \ldots, d\}$, $s_l := (1-t)t'_l$ and $s := tt'_l$. Clearly, $s > 0$, $s_j > 0$ for all $j = 1, \ldots, d$.

For the other inclusion let $x = b_0 + \sum_{j=1}^{d} s_j\vec{w}_j + s\vec{v}_l$, for $s_j, s > 0$ and $j = 1, \ldots, d$. Put $t'_j := s_j$ for $j \in \{1, \ldots, l-1, l+1, \ldots, d\}$, $(1-t)t'_l := s_l$, and $tt'_l := s$. From $t = \frac{s}{s_l+s}$ it follows that $0 < t < 1$.

2.2. It is now obvious how to derive from the assumption that the lemma holds for (d, e)-cells, that it also holds for $(d+1, e)$-cells.

□

Remark. *As the convex hull of a set $\{a_0, \ldots, a_k\} \subseteq R^n$ is the set of all points $\sum_{i=0}^{k} t_i a_i$ with $\sum_{i=0}^{k} t_i = 1$, $t_i \geq 0$, it is now easy to see that a basic cell $\langle a_0, \ldots, a_{d-e}; \vec{u}_1, \ldots, \vec{u}_e\rangle$ is the union of the interiors of the convex hulls of the sets $\{a_0, \ldots, a_{d-e}, a_0 + t\vec{u}_1, \ldots, a_0 + t\vec{u}_e\}$ for $t \to \infty$, $t > 0$.*

In the following definition "cell" is used in the sense of Definition 2.1.

Definition 4.3. *(Decomposition of a Cell)*

1. Every partition of a cell $B \subseteq R$ into finitely many cells is a decomposition of B.

2. Let $m \geq 1$. A decomposition of a cell $B \subseteq R^{m+1}$ is a partition D of B into finitely many cells such that $\pi\{D\} = \{\pi(P); P \in D\}$ is a decomposition of $\pi(B) \subseteq R^m$, where $\pi : R^{m+1} \to R^m$ is the projection onto the first m coordinates.

Lemma 4.4. *For each S-definable cell B there is a decomposition of B into basic cells.*

Proof. Let B, B_0 be cells definable in S such that $B = (f, g)_{B_0}$, or $B = \Gamma(f|B_0)$. By Lemma 2.4 and by Theorem 2.2, we may assume that f, g are affine functions, or constantly $-\infty$, ∞ respectively. Also note that if D is a decomposition of B_0 into basic cells, then $D' = \{\Gamma(f|B_i); B_i \in D\}$ is a decomposition of $B = \Gamma(f|B_0)$ into basic cells. The proof of the lemma is by induction on the dimension of B.

I. For $\dim B = 0$ and $\dim B = 1$ the lemma holds trivially.

II. Suppose the lemma holds for cells of dimension less or equal to d. Let $B_0 = \langle a_0, \ldots, a_{d-e}; \vec{u}_1, \ldots, \vec{u}_e \rangle \subseteq R^n$ and let $B = (f, g)_{B_0}$. If f is constantly $-\infty$ and g is an affine function, or if g is constantly $+\infty$ and f is an affine function, then B itself is a basic cell. If f is $-\infty$ and g is $+\infty$, then we can decompose B into the basic cells $(f, o)_{B_0}$, B_0, $(o, g)_{B_0}$, where $o : R^n \to R$ denotes the function constantly 0.

So let $B = (f, g)_{B_0}$ with

$$\begin{array}{rcl} \Gamma(f|B_0) & = & \langle b_0, \ldots, b_{d-e}; \vec{v}_1, \ldots, \vec{v}_e \rangle, \\ \Gamma(g|B_0) & = & \langle c_0, \ldots, c_{d-e}; \vec{w}_1, \ldots, \vec{w}_e \rangle. \end{array}$$

We may assume that $b_p \neq c_p$ iff $p \in \{0, \ldots, r\}$, where $-1 \leq r \leq (d - e)$, and that $\{\vec{v}_1, \ldots, \vec{v}_e, \vec{w}_q\}$ is linearly independent iff $q \in \{1, \ldots, s\}$, where $0 \leq s \leq e$.

