On global attraction to solitary waves for the Klein-Gordon field coupled to several nonlinear oscillators

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Abstract

The global attraction is established for all finite energy solutions to a model $\mathbf{U}(1)$-invariant nonlinear Klein-Gordon equation in one dimension coupled to a finite number of nonlinear oscillators: We prove that *each finite energy solution* converges as $t \to \pm\infty$ to the set of all "nonlinear eigenfunctions" of the form $\phi(x)e^{-i\omega t}$ if all oscillators are strictly nonlinear, and the distances between neighboring oscillators are sufficiently small.

Our approach is based on the analysis of *omega-limit trajectories* which form the global attractor. We show that their time spectrum is a priori compact. Then the nonlinear spectral analysis based on the Titchmarsh convolution theorem allows to reduce the time-spectrum to one point. This implies that each omega-limit trajectory is a solitary wave. Physically, the *global attraction* is caused by the nonlinear energy transfer from lower harmonics to the continuous spectrum and subsequent dispersive radiation. The Titchmarsh theorem allows to prove that this energy transfer and radiation are absent only for the solitary waves.

To check the sharpness of our conditions, we construct counterexamples showing the global attractor can contain "multifrequency solitary waves" if the distance between oscillators is large or if some of them are linear.

1 Introduction

The long time asymptotics for nonlinear wave equations have been the subject of intensive research, starting with the pioneering papers by Segal [Seg63a, Seg63b], Strauss [Str68], and Morawetz and Strauss [MS72], where the nonlinear scattering and the local attraction to zero solution were proved. Local attraction to solitary waves, or *asymptotic stability*, in $\mathbf{U}(1)$-invariant dispersive systems was addressed in [SW90, BP93, SW92, BP95] and then developed in [PW97, SW99, Cuc01a, Cuc01b, BS03, Cuc03]. Global attraction to *static*, stationary solutions in the dispersive systems *without* $\mathbf{U}(1)$ *symmetry* was established in [Kom91, Kom95, KV96, KSK97, Kom99, KS00].

We would like to have the dynamical description of the Bohr transitions to quantum stationary states in coupled nonlinear systems of Quantum Physics. This suggests investigation of the global attractors in nonlinear Hamiltonian hyperbolic equations with $\mathbf{U}(1)$-symmetry (see [KK07] for the discussion). The first result about the global attraction to solitary waves in a model with these properties was obtained in [KK06, KK07], where we considered the Klein-Gordon equation coupled to one nonlinear oscillator.

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We are aware of only one other recent advance [Tao07] in the field of nonzero global attractors for Hamiltonian PDEs. In that paper, the global attraction for the nonlinear Schrödinger equation in dimensions $n \geq 5$ was considered. The dispersive wave was explicitly specified using the rapid decay of local energy in higher dimensions. The global attractor was proved to be compact, but it was neither identified with the set of solitary waves nor was proved to be of finite dimension [Tao07, Remark 1.18].

In the present paper, we prove the attraction to the set of solitary waves for all finite energy solutions to the Klein-Gordon equation coupled to any finite number of nonlinear oscillators. For the proof, we develop an approach of the spectral inflation [KK07] justified by the Titchmarsh Convolution Theorem. This justification requires new arguments and appropriate conditions. We demonstrate the sharpness of these conditions constructing counterexamples.

Our model is based on the complex Klein-Gordon field $\psi(x,t)$, interacting with N nonlinear oscillators located at the points $X_1 < X_2 < \ldots < X_N$:

$$\ddot{\psi} = \psi'' - m^2\psi + \sum_J \delta(x - X_J) F_J(\psi(X_J, t)), \qquad x \in \mathbb{R}, \tag{1.1}$$

where $m > 0$ and F_J are nonlinear functions describing nonlinear oscillators at the points X_J. The dots stand for the derivatives in t, and the primes for the derivatives in x. All derivatives and the equation are understood in the sense of distributions. We assume that equation (1.1) is $\mathbf{U}(1)$-invariant; that is,

$$F_J(e^{i\theta}\psi) = e^{i\theta} F_J(\psi), \qquad \theta \in \mathbb{R}, \quad \psi \in \mathbb{C}, \quad 1 \leq J \leq N. \tag{1.2}$$

This symmetry leads to the charge conservation and to the existence of the solitary wave solutions, which are finite energy solutions of the following form:

$$\psi_\omega(x,t) = \phi_\omega(x) e^{-i\omega t}, \quad \omega \in \mathbb{R}, \quad \phi_\omega \in H^1(\mathbb{R}). \tag{1.3}$$

Above, $H^1(\mathbb{R})$ denotes the Sobolev space.

Definition 1.1. S is the set of all functions $\phi_\omega(x) \in H^1(\mathbb{R})$ with $\omega \in \mathbb{R}$, so that $\phi_\omega(x)e^{-i\omega t}$ is a solution to (1.1).

Note that S also contains the zero solution.

Generically, the factor-space $S/\mathbf{U}(1)$ is isomorphic to a finite union of one-dimensional intervals. The set of all solitary waves for equation (1.1) is described in Proposition 2.8. Typically, such solutions exist for ω from an interval or a collection of intervals of the real line.

Our main result is the following long-time asymptotics: In the case when all oscillators are polynomial and strictly nonlinear (see Assumptions 2.1 and 2.2 below) and all distances $|X_{J+1} - X_J|$ are sufficiently small, we prove that any finite energy solution converges to the set S of all solitary waves:

$$\psi(\cdot, t) \longrightarrow S, \qquad t \to \pm\infty, \tag{1.4}$$

where the convergence holds in local energy seminorms.

Let us give a brief sketch of our approach. We introduce a concept of the omega-limit trajectories $\beta(x,t)$ which play a crucial role in the proof. We define omega-limit trajectories as the limits

$$\psi(x, t + s_j) \to \beta(x,t), \qquad (x,t) \in \mathbb{R}^2,$$

for some sequence of times $s_j \to +\infty$. We will prove that all omega-limit trajectories are solitary waves, thus finishing the proof. To complete this program, we study the time spectrum of solutions, that is, their Fourier-Laplace transform in time. We need to prove that $\beta(x,t) = \phi_\omega(x)e^{-i\omega t}$, that is, that the time spectrum of β consists of at most one frequency. First, we show that the spectrum of the solution at $x = X_1$ and $x = X_N$ is absolutely continuous for $|\omega| > m$. At the points $x \in (X_1, X_N)$, the nonlinearity may extend the singular part of the spectrum to be at most $[-\Lambda, \Lambda]$, for some bounded Λ. Outside of this interval, the spectrum is absolutely continuous. This allows to prove that the spectrum of any omega-limit trajectory at $x = X_1$ and $x = X_N$ is contained in $[-m, m]$, while at the points $x \in (X_1, X_N)$ the spectrum is contained in $[-\Lambda, \Lambda]$. The next important observation is that each omega-limit trajectory is also a solution to the original nonlinear Klein-Gordon equation. This allows to apply the Titchmarsh theorem and prove that the spectrum of any omega-limit trajectory at all points $x \in \mathbb{R}$ consists of at most one frequency. At this last step, one needs the assumptions that the oscillators are strictly nonlinear and located sufficiently close to one another.

The requirement that the nonlinearities F_J are polynomial allows us to apply the Titchmarsh theorem which is vital in the proof. We construct counterexamples showing the sharpness of our assumptions for the global attraction to the solitary waves. Namely, for $N = 2$, we construct multifrequency solitary waves in the case when the distance $|X_2 - X_1|$ is sufficiently large or one of the oscillators is linear.

Let us mention that in the case of N oscillators, considered in this paper, the general plan of the proof is similar to the case of one oscillator (see [KK06, KK07]). However, the justifications of all steps are based on new arguments. In particular, the application of the Titchmarsh theorem required a new construction.

Our paper is organized as follows. In Section 2, we formulate our main results. In Section 3, we separate the first dispersive component. In Sections 4 and 5, we construct spectral representation for the remaining component, and prove absolute continuity of its spectrum for high frequencies. In Sections 6, we separate the second dispersive component corresponding to the high frequencies and establish compactness for the remaining *bound* component with the bounded spectrum. In Section 7, we study omega-limit trajectories of the solution. In Section 8 we collect counterexamples, and in Appendix A we establish global well-posedness.

