SCATTERING THEORY FOR JACOBI OPERATORS WITH STEPLIKE QUASI-PERIODIC BACKGROUND

IRYNA EGOROVA, JOHANNA MICHOR, AND GERALD TESCHL

Abstract. We develop direct and inverse scattering theory for Jacobi operators with steplike quasi-periodic finite-gap background in the same isospectral class. We derive the corresponding Gel'fand-Levitan-Marchenko equation and find minimal scattering data which determine the perturbed operator uniquely. In addition, we show how the transmission coefficients can be reconstructed from the eigenvalues and one of the reflection coefficients.

1. Introduction

In this paper we consider direct and inverse scattering theory for Jacobi operators with steplike quasi-periodic finite-gap background in the same isospectral class using the Marchenko [16] approach.

Scattering theory for Jacobi operators with a constant background is a classical topic first developed on an informal level by Case in [4]. The first rigorous results were established by Guseinov [12] with further extensions by Teschl [19], [20]. The case of a quasi-periodic finite-gap background was recently investigated by us in [8] (see also [23] respectively [9], [17] for applications to the Toda hierarchy).

Our motivation for the investigation of a steplike situation is twofold. First of all, steplike potentials are a simple model in quantum mechanics which have attracted renewed interest due to their possible applications in mesoscopic solid state structures. We refer for example to [6] where the discrete one dimensional Schrödinger equation is used as a simple one-band tight binding model to explain some of the essential qualitative properties of the Wannier-Stark ladders. The interested reader should also consult [11] or [18] and the references therein for further information.

Our second motivation is the study of solitons on (quasi-)periodic backgrounds. While solitons on quasi-periodic backgrounds are well investigated objects, not much about their stability is known. In fact, as pointed out only recently in [13], the general believe that the stability problem for solitons on quasi-periodic backgrounds is similar to the one for solitons on constant background is wrong (for a detailed analysis using Riemann-Hilbert techniques see [14]). This is related to the fact that solitons on quasi-periodic backgrounds give rise to different spatial asymptotics which naturally leads to the type of operators studied here. Hence our results form the basis for an investigation of solitons on quasi-periodic backgrounds via the inverse scattering transform. For further details we refer to [10].

2000 *Mathematics Subject Classification.* Primary 47B36, 81U40; Secondary 34L25, 39A11.
Key words and phrases. Inverse scattering, Jacobi operators, periodic, steplike.
Work supported by the Austrian Science Fund (FWF) under Grant No. P17762 and INTAS Research Network NeCCA 03-51-6637.

Though the case with different but constant background is well understood by now (see [7] and [3], [5], [22] for applications to the Toda lattice), the case with different (quasi-)periodic background still is in its infancy. First results for the case of two period two operators with a special choice for the respective spectra have been obtained in [1]. However, to the best of our knowledge the general problem is still open (even for the case of one-dimensional Schrödinger operators – see [11] and the references therein for a recent account). Our aim is to fill this gap for the case of isospectral background operators.

After recalling some necessary facts on algebro-geometric quasi-periodic finite-gap operators in Section 2, we construct the transformation operators and investigate the properties of the scattering data in Section 3. In particular, we show how both transmission coefficients can be reconstructed from the eigenvalues and the corresponding reflection coefficients. In Section 4 we derive the Gel'fand-Levitan-Marchenko equation and show that it uniquely determines the operator. In addition, we formulate necessary conditions for the scattering data to uniquely determine our Jacobi operator. Our final Section 5 shows that our necessary conditions for the scattering data are also sufficient.

2. Quasi-periodic finite-gap operators

As a preparation for our next section we first need to recall some facts on quasi-periodic finite-gap Jacobi operators which contain all periodic operators as a special case. We refer to [20, Chapter 9].

Let $\mathbb{M}$ be the Riemann surface associated with the following function

$$R_{2g+2}^{1/2}(z) = -\prod_{j=0}^{2g+1} \sqrt{z - E_j}, \qquad E_0 < E_1 < \cdots < E_{2g+1}, \tag{2.1}$$

where $g \in \mathbb{N}$ and $\sqrt{.}$ is the standard root with branch cut along $(-\infty, 0)$. $\mathbb{M}$ is a compact, hyperelliptic Riemann surface of genus g. A point on $\mathbb{M}$ is denoted by $p = (z, \pm R_{2g+2}^{1/2}(z)) = (z, \pm)$, $z \in \mathbb{C}$, or $p = \infty_{\pm}$, and the projection onto $\mathbb{C} \cup \{\infty\}$ by $\pi(p) = z$. The sets $\Pi_{\pm} = \{(z, \pm R_{2g+2}^{1/2}(z)) \mid z \in \mathbb{C} \backslash \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}]\} \subset \mathbb{M}$ are called upper, lower sheet, respectively.

Let $\{a_j, b_j\}_{j=1}^{g}$ be loops on the Riemann surface $\mathbb{M}$ representing the canonical generators of the fundamental group $\pi_1(\mathbb{M})$. We require a_j to surround the points E_{2j-1}, E_{2j} (thereby changing sheets twice) and b_j to surround E_0, E_{2j-1} counter-clockwise on the upper sheet, with pairwise intersection indices given by

$$a_j \circ a_k = b_j \circ b_k = 0, \qquad a_j \circ b_k = \delta_{jk}, \qquad 1 \le j, k \le g. \tag{2.2}$$

The corresponding canonical basis $\{\zeta_j\}_{j=1}^{g}$ for the space of holomorphic differentials can be constructed by

$$\underline{\zeta} = \sum_{j=1}^{g} \underline{c}(j) \frac{\pi^{j-1} d\pi}{R_{2g+2}^{1/2}}, \tag{2.3}$$

where the constants $\underline{c}(.)$ are given by

$$c_j(k) = C_{jk}^{-1}, \qquad C_{jk} = \int_{a_k} \frac{\pi^{j-1} d\pi}{R_{2g+2}^{1/2}} = 2 \int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1} dz}{R_{2g+2}^{1/2}(z)} \in \mathbb{R}.$$

The differentials fulfill

$$\int_{a_j} \zeta_k = \delta_{j,k}, \qquad \int_{b_j} \zeta_k = \tau_{j,k}, \qquad \tau_{j,k} = \tau_{k,j}, \qquad 1 \le j, k \le g. \tag{2.4}$$

Finally we will need ω_{pq}, the normalized Abelian differential of the third kind with poles at p and q, that is, ω_{pq} has vanishing a-periods and first order poles with residues $+1$, -1 at p, q, respectively.

Now pick g numbers (the Dirichlet eigenvalues)

$$(\hat{\mu}_j)_{j=1}^g = (\mu_j, \sigma_j)_{j=1}^g \tag{2.5}$$

whose projections lie in the spectral gaps, that is, $\mu_j \in [E_{2j-1}, E_{2j}]$. Associated with these numbers is the divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ which is one at the points $\hat{\mu}_j$ and zero else. Using this divisor we introduce

$$\begin{aligned} \underline{z}(p,n) &= \underline{\hat{A}}_{p_0}(p) - \underline{\hat{\alpha}}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) - n\underline{\hat{A}}_{\infty_-}(\infty_+) - \underline{\hat{\Xi}}_{p_0} \in \mathbb{C}^g, \\ \underline{z}(n) &= \underline{z}(\infty_+, n), \end{aligned} \tag{2.6}$$

where $\underline{\Xi}_{p_0}$ is the vector of Riemann constants and $\underline{A}_{p_0}$ ($\underline{\alpha}_{p_0}$) is Abel's map (for divisors). The hat indicates that we regard it as a (single-valued) map from $\hat{\mathbb{M}}$ (the fundamental polygon associated with $\mathbb{M}$) to $\mathbb{C}^g$. We recall that the function $\theta(\underline{z}(p,n))$ has precisely g zeros $\hat{\mu}_j(n)$ (with $\hat{\mu}_j(0) = \hat{\mu}_j$), where $\theta(\underline{z})$ is the Riemann theta function of $\mathbb{M}$.