Claim Let

$$\begin{array}{rcl} \mathrm{S}_1 & = & \{\langle b_0, \ldots, b_p, c_p, \ldots c_{d-e}; \vec{w}_1, \ldots, \vec{w}_e \rangle : p \in \{0, \ldots, r\} \neq \emptyset\}, \\ \mathrm{S}_2 & = & \{\langle b_0, \ldots, b_{d-e}; \vec{v}_1, \ldots, \vec{v}_q, \vec{w}_q, \ldots, \vec{w}_e \rangle : q = \{1, \ldots, s\} \neq \emptyset\}, \end{array}$$

and F_1 consists of all basic cells of the form

$$\langle b_0, \ldots, b_p, c_{p+1}, \ldots, c_{d-e}; \vec{w}_1, \ldots, \vec{w}_e \rangle,$$

for $p, p+1 \in \{0, \ldots, r\}$, or $p = r$ and $s \geq 1$, and F_2 consists of all basic cells of the form

$$\langle b_0, \ldots, b_{d-e}; \vec{v}_1, \ldots, \vec{v}_q, \vec{w}_{q+1}, \ldots, \vec{w}_e \rangle,$$

such that $q, q+1 \in \{1, \ldots, s\}$, or $q + 1 = 1$ and $r \geq 0$. Then $D = \mathrm{S}_1 \cup \mathrm{S}_2 \cup \mathrm{F}_1 \cup \mathrm{F}_2$ is a decomposition of B into basic cells.

Clearly, if $B \in D$ then B is a basic cell and $\pi(B) = B_0$. The proof of the claim is now a corollary of the remark made directly before Definition 4.3 and the following lemma which is proved in [1], p.122 (it is of course also possible to generalize the proof of Lemma 4.5 in a completely straightforward way to our setting). Below we assume that $(a_0, \ldots, a_d)$ is a d-simplex in R^n, $r_i, s_i \in R$, $r_i \leq s_i$ for $i = 0, \ldots, d$, and $r_j < s_j$ for some j, and that $b_i := (a_i, r_i)$, $c_i := (a_i, s_i) \in R^{n+1}$.

Lemma 4.5. *Let L consist of all $(d+1)$-simplexes $(b_0, \ldots, b_i, c_i, \ldots, c_d)$ with $b_i \neq c_i$, and all faces of these $(d+1)$-simplexes. Then L is a closed complex and*

$$\begin{array}{rcl} |L| & = & \{t(t_0 b_0 + \cdots + t_d b_d) + (1-t)(t_0 c_0 + \ldots t_d c_d) : \\ & & 0 \leq t \leq 1,\ t_i \geq 0,\ \sum t_i = 1\} \\ & = & \textit{convex hull of } \{b_0, \ldots, b_d, c_0, \ldots c_d\}. \end{array}$$

□

The next corollary follows immediately by Theorem 2.2 and Lemma 4.4.

Corollary 4.6. *Every semilinear set can be decomposed into a finite number of basic cells.*

An affine bijection is a bijection $f : R^m \to R^n$ such that $f = (f_1, \dots, f_n)$, with $f_i : R^m \to R$ an affine function for every $i = 1, \dots, n$. We show that affine bijections map (d, e)-basic cells onto sets that are "almost" (d, e)-basic cells: they preserve all their characteristic features except their positions in the ambient space. Images of basic cells under affine bijections are called (d, e)-sets below.

Definition 4.7. *A set $X \subseteq R^n$ is a (d, e)-*set *if*

$$X = \{\sum_{i=0}^{d-e} t_i a_i + \sum_{j=1}^{e} t'_j \vec{u}_j \ : \ t_i, t'_j \in R, \ \sum_{i=0}^{d-e} t_i = 1, \ t_i, t'_j > 0\},$$

where $a_i \in M^n$, for $i = 0, \dots, d-e$, $\vec{u}_j$ is a vector in M^n, for $j = 1, \dots, e$, and the set of points $\{a_0, \dots, a_{d-e}, a_0 + \vec{u}_1, \dots, a_0 + \vec{u}_e\}$ is affine independent.

Lemma 4.8. *If X is a (d, e)-set and f is an affine bijection on X, then $f(X)$ is a (d, e)-set.*

Proof. Let $X = \{\sum_{i=0}^{d-e} t_i a_i + \sum_{j=1}^{e} t'_j \vec{u}_j : \ \sum_{i=0}^{d-e} t_i = 1, \ t_i, t'_j > 0\} \subseteq R^m$ be a (d, e)-set, and let $f : X \to f(X) \subseteq R^n$ be an affine bijection given by

$$f(x) = Ax + \lambda,$$

where A is an $n \times m$-matrix, and λ is an $n \times 1$ matrix, both with entries in R. We put $b_i := Aa_i + \lambda$ for $i = 0, \dots, d-e$, and $\vec{v}_j := A\vec{u}_j$ for $j = 1, \dots, e$. It is easily checked that for $x = \sum_{i=0}^{d-e} t_i a_i + \sum_{j=1}^{e} t'_j \vec{u}_j \in X$,

$$f(x) = \sum_{i=0}^{d-e} t_i b_i + \sum_{j=1}^{e} t'_j \vec{v}_j.$$

Since f is a definable bijection, $\dim X = \dim f(X)$. Hence the set

$$\{b_0, \dots, b_{d-e}, b_0 + \vec{v}_1, \dots, b_0 + \vec{v}_e\}$$

is affine independent. So $f(X)$ is a (d, e)-set.