2 Main results

Model

We consider the Cauchy problem for the Klein-Gordon equation with the nonlinearity concentrated at the points $X_1 < X_2 < \ldots < X_N$:

$$\begin{cases} \ddot\psi(x,t) = \psi''(x,t) - m^2\psi(x,t) + \sum_J \delta(x - X_J) F_J(\psi(X_J,t)), & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ \psi|_{t=0} = \psi_0(x), \qquad \dot\psi|_{t=0} = \pi_0(x). \end{cases} \tag{2.1}$$

If we identify a complex number $\psi = u + iv \in \mathbb{C}$ with the two-dimensional vector $(u, v) \in \mathbb{R}^2$, then, physically, equation (2.1) describes small crosswise oscillations of the infinite string in three-dimensional space (x, u, v) stretched along the x-axis. The string is subject to the action of an "elastic force" $-m^2\psi(x,t)$ and coupled to nonlinear oscillators of forces $F_J(\psi)$ attached at the points X_J. We denote by $\mathcal{X}$ the set of all the locations of oscillators:

$$\mathcal{X} = \{X_1, X_2, \ldots, X_N\}. \tag{2.2}$$

We will assume that the oscillator forces F_J admit real-valued potentials:

$$F_J(\psi) = -\nabla U_J(\psi), \quad \psi \in \mathbb{C}, \qquad U_J \in C^2(\mathbb{C}), \tag{2.3}$$

where the gradient is taken with respect to $\operatorname{Re}\psi$ and $\operatorname{Im}\psi$. We define $\Psi(t) = \begin{bmatrix} \psi(x,t) \\ \pi(x,t) \end{bmatrix}$ and write the Cauchy problem (2.1) in the vector form:

$$\dot\Psi(t) = \begin{bmatrix} 0 & 1 \\ \partial_x^2 - m^2 & 0 \end{bmatrix} \Psi(t) + \sum_J \delta(x - X_J) \begin{bmatrix} 0 \\ F_J(\psi) \end{bmatrix}, \qquad \Psi|_{t=0} = \Psi_0 \equiv \begin{bmatrix} \psi_0 \\ \pi_0 \end{bmatrix}. \tag{2.4}$$

Equation (2.4) formally can be written as a Hamiltonian system,

$$\dot\Psi(t) = \mathcal{J}\, D\mathcal{H}(\Psi), \qquad \mathcal{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{2.5}$$

where $D\mathcal{H}$ is the variational derivative of the Hamilton functional

$$\mathcal{H}(\Psi) = \frac{1}{2} \int\limits_{\mathbb{R}} \left(|\pi|^2 + |\psi'|^2 + m^2|\psi|^2\right) dx + \sum_J U_J(\psi(X_J)), \qquad \Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix}. \tag{2.6}$$

We assume that the potentials $U_J(\psi)$ are $\mathbf{U}(1)$-invariant, where $\mathbf{U}(1)$ stands for the unitary group $e^{i\theta}$, $\theta \in \mathbb{R} \bmod 2\pi$. Namely, we assume that there exist $u_J \in C^2(\mathbb{R})$ such that

$$U_J(\psi) = u_J(|\psi|^2), \qquad \psi \in \mathbb{C}, \quad 1 \le J \le N. \tag{2.7}$$

Remark 2.1. In the context of the model of the infinite string in $\mathbb{R}^3$ that we described after (2.1), the assumption (2.7) means that the potentials $U_J(\psi)$ are rotation-invariant with respect to the x-axis.

Conditions (2.3) and (2.7) imply that

$$F_J(\psi) = \alpha_J(|\psi|^2)\psi, \qquad \psi \in \mathbb{C}, \tag{2.8}$$

where $\alpha_J(\cdot) = -2u_J'(\cdot) \in C^1(\mathbb{R})$ are real-valued. Therefore, (1.2) holds. Since (2.4) is $\mathbf{U}(1)$-invariant, the Nöther theorem formally implies that the *charge functional*

$$\mathcal{Q}(\Psi) = \frac{i}{2}\int_{\mathbb{R}} \left(\overline{\psi}\pi - \overline{\pi}\psi\right)\,dx, \qquad \Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix}, \tag{2.9}$$

is conserved for solutions $\Psi(t)$ to (2.4).

Let us introduce the phase space $\mathscr{E}$ of finite energy states for equation (2.1). Denote by L^2 the complex Hilbert space $L^2(\mathbb{R})$ with the norm $\|\cdot\|_{L^2}$, and denote by $\|\cdot\|_{L^2_R}$ the norm in $L^2(-R,R)$ for $R > 0$.

Definition 2.2. (*i*) $\mathscr{E}$ is the Hilbert space of the states $\Psi = (\psi, \pi)$, with the norm

$$\|\Psi\|^2_{\mathscr{E}} := \|\pi\|^2_{L^2} + \|\psi'\|^2_{L^2} + m^2\|\psi\|^2_{L^2}. \tag{2.10}$$

(*ii*) $\mathscr{E}_F$ is the space $\mathscr{E}$ endowed with the Fréchet topology defined by local energy seminorms

$$\|\Psi\|^2_{\mathscr{E},R} := \|\pi\|^2_{L^2(-R,R)} + \|\psi'\|^2_{L^2(-R,R)} + m^2\|\psi\|^2_{L^2(-R,R)}, \qquad R > 0. \tag{2.11}$$

Remark 2.3. The space $\mathscr{E}_F$ is metrizable. The metric could be introduced by

$$\operatorname{dist}(\Psi, \Phi) = \sum_{R=1}^{\infty} 2^{-R}\|\Psi - \Phi\|_{\mathscr{E},R}. \tag{2.12}$$

Equation (2.4) is formally a Hamiltonian system with the phase space $\mathscr{E}$ and the Hamilton functional $\mathcal{H}$. Both $\mathcal{H}$ and $\mathcal{Q}$ are continuous functionals on $\mathscr{E}$. Let us note that $\mathscr{E} = H^1 \oplus L^2$, where H^1 denotes the Sobolev space

$$H^1 = H^1(\mathbb{R}) = \{\psi(x) \in L^2(\mathbb{R}) :\ \psi'(x) \in L^2(\mathbb{R})\}.$$

We introduced into (2.10) the factor $m^2 > 0$, to have a convenient relation $\mathcal{H}(\psi, \dot\psi) = \frac{1}{2}\|(\psi, \dot\psi)\|^2_{\mathscr{E}} + \sum_J U_J(\psi(X_J))$.

Global well-posedness

To have a priori estimates available for the proof of the global well-posedness, we assume that

$$U_J(\psi) \ge A_J - B_J|\psi|^2 \quad \text{for } \psi \in \mathbb{C}, \quad \text{where} \quad A_J \in \mathbb{R}, \quad B_J \ge 0, \quad 1 \le J \le N; \quad \sum_J B_J < m. \tag{2.13}$$

Theorem 2.4. *Let $F_J(\psi)$ satisfy conditions (2.3) and (2.7):*

$$F_J(\psi) = -\nabla U_J(\psi), \qquad U_J(\psi) = u_J(|\psi|^2), \qquad u_J(\cdot) \in C^2(\mathbb{R}).$$

Additionally, assume that (2.13) holds. Then:

(*i*) *For every $\Psi_0 \in \mathscr{E}$ the Cauchy problem (2.4) has a unique solution $\Psi(t)$ such that $\Psi \in C(\mathbb{R}, \mathscr{E})$.*

(*ii*) *The map $W(t) :\ \Psi_0 \mapsto \Psi(t)$ is continuous in $\mathscr{E}$ for each $t \in \mathbb{R}$.*

(*iii*) *The energy and charge are conserved: $\mathcal{H}(\Psi(t)) = \mathrm{const}$, $\mathcal{Q}(\Psi(t)) = \mathrm{const}$, $t \in \mathbb{R}$.*

(*iv*) *The following* a priori *bound holds: $\|\Psi(t)\|_{\mathscr{E}} \le C(\Psi_0)$, $t \in \mathbb{R}$.*

We prove this Theorem in Appendix A.

Solitary waves and the main theorem

Definition 2.5. (*i*) The solitary waves of equation (2.1) are solutions of the form

$$\psi(x,t)=\phi_\omega(x)e^{-i\omega t},\qquad \text{where}\quad \omega\in\mathbb{R},\quad \phi_\omega\in H^1(\mathbb{R}). \tag{2.14}$$

(*ii*) The solitary manifold is the set $\mathbf{S}=\big\{(\phi_\omega,-i\omega\phi_\omega)\colon \omega\in\mathbb{R},\ \phi_\omega\in H^1(\mathbb{R})\big\}\subset\mathscr{E}$.

Remark 2.6. (*i*) Identity (1.2) implies that the set **S** is invariant under multiplication by $e^{i\theta}$, $\theta\in\mathbb{R}$.

(*ii*) Let us note that for any $\omega\in\mathbb{R}$ there is a zero solitary wave with $\phi_\omega(x)\equiv 0$ since $F_J(0)=0$ by (2.8).

(*iii*) According to (2.8), $\alpha_J(|C|^2)=F_J(C)/C\in\mathbb{R}$ for any $C\in\mathbb{C}\backslash 0$.

Definition 2.7. The function $F_J(\psi)$ is *strictly nonlinear* if the equation $\alpha_J(C^2)=a$ has a discrete (or empty) set of positive roots C for each particular $a\in\mathbb{R}$.

The following proposition provides a concise description of all solitary waves. Formally this proposition is not necessary for our exposition.

Proposition 2.8. *Assume that $F_J(\psi)$ satisfy (1.2) and that $F_J(\psi)$, $1\le J\le N$, are strictly nonlinear in the sense of Definition 2.7. Then all solitary wave solutions to (2.1) are given by (2.14) with*

$$\phi_\omega(x)=\sum_J C_J e^{-\kappa(\omega)|x-X_J|},\qquad \kappa(\omega)=\sqrt{m^2-\omega^2}, \tag{2.15}$$

where $\omega\in[-m,m]$ and $C_J\in\mathbb{C}$, $1\le J\le N$, satisfy the following relations:

$$2\kappa(\omega)C_J=F_J\Big(\sum_K C_K e^{-\kappa(\omega)|X_J-X_K|}\Big). \tag{2.16}$$

Remark 2.9. By (2.15), $\omega=\pm m$ can only correspond to zero solution.

The proof of this Proposition repeats the proof of a similar result for the case $N=1$ in [KK07].

As we mentioned before, we need to assume that the nonlinearities are nonlinear polynomials. This condition is crucial in our argument: It will allow to apply the Titchmarsh convolution theorem.

Let us formulate all the assumptions which we need to formulate the main result.

