A note on quantifier elimination in o-minimal fields with convex valuations

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Abstract

In [1], Ealy and Maříková proved the following: Let R be an o-minimal field in a language $\mathcal{L}$, let V be a convex subring, and let (R_0, V_0) be an elementary substructure of (R, V). Then (R, V) is model-complete in the language $\mathcal{L}_{R_0} \cup \{V\}$ relative to quantifier elimination in R and provided that the residue field of (R, V) with structure induced by R is o-minimal.

Here we show that "is model-complete" above can be replaced by "eliminates quantifiers". Consequently, $Th(R, V)$ is universally axiomatisable and has definable Skolem functions.

1 Introduction

We use the same set-up as in [1]: We let R be an o-minimal field (i.e. an o-minimal expansion of a real closed field), and V a convex subring of R with corresponding residue field $\mathbf{k}$. We assume (unless indicated otherwise) that the structure on $\mathbf{k}$ induced from R is o-minimal. By [1], the structure on $\mathbf{k}$ induced from R is the same as the structure induced from (R, V).

We let (R_0, V_0) be an elementary substructure of (R, V), and we consider (R, V) in the language $\mathcal{L}_{R_0} \cup \{V\}$, where $\mathcal{L}$ is the language of R. (Thus, $\mathcal{L}_{R_0} \cup \{V\}$ is the language $\mathcal{L}$ expanded by a predicate for the convex subring V and constants for all elements of R_0.) We further assume that $\mathcal{L}$ is such that $\mathrm{Th}(R)$ eliminates quantifiers in $\mathcal{L}$, and that $\mathcal{L}$ and $\mathrm{Th}(R)$ have been expanded by definitions as follows: for each $\mathcal{L}$-formula $\phi(x_1,\dots,x_n,y)$ such that

$$\mathrm{Th}(R) \vdash \forall x_1 \dots \forall x_n\ \exists! y\ \phi(x_1,\dots,x_n,y)$$

we add a new function symbol f to $\mathcal{L}$ and the axiom

$$\phi(x_1,\dots,x_n,f(x_1,\dots,x_n))$$

to the theory $\mathrm{Th}(R)$. Then $\mathrm{Th}(R)$ has a universal axiomatization and any substructure of a model of $\mathrm{Th}(R)$ is an elementary substructure. (NB the requirement that $\mathcal{L}$ contains function symbols for definable Skolem functions in R is not part of the set-up in [1]).

We shall denote the theory of (R,V) (in the language $\mathcal{L}_{R_0} \cup V$) by T, and we will use the following results proved in [1]. In the first theorem, o-minimality of $\mathbf{k}$ is not needed, and (R,V) is considered as a structure in the language $\mathcal{L} \cup \{V\}$ (i.e. there is no need to expand by constants for all elements of an elementary substructure). For $a \in \mathcal{R} \succeq R$, we denote by $R\langle a\rangle$ the (elementary) substructure of $\mathcal{R}$ generated by a over R.

Theorem 1.1 *Let $\mathcal{R}$ be an elementary extension of R, let $a \in \mathcal{R}$, and let $W \subseteq R\langle a\rangle$ be such that $(R,V) \subseteq (R\langle a\rangle, W)$. Then $(R,V) \preceq (R\langle a\rangle, W)$ iff there are no R-definable functions f and g such that $f(a) \in V$ and $g(a) > V$ and $V < f(a), g(a) < R^{>V}$.*

Theorem 1.2 *T is model complete.*

2 Substructures of models of T

We let p be the type $x > V$ and $x < r$ for all $r > V$. By $p|_{R_0}$ we denote the restriction of p to R_0. By $\widehat{p|_{R_0}}$ we denote the realization of $p|_{R_0}$ in R.

Lemma 2.1 *There is no R_0-definable decreasing function f such that $f(\widehat{p|_{R_0}}) = \widehat{p|_{R_0}}$.*

Proof: A decreasing function with the above property would have to have a fixed point $x \in \widehat{p|_{R_0}}$, so $x \in R_0$.

$\square$

Note that the above lemma holds for any elementary substructure of (R, V) in place of (R_0, V_0).

Lemma 2.2 *Let $a \in R$, and let $V_a = V \cap R_0\langle a \rangle$. Then*

$$(R_0, V_0) \preceq (R_0\langle a \rangle, V_a) \preceq (R, V).$$

Proof: We have $(R_0, V_0) \subseteq (R_0\langle a \rangle, V_a) \subseteq (R, V)$. By Theorem 1.1, if $(R_0\langle a \rangle, V_a)$ fails to be an elementary extension of (R_0, V_0), then there are R_0-definable functions f, g such that $f(a) \in V_a$, $g(a) > V_a$, and $V_0 < f(a), g(a) < R_0^{>V_0}$. We set $h = g \circ f^{-1}$ and $b = f(a)$. Then $b \in V_a$, $h(b) > V_a$, and $V_0 < b, h(b) < R_0^{>V_0}$. By monotonicity, h is continuous and strictly monotone on an R_0-definable interval with left endpoint in V_0 and right endpoint in $R_0^{>V_0}$. Since $h(\widehat{p|_{R_0}}) = \widehat{p|_{R_0}}$, h cannot be decreasing by Lemma 2.1. So h is increasing. Since $V_a = V \cap R_0\langle a \rangle$, we have $b \in V$ and $h(b) > V$. So $(R, V) \models \exists x \in V\ h(x) > V$. Hence $(R_0, V_0) \models \exists x \in V_0\ h(x) > V_0$ – but this yields a contradiction with h being increasing. It follows that $(R_0, V_0) \preceq (R_0\langle a \rangle, V_a)$. In particular, $\mathrm{Th}(R_0\langle a \rangle, V_a) = T$, so $(R_0\langle a \rangle, V_a) \preceq (R, V)$ by Theorem 1.2. $\square$

Using induction and the fact that the union of an elmentary chain is an elementary extension, the above lemma implies that any substructure of (R, V) is an elementary substructure of (R, V):

Lemma 2.3 *Let $(R', V') \subseteq (R, V)$. Then $(R', V') \preceq (R, V)$.*

Corollary 2.4 *T is universally axiomatizable.*

Recall that a for a model complete theory, quantifier elimination is equivalent to $T^{\forall}$ having the amalgamation property. By the above corollary, $T^{\forall} = T$, and so we have the following.

Theorem 2.5 *T admits QE.*

Corollary 2.6 *T has definable Skolem functions.* [1]

[1]In [2], Laskowski and Shaw claim that the expansion of an o-minimal field by a valuational cut has definable Skolem functions, but their proof is wrong. So, to my knowledge, this corollary is new.

PROOF: T eliminates quantifiers and has a universal axiomatization. $\square$

Corollary 2.7 *If $f : R \to R$ is (R, V)-definable, then there are R-definable functions $f_1, \ldots, f_k : R \to R$ such that for each $a \in R$ $f(a) = f_i(a)$ for some $i \in \{1, \ldots, k\}$.*

PROOF: This follows from T having definable Skolem functions and a universal axiomatisation. $\square$

References

[1] C. F. Ealy, J. Maříková, *Model-completeness of o-minimal fields with convex valuations*, J. Symb. Log. 80 (2015), no. 1, 234-250.

[2] Laskowski, Shaw *Definable choice for a class of weakly o-minimal theories*, Archive for Math Logic 55 (2016), no. 5-6, 735–748.