□

Definition 4.9. *Let M be a (d_1, e_1)-set and N a (d_2, e_2)-set. We say that M, N are* equal *if $d_1 = d_2$ & $e_1 = e_2$.*

Corollary 4.10. *Let $X, Y \in \mathrm{Def}^\infty(S)$. Then $X \sim Y$ iff there are partitions of X, Y into the same numbers of equal (d, e)-sets.*

Proof. It is clear from the proof of Lemma 4.8 how to define a bijection between two equal (d, e)-sets.

So let f be an S-definable bijection between X and Y. By Lemma 2.4, f is a finite union of affine bijections defined on basic semilinear sets $X_1, \dots X_g$, which form a partition of X. We take a decomposition D of X into basic cells such that D is a refinement of $\{X_1, \dots X_g\}$ and apply Lemma 4.8.

□

5 The Grothendieck Ring of S

It follows by Corollary 4.10 that S-definable sets are finite unions of points and sets generated by bounded and unbounded intervals $(a, b) = i$ and $(a, \infty) = p$, where $a, b \in R$. Note that for any Euler characteristic χ on S with values in a commutative ring with unity K, $\chi(\{a\}) = 1$ and $\chi((a, b)) = -1$, since $(a, b) \cup \{b\} \cup (b, c) \sim (a, c)$. Let us denote $\chi((a, \infty)) \in K$ by x. We may then think of the χ-values of (d, e)-sets as of monomials, and of definable sets as polynomials from $\mathbb{Z}[x]$. The question is, under what conditions can we add a set to two sets not definably equivalent in order to obtain definably equivalent ones? In other words, when are two non-equivalent sets forced to have the same Euler characteristic? That is, which polynomials in $\mathbb{Z}[x]$ are forced to equal to 0? Surely $x^2 = x^2 + x + x^2$, because we can partition $p \times p$ into (sets definably equivalent to) $p \times p,\ p,\ p \times p$. This partition shows that $x(x + 1) = 0$ has to be true in $K_0(S)$. It follows that in order to obtain the Grothendieck ring $K_0(S)$, we have to factorize $\mathbb{Z}[x]$ at least by the ideal generated by the polynomial $x(x + 1)$. In this section we prove that $\mathbb{Z}[x]/(x(x + 1))$ is already $K_0(S)$.

Remark. *By the Chinese Remainder Theorem,*

$$\mathbb{Z}[x]/(x(x + 1)) \cong \mathbb{Z}[x]/(x) \oplus \mathbb{Z}[x]/(x + 1) \cong \mathbb{Z}^2,$$

and $\mathbb{Z}[x]/(x(x + 1)) \cong \mathbb{Z}^2$ via $\iota : [f(x)] \mapsto (f(0), f(-1))$. So the Euler characteristic of a (d, e)-set X as calculated in $\mathbb{Z}^2$ is $(-1)^{d-e} = ((-1)^d, (-1)^d)$ if $e = 0$, and $(-1)^{d-e}x^e = (0, (-1)^d)$ if $e > 0$.

Definition 5.1. *Let $X \in \mathrm{Def}^\infty(S)$, and let P be a finite partition of X into (d, e)-sets. We define a function $\chi_P : \mathrm{Def}^\infty(S) \to \mathbb{Z}^2$ as follows.*

1. *If X is a (d,e)-set, then*

$$\begin{array}{rcll} \chi_P(X) & = & ((-1)^d, (-1)^d) & \text{if} \quad e = 0, \text{ and} \\ \chi_P(X) & = & (0, (-1)^d) & \text{if} \quad e > 0. \end{array}$$

2. *If $X \in \mathrm{Def}^\infty(S)$, $X \neq \emptyset$, then*

$$\chi_P(X) = \sum_{M \in P} \chi_P(M).$$

For $X = \emptyset$ we put $\chi_P(X) = (0,0)$.

After we show that χ_P is independet of the given partition P, we shall omit the index P.

Lemma 5.2. *If $X \in \mathrm{Def}^\infty(S)$ and P_1, P_2 are finite partitions of X into (d,e)-sets, then $\chi_{P_1}(X) = \chi_{P_2}(X)$.*

Proof. First, let X be a (d,e)-set. Note that every partition of X into finitely many (d,e)-sets is obtained by applying the operations from 1. and 2. finitely many times.