Assumption 2.1. For all $1\le J\le N$,

$$F_J(\psi)=-\nabla U_J(\psi),\quad \text{where}\quad U_J(\psi)=\sum_{n=0}^{p_J}u_{J,n}|\psi|^{2n},\qquad u_{J,n}\in\mathbb{R}. \tag{2.17}$$

Assumption 2.2. For all $1\le J\le N$, we have

$$u_{J,p_J}>0\quad\text{and}\quad p_J\ge 2. \tag{2.18}$$

Assumptions 2.1 and 2.2 guarantee that all nonlinearities F_J are strictly nonlinear and satisfy (2.3), (2.7), and also that the bound (2.13) takes place.

We introduce the following quantities:

$$\mu_1=m,\quad \mu_{J+1}=(2p_J-1)\mu_J;\qquad \mu'_N=m,\quad \mu'_J=(2p_{J+1}-1)\mu'_{J+1},\qquad 1\le J\le N-1, \tag{2.19}$$

where p_J are exponentials from (2.17). We also denote

$$\Lambda=\max_{1\le J\le N}(2p_J-1)M_J,\qquad \text{where}\quad M_J=\min(\mu_J,\mu'_J). \tag{2.20}$$

Assumption 2.3. The intervals $[X_J,X_{J+1}]$, $1\le J\le N-1$, are small enough so that

$$\Lambda<\sqrt{\frac{\pi^2}{|X_{J+1}-X_J|^2}+m^2},\qquad 1\le J\le N-1. \tag{2.21}$$

Our main result is the following theorem.

Theorem 2.10 (Main Theorem). *Let Assumptions 2.1, 2.2, and 2.3 hold. Then for any $\Psi_0 \in \mathscr{E}$ the solution $\Psi(t) \in C(\mathbb{R}, \mathscr{E})$ to the Cauchy problem (2.4) converges to* $\mathbf{S}$*:*

$$\lim_{t\to\pm\infty} \operatorname{dist}(\Psi(t), \mathbf{S}) = 0, \tag{2.22}$$

where $\operatorname{dist}(\Psi, \mathbf{S}) := \inf_{\boldsymbol{s}\in\mathbf{S}} \operatorname{dist}(\Psi, \boldsymbol{s})$*, and* dist *is introduced in (2.12).*

Remark 2.11. (*i*) The solution $\Psi(t)$ exists by Theorem 2.4 since Assumptions 2.1 and 2.2 guarantee that conditions (2.3), (2.7), and (2.13) hold.

(*ii*) It suffices to prove Theorem 2.10 for $t \to +\infty$.

(*iii*) In Sections 8.1 and 8.2, we construct counterexamples to the convergence (2.22) in the case when Assumption 2.2 or Assumption 2.3 are not satisfied.

(*iv*) For the real initial data, we obtain a real-valued solution $\psi(t)$ to (2.1). Therefore, the convergence (2.22) of $\Psi(t) = (\psi(t), \dot\psi(t))$ to the set of pairs $(\phi_\omega, -i\omega\phi_\omega)$ with $\omega \in \mathbb{R}$ implies that $\Psi(t)$ locally converges to zero:

$$\lim_{t\to\infty} \operatorname{dist}(\Psi(t), 0) = 0.$$

3 Separation of dispersive component

Let us split the solution $\psi(x,t)$ into two components, $\psi(x,t) = \chi(x,t) + \varphi(x,t)$, which are defined for all $t \in \mathbb{R}$ as solutions to the following Cauchy problems:

$$\ddot\chi(x,t) = \chi''(x,t) - m^2\chi(x,t), \qquad (\chi, \dot\chi)|_{t=0} = (\psi_0(x), \pi_0(x)), \tag{3.1}$$

$$\ddot\varphi(x,t) = \varphi''(x,t) - m^2\varphi(x,t) + \sum_J \delta(x - X_J) f_J(t), \qquad (\varphi, \dot\varphi)|_{t=0} = (0,0), \tag{3.2}$$

where $(\psi_0(x), \pi_0(x))$ is the initial data from (2.1), and

$$f_J(t) := F_J(\psi(X_J, t)), \qquad t \in \mathbb{R}. \tag{3.3}$$

The following lemma is proved in [KK07, Lemma 3.1].

Lemma 3.1. *There is a local energy decay for* χ*:*

$$\lim_{t\to\infty} \|(\chi(\cdot,t), \dot\chi(\cdot,t))\|_{\mathscr{E},R} = 0, \qquad \forall R > 0. \tag{3.4}$$

Let $k(\omega)$ be the analytic function with the domain $D := \mathbb{C}\backslash((-\infty, -m] \cup [m, +\infty))$ such that

$$k(\omega) = \sqrt{\omega^2 - m^2}, \qquad \operatorname{Im} k(\omega) > 0, \qquad \omega \in D. \tag{3.5}$$

Let us also denote its limit values for $\omega \in \mathbb{R}$ by

$$k_\pm(\omega) := k(\omega \pm i0), \qquad \omega \in \mathbb{R}. \tag{3.6}$$

As illustrated on Figure 1 (where all square roots take positive values), we have

$$k_-(\omega) = k_+(\omega) \quad \text{for} \quad -m \le \omega \le m, \qquad k_-(\omega) = -k_+(\omega) \quad \text{for} \quad \omega \in \mathbb{R}\backslash(-m, m), \tag{3.7}$$

and also

$$\omega\, k_+(\omega) \ge 0 \qquad \text{for} \quad \omega \in \mathbb{R}\backslash(-m, m). \tag{3.8}$$

We set $\mathcal{F}_{t\to\omega}[g(t)] = \int_{\mathbb{R}} e^{i\omega t} g(t)\, dt$ for a function $g(t)$ from the Schwartz space $\mathscr{S}(\mathbb{R})$. Let us study the Fourier transform $\hat\chi(x,\omega) := \mathcal{F}_{t\to\omega}[\chi(x,t)]$, which is a continuous function of x valued in tempered distributions.

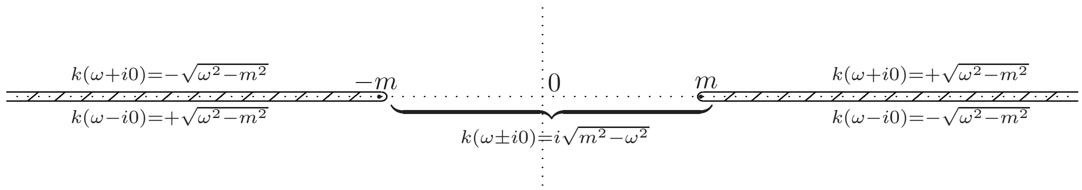

Figure 1: Domain D and the values of $k_\pm(\omega) := k(\omega \pm i0)$, $\omega \in \mathbb{R}$.

Lemma 3.2. • *$\hat{\chi}(x,\omega)$ is a continuous function of $x \in \mathbb{R}$ with values in $L^1_{\rm loc}(\mathbb{R})$, and*

$$\hat{\chi}(x,\omega) = 0, \qquad |\omega| < m. \tag{3.9}$$

• *The following bound holds:*

$$\sup_{x\in\mathbb{R}} \int_{|\omega|>m} |\hat{\chi}(x,\omega)|^2 \omega\, k_+(\omega)\, d\omega < \infty. \tag{3.10}$$

Proof. Set $\omega(k) = \operatorname{sgn} k \sqrt{m^2+k^2}$ for $k \in \mathbb{R}$. Note that the function $k_+(\omega)$ for $|\omega| > m$ is inverse to the function $\omega(k)$, $k \neq 0$. We have:

$$\chi(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ikx} \Big[\hat{\psi}_0(k)\cos(\omega(k)t) + \hat{\pi}_0(k)\frac{\sin(\omega(k)t)}{\omega(k)}\Big]\, dk. \tag{3.11}$$

Hence, for the Fourier transform of $\chi(x,t)$, we obtain, for any $x \in \mathbb{R}$:

$$\begin{aligned}
\hat{\chi}(x,\omega) &= \int_{\mathbb{R}} e^{-ikx} \Big[\hat{\psi}_0(k)\frac{\delta(\omega-\omega(k))+\delta(\omega+\omega(k))}{2} + \hat{\pi}_0(k)\frac{\delta(\omega-\omega(k))-\delta(\omega+\omega(k))}{2i\omega(k)}\Big]\, dk \\
&= \int_{|\omega'|>m} e^{-ik_+(\omega')x} \Big[\hat{\psi}_0(k_+(\omega'))\frac{\delta(\omega-\omega')+\delta(\omega+\omega')}{2} + \hat{\pi}_0(k_+(\omega'))\frac{\delta(\omega-\omega')-\delta(\omega+\omega')}{2i\omega'}\Big]\, \frac{\omega'\, d\omega'}{k_+(\omega')}.
\end{aligned}$$

The above relation is understood in the sense of distributions of $\omega \in \mathbb{R}$. We used the substitution $k = k_+(\omega')$. Now (3.9) is obvious. Evaluating the last integral, we get:

$$\hat{\chi}(x,\omega) = \frac{\omega}{2k_+(\omega)} \Big\{ e^{-ik_+(\omega)x}\hat{\psi}_0(k_+(\omega)) + e^{ik_+(\omega)x}\hat{\psi}_0(-k_+(\omega)) + e^{-ik_+(\omega)x}\frac{\hat{\pi}_0(k_+(\omega))}{i\omega} - e^{ik_+(\omega)x}\frac{\hat{\pi}_0(-k_+(\omega))}{i\omega} \Big\}, \qquad |\omega| > m.$$