With this notation our quasi-periodic finite-gap operator is given by

$$H_q f(n) = a_q(n) f(n+1) + a_q(n-1) f(n-1) + b_q(n) f(n), \tag{2.7}$$

where

$$\begin{aligned} a_q(n)^2 &= \tilde{a}^2 \frac{\theta(\underline{z}(n+1))\theta(\underline{z}(n-1))}{\theta(\underline{z}(n))^2}, \\ b_q(n) &= \tilde{b} + \sum_{j=1}^g c_j(g) \frac{\partial}{\partial w_j} \ln\Big(\frac{\theta(\underline{w} + \underline{z}(n))}{\theta(\underline{w} + \underline{z}(n-1))} \Big)\Big|_{\underline{w}=0}. \end{aligned} \tag{2.8}$$

The constants $\tilde{a}$, $\tilde{b}$, $c_j(g)$ depend only on the Riemann surface (see [20, Section 9.2]).

Introduce the Baker-Akhiezer function

$$\psi_q(p,n) = C(n,0) \frac{\theta(\underline{z}(p,n))}{\theta(\underline{z}(p,0))} \exp\Big(n \int_{p_0}^p \omega_{\infty_+,\infty_-} \Big),$$

where $C(n,0)$ is real-valued,

$$C(n,0)^2 = \frac{\theta(\underline{z}(0))\theta(\underline{z}(-1))}{\theta(\underline{z}(n))\theta(\underline{z}(n-1))}, \tag{2.9}$$

and $\omega_{\infty_+,\infty_-}$ is the Abelian differential of the third kind with poles at ∞_+ respectively ∞_-. The two branches $\psi_{q,\pm}(z,n) = \psi_q(p,n)$, $p = (z, \pm)$, of the Baker-Akhiezer function are solutions (in the weak sense) of

$$H_q \psi_{q,\pm}(z,n) = z \psi_{q,\pm}(z,n)$$

which are linearly independent away from the branch points. In fact, their Wronskian is given by

$$W_q(\psi_{q,-}(z),\psi_{q,+}(z)) = a_q(n)\,(\psi_{q,-}(z,n),\psi_{q,+}(z,n+1)-\psi_{q,-}(z,n+1),\psi_{q,+}(z,n))$$
$$= \frac{R_{2g+2}^{1/2}(z)}{\prod_{j=1}^{g}(z-\mu_j)}. \tag{2.10}$$

It is well-known that the spectrum of H_q consists of $g+1$ bands

$$\sigma(H_q) = \bigcup_{j=0}^{g}[E_{2j},E_{2j+1}]. \tag{2.11}$$

Note that $\psi_{q,\pm}(z,n)$ are discontinuous along the spectrum and we set $\psi_{q,\pm}(\lambda,n) = \lim_{\varepsilon\downarrow 0}\psi_{q,\pm}(\lambda+\mathrm{i}\varepsilon,n)$, $\lambda\in\sigma(H_q)$. This implies $\lim_{\varepsilon\downarrow 0}\psi_{q,\pm}(\lambda-\mathrm{i}\varepsilon,n) = \overline{\psi_{q,\pm}(\lambda,n)} = \psi_{q,\mp}(\lambda,n)$, $\lambda\in\sigma(H_q)$. For further information and proofs we refer to [20, Chapter 9].

3. Scattering data

Consider two quasi-periodic finite-gap operators $H_q^\pm$ associated with the sequences $a_q^\pm$, $b_q^\pm$ in the same isospectral class,

$$\sigma(H_q^+) = \sigma(H_q^-) \equiv \Sigma = \bigcup_{j=0}^{g}[E_{2j},E_{2j+1}], \tag{3.1}$$

but with possibly different Dirichlet data $\{\hat\mu_j^\pm\}_{j=1}^g$. We will add $\pm$ as a superscript to all data introduced in Section 2 to distinguish between the corresponding data of H_q^+ and H_q^-. To avoid excessive sub/superscripts we abbreviate

$$\psi_q^\pm(z,n) = \psi_{q,\pm}^\pm(z,n) \quad\text{and}\quad \breve\psi_q^\pm(z,n) = \psi_{q,\mp}^\pm(z,n), \tag{3.2}$$

that is, $\psi_q^\pm(z,n)$ is the solution of $H_q^\pm$ decaying near $\pm\infty$ and $\breve\psi_q^\pm(z,n)$ is the solution of $H_q^\pm$ decaying near $\mp\infty$. Note that for $\lambda\in\Sigma$ we have $\breve\psi_q^\pm(\lambda,n) = \overline{\psi_q^\pm(\lambda,n)}$.

In addition, we split the set of Dirichlet eigenvalues into three parts:

$$\begin{aligned} M^\pm &= \{\mu_j^\pm \,|\, \mu_j^\pm\in\mathbb{R}\backslash\Sigma \text{ is a pole of } \psi_q^\pm(z,1)\},\\ \breve M^\pm &= \{\mu_j^\pm \,|\, \mu_j^\pm\in\mathbb{R}\backslash\Sigma \text{ is a pole of } \breve\psi_q^\pm(z,1)\},\\ \tilde M^\pm &= \{\mu_j^\pm \,|\, \mu_j^\pm\in\partial\Sigma\}, \end{aligned} \tag{3.3}$$

and introduce

$$\hat\psi_q^\pm(z,n) = \Big(\prod_{\mu\in M^\pm}(z-\mu)\Big)\psi_q^\pm(z,n), \tag{3.4}$$

which are holomorphic for all $z\in\mathbb{C}\backslash\Sigma$.

Let $a(n)$, $b(n)$ be sequences satisfying

$$\sum_{n=0}^{\pm\infty}|n|\Big(|a(n)-a_q^\pm(n)|+|b(n)-b_q^\pm(n)|\Big)<\infty \tag{3.5}$$

and denote the corresponding operator by H.

The special case $H_q^- = H_q^+$ has been exhaustively studied in [8] (see also [23]) and several results are straightforward generalizations. In such situations we will simply refer to [8] and only point out possible differences. In fact, we begin by recalling two theorems from [8]:

Theorem 3.1. *Assume (3.5). Then there exist weak solutions $\psi_\pm(z,.)$ of $H\psi = z\psi$ satisfying*

$$\lim_{n\to\pm\infty} |w(z)^{\mp n}(\psi_\pm(z,n) - \psi_q^\pm(z,n))| = 0, \tag{3.6}$$

where $w(z) = \exp(\int_{p_0}^{(z,+)} \omega_{\infty_+,\infty_-})$ is the quasi-momentum map. Moreover, $\psi_\pm(z,.)$ are continuous (resp. holomorphic) with respect to z whenever $\psi_q^\pm(z,.)$ are and
(3.7)

$$\psi_\pm(z,n) = \frac{z^{\mp n}}{A_\pm(n)}\Big(\prod_{j=0}^{n-1}{}^{*}\; a_q^\pm(j)\Big)^{\pm 1}\Big(1 + \Big(B_\pm(n) \pm \sum_{j=1}^{n}{}^{*}\; b_q^\pm(j - {\scriptstyle 0 \atop 1})\Big)\frac{1}{z} + O(\frac{1}{z^2})\Big),$$

where

$$\begin{aligned} A_+(n) &= \prod_{j=n}^{\infty}\frac{a(j)}{a_q^+(j)}, \qquad B_+(n) = \sum_{m=n+1}^{\infty}(b_q^+(m) - b(m)),\\ A_-(n) &= \prod_{j=-\infty}^{n-1}\frac{a(j)}{a_q^-(j)}, \qquad B_-(n) = \sum_{m=-\infty}^{n-1}(b_q^-(m) - b(m)). \end{aligned} \tag{3.8}$$

Defining $\hat\psi_\pm(z,n)$ using $\hat\psi_q^\pm(z,n)$ (instead of $\psi_q^\pm(z,n)$), we have that $\hat\psi_\pm(z,n)$ are holomorphic in $\mathbb{C}\backslash\Sigma$.