1. "Cutting" X by a $(d-1,0)$-set and obtaining a partition P of X into a (d,0)-, a $(d-1,0)$-, and a (d,e)-set.

2. "Cutting" X by a $(d-1,m)$-set and obtaining a partition P of X into a (d,l)-, a $(d-1,m)$-, and a (d,e)-set, where $e,l,m > 0$.

In both cases it is easily checked that $\chi(X) = \chi_P(X)$.

So let $X \in \mathrm{Def}^\infty(S)$, and let P_1, P_2 be finite partitions of X into (d,e)-sets. By Lemma 4.4 there is a decomposition D of X into basic cells, and we may assume that D partitiones every (d,e)-set in $P_1 \cup P_2$. Then

$$\chi_{P_1}(X) = \sum_{N \in P_1} \sum_{B \in D|N} \chi(B) = \chi_D(X) = \sum_{N \in P_2} \sum_{B \in D|N} \chi(B) = \chi_{P_2}(X).$$

□

Theorem 5.3. *$\chi/\mathbb{Z}^2$ is the universal Euler characteristic on S.*

Proof. By Corollary 4.10, if $X, Y \in \mathrm{Def}^\infty(S)$, $X \sim Y$, then $\chi(X) = \chi(Y)$. It is immediate from the definition that $\chi(\{a\}) = 1_Z = (0,1)$, and $\chi(X \dot\cup Y) = \chi(X) + \chi(Y)$, for $X, Y \in \mathrm{Def}^\infty(S)$. So it is left to see that if $X, Y \in \mathrm{Def}^\infty(S)$, then $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$. It is enough to check this for (d,e)-sets and left to the reader. Hence χ is an Euler characteristic on S.

That χ is universal follows now immediately from the considerations made in the beginning of this section. □

Corollary 5.4. *S satisfies ontoPHP.*

Note that Corollary 5.4 follows already from the fact that the standard o-minimal Euler characteristic is nontrivial.

Lemma 5.5. *$\chi/\mathbb{Z}^2$ is a strong Euler characteristic on S.*

Proof. Suppose $X \subseteq R^m$, $Y \subseteq R^n$ and $f : X \to Y$ is an S-definable function, such that for every $y \in Y$, $\chi(f^{-1}\{y\}) = c$ for a fixed $c \in \mathbb{Z}^2$. After possibly permuting the coordinates, we may assume that $\mathrm{rng}(f) \subseteq \pi^n(\Gamma(f))$, where $\pi^n : R^{n+m} \to R^n$ is the projection onto the first n coordinates. Let D be a decomposition of $\Gamma(f)$ into basic cells, and let $B \in D$, $B_0 = \pi^n(B)$, so $B_0 \subseteq \mathrm{rng}(f)$. We let $\dim(B_0) = d_1$, and we fix a $y_0 \in B_0$. Then $\dim(B) = d_1 + d_2$, where $d_2 = \dim\{x : (y_0, x) \in B\}$.

1. If $\mathrm{nid}(B_0) > 0$, then also $\mathrm{nid}(B) > 0$, and

$$\chi(B) = (0, (-1)^{d_1+d_2}) = \chi(B_0) \cdot \chi(\{x : (y_0, x) \in B\}).$$

2. If $\mathrm{nid}(B_0) = 0$ and also $\mathrm{nid}(B) = 0$, then

$$\chi(B) = ((-1)^{d_1+d_2}, (-1)^{d_1+d_2}) = \chi(B_0) \cdot \chi(\{x : (y_0, x) \in B\}).$$

If $\mathrm{nid}(B_0) = 0$ and $\mathrm{nid}(B) > 0$, then

$$\chi(B) = (0, (-1)^{d_1+d_2}) = \chi(B_0) \cdot \chi(\{x : (y_0, x) \in B\}).$$

In the sums below we assume that $D = \{B_{ij} : i = 1, \dots k \text{ and } j = 1, \dots, e_i\}$, $B_l = \pi^n(B_{ij})$ iff $l = i$ and $j \in \{1, \dots, e_i\}$, and $y_i \in B_i$ for $i = 1, \dots, k$.

$$\begin{aligned}
\chi(\Gamma(f)) &= \textstyle\sum_{i=1}^k \sum_{j=1}^{e_i} \chi(B_{ij}) = \\
&= \textstyle\sum_{i=1}^k \sum_{j=1}^{e_i} \chi(B_i) \cdot \chi(\{x : (y_i, x) \in B_{ij})\} = \\
&= \textstyle\sum_{i=1}^k \chi(B_i) \cdot c = \chi(\mathrm{rng}(f)) \cdot c.
\end{aligned}$$

Now $\chi(\Gamma(f)) = \chi(\mathrm{dom}(f))$, because f is piecewise affine. Hence $\chi(X) = \chi(Y) \cdot c$.