We took into account that $k_+(-\omega) = -k_+(\omega)$ for $\omega \in \mathbb{R}\backslash(-m,m)$ (see (3.7)). Thus, we have:

$$\int\limits_{|\omega|>m} |\hat{\chi}(x,\omega)|^2 \omega\, k_+(\omega)\, d\omega \le \int\limits_{|\omega|>m} \Big[\frac{\omega^2|\hat{\psi}_0(k_+(\omega))|^2}{k_+^2(\omega)} + \frac{|\hat{\pi}_0(k_+(\omega))|^2}{k_+^2(\omega)}\Big]\omega\, k_+(\omega)\, d\omega = \int\limits_{\mathbb{R}} \Big[|\hat{\psi}_0(k)|^2 + \frac{|\hat{\pi}_0(k)|^2}{\omega^2(k)}\Big]\omega^2(k)\, dk.$$

The finiteness of the right-hand side follows from the finiteness of the energy of the initial data (ψ_0, π_0):

$$\|(\psi_0,\pi_0)\|^2_{\mathscr{E}} = \frac{1}{2\pi} \int_{\mathbb{R}} \Big[\omega^2(k)|\hat{\psi}_0(k)|^2 + |\hat{\pi}_0(k)|^2\Big]\, dk < \infty.$$

□

4 Spectral representation

The function $\varphi(x,t) = \psi(x,t) - \chi(x,t)$ satisfies the following Cauchy problem:

$$\ddot{\varphi}(x,t) = \varphi''(x,t) - m^2\varphi(x,t) + \sum_J \delta(x - X_J) f_J(t), \qquad (\varphi,\dot{\varphi})\big|_{t=0} = (0,0), \tag{4.1}$$

with $f_J(t)$ defined in (3.3). Note that $\psi(X_J,\cdot)\in C_b(\mathbb{R})$ for $1\le J\le N$ by the Sobolev embedding, since $(\psi(x,t),\dot\psi(x,t))\in C_b(\mathbb{R},\mathscr{E})$ by Theorem 2.4 (*iv*). Hence, $f_J(t)\in C_b(\mathbb{R})$. On the other hand, since $\chi(x,t)$ is a finite energy solution to the free Klein-Gordon equation, we also have

$$(\chi(x,t),\dot\chi(x,t))\in C_b(\mathbb{R},\mathscr{E}). \tag{4.2}$$

Therefore, the function $\varphi(x,t)=\psi(x,t)-\chi(x,t)$ satisfies

$$(\varphi(x,t),\dot\varphi(x,t))\in C_b(\mathbb{R},\mathscr{E}). \tag{4.3}$$

The Fourier transform

$$\hat\varphi(x,\omega)=\mathcal{F}_{t\to\omega}[\varphi(x,t)],\qquad (x,\omega)\in\mathbb{R}^2, \tag{4.4}$$

is a continuous function of $x\in\mathbb{R}$ with values in tempered distributions of $\omega\in\mathbb{R}$. It satisfies the following equation (Cf. (4.1)):

$$-\omega^2\hat\varphi(x,\omega)=\hat\varphi''(x,\omega)-m^2\hat\varphi(x,\omega)+\sum_J\delta(x-X_J)\hat f_J(\omega),\qquad (x,\omega)\in\mathbb{R}^2. \tag{4.5}$$

We are going to construct a representation for the solution $\hat\varphi(x,\omega)$ in a form suitable for our purposes.

Lemma 4.1. *$\hat\varphi$ is a smooth function of $x\in\mathbb{R}\backslash\mathcal{X}$ (where $\mathcal{X}=\{X_1,\,X_2,\,\dots,\,X_N\}$), with values in tempered distributions of $\omega\in\mathbb{R}$, and there exist quasimeasures $\hat\Phi_J^\pm$, $1\le J\le N$, and $\hat\Theta_J$, $1\le J\le N-1$, so that*

$$\hat\varphi(x,\omega)=\begin{cases}\hat\Phi_1^+(\omega)e^{-ik_+(\omega)(x-X_1)}+\hat\Phi_1^-(\omega)e^{-ik_-(\omega)(x-X_1)}, & x\le X_1,\\ \hat\Phi_J(\omega)\cos(k_+(\omega)(x-X_J))+\hat\Theta_J(\omega)\frac{\sin(k_+(\omega)(x-X_J))}{k_+(\omega)}, & x\in[X_J,X_{J+1}],\quad 1\le J\le N-1,\\ \hat\Phi_N^+(\omega)e^{ik_+(\omega)(x-X_N)}+\hat\Phi_N^-(\omega)e^{ik_-(\omega)(x-X_N)}, & x\ge X_N,\end{cases} \tag{4.6}$$

where $\hat\Phi_J(\omega):=\hat\Phi_J^+(\omega)+\hat\Phi_J^-(\omega)$.

Remark 4.2. A tempered distribution $\mu(\omega)\in\mathscr{S}'(\mathbb{R})$ is called a *quasimeasure* if $\check\mu(t)=\mathscr{F}^{-1}_{\omega\to t}[\mu(\omega)]\in C_b(\mathbb{R})$. For more details, see [KK07, Appendix B].

Remark 4.3. The representation (4.6) implies that

$$\hat\Phi_J(\omega)=\hat\varphi(X_J,\omega),\quad 1\le J\le N, \tag{4.7}$$

$$\hat\Phi_1^+(\omega)+\hat\Phi_1^-(\omega)=\hat\Phi_1(\omega)=\hat\varphi(X_1,\omega),\qquad \hat\Phi_N^+(\omega)+\hat\Phi_N^-(\omega)=\hat\varphi(X_N,\omega), \tag{4.8}$$

and also that

$$\hat\varphi'(X_J+0,\omega)=\hat\Theta_J(\omega),\quad 1\le J\le N-1. \tag{4.9}$$

Proof. **Step 1: Complex Fourier-Laplace transform.** We denote

$$f_J^\pm(t):=\theta(\pm t)f_J(t)=\theta(t)F_J(\psi(X_J,t)) \tag{4.10}$$

and split $\varphi(x,t)$ into

$$\varphi(x,t)=\varphi^+(x,t)+\varphi^-(x,t),\qquad \text{where}\quad \varphi^\pm(x,t):=\theta(\pm t)\varphi(x,t). \tag{4.11}$$

Then $\varphi^\pm(x,t)$ satisfy

$$\ddot\varphi^\pm(x,t)=\partial_x^2\varphi^\pm(x,t)-m^2\varphi^\pm(x,t)+\sum_J\delta(x-X_J)f_J^\pm(t),\qquad t\in\mathbb{R}, \tag{4.12}$$

since $(\varphi^\pm,\dot\varphi^\pm)|_{t=0}=(0,0)$. Let us analyze the complex Fourier-Laplace transforms of $\varphi^\pm(x,t)$:

$$\tilde\varphi^\pm(x,\omega)=\mathcal{F}_{t\to\omega}[\theta(\pm t)\varphi(x,t)]:=\int_0^\infty e^{i\omega t}\theta(\pm t)\varphi(x,t)\,dt,\quad \omega\in\mathbb{C}^\pm, \tag{4.13}$$

where $\mathbb{C}^\pm:=\{z\in\mathbb{C}:\ \pm\mathrm{Im}\,z>0\}$. Due to (4.3), $\tilde\varphi^\pm(\cdot,\omega)$ are H^1-valued analytic functions of $\omega\in\mathbb{C}^\pm$. In what follows, we will consider φ^+; the function φ^- considered in the same way.

Equation (4.12) implies that $\tilde\varphi^+$ satisfies

$$-\omega^2\tilde\varphi^+(x,\omega)=\partial_x^2\tilde\varphi^+(x,\omega)-m^2\tilde\varphi^+(x,\omega)+\sum_J\delta(x-X_J)\tilde f_J^+(\omega),\quad \omega\in\mathbb{C}^+. \tag{4.14}$$

The fundamental solutions $G_\pm(x,\omega)=\dfrac{e^{\pm ik(\omega)|x|}}{\pm 2ik(\omega)}$ satisfy

$$G''_\pm(x,\omega)+(\omega^2-m^2)G_\pm(x,\omega)=\delta(x),\qquad \omega\in\mathbb{C}^+.$$

The solution $\tilde\varphi^+(x,\omega)$ could be written as a linear combination of these fundamental solutions. We use the standard "limiting absorption principle" for the selection of the appropriate fundamental solution: Since $\tilde\varphi^+(\cdot,\omega)\in H^1$ for $\omega\in\mathbb{C}^+$, only G_+ is acceptable, because for $\omega\in\mathbb{C}^+$ the function $G_+(\cdot,\omega)$ is in H^1 by definition (3.5), while G_- is not. This suggests the following representation:

$$\tilde\varphi^+(x,\omega)=-\sum_J\tilde f_J^+(\omega)G_+(x-X_J,\omega)=-\sum_J\tilde f_J^+(\omega)\frac{e^{ik(\omega)|x-X_J|}}{2ik(\omega)},\qquad \omega\in\mathbb{C}^+. \tag{4.15}$$

The proof is straightforward since (4.15) belongs to $H^1(\mathbb{R})$ for $\omega\in\mathbb{C}^+$ while the solution to (4.14) which is an H^1-valued analytic function in ω is unique. For $x\le X_1$, the relation (4.15) yields

$$\tilde\varphi^+(x,\omega)=-\sum_J\tilde f_J^+(\omega)\frac{e^{-ik(\omega)(x-X_J)}}{2ik(\omega)}=e^{-ik(\omega)(x-X_1)}\tilde\varphi^+(X_1,\omega),\qquad x\le X_1,\quad \omega\in\mathbb{C}^+. \tag{4.16}$$