Theorem 3.2. *Assume (3.5). Then we have $\sigma_{ess}(H) = \Sigma$, the point spectrum of H is finite and confined to the spectral gaps of $H_q^\pm$, that is, $\sigma_p(H) = \{\rho_j\}_{j=1}^q \subset \mathbb{R}\backslash\Sigma$. Furthermore, the essential spectrum of H is purely absolutely continuous.*

It will be convenient to identify $\mathbb{C}\backslash\Sigma$ with Π_+ and regard the functions $\psi_\pm$, $\psi_q^\pm$ as functions on Π_+. Furthermore, we extend them by continuity to $\partial\Pi_+$, that is, $\psi_+((\lambda,\pm),n) = \lim_{\varepsilon\downarrow 0}\psi_+(\lambda \pm \mathrm{i}\varepsilon, n)$, etc, for $\lambda\in\Sigma$. Consequently we have $\psi_q^\pm(p^*,n) = \overline{\psi_q^\pm(p,n)}$ and $\psi_\pm(p^*,n) = \overline{\psi_\pm(p,n)}$ for $p\in\partial\Pi_+$.

Note: Unfortunately this disagrees with our previous definition of ψ_q^- which was originally defined as the branch on Π_-. Moreover, one should emphasize that $\psi_\pm(p,n)$ does not have a continuation to Π_- in general.

Using the fact that $\psi_q^\pm(p,n)$ form an orthonormal basis for $L^2(\partial\Pi_+, d\omega^\pm)$, where

$$d\omega^\pm = \frac{\prod_{j=1}^g(\pi - \mu_j^\pm)}{R_{2g+2}^{1/2}}d\pi, \tag{3.9}$$

we can define

$$\begin{aligned} K_\pm(n,m) &= \int_{\partial\Pi_+}\psi_\pm(p,n)\overline{\psi_q^\pm(p,m)}d\omega^\pm\\ &= 2\mathrm{Re}\int_\Sigma \psi_\pm(\lambda,n)\overline{\psi_q^\pm(\lambda,m)}d\omega^\pm. \end{aligned} \tag{3.10}$$

Then $K_\pm(n,m)$ satisfy ([8]):

Lemma 3.3. *Assume (3.5). The Jost solutions $\psi_\pm(z,n)$ can be represented as*

$$\psi_\pm(z,n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m)\psi_q^\pm(z,m), \tag{3.11}$$

where the kernels $K_\pm(n,.)$ satisfy $K_\pm(n,m)=0$ for $\pm m < \pm n$ and

$$|K_\pm(n,m)| \le C_\pm(n) \sum_{j=[\frac{m\pm n}{2}]\pm 1}^{\pm\infty} \Big(|a(j)-a_q^\pm(j)| + |b(j)-b_q^\pm(j)|\Big), \quad \pm m > \pm n > 0. \tag{3.12}$$

The functions $C_\pm(n) > 0$ decrease as $n \to \pm\infty$ and depend only on $H_q^\pm$ and the value of the sums in (3.5).

Associated with $K_\pm(n,m)$ is the operator

$$(\mathcal{K}_\pm f)(n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m) f(m), \qquad f \in \ell^\infty_\pm(\mathbb{Z},\mathbb{C}), \tag{3.13}$$

which acts as a transformation operator for the pair τ, $\tau_q^\pm$.

Theorem 3.4. *Let $\tau_q^\pm$ and τ be the quasi-periodic and perturbed Jacobi difference expressions, respectively. Then*

$$\tau \mathcal{K}_\pm f = \mathcal{K}_\pm \tau_q^\pm f, \qquad f \in \ell^\infty_\pm(\mathbb{Z},\mathbb{C}). \tag{3.14}$$

Furthermore, for $n \in \mathbb{Z}$ we have

$$\begin{aligned} \frac{a(n)}{a_q^+(n)} &= \frac{K_+(n+1,n+1)}{K_+(n,n)} \\ \frac{a(n)}{a_q^-(n)} &= \frac{K_-(n,n)}{K_-(n+1,n+1)}, \\ b(n) - b_q^+(n) &= a_q^+(n)\frac{K_+(n,n+1)}{K_+(n,n)} - a_q^+(n-1)\frac{K_+(n-1,n)}{K_+(n-1,n-1)} \\ b(n) - b_q^-(n) &= a_q^-(n-1)\frac{K_-(n,n-1)}{K_-(n,n)} - a_q^-(n)\frac{K_-(n+1,n)}{K_-(n+1,n+1)}. \end{aligned} \tag{3.15}$$

Next we define the coefficients of the scattering matrix via the scattering relations

$$\psi_\mp(\lambda,n) = \alpha_\pm(\lambda)\overline{\psi_\pm(\lambda,n)} + \beta_\pm(\lambda)\psi_\pm(\lambda,n), \qquad \lambda \in \Sigma, \tag{3.16}$$

where

$$\begin{aligned} \alpha_\pm(\lambda) &= \frac{W(\psi_\pm(\lambda),\psi_\mp(\lambda))}{W(\psi_\pm(\lambda),\overline{\psi_\pm(\lambda)})} = \frac{\prod_{j=1}^g(\lambda-\mu_j^\pm)}{R_{2g+2}^{1/2}(\lambda)} W(\psi_-(\lambda),\psi_+(\lambda)), \\ \beta_\pm(\lambda) &= \frac{W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)})}{W(\psi_\pm(\lambda),\overline{\psi_\pm(\lambda)})} = \mp\frac{\prod_{j=1}^g(z-\mu_j^\pm)}{R_{2g+2}^{1/2}(z)} W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)}), \end{aligned} \tag{3.17}$$

and $W_n(f,g) = a(n)(f(n)g(n+1) - f(n+1)g(n))$ denotes the Wronskian. Transmission $T_\pm(\lambda)$ and reflection $R_\pm(\lambda)$ coefficients are then defined by

$$T_\pm(\lambda) = \alpha_\pm^{-1}(\lambda), \qquad R_\pm(\lambda) = \frac{\beta_\pm(\lambda)}{\alpha_\pm(\lambda)} = \frac{W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)})}{W(\psi_\pm(\lambda),\psi_\mp(\lambda))}. \tag{3.18}$$

Using

$$T_\pm(z) = \frac{R_{2g+2}^{1/2}(z)}{\prod_{j=1}^g (z-\mu_j^\pm)} \frac{1}{W(\psi_-(z),\psi_+(z))} \tag{3.19}$$

we see that $T_\pm(z)$ admit a meromorphic extension to $\mathbb{C}\backslash\Sigma$. Since $W(\hat\psi_-(z),\hat\psi_+(z))$ is holomorphic in $\mathbb{C}\backslash\Sigma$ with simple zeros at the eigenvalues ρ_k (see [20, Section 2.2]) we obtain the following behavior:

$$T_\pm(z) = \frac{\prod_{\mu_j^\mp\in M^\mp}(z-\mu_j^\mp)}{\prod_{\mu_j^\pm\in \bar M^\pm}(z-\mu_j^\pm)} \frac{D_\pm(z)}{\prod_{k=1}^q (z-\rho_k)}, \tag{3.20}$$

where $D_\pm(z)$ are holomorphic and do not vanish in $\mathbb{C}\backslash\Sigma$. That is, in general the transmission coefficients have simple poles at the eigenvalues ρ_j of H. In addition, there are simple poles at $\mu_j^\pm\in\bar M^\pm$ and simple zeros at $\mu_j^\mp\in M^\mp$. A pole at $\mu_j^\pm$ could cancel with a zero at $\mu_j^\mp$ or could give a second order pole if $\mu_j^\pm=\rho_k$.