□

6 Abstract Dimension on S

Recall that by Lemma 4.4 and Corollary 4.10, we may identify the elements of $\mathrm{Def}^\infty(S)$ with polynomials from $\mathbb{N}[x, y]$. In particular, $\sum_{i,j} a_{ij} x^i y^j \in \mathbb{N}[x, y]$ corresponds to $X \in \mathrm{Def}^\infty(S)$ such that there is a partition P of X into (d, e)-sets with the number of (i, j)-sets in P being a_{ij}.

Definition 6.1. *We define an ordering* $\preceq$ *on* $\mathbb{N}[x,y]$ *as follows.*

1. *Let* $a_{de}x^dy^e, a_{kl}x^ky^l \in \mathbb{N}[x,y]$, *with* $a_{de}, a_{kl} \neq 0$. *Then*

$$a_{de}x^dy^e \preceq a_{kl}x^ky^l \text{ if } d \leq k \,\&\, e \leq l.$$

2. *Let* $f, g \in \mathbb{N}[x,y]$, *then*

$$f \preceq g \text{ if for every } m \in \max f \text{ there is } m' \in \max g \text{ with } m \preceq m',$$

where $\max f$ *denotes the set of* $\preceq$*-maximal monomials of* f *with omitted coefficients.*

3. *If* o *stands for the zero polynomial, then* $o \preceq f$, *for every* $f \in \mathbb{N}[x,y]$.

Observe that for $X \in \mathrm{Def}^\infty(S)$, and $f_1, f_2 \in \mathbb{N}[x,y]$ two representations of X corresponding to finite partitions P_1, P_2 of X into (d,e)-sets, $\max f = \max g$ (see e.g. the proof of Lemma 5.2).

Definition 6.2. *We let* D *be the semiring*

$$\langle D, 0_D, 1_D, +_D, \cdot_D \rangle,$$

where $D \subseteq \mathbb{N}[x,y]$ *such that* $f \in D$ *iff all coefficients of* f *are 1 and for every two distinct monomials* x^dy^e, x^ky^l *of* f, $\neg x^dy^e \preceq x^ky^l$ *and* $\neg x^ky^l \preceq x^dy^e$.

The operations $+_D$, $\cdot_D$ *are defined for* $f, g \in D$ *by*

$$\begin{aligned} f +_D g &= \sum \max(f+g), \\ f \cdot_D g &= \sum \max(f \cdot g), \end{aligned}$$

where the symbol $\sum \max f$ *means that we sum up the* $\preceq$*-maximal elements of* f. *We put* $0_D := o$, *the zero-polynomial, and* $1_D := 1$.

Definition 6.3. *Let* $[X] \in \widetilde{\mathrm{Def}}(S)$, *and let* $f \in \mathbb{N}[x,y]$ *correspond to* X. *We define a semiring homomorphism* $\mathrm{DIM} : \widetilde{\mathrm{Def}}(S) \to D$ *by*

$$\mathrm{DIM}([X]) := \sum \max f.$$

Note that by Corollary 4.10 and by the remark directly preceding Definition 6.2, DIM is well-defined. Clearly, DIM is indeed a semiring homomorphism.

Lemma 6.4. $\mathrm{DIM} \circ [\ ]$ *is the universal dimension on* S.

Proof. It is immediate from the definition of DIM that for $[X] \in \widetilde{\mathrm{Def}}(S)$, $\mathrm{DIM}([X]) = \mathrm{DIM}([X]) + \mathrm{DIM}([X])$. That $\mathrm{DIM} \circ [\ ]$ is universal follows from the fact that a (d, e)-set can be S-definably embedded into a (k, l)-set iff $k \geq d$ and $l \geq e$.

□

Theorem 6.5. *Let* $X, Y \in \mathrm{Def}^{\infty}(S)$. *Then*

$$X \sim Y \textit{ iff } \chi(X) = \chi(Y) \;\&\; \mathrm{DIM}([X]) = \mathrm{DIM}([Y]).$$

Proof. The left to right direction follows by Corrolary 4.10.