For $x\in[X_J,X_{J+1}]$, $1\le J\le N-1$, the relation (4.15) implies that

$$\tilde\varphi^+(x,\omega)=\tilde\Phi_J^+(\omega)\cos(k(\omega)(x-X_J))+\tilde\Theta_J^+(\omega)\frac{\sin(k(\omega)(x-X_J))}{k(\omega)},\qquad x\in[X_J,X_{J+1}],\quad \omega\in\mathbb{C}^+, \tag{4.17}$$

where $\tilde\Phi_J^+$ and $\tilde\Theta_J^+$, $1\le J\le N-1$, are analytic functions of $\omega\in\mathbb{C}^+$. We note that, by (4.15),

$$\tilde\Phi_J^+(\omega)=\tilde\varphi^+(X_J,\omega),\qquad \tilde\Theta_J^+(\omega)=\partial_x\tilde\varphi^+(X_J+0,\omega)=-\sum_{J'}\operatorname{sgn}(X_J-X_{J'})\tilde f_{J'}^+(\omega)\frac{e^{ik(\omega)|X_J-X_{J'}|}}{2}. \tag{4.18}$$

Step 2: Traces on real line. Now we need to extend the relations (4.16) and (4.17) to $\omega\in\mathbb{R}$. The Fourier transform $\hat\varphi^+(x,\omega):=\mathcal{F}_{t\to\omega}[\theta(t)\varphi(x,t)]$ is a tempered H^1-valued distribution of $\omega\in\mathbb{R}$ by (4.3). It is the boundary value of the analytic function $\tilde\varphi^+(x,\omega)$, in the following sense:

$$\hat\varphi^+(x,\omega)=\lim_{\varepsilon\to0+}\tilde\varphi^+(x,\omega+i\varepsilon),\qquad \omega\in\mathbb{R}, \tag{4.19}$$

where the convergence is in the space of tempered distributions $\mathscr{S}'(\mathbb{R},H^1(\mathbb{R}))$. Indeed,

$$\tilde\varphi^+(x,\omega+i\varepsilon)=\mathcal{F}_{t\to\omega}[\theta(t)\varphi(x,t)e^{-\varepsilon t}],\qquad \theta(t)\varphi(x,t)e^{-\varepsilon t}\underset{\varepsilon\to0+}{\longrightarrow}\theta(t)\varphi(x,t),$$

where the convergence holds in $\mathscr{S}'(\mathbb{R},H^1(\mathbb{R}))$. Therefore, (4.19) holds by the continuity of the Fourier transform $\mathcal{F}_{t\to\omega}$ in $\mathscr{S}'(\mathbb{R})$.

The distributions $\hat\Phi_J^+(\omega)$, $\hat\Theta_J^+(\omega)\in\mathscr{S}'(\mathbb{R})$, $\omega\in\mathbb{R}$, are defined as the boundary values of the functions $\tilde\Phi_J^+(\omega)$ and $\tilde\Theta_J^+(\omega)$ analytic in $\omega\in\mathbb{C}^+$:

$$\hat\Phi_J^+(\omega)=\lim_{\varepsilon\to0+}\tilde\Phi_J^+(\omega+i\varepsilon),\qquad \omega\in\mathbb{R},\quad 0\le J\le N, \tag{4.20}$$

$$\hat\Theta_J^+(\omega)=\lim_{\varepsilon\to0+}\tilde\Theta_J^+(\omega+i\varepsilon),\qquad \omega\in\mathbb{R},\quad 1\le J\le N-1. \tag{4.21}$$

The above convergence holds in the space of quasimeasures by (4.18), since $\tilde\varphi^+(X_J,\omega)$ and $\tilde f_J^+(\omega)$ are quasimeasures (see Remark 4.2) while the exponential factors in (4.18) are multiplicators in the space of quasimeasures [KK07, Appendix B]. Therefore, the formulas (4.17) with $1\le J\le N-1$ imply, in the limit $\operatorname{Im}\omega\to0+$, that

$$\hat\varphi^+(x,\omega)=\hat\Phi_J^+(\omega)\cos(k(\omega+i0)(x-X_J))+\hat\Theta_J^+(\omega)\frac{\sin(k(\omega+i0)(x-X_J))}{k(\omega+i0)},\qquad x\in[X_J,X_{J+1}],\quad \omega\in\mathbb{R}, \tag{4.22}$$

since $\cos(k(\omega+i0)(x-X_J))$ and $\frac{\sin(k(\omega+i0)(x-X_J))}{k(\omega+i0)}$ are smooth functions of $\omega\in\mathbb{R}$. Similar representation holds for $\hat{\varphi}^-(x,\omega)$. Therefore, the representation (4.6) follows for $X_1\le x\le X_N$.

The formula (4.6) for $x\le X_1$ follows from taking the limit $\operatorname{Im}\omega\to 0+$ in the expression (4.16) for $\tilde{\varphi}^+(x,\omega)$ and the limit $\operatorname{Im}\omega\to 0-$ in a similar expression for $\tilde{\varphi}^-(x,\omega)$:

$$\tilde{\varphi}^-(x,\omega)=-\sum_J \tilde{f}_J^-(\omega)\frac{e^{-ik(\omega)(x-X_J)}}{2ik(\omega)}=e^{-ik(\omega)(x-X_1)}\tilde{\varphi}^-(X_1,\omega),\qquad x\le X_1,\quad \omega\in\mathbb{C}^-, \tag{4.23}$$

and then taking the sum of the resulting expressions. This justifies (4.6) for $x\le X_1$. Similarly we justify (4.6) for $x\ge X_N$. □

5 Absolute continuity of the spectrum

Lemma 5.1. *The distributions $\hat{\Phi}_1^\pm(\omega)$, $\hat{\Phi}_N^\pm(\omega)$ are absolutely continuous for $|\omega|>m$, and moreover*

$$\int_{|\omega|>m}\Big[|\hat{\Phi}_1^\pm(\omega)|^2+|\hat{\Phi}_N^\pm(\omega)|^2\Big]\,\omega\,k_+(\omega)\,d\omega<\infty, \tag{5.1}$$

where $\omega\,k_+(\omega)\ge 0$ by (3.8).

The bound for each of $\hat{\Phi}_1^\pm(\omega)$, $\hat{\Phi}_N^\pm(\omega)$ is obtained verbatim by applying the proof of [KK07, Proposition 3.3].

Proposition 5.2. *The distributions $\hat{\Phi}_J(\omega)$, $1\le J\le N$, and $\hat{\Theta}_J(\omega)$, $1\le J\le N-1$, are absolutely continuous for $|\omega|>\mu_J$ and $|\omega|>(2p_J-1)\mu_J$, respectively, with μ_J defined in (2.19). Moreover, for any $\epsilon>0$,*

$$\int_{|\omega|>\mu_J+\epsilon}|\hat{\Phi}_J(\omega)|^2\omega^2\,d\omega<\infty,\quad 1\le J\le N;\qquad \int_{|\omega|>(2p_J-1)\mu_J+\epsilon}|\hat{\Theta}_J(\omega)|^2\,d\omega<\infty,\quad 1\le J\le N-1. \tag{5.2}$$

Proof. We will use induction, proving the absolute continuity of $\hat{\varphi}(X_J,\omega)$ and $\partial_x\hat{\varphi}(X_J\pm 0,\omega)$ starting with $J=1$ and going to $J=N$. By Lemma 4.1, $\hat{\varphi}(X_1,\omega)=\hat{\Phi}_1(\omega)=\hat{\Phi}_1^+(\omega)+\hat{\Phi}_1^-(\omega)$ and $\partial_x\hat{\varphi}(X_1-0,\omega)=-ik_+(\omega)\hat{\Phi}_1^+(\omega)-ik_-(\omega)\hat{\Phi}_1^-(\omega)$. Hence, Lemma 5.1 implies that, for any $\epsilon>0$,

$$\int_{|\omega|>m+\epsilon}|\hat{\varphi}(X_1,\omega)|^2\omega^2\,d\omega<\infty,\qquad \int_{|\omega|>m+\epsilon}|\hat{\varphi}'(X_1-0,\omega)|^2\,d\omega<\infty. \tag{5.3}$$

Now assume that for some $1\le J<N$ and for any $\epsilon>0$ we have:

$$\int_{|\omega|>\mu_J+\epsilon}|\hat{\varphi}(X_J,\omega)|^2\omega^2\,d\omega<\infty,\qquad \int_{|\omega|>\mu_J+\epsilon}|\hat{\varphi}'(X_J-0,\omega)|^2\,d\omega<\infty. \tag{5.4}$$

Lemma 4.1 and equation (4.5) yield the jump condition

$$\hat{\Theta}_J(\omega)=\hat{\varphi}'(X_J+0,\omega)=\hat{\varphi}'(X_J-0,\omega)-\hat{f}_J(\omega),\qquad \omega\in\mathbb{R}, \tag{5.5}$$

where $f_J(t)=F_J(\psi(X_J,t))$ by (3.3).