The Plücker identity (c.f. [20, (2.169)]) implies

$$\alpha_+(\lambda)\overline{\alpha_-(\lambda)} = 1-\beta_+(\lambda)\beta_-(\lambda), \tag{3.21}$$

and using

$$\alpha_+(\lambda)=\alpha_-(\lambda)\prod_{j=1}^g \frac{\lambda-\mu_j^+}{\lambda-\mu_j^-}, \qquad \overline{\beta_+}(\lambda) = -\beta_-(\lambda)\prod_{j=1}^g \frac{\lambda-\mu_j^+}{\lambda-\mu_j^-}, \tag{3.22}$$

we obtain

$$|\alpha_\pm(\lambda)|^2 = \prod_{j=1}^g \frac{\lambda-\mu_j^\pm}{\lambda-\mu_j^\mp} + |\beta_\pm(\lambda)|^2. \tag{3.23}$$

The norming constants $\gamma_{\pm,j}$ corresponding to $\rho_j\in\sigma_p(H)$ are given by

$$\gamma_{\pm,j}^{-1} = \sum_{n\in\mathbb{Z}} |\hat\psi_\pm(\rho_j,n)|^2, \qquad 1\le j\le q. \tag{3.24}$$

Moreover, we set $\hat\psi_\pm(\rho_j,.) = c_j^\pm\hat\psi_\mp(\rho_j,.)$ with $c_j^+c_j^-=1$.

Lemma 3.5. *The coefficients $T_\pm(\lambda)$, $R_\pm(\lambda)$ are continuous for $\lambda\in\Sigma$ except at possibly the band edges E_j, and fulfill*

$$T_+(\lambda)\overline{T_-(\lambda)} + |R_\pm(\lambda)|^2 = 1, \qquad \lambda\in\Sigma, \tag{3.25}$$

$$T_\pm(\lambda)\overline{R_\pm(\lambda)} + \overline{T_\pm(\lambda)}R_\mp(\lambda) = 0, \qquad \lambda\in\Sigma. \tag{3.26}$$

In particular,

$$|T_\pm(\lambda)|^2 \prod_{j=1}^g \frac{\lambda-\mu_j^\pm}{\lambda-\mu_j^\mp} + |R_\pm(\lambda)|^2 = 1, \tag{3.27}$$

and hence $|R_\pm(\lambda)|^2\le 1$ with equality only possibly at the band edges E_j, where

$$\begin{aligned} \lim_{z\to E} R_{2g+2}^{1/2}(z)\tfrac{R_\pm(z)+1}{T_\pm(z)} &= 0, \qquad E\neq\{\mu_j^+,\mu_j^-\},\\ \lim_{z\to E} R_{2g+2}(z)\tfrac{R_\pm(z)+1}{T_\pm(z)} &= 0, \qquad E=\mu_j^\mp\neq\mu_j^\pm,\\ \lim_{z\to E} \tfrac{R_\pm(z)-1}{T_\pm(z)} &= 0, \qquad E=\mu_j^\pm\neq\mu_j^\mp,\\ \lim_{z\to E} R_{2g+2}^{1/2}(z)\tfrac{R_\pm(z)-1}{T_\pm(z)} &= 0, \qquad E=\mu_j^+=\mu_j^-. \end{aligned} \tag{3.28}$$

The transmission coefficients $T_\pm(\lambda)$ have a meromorphic continuation with

$$\operatorname{Res}_{z=\rho_k} \frac{\prod_{\mu_j^\mp \in M^\mp}(z-\mu_j^\mp)}{\prod_{\mu_j^\pm \in \bar{M}^\pm \cup \tilde{M}^\pm}(z-\mu_j^\pm)} T_\pm(z) = -\sqrt{\gamma_{+,k}\gamma_{-,k}} R_{2g+2}^{1/2}(\rho_k). \tag{3.29}$$

In addition, $T_\pm(z) \in \mathbb{R}$ as $z \in \mathbb{R}\backslash\Sigma$ and

$$T_\pm(\infty) = \frac{1}{K_+(0,0)K_-(0,0)}. \tag{3.30}$$

Proof. To prove (3.28), recall first that if $E \notin \tilde{M}^\pm$, then $\psi_\pm(\lambda,n) - \overline{\psi_\pm(\lambda,n)} \to 0$ as $\lambda \to E$, and $\psi_\pm(E,n)$ are bounded. If $E \in \tilde{M}^\pm$, then $R_{2g+2}^{1/2}(\lambda)\psi_\pm(\lambda,n)$ and $\psi_\pm(\lambda,n) + \overline{\psi_\pm(\lambda,n)}$ are bounded at E.

Thus using definition (3.17), we have as $E \notin \tilde{M}^- \cup \tilde{M}^+$

$$R_{2g+2}^{1/2}(\lambda)\frac{R_\pm(\lambda)+1}{T_\pm(\lambda)} = \prod_{j=1}^{g}(\lambda-\mu_j^\pm)\big(W(\psi_-(\lambda),\psi_+(\lambda)) \mp W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)})\big) \to 0.$$

If $E \in \tilde{M}^\pm\backslash\tilde{M}^\mp$, then

$$\frac{R_\pm(\lambda)-1}{T_\pm(\lambda)} = \pm\frac{\prod_{j=1}^{g}(\lambda-\mu_j^\pm)}{R_{2g+2}^{1/2}(\lambda)} W(\psi_\pm(\lambda)+\overline{\psi_\pm(\lambda)},\psi_\mp(\lambda)) \to 0.$$

In the same manner one proves the remaining two equalities in (3.28).

By [20, (2.33)],

$$W'(\hat{\psi}_-(z),\hat{\psi}_+(z))\Big|_{z=\rho_j} = \frac{-1}{\sqrt{\gamma_{-,j}\gamma_{+,j}}}, \tag{3.31}$$

which proves (3.29). Finally, (3.30) follows from (3.7). □

Observe that (3.25) implies $|R_-(\lambda)| = |R_+(\lambda)|$.

The sets

$$S_\pm(H) = \{R_\pm(\lambda), \lambda \in \Sigma;\ (\rho_j,\gamma_{\pm,j}), 1 \le j \le q\} \tag{3.32}$$

are called left/right scattering data for H.