For the other implication, let $X, Y \in \mathrm{Def}^{\infty}(S)$ be such that $\chi(X) = \chi(Y)$ and $\mathrm{DIM}([X]) = \mathrm{DIM}([Y])$. Note that every (d, e)-set X with $d, e > 0$ can be partitioned into a (d, e)-set, a $(d-1, e-1)$-set and a $(d, e-1)$-set, that if $d = e > 1$ then X can be partitioned into two (d, d)-sets and a $(d-1, d-1)$-set, and that if $0 < e < d$, then X can be partitioned into two (d, e)-sets and a $(d-1, e)$-set. [1]

Let P_1, P_2 respectively, be finite partitions of X, Y respectively, into (d, e)-sets. We may write informally

$$\begin{aligned} X \;=\;& a_{00} + \\ & a_{10}x + a_{11}xy + \\ & a_{20}x^2 + a_{21}x^2y + a_{22}x^2y^2 + \\ & \cdots \end{aligned}$$

and

$$\begin{aligned} Y \;=\;& b_{00} + \\ & b_{10}x + b_{11}xy + \\ & b_{20}x^2 + b_{21}x^2y + b_{22}x^2y^2 + \\ & \cdots, \end{aligned}$$

where a_{de} denotes the number of (d, e)-sets in P_1 and b_{de} denotes the number of (d, e)-sets in P_2. To prove the theorem, it is enough to construct partitions P_1', P_2' of X, Y into (d, e)-sets such that $a_{de} = b_{de}$ for all d, e.

So let (k, l) be such that there is a $\preceq$-maximal (k, l)-set in P_1, P_2. Note that using the partitions given in the beginning of the proof, we may assume

[1] More concretely: we let $B = \langle a_0, \ldots, a_{d-e}; \vec{u}_1, \ldots, \vec{u}_e \rangle$. If $d, e > 0$, then $B = \langle a_0, a_1, \ldots, a_{d-e}, a_0 + \vec{u}_1; \vec{u}_2, \ldots, \vec{u}_e \rangle \cup \langle a_1, \ldots, a_{d-e}, a_0 + \vec{u}_1; \vec{u}_2, \ldots, \vec{u}_e \rangle$ $\cup \langle a_1, \ldots, a_{d-e}, a_0 + \vec{u}_1; \vec{u}_1, \ldots, \vec{u}_e \rangle$, if $d = e > 1$, then $B = \langle a_0; \vec{u}_1 + \vec{u}_2, \vec{u}_2, \ldots, \vec{u}_d \rangle \cup \langle a_0; \vec{u}_1, \vec{u}_1 + \vec{u}_2, \vec{u}_3, \ldots, \vec{u}_d \rangle \cup \langle a_0; \vec{u}_1 + \vec{u}_2, \vec{u}_3, \ldots, \vec{u}_d \rangle$, and finally if $0 < e < d$, then $B = \langle \frac{a_0 + a_1}{2}, a_1, \ldots, a_{d-e}; \vec{u}_1, \ldots, \vec{u}_e \rangle \cup \langle a_0, \frac{a_0 + a_1}{2}, a_3, \ldots, a_{d-e}; \vec{u}_1, \ldots, \vec{u}_e \rangle \cup \langle \frac{a_0 + a_1}{2}, a_3, \ldots, a_{d-e}; \vec{u}_1, \ldots, \vec{u}_e \rangle$.

that $a_{de}, b_{de} \neq 0$ for all $(d,e) \preceq (k,l)$. If $a_{kl} > b_{kl}$ (the case $a_{kl} < b_{kl}$ is treated similarly), then clearly $(k,l) \neq (0,0)$ and $(k,l) \neq (1,1)$. If $k = l$ then replace b_{kk} by $(b_{kk}+1)$, and $b_{(k-1)(k-1)}$ by $(b_{(k-1)(k-1)}+1)$ (this corresponds to a partition of a (k,k)-set into two (k,k)-sets and a $(k-1,k-1)$-set). If $0 < l < k$ then replace b_{kl} by $(b_{kl}+1)$ and $b_{(k-1)l}$ by $(b_{(k-1)l}+1)$ (which corresponds to a partition of a (k,l)-set into two (k,l)-sets and a $(k-1,l)$-set). Repeat this procedure as long as $a_{kl} > b_{kl}$, once $a_{kl} = b_{kl}$, compare $a_{k(l-1)}, b_{k(l-1)}$ and do the same, once $a_{k0} = b_{k0}$, compare $a_{(k-1)l}, b_{(k-1)l}$ and proceed as above.

Since $\chi(X), \chi(Y)$ remain the same independently of the partitions of X and Y into (d,e)-sets, eventually we obtain $a_{00} = b_{00}$. After applying the above procedure to all $\preceq$-maximal (d,e)-sets in P_1, P_2, we obtain the desired partitions P_1', P_2'.