Lemma 5.3. *For any $\epsilon>0$ the following inequality holds:*

$$\int_{|\omega|>(2p_J-1)(\mu_J+2\epsilon)}|\hat{f}_J(\omega)|^2\,d\omega<\infty. \tag{5.6}$$

Proof. Let $\zeta_J(\omega)\in C_0^\infty(\mathbb{R})$ be such that $\zeta_J(\omega)\equiv 1$ for $|\omega|\le\mu_J+\epsilon$ and $\zeta_J(\omega)\equiv 0$ for $|\omega|\ge\mu_J+2\epsilon$. We denote $\psi(X_J,t)$ by $\uppsi_J(t)$, and split it into

$$\uppsi_J(t)=\uppsi_{J,b}(t)+\uppsi_{J,d}(t), \tag{5.7}$$

where the functions in the right-hand side are defined by their Fourier transforms:

$$\hat{\uppsi}_{J,b}(\omega)=\zeta_J(\omega)\hat{\uppsi}_J(\omega)=\zeta_J(\omega)\hat{\psi}(X_J,\omega),\qquad \hat{\uppsi}_{J,d}(\omega)=(1-\zeta_J(\omega))\hat{\uppsi}_J(\omega)=(1-\zeta_J(\omega))\hat{\psi}(X_J,\omega). \tag{5.8}$$

By Lemma 3.2 and by (5.4), we have

$$\int_{\mathbb{R}} |(1-\zeta_J(\omega))\hat{\chi}(X_J,\omega)|^2\,\omega^2\,d\omega<\infty, \qquad \int_{\mathbb{R}} |(1-\zeta_J(\omega))\hat{\varphi}(X_J,\omega)|^2\,\omega^2\,d\omega<\infty. \tag{5.9}$$

Since $\hat{\psi}_{J,d}(\omega)=(1-\zeta_J(\omega))(\hat{\chi}(X_J,\omega)+\hat{\varphi}(X_J,\omega))$, we also have

$$\int_{\mathbb{R}} \left|(1-\zeta_J(\omega))\hat{\psi}_J(\omega)\right|^2\omega^2\,d\omega<\infty,$$

proving that

$$\psi_{J,d}(t)\in H^1(\mathbb{R}). \tag{5.10}$$

For $\hat{f}_J(\omega)=\mathcal{F}_{t\to\omega}[F_J(\psi_J(t))]=\mathcal{F}_{t\to\omega}[F_J(\psi(X_J,t))]$, taking into account (2.17) and (5.7), we have:

$$\begin{aligned}\hat{f}_J(\omega) &= -\sum_{n=1}^{p_J}2n\,u_{J,n}\,\underbrace{(\hat{\psi}_J*\hat{\overline{\psi}}_J)*\ldots*(\hat{\psi}_J*\hat{\overline{\psi}}_J)}_{n-1}*\hat{\psi}_J\\ &= \;.\,.\,.\,.\,.\;-\sum_{n=1}^{p_J}2n\,u_{J,n}\,\underbrace{(\hat{\psi}_{J,b}*\hat{\overline{\psi}}_{J,b})*\ldots*(\hat{\psi}_{J,b}*\hat{\overline{\psi}}_{J,b})}_{n-1}*\hat{\psi}_{J,b},\end{aligned} \tag{5.11}$$

where the dots in the right-hand side denote the convolutions of $\hat{\psi}_{J,b}$, $\hat{\overline{\psi}}_{J,b}$, $\hat{\psi}_{J,d}$, and $\hat{\overline{\psi}}_{J,d}$ that contain at least one of $\hat{\psi}_{J,d}$, $\hat{\overline{\psi}}_{J,d}$. Since $\psi_{J,b}(t)$, $\psi_{J,d}(t)$ are bounded while $\psi_{J,d}(t)\in H^1(\mathbb{R})$ by (5.10), all these terms belong to $L^2(\mathbb{R})$. Finally, since $\operatorname{supp}\hat{\psi}_{J,b}\subset[-\mu_J-2\epsilon,\mu_J+2\epsilon]$, the convolutions under the summation sign in the right-hand side of (5.11) are supported inside $[-(2p_J-1)(\mu_J+2\epsilon),(2p_J-1)(\mu_J+2\epsilon)]$ and do not contribute into the integral (5.6). □

Using (5.4) and Lemma 5.3 to estimate the norms of $\partial_x\hat{\varphi}(X_J-0,\omega)$ and $\hat{f}_J(\omega)$ in the right-hand side in the relation (5.5), we conclude that

$$\int\limits_{|\omega|>(2p_J-1)(\mu_J+2\epsilon)} |\hat{\varphi}'(X_J+0,\omega)|^2\,d\omega<\infty. \tag{5.12}$$

Now the inequalities

$$\int\limits_{|\omega|>(2p_J-1)(\mu_J+2\epsilon)} |\hat{\varphi}(X_{J+1},\omega)|^2\omega^2\,d\omega<\infty, \qquad \int\limits_{|\omega|>(2p_J-1)(\mu_J+2\epsilon)} |\hat{\varphi}'(X_{J+1}-0,\omega)|^2\,d\omega<\infty \tag{5.13}$$

follow from the representation (4.6) for $x\in[X_J,X_{J+1}]$, where we apply the first inequality from (5.4) and the inequality (5.12). Therefore, starting with (5.3), one shows by induction that (5.4) holds for all $1\le J\le N$. The estimates on $\hat{\Phi}_J(\omega)=\hat{\varphi}(X_J,\omega)$ and $\hat{\Theta}_J(\omega)=\hat{\varphi}'(X_J+0,\omega)$ stated in the Proposition follow from (5.4) and (5.12), respectively. This finishes the proof of Proposition 5.2. □

Corollary 5.4. *The distributions $\hat{\Phi}_J(\omega)=\hat{\varphi}(X_J,\omega)$, $1\le J\le N$, are absolutely continuous for $|\omega|>M_J$, while $\hat{\Theta}_J(\omega)=\partial_x\hat{\varphi}(X_J+0,\omega)$, $1\le J\le N-1$, are absolutely continuous for $|\omega|>(2p_J-1)M_J$, where $M_J:=\min(\mu_J,\mu_J')$ is defined in (2.20).*

Proof. In the proof of Proposition 5.2, we could as well proceed from $J=N$ to $J=1$, proving the result stated in the Corollary. □

6 Compactness

Second dispersive component

Let $\zeta(\omega)\in C_0^\infty(\mathbb{R})$ be such that $\zeta(\omega)\equiv 1$ for $|\omega|<\Lambda$, where Λ is from (2.20). Define $\varphi_d(x,t)$ by its Fourier transform:

$$\hat{\varphi}_d(x,\omega):=(1-\zeta(\omega))\hat{\varphi}(x,\omega)\quad x\in\mathbb{R},\;\;\omega\in\mathbb{R}. \tag{6.1}$$

Lemma 6.1. *$\varphi_d(x,t)$ is a bounded continuous function of $t\in\mathbb{R}$ with values in $H^1(\mathbb{R})$:*

$$\varphi_d(x,t)\in C_b(\mathbb{R},H^1(\mathbb{R})). \tag{6.2}$$

The local energy decay holds for $\varphi_d(x,t)$:

$$\lim_{t\to\infty}\|(\varphi_d,\dot\varphi_d)\|_{\mathscr{E},R}=0,\qquad \forall R>0. \tag{6.3}$$

Proof. We generalize the proof of [KK07, Proposition 3.6]. By Lemma 4.1,

$$\hat\varphi_d(x,\omega)=\begin{cases}(1-\zeta(\omega))\left[\hat\Phi_1^+(\omega)e^{-ik_+(\omega)(x-X_1)}+\hat\Phi_1^-(\omega)e^{-ik_-(\omega)(x-X_1)}\right], & x\le X_1,\\ (1-\zeta(\omega))\hat\Phi_J(\omega)\cos(k_+(\omega)(x-X_J))+(1-\zeta(\omega))\hat\Theta_J(\omega)\frac{\sin(k_+(\omega)(x-X_J))}{k_+(\omega)(x-X_J)}, & x\in[X_J,X_{J+1}],\\ (1-\zeta(\omega))\left[\hat\Phi_N^+(\omega)e^{ik_+(\omega)(x-X_N)}+\hat\Phi_N^-(\omega)e^{ik_-(\omega)(x-X_N)}\right], & x\ge X_N.\end{cases} \tag{6.4}$$

Each of the functions entering the above expression, considered on the whole real line, corresponds to a finite energy solution to a linear Klein-Gordon equation, satisfying the properties stated in the lemma. For example, define $u(x,t)$ by its Fourier transform:

$$\hat u(x,\omega):=(1-\zeta(\omega))\hat\Phi_1(\omega)\cos(k_+(\omega)(x-X_1)),\qquad x\in\mathbb{R}.$$

Then $u(x,t)$ is a solution to a linear Klein-Gordon equation, and, by Proposition 5.2, the corresponding initial data are of finite energy:

$$(u(x,0),\dot u(x,0))\in\mathscr{E}.$$

Hence $u(x,t)\in C_b(\mathbb{R},H^1(\mathbb{R}))$ and satisfies the local energy decay of the form (6.3) (see [KK07, Lemma 3.1]. This finishes the proof. □

Compactness for the bound component

We introduce the bound component of $\varphi(x,t)$ by

$$\varphi_b(x,t)=\varphi(x,t)-\varphi_d(x,t)=\psi(x,t)-\chi(x,t)-\varphi_d(x,t),\qquad x\in\mathbb{R},\ \ t\in\mathbb{R}. \tag{6.5}$$

By Lemma 6.1,

$$\varphi_b(x,t)\in C_b(\mathbb{R},H^1(\mathbb{R})). \tag{6.6}$$

Lemma 4.1 and (6.1), (6.5) imply the multiplicative relation

$$\hat\varphi_b(x,\omega)=\begin{cases}\zeta(\omega)\big[\hat\Phi_1^+(\omega)e^{-ik_+(\omega)(x-X_1)}+\hat\Phi_1^-(\omega)e^{-ik_-(\omega)(x-X_1)}\big], & x\le X_1,\\ \zeta(\omega)\big[\hat\Phi_J(\omega)\cos(k_+(\omega)(x-X_J))+\hat\Theta_J(\omega)\frac{\sin(k_+(\omega)(x-X_J))}{k_+(\omega)}\big], & x\in[X_J,X_{J+1}],\\ \zeta(\omega)\big[\hat\Phi_N^+(\omega)e^{ik_+(\omega)(x-X_N)}+\hat\Phi_N^-(\omega)e^{ik_-(\omega)(x-X_N)}\big], & x\ge X_N.\end{cases} \tag{6.7}$$

By (6.6), the functions

$$\varphi_{b,J}(t):=\varphi_b(X_J,t)=\varphi(X_J,t)-\varphi_d(X_J,t)$$

are bounded and continuous. Therefore, $\hat\varphi_b(X_J,\cdot)\in\mathscr{S}'(\mathbb{R})$ are quasimeasures (see Remark 4.2).