Theorem 3.6. *The transmission coefficients can be reconstructed from one reflection coefficient and the eigenvalues via*

$$\begin{aligned} T_+(z) =& \sqrt{\frac{\theta(\underline{z}^+(\infty_+,0))\theta(\underline{z}^+(\infty_-,0))}{\theta(\underline{z}^-(\infty_+,0))\theta(\underline{z}^-(\infty_-,0))}} \frac{\theta(\underline{z}^-(\hat{z}^*,0))}{\theta(\underline{z}^+(\hat{z}^*,0))} \times \\ & \times \left(\prod_{j=1}^{q} \exp\left(-\int_{E(\rho_j)}^{\hat{\rho}_j} \omega_{\hat{z}\hat{z}^*}\right)\right) \exp\left(\frac{1}{2\pi\mathrm{i}}\int_\Sigma \ln(1-|R_\pm|^2)\omega_{\hat{z}\hat{z}^*}\right), \end{aligned} \tag{3.33}$$

$$\begin{aligned} T_-(z) =& \sqrt{\frac{\theta(\underline{z}^-(\infty_+,0))\theta(\underline{z}^-(\infty_-,0))}{\theta(\underline{z}^+(\infty_+,0))\theta(\underline{z}^+(\infty_-,0))}} \frac{\theta(\underline{z}^+(\hat{z},0))}{\theta(\underline{z}^-(\hat{z},0))} \times \\ & \times \left(\prod_{j=1}^{q} \exp\left(-\int_{E(\rho_j)}^{\hat{\rho}_j} \omega_{\hat{z}\hat{z}^*}\right)\right) \exp\left(\frac{1}{2\pi\mathrm{i}}\int_\Sigma \ln(1-|R_\pm|^2)\omega_{\hat{z}\hat{z}^*}\right), \end{aligned} \tag{3.34}$$

where $\hat{z} = (z, +)$, the integral over Σ is taken on the upper sheet, and $E(\rho)$ is E_0 if $\rho < E_0$, either E_{2j-1} or E_{2j} if $\rho \in (E_{2j-1}, E_{2j})$, $1 \le j \le g$, and E_{2g+1} if $\rho > E_{2g+1}$.

Furthermore, the phase shift between H_q^+ and H_q^- can also be computed from one reflection coefficient and the eigenvalues

$$\underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}^+}) - \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}^-}) = \sum_{j=1}^{q} \int_{\hat{\rho}_j^*}^{\hat{\rho}_j} \underline{\zeta} - \frac{1}{2\pi\mathrm{i}} \int_{\partial\Pi_+} \ln(1 - |R_\pm|^2)\underline{\zeta}. \tag{3.35}$$

Proof. First of all note that $\operatorname{Im}\int \omega_{pp^*}$ is the Green's function of Π_+ and that

$$B(z,\rho) = \exp\left(\int_{E(\rho)}^{\hat{\rho}} \omega_{\hat{z}\hat{z}^*}\right) \tag{3.36}$$

is the Blaschke factor (see [21]). That is, $B(z,\rho)$ is a multivalued holomorphic function, which vanishes at $z = \rho$ and satisfies $|B(\lambda,\rho)| = 1$ for $\lambda \in \Sigma$.

We start by considering the multivalued function

$$t_+(z) = \sqrt{\frac{\theta(\underline{z}^-(\infty_+,0))\theta(\underline{z}^-(\infty_-,0))}{\theta(\underline{z}^+(\infty_+,0))\theta(\underline{z}^+(\infty_-,0))}} \frac{\theta(\underline{z}^+(\hat{z}^*,0))}{\theta(\underline{z}^-(\hat{z}^*,0))} \left(\prod_{j=1}^{q} B(z,\rho_j)\right) T_+(z) \tag{3.37}$$

which has neither zeros nor poles on Π_+. Using the same argument as in [20, eq. (9.27)] one verifies that $\theta(\underline{z}^+(\infty_\pm,0))/\theta(\underline{z}^-(\infty_\pm,0))$ is positive. The absolute value of $t_+(z)$ is single-valued and, using $\overline{\underline{z}^\pm(\hat{\lambda},n)} = \underline{z}^\pm(\hat{\lambda}^*,n) \mod \mathbb{Z}^g$ for $\hat{\lambda} = (\lambda,+)$ with $\lambda \in \Sigma$, one obtains

$$\begin{aligned} |t_+(\lambda)|^2 &= \frac{\theta(\underline{z}^-(\infty_+,0))\theta(\underline{z}^-(\infty_-,0))}{\theta(\underline{z}^+(\infty_+,0))\theta(\underline{z}^+(\infty_-,0))} \frac{\theta(\underline{z}^+(\hat{\lambda}^*,0))}{\theta(\underline{z}^-(\hat{\lambda}^*,0))} \frac{\theta(\underline{z}^+(\hat{\lambda},0))}{\theta(\underline{z}^-(\hat{\lambda},0))} |T_+(\lambda)|^2 \\ &= \frac{\prod(\lambda - \mu_j^+)}{\prod(\lambda - \mu_j^-)} |T_+(\lambda)|^2. \end{aligned} \tag{3.38}$$

Here the last equality follows since the ratio of theta functions extends to a (single-valued) meromorphic function on $\mathbb{C}$ which is one at ∞. Hence $|t_+(\lambda)|^2 = 1 - |R_\pm(\lambda)|^2$ for $\lambda \in \Sigma$ by (3.27) and $\ln|t_+(z)|$ can be reconstructed from its boundary values using Green's function:

$$|t_+(z)| = \exp\left(\operatorname{Re}\frac{1}{2\pi\mathrm{i}} \int_\Sigma \ln(1 - |R_\pm|^2)\omega_{\hat{z}\hat{z}^*}\right). \tag{3.39}$$

So (3.33) holds at least when taking absolute values. But since both sides are meromorphic, they can only differ by a factor of absolute value one. Since both sides are positive at ∞_+, they are equal. The claim for $T_-(z)$ follows analogously.

To show (3.35) we use that, since $T_\pm$ are single-valued, there is no jump when we go around b-cycles. Hence the jump when going around b_ℓ of all factors must add up to zero, which is just (3.35). □

Combining this result with (3.29) we obtain:

Corollary 3.7. *One of the scattering data $S_-(H)$ or $S_+(H)$ determines the other.*

4. The Gel'fand-Levitan-Marchenko equation

Finally we want to derive the Gel'fand-Levitan-Marchenko equation and show that one of the sets $S_-(H)$ or $S_+(H)$ uniquely determines H.

Theorem 4.1. *The kernel $K_\pm(n,m)$ of the transformation operator satisfies the Gel'fand-Levitan-Marchenko equation*

$$K_\pm(n,m) + \sum_{l=n}^{\pm\infty} K_\pm(n,l) F^\pm(l,m) = \frac{\delta(n,m)}{K_\pm(n,n)}, \qquad \pm m \geq \pm n, \tag{4.1}$$

where

$$\begin{aligned} F^\pm(m,n) =& 2\mathrm{Re} \int_\Sigma R_\pm(\lambda) \psi_q^\pm(\lambda,m) \psi_q^\pm(\lambda,n) d\omega^\pm \\ &+ \sum_{j=1}^q \gamma_{\pm,j} \hat{\psi}_q^\pm(\rho_j,n) \hat{\psi}_q^\pm(\rho_j,m). \end{aligned} \tag{4.2}$$

Proof. As already done for $\psi_\pm$, we regard $T_\pm(z)$ as functions on Π_+ and $R_\pm(\lambda)$ as functions on $\partial\Pi_+$ by setting $R_\pm((\lambda,+)) = R_\pm(\lambda)$ respectively $R_\pm((\lambda,-)) = \overline{R_\pm(\lambda)}$.