□

Corollary 6.6. *Let M be an o-minimal expansion of a real closed field and let X, Y be two bounded semilinear sets. If X, Y are definably equivalent in M, then X, Y are already S-definably equivalent.*

Proof. Two bounded semilinear sets have the same universal Euler characteristic iff they have the same o-minimal Euler characteristic, and they have the same universal abstract dimension iff they have the same o-minimal dimension.

□

7 Semibounded Sets

As shown in [6] (for the reals), [8] (generally) there is exactly one o-minimal structure properly between S and R, namely

$$B = (R, +, <, \{\lambda_r : r \in R\}, \mathcal{B}),$$

where $\mathcal{B}$ is a bounded semialgebraic set that is not semilinear (standardly, $\mathcal{B}$ is taken to be the graph of multiplication in R restricted to $[0,1]^2$). Note that then all bounded R-definable sets are B-definable. We call the B-definable sets *semibounded*.

From our results on semilinear sets and from the Structure Theorem from [2], [7] (in [7] the author is working over the reals, in [2] a general version of the Structure Theorem is proved) we shall derive that $K_0(B) = K_0(S)$. We use here a more restricted definition of semibounded set than in [2], so we begin by stating the definitions and results from [2], [7] explicitly in the form we need them.

Definition 7.1. *1. Let $X \subseteq R^n$ be a B-definable set. We call X* almost linear *if there are nonzero vectors $\vec{v}_1, \ldots, \vec{v}_k$ in R^n, $k \geq 0$, and a bounded B-definable cell $C \subseteq R^n$ such that*

$$X = \{c + \sum_{j=1}^{k} t_j \vec{v}_j \ : \ c \in C, t_j > 0\}.$$

An almost linear set X is in normal form *if for each $x \in X$ there are unique $t_1, \ldots, t_k > 0$ and a unique $c \in C$ such that $x = c + \sum_{j=1}^{k} t_j \vec{v}_j$ (in particular the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent).*

2. Let $X \subseteq R^n$ be almost linear in normal form and let $f : X \to R$ be a continuous map. We say that f is almost linear with respect to X *if there is a map $\hat{f} : \hat{X} \to R$, where $\hat{X} = \{c + \sum_{j=1}^{k} t_j \vec{v}_j \ : \ c \in C, t_j \geq 0\}$, such that $\hat{f}$ extends f, $\hat{f}|C$ is a bounded function and, for some scalars $m_1, \ldots, m_k$,*

$$\hat{f}(c + \sum_{j=1}^{k} t_j \vec{v}_j) = \hat{f}(c) + \sum_{j=1}^{k} t_j m_j,$$

for all $c \in C$ and all $t_j \geq 0$, $j = 1, \ldots, k$.

Note that the function $\hat{f}$ in the definition above is unique, that if X is almost linear in normal form and f is linear with respect to X, then $\Gamma(f|X)$ is almost linear in normal form, and that $\dim X = \dim C + k$ (where dim is the standard o-minimal dimension).

Theorem 7.2. *(Structure Theorem) Let $X \subseteq R^n$ be B-definable. Then*

1. X can be partitioned into finitely many almost linear sets in normal form.

2. if X is the graph of a B-definable function $f : Y \to R$ for some $Y \subseteq R^{n-1}$, then Y can be partitioned into finitely many almost linear sets with respect to f.

Note that it is implicit in the Structure Theorem that there is no B-definable bijection between a bounded and an unbounded interval.

Lemma 7.3. *Every semibounded set is B-equivalent to a semilinear set.*

Proof. Due to the Structure Theorem it suffices to show that every almost linear set in normal form is B-equivalent to a semilinear set. We claim that the almost linear set in normal form

$$X = \{c + \sum_{j=1}^{e} t'_j \vec{v}_j \ : \ c \in C \ , \ t'_j > 0\}$$

is B-equivalvalent to a (d,e)-set of the form

$$Y=\{\sum_{i=0}^{d-e} t_i a_i + \sum_{j=1}^{e} t'_j \vec{u}_j \;:\; \sum_{i=0}^{d-e} t_i = 1\,,\; t_i, t'_j > 0\},$$

where $d = \dim C + e$. Since all the bounded semialgebraic sets are B-definable, all the bounded semibounded cells of same dimension are B-definably equivalent by Theorem 3.9. Let f be a semibounded bijection between $\{\sum_{i=0}^{d-e} t_i a_i \,:\; \sum_{i=0}^{d-e} t_i = 1,\; t_i > 0\}$ and C. We define a bijection $f' : Y \to X$ by

$$f'(\sum_{i=0}^{d-e} t_i a_i + \sum_{j=1}^{e} t'_j \vec{u}_j) = f(\sum_{i=0}^{d-e} t_i a_i) + \sum_{j=1}^{e} t'_j f'(\vec{u}_j),$$

where $f'(\vec{u}_j) = \vec{v}_j$, for all $j = 1, \ldots, e$.