Proposition 6.2. (*i*) *The function $\varphi_b(x,t)$ is smooth for $x\in\mathbb{R}\backslash\mathcal{X}$ (where $\mathcal{X}=\{X_1,\,X_2,\,\dots,\,X_N\}$) and $t\in\mathbb{R}$.*

(*ii*) *For any $R>0$,*

$$\sup_{|x|\le R,\,x\notin\mathcal{X}}\ \sup_{t\in\mathbb{R}}|\partial_x^m\partial_t^n\varphi_b(x,t)|<\infty. \tag{6.8}$$

The argument repeats the proof of Proposition [KK07, Proposition 4.1].

Remark 6.3. Let us note that the bounds (6.8) are independent of x and remain valid for $x\notin\mathcal{X}$, although the derivatives $\partial_x^m\partial_t^n\varphi_b(x,t)$ with $m\ne 0$ may have jumps at $x=X_J$. (Note that this is the case for the solitary waves in (2.15).)

We now may deduce the compactness of the set of translations of the bound component, $\{\varphi_b(x,s+t)\colon s\ge 0\}$.

Corollary 6.4. (i) *By the Ascoli-Arzelà Theorem, for any sequence $s_j \to \infty$ there exists a subsequence $s_{j'} \to \infty$ such that*

$$\varphi_b(x, s_{j'} + t) \to \beta(x,t), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{6.9}$$

and also for any nonnegative integers m and n,

$$\partial_x^m \partial_t^n \varphi_b(x, s_{j'} + t) \to \partial_x^m \partial_t^n \beta(x,t), \qquad x \notin \mathcal{X}, \quad t \in \mathbb{R}, \tag{6.10}$$

for some $\beta(x,t) \in C_b(\mathbb{R}, H^1(\mathbb{R}))$. The convergence in (6.9) and (6.10) is uniform in x and t as long as $|x|+|t| \le R$, for any $R > 0$. The convergence in (6.10) also holds for $x = X_J \pm 0$.

(ii) *By the Fatou Lemma,*

$$\sup_{t\in\mathbb{R}} \|\beta(\cdot,t)\|_{H^1} < \infty. \tag{6.11}$$

We call *omega-limit trajectory* any function $\beta(x,t)$ that can appear as a limit in (6.9), (6.10).

Remark 6.5. Previous analysis demonstrates that the long-time asymptotics of the solution $\psi(x,t)$ in $\mathscr{E}_F$ depends only on the singular component $\varphi(x,t)$. Due to Corollary 6.4, to conclude the proof of Theorem 2.10, it suffices to check that every omega-limit trajectory belongs to the set of solitary waves; that is,

$$\beta(x,t) = \phi_{\omega_+}(x) e^{-i\omega_+ t} \qquad \text{for some } \omega_+ \in [-m, m]. \tag{6.12}$$

7 Nonlinear spectral analysis

Bounds for the spectrum

By Lemmas 3.1 and 6.1, the dispersive components $\chi(\cdot,t)$ and $\varphi_d(\cdot,t)$ converge to zero in $\mathscr{E}_F$ as $t \to \infty$. On the other hand, by Corollary 6.4, the bound component $\varphi_b(x, t + s_{j'})$ converges to $\beta(x,t)$ as $j' \to \infty$, uniformly in every compact set of the plane $\mathbb{R}^2$. Hence, $\psi(x, t + s_{j'}) = \varphi_b(x, t + s_{j'}) + \chi(x, t + s_{j'}) + \varphi_d(x, t + s_{j'})$ also converges to $\beta(x,t)$ uniformly in every compact set of the plane $\mathbb{R}^2$. Therefore, taking the limit in equation (2.1), we conclude that the omega-limit trajectory $\beta(x,t)$ also satisfies the same equation:

$$\ddot\beta(x,t) = \beta''(x,t) - m^2\beta(x,t) + \sum_J \delta(x - X_J) F_J(\beta). \tag{7.1}$$

Remark 7.1. Note that the bound component $\varphi_b(x,t)$ itself generally does not satisfy equation (7.1).

Taking the Fourier transform of β in time, we see by (6.10) that $\hat\beta(x,\omega)$ is a continuous function of $x \in \mathbb{R}$, smooth for $x \in \mathbb{R}\backslash\mathcal{X}$, with values in tempered distributions of $\omega \in \mathbb{R}$, and that it satisfies the corresponding stationary equation

$$-\omega^2\hat\beta(x,\omega) = \hat\beta''(x,\omega) - m^2\hat\beta(x,\omega) + \sum_J \delta(x - X_J)\hat g_J(\omega), \qquad (x,\omega) \in \mathbb{R}^2, \tag{7.2}$$

valid in the sense of tempered distributions of $(x,\omega) \in \mathbb{R}^2$, where $\hat g_J(\omega)$ are the Fourier transforms of the functions

$$g_J(t) := F_J(\beta(X_J, t)), \qquad 1 \le J \le N. \tag{7.3}$$

We also denote

$$\mathrm{\beta}_J(t) := \beta(X_J, t), \qquad \Sigma_J := \operatorname{supp} \hat{\mathrm{\beta}}_J, \qquad 1 \le J \le N. \tag{7.4}$$

From (6.7), we know that the spectrum of $\varphi_b(x,t)$ is bounded for all $x \in \mathbb{R}$. Hence, the convergence (6.10) implies that the spectrum of $\beta(x,t)$ is also bounded. We will need more precise bounds on the size of the spectrum of β:

Lemma 7.2. (i) $\Sigma_J := \operatorname{supp} \hat{\mathrm{\beta}}_J \subset [-M_J, M_J], \quad 1 \le J \le N$;

(ii) $\operatorname{supp} \hat\beta'(X_J + 0, \omega) \subset [-(2p_J - 1)M_J, (2p_J - 1)M_J], \quad 1 \le J \le N - 1$, *with $M_J > 0$ defined in (2.20).*

Proof. We have the relation

$$\varphi_b(x, s_j + t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} e^{-i\omega s_j} \hat{\varphi}_b(x, \omega)\, d\omega, \qquad x \in \mathbb{R}, \quad t \in \mathbb{R},$$

where the integral is understood as the pairing of a smooth function (oscillating exponent) with a compactly supported distribution. Then the convergence (6.9) implies that

$$e^{-i\omega s_{j'}} \hat{\varphi}_b(x, \omega) \to \hat{\beta}(x, \omega), \qquad x \in \mathbb{R}, \quad s_{j'} \to \infty, \tag{7.5}$$

in the sense of quasimeasures. Since $\hat{\varphi}_b(X_J, \omega)$ is locally L^2 for $|\omega| > M_J$ by Corollary 5.4, the convergence (7.5) at $x = X_J$ shows that $\hat{\upbeta}_J(\omega) := \hat{\beta}(X_J, \omega)$ vanishes for $|\omega| > M_J$. This proves the first statement of the lemma.

The second statement is proved similarly. Namely, the convergence (6.10) implies that

$$e^{-i\omega s_{j'}} \partial_x \hat{\varphi}_b(X_J + 0, \omega) \to \partial_x \hat{\beta}(X_J + 0, \omega), \qquad s_{j'} \to \infty, \tag{7.6}$$

in the sense of quasimeasures. Since $\hat{\varphi}_b'(X_J + 0, \omega)$ is locally L^2 for $|\omega| > (2p_J - 1)M_J$ by Corollary 5.4, the convergence (7.6) shows that $\hat{\beta}'(X_J + 0, \omega)$ vanishes for $|\omega| > (2p_J - 1)M_J$. □

We denote

$$\kappa(\omega) := -ik_+(\omega), \qquad \omega \in \mathbb{R}, \tag{7.7}$$

where $k_+(\omega)$ was introduced in (3.6). We then have $\operatorname{Re} \kappa(\omega) \geq 0$, and also

$$\kappa(\omega) = \sqrt{\omega^2 - m^2} > 0 \quad \text{for} \quad -m < \omega < m,$$

in accordance with (2.15).

Proposition 7.3. *The distribution $\hat{\beta}(x, \omega)$ admits the following representation:*

$$\hat{\beta}(x, \omega) = \begin{cases} \hat{\upbeta}_1(\omega) e^{\kappa(\omega)(x - X_1)}, & x \leq X_1, \\ \hat{\upbeta}_J(\omega) \cosh(\kappa(\omega)(x - X_J)) + \hat{\beta}'(X_J + 0, \omega) \frac{\sinh(\kappa(\omega)(x - X_J))}{\kappa(\omega)}, & x \in [X_J, X_{J+1}], \quad 1 \leq J \leq N - 1, \\ \hat{\upbeta}_N(\omega) e^{-\kappa(\omega)(x - X_N)}, & x \geq X_N. \end{cases} \tag{7.8}$$

Proof. By (7.5), the middle line in (7.8) follows from the representation (4.6) since the multiplicators are smooth bounded functions of $\omega \in \mathbb{R}$. Taking the limit in the first line of (4.6), we obtain the first line in (7.8) since $\Sigma_1 \subset [-m, m]$ by Lemma 7.2, while $k_+(\omega) = k_-(\omega) = i\kappa(\omega)$ for $-m \leq \omega \leq m$ (Cf. (3.7), (7.7)). Similarly we explain the last line in (7.8). □

Reduction to point spectrum

Proposition 7.4. *Any omega-limit trajectory $\beta(x, t)$ is a solitary wave:*

$$\beta(x, t) = \phi(x) e^{-i\omega_+ t} \quad \text{with} \quad \omega_+ \in [-m, m] \quad \text{and} \quad \phi(x) \in H^1(\mathbb{R}).$$

Proof. The proof is based on the following lemmas.