Computing the Fourier coefficients of

$$T_\pm(\lambda)\psi_\mp(\lambda,n) = R_\pm(\lambda)\psi_\pm(\lambda,n) + \overline{\psi_\pm(\lambda,n)} \tag{4.3}$$

as in [8, Sec. 7] one obtains by (3.7), (3.9), (3.19), (3.30), (3.31), $\hat{\psi}_-(\rho_j,n) = c_j^- \hat{\psi}_+(\rho_j,n)$, and the residue theorem

$$\begin{aligned} \int_{\partial\Pi_+} T_+(p)\psi_-(p,n)\psi_q^+(p,m) d\omega^+ &= \int_{\partial\Pi_+} \frac{\psi_-(p,n)\psi_q^+(p,m)}{W(\psi_-(p),\psi_+(p))} d\pi \\ &= \frac{\delta(n,m)}{K_+(n,n)} + \sum_{j=1}^q \mathrm{Res}_{\rho_j} \left(\frac{\hat{\psi}_-(p,n)\hat{\psi}_q^+(p,m)}{W(\hat{\psi}_-(p),\hat{\psi}_+(p))} \right) \\ &= \frac{\delta(n,m)}{K_+(n,n)} - \sum_{j=1}^q \gamma_{+,j} \hat{\psi}_+(\rho_j,n) \hat{\psi}_q^+(\rho_j,m). \end{aligned}$$

The right hand side follows analogous to [8]. □

Note that while the scattering data depend on the particular normalization chosen for $\psi_q^\pm$, the kernel of Gel'fand-Levitan-Marchenko equation is of course independent of this normalization.

Theorem 4.2. *For $n \in \mathbb{Z}$, the Gel'fand-Levitan-Marchenko operator*

$$\mathcal{F}_n^\pm : \ell^2 \to \ell^2, \qquad \mathcal{F}_n^\pm f(j) = \sum_{l=0}^\infty F^\pm(n \pm l, n \pm j) f(l), \tag{4.4}$$

is Hilbert-Schmidt. Moreover, $1 + \mathcal{F}_n^\pm$ is positive and hence invertible.

In particular, the Gel'fand-Levitan-Marchenko equation (4.1)

$$(1 + \mathcal{F}_n^\pm) K_\pm(n, n \pm .) = (K_\pm(n,n))^{-1} \delta_0 \tag{4.5}$$

has a unique solution and $S_+(H)$ or $S_-(H)$ uniquely determine H.

Proof. The proof can be done as in [8, Theorem 7.5], since by equation (3.27) we have $|R_\pm(\lambda)| < 1$ for $\lambda \in \Sigma \backslash \partial\Sigma$. □

To finish the direct scattering step we summarize the properties of $S_\pm(H)$.

Hypothesis H. 4.3. *The scattering data*

$$S_\pm(H) = \{R_\pm(\lambda), \lambda \in \Sigma; (\rho_j, \gamma_{\pm,j}), 1 \le j \le q\}$$

satisfy the following conditions:

(i) The reflection coefficients $R_\pm(\lambda)$ are continuous except possibly at the band edges E and fulfill $|R_\pm(\lambda)| < 1$ for $\lambda \in \Sigma \backslash \partial\Sigma$.

The Fourier coefficients

$$\tilde F^\pm(l,m) = 2\mathrm{Re} \int_\Sigma R_\pm(\lambda) \psi_q^\pm(\lambda, l) \psi_q^\pm(\lambda, m) d\omega^\pm \tag{4.6}$$

satisfy

$$|\tilde F^\pm(n,m)| \le \sum_{j=n+m}^{\pm\infty} q(j), \qquad q(j) \ge 0, \qquad |j| q(j) \in \ell^1(\mathbb{Z}),$$
$$\sum_{n=n_0}^{\pm\infty} |n| \big| \tilde F^\pm(n,n) - \tilde F^\pm(n \pm 1, n \pm 1) \big| < \infty,$$
$$\sum_{n=n_0}^{\pm\infty} |n| \big| a_q^\pm(n) \tilde F^\pm(n, n+1) - a_q^\pm(n-1) \tilde F^\pm(n-1, n) \big| < \infty.$$

(ii) The values $\rho_j \in \mathbb{R} \backslash \Sigma$ are distinct and $\gamma_{\pm,j} \ge 0$ for $1 \le j \le q$.

(iii) $\ln(1 - |R_\pm|^2)$ *is integrable on Σ and $T_\pm(\lambda)$ defined via (3.33), (3.34) extend to single-valued functions on Π_+ with*

$$T_-(\lambda) = T_+(\lambda) \prod_{j=1}^{g} \frac{\lambda - \mu_j^+}{\lambda - \mu_j^-}. \tag{4.7}$$

(iv) Transmission and reflection coefficients satisfy (3.28), (3.30), and the consistency conditions

$$\frac{R_\mp(\lambda)}{\overline{R_\pm(\lambda)}} = - \frac{T_\pm(\lambda)}{\overline{T_\pm(\lambda)}}, \qquad \gamma_{+,j}\, \gamma_{-,j} = \frac{\Big(\mathrm{Res}_{\rho_j} \frac{\prod_{\mu_j^\mp \in M^\mp} (z - \mu_j^\mp)}{\prod_{\mu_j^\pm \in \hat M^\pm \cup \check M^\pm} (z - \mu_j^\pm)} T_\pm(z)\Big)^2}{R_{2g+2}(\rho_j)}.$$

5. Inverse scattering theory

In this section we reconstruct the operator H from a given set S_+ or S_- and given quasi-periodic Jacobi operators $H_q^\pm$.

If $S_\pm$ (satisfying H.4.3 (i)–(ii)) and $H_q^\pm$ are known, we can construct $F^\pm(l,m)$ via formula (4.2) and thus derive the Gel'fand-Levitan-Marchenko equation, which has a unique solution by Theorem 4.2. We obtain that

$$K_\pm(n,n) = \langle \delta_0, (1 + \mathcal{F}_n^\pm)^{-1} \delta_0 \rangle^{1/2}$$
$$K_\pm(n, n \pm j) = \frac{1}{K_\pm(n,n)} \langle \delta_j, (1 + \mathcal{F}_n^\pm)^{-1} \delta_0 \rangle. \tag{5.1}$$

Since $1+\mathcal{F}_n^{\pm}$ is positive, $K_\pm(n,n)$ is positive and we can set (see Theorem 3.4)

$$\begin{aligned} a_+(n) &= a_q^+(n)\frac{K_+(n+1,n+1)}{K_+(n,n)},\\ a_-(n) &= a_q^-(n)\frac{K_-(n,n)}{K_-(n+1,n+1)},\\ b_+(n) &= b_q^+(n) + a_q^+(n)\frac{K_+(n,n+1)}{K_+(n,n)} - a_q^+(n-1)\frac{K_+(n-1,n)}{K_+(n-1,n-1)},\\ b_-(n) &= b_q^-(n) + a_q^-(n-1)\frac{K_-(n,n-1)}{K_-(n,n)} - a_q^-(n)\frac{K_-(n+1,n)}{K_-(n+1,n+1)}. \end{aligned} \tag{5.2}$$

Let H_+, H_- be the associated Jacobi operators. As in [8] one proves

Lemma 5.1. *Suppose a given set $S_\pm$ satisfies H.4.3 (i)–(ii). Then the sequences defined in (5.2) satisfy $n|a_\pm(n)-a_q^\pm(n)|$, $n|b_\pm(n)-b_q^\pm(n)| \in \ell^1_\pm(\mathbb{N})$.*

Moreover, $\psi_\pm(\lambda,n)=\sum_{m=n}^{\pm\infty}K_\pm(n,m)\psi_q^\pm(\lambda,m)$, where $K_\pm(n,m)$ is the solution of the Gel'fand-Levitan-Marchenko equation, satisfies $\tau_\pm\psi_\pm=\lambda\psi_\pm$.