□

Definition 7.4. *Let* $X = \{c + \sum_{j=1}^{e} t_j \vec{v}_j \;:\; c \in C_1\,,\; t_j > 0\}$ *and let* $Y = \{c + \sum_{j=1}^{l} t_j \vec{w}_j \;:\; c \in C_2\,,\; t_j > 0\}$ *be almost linear sets in normal form. We say that* X, Y *are* equal *if* $\dim C_1 = \dim C_2$ *and* $e = l$.

Lemma 7.5. *Let* $X, Y \in \mathrm{Def}^{\infty}(B)$. *Then* $X \sim_B Y$ *iff there are partitions of* X, Y *into the same numbers of equal almost linear sets in normal form.*

Proof. The right to left implication follows by the proof of Lemma 7.3 and by Corollary 4.10.

For the other implication, let f be a B-definable bijection between X and Y. We may assume that $X, Y \subseteq R^n$ and that f is given by the tuple of functions $(f_1, \ldots, f_n)$, where $f_i : R^n \to R$, $i = 1, \ldots, n$. By the Structure Theorem, there is a finite partition P of X into almost linear sets in normal form, such that $f_i|M$ is almost linear with respect to M, for each $M \in P$ and for every $i = 1, \ldots, n$. Let $M = \{c + \sum_{j=1}^{e} t_j \vec{u}_j : c \in C, t_j > 0\}$. Then for every $x \in M$,

$$\begin{aligned} f_1(x) &= \hat{f}_1(c) + \textstyle\sum_{j=1}^{e} t_j m_{1j} \\ &\vdots \\ f_n(x) &= \hat{f}_n(c) + \textstyle\sum_{j=1}^{e} t_j m_{nj}, \end{aligned}$$

where m_{ij} are scalars, $i = 1, \ldots, n$. Clearly, $\hat{f}(C)$ is a cell of dimension $\dim C$, and f(X) is an almost linear set in normal form.

□

By Lemma 7.5, $K_0(B)$ is a quotient of $\mathbb{Z}^2$. We shall show that there are no semilinear sets, that are semiboundedly but not semilinearly equivalent, hence that $K_0(B) = \mathbb{Z}^2$.

Theorem 7.6. *If $X, Y \in \mathrm{Def}^\infty(S)$ and $X \sim_B Y$, then $X \sim_S Y$.*

Proof. By Theorem 6.5, it is enough to show that $\chi(X) = \chi(Y)$ (as defined in 5.1) and $\mathrm{DIM}(X) = \mathrm{DIM}(Y)$. Since semibounded bijections preserve the standard o-minimal dimension and the number of infinite directions, there has to be $\mathrm{DIM}(X) = \mathrm{DIM}(Y)$.

Assume $\chi(X) \neq \chi(Y)$ for χ as defined in 5.1. The polynomials $\chi(X)$, $\chi(Y)$ are equal in $K_0(B)$. Take any partitions P_1, P_2 of X, Y into (d,e)-sets, we have $\chi_{P_1}(X) \neq \chi_{P_2}(Y)$ in $K_0(S)$. There have to be refinements P_1', P_2' of P_1, P_2 that partition X, Y into almost linear sets in normal form such that P_1', P_2' contain the same numbers of equal almost linear sets in normal form. So $\chi_{P_1'}(X) = \chi_{P_2'}(Y)$ also in $K_0(S)$ (when calculating we regard a (d,e)-almost linear set in normal form as a (d,e)-set) but this is a contradiction with the fact that any such P_1', P_2' preserve the χ-values.

$\square$

Corollary 6.6 and Theorem 7.6 are related to the Hauptvermutung Theorem of Shiota [11]. He proves that two semilinear sets are semilinearly equivalent assuming that they are equivalent in an $\mathfrak{X}$-system (which is a geometric category that corresponds - when restricted to bounded sets - to o-minimal expansions of the real field), but also assuming that they are compact. For the relation between o-minimal structures and $\mathfrak{X}$-systems see [10].

Corollary 7.7. $K_0(B) = K_0(S)$.

Corollary 7.8. *The dimension semiring* $\mathrm{D}(B)$ *is isomorphic to* $\mathrm{D}(S)$.

Remark. *The universal Euler characteristic on B is a strong Euler characteristic on B. The same proof as in the semilinear case goes through.*

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