We set

$$W(\lambda) = \frac{R_{2g+2}^{1/2}(\lambda)}{T_\pm(\lambda)\prod_{j=1}^g(\lambda-\mu_j^\pm)}. \tag{5.3}$$

According to H.4.3 (iii), this function is holomorphic in $\mathbb{C}\backslash(M^\pm\cup\bar{M}^\pm\cup\partial\Sigma)$.

Now we can prove the main result of this section.

Theorem 5.2. *Hypothesis H.4.3 is necessary and sufficient for sets $S_\pm$ to be the left/right scattering data of a unique Jacobi operator H associated with sequences a, b satisfying (3.5).*

Proof. Necessity has been established in the previous section. By Lemma 5.1, we know existence of sequences $a_\pm$, $b_\pm$ and corresponding solutions $\psi_\pm(z,n)$ associated with S_+ (or S_-). Hence it remains to establish $a_+(n)=a_-(n)$ and $b_+(n)=b_-(n)$.

We study the following part of the GLM-equation

$$\sum_{m\in\mathbb{Z}}\Phi_+(n,m)\overline{\psi_q^+(\lambda,m)}, \qquad \Phi_+(n,.) := \sum_{l=n}^{\infty}K_+(n,l)\tilde{F}^+(l,.)\in\ell^1_+(\mathbb{Z}) \tag{5.4}$$

and obtain as in [8, Theorem 8.2]

$$T_+(\lambda)h_-(\lambda,n) = \overline{\psi_+(\lambda,n)} + R_+(\lambda)\psi_+(\lambda,n), \qquad \lambda\in\Sigma, \tag{5.5}$$

where

$$\begin{aligned} h_-(\lambda,n) = \frac{\overline{\psi_q^+(\lambda,n)}}{T_+(\lambda)}\Bigg(&\frac{1}{K_+(n,n)} + \sum_{m=-\infty}^{n-1}\Phi_+(n,m)\frac{\overline{\psi_q^+(\lambda,m)}}{\overline{\psi_q^+(\lambda,n)}}\\ &+\sum_{j=1}^q\gamma_{+,j}\hat{\psi}_+(\rho_j,n)\frac{W_{n-1}(\hat{\psi}_q^+(\rho_j),\overline{\psi_q^+(\lambda)})}{(\lambda-\rho_j)\overline{\psi_q^+(\lambda,n)}}\Bigg). \end{aligned} \tag{5.6}$$

Clearly, using $\bar{\psi}_q^\pm(\lambda,n)=\overline{\psi_q^\pm(\lambda,n)}$, the function $h_+(\lambda,n)$ extends to a meromorphic function in $\mathbb{C}\backslash\Sigma$.

Similar we obtain a function $h_+(z,n)$. The functions $h_\mp(z,n)$ have simple poles at $\mu_j^\mp \in M^\mp$ and the following limits

$$h_\mp(z,n) = \frac{z^{\pm n}}{K_\pm(n,n)T_\pm(\infty)}\Big(\prod_{j=0}^{n-1} a_q^\pm(j)\Big)^{\mp 1}\Big(1+O(\frac{1}{z})\Big), \quad z\to\infty, \tag{5.7}$$

$$\lim_{z\to\rho_j} h_\mp(z,n) = \frac{\gamma_{\pm,j}\hat\psi_\pm(\rho_j,n)}{\sqrt{\gamma_{+,j}\gamma_{-,j}}\prod_{\mu\in M^\mp}(\rho_j-\mu)}. \tag{5.8}$$

By virtue of the consistency condition $T_\pm(\lambda)\overline{R_\pm(\lambda)} = -\overline{T_\pm(\lambda)}R_\mp(\lambda)$ we obtain

$$\overline{h_\pm(\lambda,n)} + R_\pm(\lambda)h_\pm(\lambda,n) = \psi_\mp(\lambda,n)T_\pm(\lambda), \qquad \lambda\in\Sigma. \tag{5.9}$$

Eliminating $R_\pm(\lambda)$ in (5.5) and (5.9) and using (5.3) yields

$$\begin{aligned} &R_{2g+2}^{-1/2}(\lambda)\prod_{j=1}^{g}(\lambda-\mu_j^\pm)\big(\overline{h_\pm(\lambda,n)}\psi_\pm(\lambda,n) - \overline{\psi_\pm(\lambda,n)}h_\pm(\lambda,n)\big)\\ &= \frac{\psi_+(\lambda,n)\psi_-(\lambda,n) - h_+(\lambda,n)h_-(\lambda,n)}{W(\lambda)} = G(\lambda,n), \qquad \lambda\in\Sigma. \end{aligned} \tag{5.10}$$

Observe that $G(\lambda,n)$ extends to a meromorphic function on $\mathbb{C}\backslash\Sigma$ which is continuous on the interior of Σ with equal real limits from above and below: $G(\lambda\pm \mathrm{i}0,n) = \overline{G(\lambda\mp \mathrm{i}0,n)} = G(\lambda\pm\mathrm{i}0,n)$. So by the Schwarz reflection principle $G(z,n)$ is holomorphic on the set $\mathbb{C}\backslash(M^\pm\cup\bar M^\pm\cup\{\rho_k\}_{k=1}^q\cup\partial\Sigma)$.

Since the difference $\psi_+\psi_- - h_+h_-$ vanishes at the points ρ_k by (5.8) and H.4.3 (iv), the poles there are removable.

Next let us investigate the behavior at the band edges. Since $G(z,n)$ has at most poles, it is sufficient to control the behavior from one direction. To this end we use (3.28) and the identity (5.10). If $E\notin\tilde M^+\cup\tilde M^-$,

$$\begin{aligned} &\lim_{\lambda\to E} R_{2g+2}^{1/2}(\lambda)\prod_{j=1}^{g}(\lambda-\mu_j^\pm)h_\mp(\lambda,n)\overline{\psi_\mp(\lambda,n)}\\ &= \lim_{\lambda\to E}\frac{R_{2g+2}^{1/2}\prod_{j=1}^g(\lambda-\mu_j^\pm)}{T_\pm}\Big(\overline{\psi_\pm} + R_\pm\psi_\pm\Big)\overline{\psi_\mp}\\ &= \lim_{\lambda\to E}\frac{R_{2g+2}^{1/2}\prod_{j=1}^g(\lambda-\mu_j^\pm)}{T_\pm}\Big((R_\pm+1)\psi_\pm + \overline{\psi_\pm} - \psi_\pm\Big)\overline{\psi_\mp} = 0. \end{aligned}$$

For $E\in\tilde M^+\cap\tilde M^-$, the functions $\psi_\pm(\lambda,.)+\overline{\psi_\pm(\lambda,.)}$ are bounded in the vicinity of $E\in\tilde M^\pm$,

$$\begin{aligned} &\lim_{\lambda\to E}\frac{R_{2g+2}^{1/2}\prod_{j=1}^g(\lambda-\mu_j^\pm)}{T_\pm}\Big(\overline{\psi_\pm}+R_\pm\psi_\pm\Big)\overline{\psi_\mp}\\ &= \frac{R_{2g+2}^{1/2}\prod_{j=1}^g(\lambda-\mu_j^\pm)}{T_\pm}(\psi_\pm+\overline{\psi_\pm})\overline{\psi_\mp} + \prod_{j=1}^g(\lambda-\mu_j^\pm)\psi_\pm\overline{\psi_\pm}\frac{R_{2g+2}^{1/2}(R_\pm-1)}{T_\pm} = 0. \end{aligned}$$

Analogously, one proves that $R_{2g+2}(z)G(z,n)$ vanishes at the points $E\in\tilde M_\pm\backslash\tilde M_\mp$. Therefore this function is holomorphic on $\mathbb{C}$ and $R_{2g+2}(z)G(z,n) = O(z-E)$ for

$E \in \partial\Sigma$. Thus $G(z,n)$ has only removable singularities and since $G(z,n) \to 0$ as $z \to \infty$, Liouville's theorem implies $G(z,n) \equiv 0$ and

$$\psi_+(z,n)\psi_-(z,n) - h_+(z,n)h_-(z,n) \equiv 0, \qquad \forall n \in \mathbb{Z}.$$

For $z \to \infty$ we obtain by (3.7) and (5.7)

$$T_+(\infty)T_-(\infty) = \left(\frac{1}{K_+(0,0)K_-(0,0)} \prod_{j=0}^{n-1} \frac{a_-(j)}{a_+(j)} \right)^2$$

and therefore by (3.30)

$$a_+(n) = a_-(n) \equiv a(n), \qquad \forall n \in \mathbb{Z}. \tag{5.11}$$

It remains to prove $b_+(n) = b_-(n)$. Proceeding as for $G(\lambda,n)$ we can show that

$$\begin{aligned} &\frac{\psi_+(\lambda,n)\psi_-(\lambda,n+1) - h_+(\lambda,n+1)h_-(\lambda,n)}{W(\lambda)} \\ &\quad = \frac{\prod_{j=1}^{g}(\lambda - \mu_j^+)}{R_{2g+2}^{1/2}(\lambda)} \big(\overline{h_+(\lambda,n+1)}\psi_+(\lambda,n) - \overline{\psi_+(\lambda,n)}h_+(\lambda,n+1)\big) \end{aligned} \tag{5.12}$$

is a constant equal to $-1/a(n)$. Thus

$$\bar{W}(z,n) := a(n)\left(\psi_+(z,n)\psi_-(z,n+1) - h_+(z,n+1)h_-(z,n)\right) = -W(z).$$

Computing the asymptotics as $z \to \infty$ (compare (3.7)) we see

$$0 = \bar{W}(z,n) - \bar{W}(z,n-1) = \frac{b_+(n) - b_-(n)}{A_+(0)A_-(0)} \tag{5.13}$$

and in particular $b_+(n) = b_-(n) \equiv b(n)$. □

Acknowledgments. I.E. gratefully acknowledges the extraordinary hospitality of the Faculty of Mathematics of the University of Vienna during two stays in 2006, where parts of this paper were written.

References

[1] J. Bazargan and I. Egorova, *Jacobi operator with step-like asymptotically periodic coefficients*, Mat. Fiz. Anal. Geom. **10**, No.3, 425–442 (2003).
[2] A. Boutet de Monvel and I. Egorova, *The Toda lattice with step-like initial data. Soliton asymptotics*, Inverse Problems **16**, No.4, 955–977 (2000).
[3] A. Boutet de Monvel, I. Egorova, and E. Khruslov, *Soliton asymptotics of the Cauchy problem solution for the Toda lattice*, Inverse Problems **13**, No.2, 223–237 (1997).
[4] K. M. Case, *Orthogonal polynomials from the viewpoint of scattering theory*, J. Math. Phys. **14**, 2166–2175 (1973).
[5] P. Deift, S. Kamvissis, T. Kriecherbauer, and X. Zhou, *The Toda rarefaction problem*, Comm. Pure Appl. Math. **49**, No.1, 35–83 (1996).
[6] A. Di Carlo, P. Vogl, and W. Pötz, *Theory of Zener tunneling and Wannier-Stark states in semiconductors*, Phys. Rev. **B50**, 8358–8377 (1994).
[7] I. Egorova, *The scattering problem for step-like Jacobi operator*, Mat. Fiz. Anal. Geom. **9**, No.2, 188–205 (2002).
[8] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with quasi-periodic background*, Comm. Math. Phys. **264-3**, 811-842 (2006).
[9] I. Egorova, J. Michor, and G. Teschl, *Inverse scattering transform for the Toda hierarchy with quasi-periodic background*, Proc. Amer. Math. Soc. **135**, 1817-1827 (2007).
[10] I. Egorova, J. Michor, and G. Teschl, *Soliton solutions of the Toda hierarchy on quasi-periodic background revisited*, preprint.

[11] F. Gesztesy, R. Nowell, and W. Pötz, W, *One-dimensional scattering theory for quantum systems with nontrivial spatial asymptotics*, Differential Integral Equations **10** , no. 3, 521–546 (1997).

[12] G.S. Guseinov, *The inverse problem of scattering theory for a second-order difference equation on the whole axis*, Soviet Math. Dokl., **17**, 1684–1688 (1976).

[13] S. Kamvissis and G. Teschl, *Stability of periodic soliton equations under short range perturbations*, Phys. Lett. A (to appear).

[14] S. Kamvissis and G. Teschl, *Stability of the periodic Toda lattice under short range perturbations*, in preparation.

[15] E. A. Kuznetsov and A. V. Mikhaĭlov, *Stability of stationary waves in nonlinear weakly dispersive media*, Soviet Phys. JETP **40**, No.5, 855–859 (1975).

[16] V.A. Marchenko, *Sturm–Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[17] J. Michor and G. Teschl, *Trace formulas for Jacobi operators in connection with scattering theory for quasi-periodic background*, Operator Theory, Analysis, and Mathematical Physics, J. Janas, et al. (eds.), 69–76, Oper. Theory Adv. Appl., 174, Birkhäuser, Basel, 2007.

[18] W. Pötz, *Scattering theory for mesoscopic quantum systems with non-trivial spatial asymptotics in one dimension*, J. Math. Phys. **36**, 1707–1740 (1995).

[19] G. Teschl, *On the initial value problem for the Toda and Kac-van Moerbeke hierarchies*, AMS/IP Studies in Advanced Mathematics **16**, 375–384 (2000).

[20] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. and Mon. **72**, Amer. Math. Soc., Rhode Island, 2000.

[21] G. Teschl, *Algebro-geometric constraints on solitons with respect to quasi-periodic backgrounds*, Bull. London Math. Soc. (to appear).

[22] S. Venakides, P. Deift, and R. Oba, *The Toda shock problem*, Comm. Pure Appl. Math. **44**, no. 8-9, 1171–1242 (1991).

[23] A. Volberg and P. Yuditskii, *On the inverse scattering problem for Jacobi Matrices with the Spectrum on an Interval, a finite system of intervals or a Cantor set of positive length*, Comm. Math. Phys. **226**, 567–605 (2002).

Kharkiv National University, 47,Lenin ave, 61164 Kharkiv, Ukraine
E-mail address: egorova@ilt.kharkov.ua

Faculty of Mathematics, Nordbergstrasse 15, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria
E-mail address: Johanna.Michor@esi.ac.at
URL: http://www.mat.univie.ac.at/~jmichor/

Faculty of Mathematics, Nordbergstrasse 15, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria
E-mail address: Gerald.Teschl@univie.ac.at
URL: http://www.mat.univie.ac.at/~gerald/