Lemma 7.5. *If $\Sigma_1 = \emptyset$, then $\beta(x, t) \equiv 0$.*

Proof. According to equation (7.2), the function $\hat{\beta}$ satisfies the following continuity and jump conditions at the point X_1:

$$\hat{\beta}(X_1 + 0, \omega) = \hat{\beta}(X_1 - 0, \omega) = \hat{\upbeta}_1(\omega), \qquad \hat{\beta}'(X_1 + 0, \omega) = \hat{\beta}'(X_1 - 0, \omega) + \hat{g}_1(\omega), \qquad \omega \in \mathbb{R}. \tag{7.9}$$

$\Sigma_1 = \emptyset$ means that $\hat{\upbeta}_1(\omega) \equiv 0$, that is, $\upbeta_1(t) \equiv 0$. Hence, $g_1(t) \equiv F_1(\upbeta_1(t)) \equiv 0$, and $\hat{g}_1(\omega) \equiv 0$. On the other hand, the first line of (7.8) implies that $\hat{\beta}(x, \omega) \equiv 0$ for $x \leq X_1$, and in particular $\hat{\beta}'(X_1 - 0, \omega) \equiv 0$. Therefore, the jump condition (7.9) implies that $\hat{\beta}'(X_1 + 0, \omega) \equiv 0$. Hence, $\hat{\beta}(x, \omega) \equiv 0$ for $x \in [X_1, X_2]$ by the middle line of (7.8). By induction, $\hat{\upbeta}_J(x, \omega) \equiv 0$. □

Now we consider the case $\Sigma_1 \neq \emptyset$.

Lemma 7.6. *If $\Sigma_1 \neq \emptyset$, then $\Sigma_1 = \{\omega_+\}$ for some $\omega_+ \in [-m, m]$.*

Proof. By Lemma 7.2, we know that $\Sigma_1 \subset [-m, m]$. To show that Σ_1 consists of a single point, we assume that, on the contrary, $\inf \Sigma_1 < \sup \Sigma_1$. By (2.17), the Fourier transform $\hat{g}_1(\omega)$ of $g_1(t) := F_1(\beta(X_1, t))$ is given by

$$\hat{g}_1 = -\sum_{n=1}^{p_1} 2n\, u_{1,n} \underbrace{(\hat{\beta}_1 * \hat{\overline{\beta}}_1) * \ldots * (\hat{\beta}_1 * \hat{\overline{\beta}}_1)}_{n-1} * \hat{\beta}_1. \tag{7.10}$$

Applying the Titchmarsh Convolution Theorem [Tit26] (see also [Lev96, p.119] and [Hör90, Theorem 4.3.3]) to the convolutions in (7.10), we obtain the following equalities:

$$\inf \operatorname{supp} \hat{g}_1 = \inf \operatorname{supp} \hat{\beta}_1 + (p_1 - 1) \inf \operatorname{supp}(\hat{\beta}_1 * \hat{\overline{\beta}}_1) = \inf \Sigma_1 + (p_1 - 1)(\inf \Sigma_1 - \sup \Sigma_1), \tag{7.11}$$

$$\sup \operatorname{supp} \hat{g}_1 = \sup \operatorname{supp} \hat{\beta}_1 + (p_1 - 1) \sup \operatorname{supp}(\hat{\beta}_1 * \hat{\overline{\beta}}_1) = \sup \Sigma_1 + (p_1 - 1)(\sup \Sigma_1 - \inf \Sigma_1), \tag{7.12}$$

where we used the relations $\inf \operatorname{supp} \hat{\overline{\beta}}_1 = -\sup \operatorname{supp} \hat{\beta}_1$, $\sup \operatorname{supp} \hat{\overline{\beta}}_1 = -\inf \operatorname{supp} \hat{\beta}_1$. Note that the Titchmarsh theorem is applicable since $\operatorname{supp} \hat{\beta}_1$ is compact by Lemma 7.2. Since we assumed that $\inf \Sigma_1 < \sup \Sigma_1$, (7.11) and (7.12) imply that $\inf \operatorname{supp} \hat{g}_1 < \inf \Sigma_1$, $\sup \operatorname{supp} \hat{g}_1 > \sup \Sigma_1$. Therefore, the jump condition (7.9) with $J = 1$ implies that

$$\inf \operatorname{supp} \hat{\beta}'(X_1 + 0, \cdot) = \inf \operatorname{supp} \hat{g}_1 < \inf \Sigma_1, \qquad \sup \operatorname{supp} \hat{\beta}'(X_1 + 0, \cdot) = \sup \operatorname{supp} \hat{g}_1 > \sup \Sigma_1. \tag{7.13}$$

The ratio $\sinh(\kappa(\omega)(X_2 - X_1))/\kappa(\omega)$ could only vanish at the points $\omega = \pm\omega_{1,n}$, where

$$\omega_{J,n} := \sqrt{\frac{\pi^2 n^2}{|X_{J+1} - X_J|^2} + m^2}, \qquad 1 \le J \le N - 1, \quad n \in \mathbb{N}.$$

Due to Assumption 2.3 and Lemma 7.2, $\operatorname{supp} \hat{\beta}'(X_1 + 0, \omega) \cap \{\pm\omega_{1,n} \colon n \in \mathbb{N}\} = \emptyset$. Hence, the middle line of (7.8) at $x = X_2 - 0$ and the inequalities (7.13) imply that

$$\inf \Sigma_2 = \inf \operatorname{supp} \hat{g}_1 < \inf \Sigma_1, \qquad \sup \Sigma_2 = \sup \operatorname{supp} \hat{g}_1 > \sup \Sigma_1. \tag{7.14}$$

We proceed by induction, proving that

$$\inf \Sigma_1 > \inf \Sigma_2 > \ldots > \inf \Sigma_N, \qquad \sup \Sigma_1 < \sup \Sigma_2 < \ldots < \sup \Sigma_N. \tag{7.15}$$

It then follows that $\inf \Sigma_N < \sup \Sigma_N$. Starting from $J = N$ and going to the left, we also prove the opposite inequalities:

$$\inf \Sigma_1 < \inf \Sigma_2 < \ldots < \inf \Sigma_N, \qquad \sup \Sigma_1 > \sup \Sigma_2 > \ldots > \sup \Sigma_N. \tag{7.16}$$

The contradiction of (7.15) and (7.16) shows that our assumption that $\inf \Sigma_1 < \sup \Sigma_1$ was false, hence $\Sigma_1 = \{\omega_+\}$ for some $\omega_+ \in [-m, m]$. ∎

Thus, $\operatorname{supp} \hat{\beta}_1(\omega) = \Sigma_1 \subset \{\omega_+\}$, with $\omega_+ \in [-m, m]$. Therefore,

$$\hat{\beta}_1(\omega) = a_1 \delta(\omega - \omega_+), \qquad \text{with some } a_1 \in \mathbb{C}. \tag{7.17}$$

Note that the derivatives $\delta^{(k)}(\omega - \omega_+)$, $k \ge 1$ do not enter the expression for $\hat{\beta}_1(\omega) = \mathcal{F}_{t\to\omega}[\beta(X_1, t)]$ since $\beta(x, t)$ is a bounded continuous function of $(x, t) \in \mathbb{R}^2$ due to the bound (6.11).

Lemma 7.7. *$\hat{\beta}(x, \omega) = a(x)\delta(\omega - \omega_+)$, where $a(x)$ is a bounded continuous function.*

Proof. For $x \le X_1$, the representation stated in the lemma follows from the first line in (7.8) and from (7.17). Let us prove this representation for $X_1 \le x \le X_2$. By (7.17), we have $\beta_1(t) := \beta(X_1, t) = a_1 e^{-i\omega_+ t}/2\pi$, hence $g_1(t) := F_1(\beta_1(t)) = b_1 e^{-i\omega_+ t}$ for some $b_1 \in \mathbb{C}$ due to the $U(1)$-invariance (1.2). Therefore, $\hat{g}_1(\omega) = 2\pi b_1 \delta(\omega - \omega_+)$. Moreover, by (7.8), we have $\hat{\beta}'(X_1 - 0, \omega) = \kappa(\omega_+) a_1 \delta(\omega - \omega_+)$. Hence, the jump condition (7.9) implies that $\hat{\beta}'(X_1 + 0, \omega) = c_1 \delta(\omega - \omega_+)$, for some $c_1 \in \mathbb{C}$. Finally, (7.8) implies that $\hat{\beta}(x, \omega) = a(x)\delta(\omega - \omega_+)$ for $x \in [X_1, X_2]$, with $a(x)$ a continuous complex-valued function of x. Proceeding by induction, we obtain similar representation for $\hat{\beta}(x, \omega)$ for all $x \in \mathbb{R}$. ∎

Now we can finish the proof of Proposition 7.4. Lemma 7.7 implies that $\beta(x,t) = \phi(x)e^{-i\omega_+ t}$, where $\phi(x) = a(x)/2\pi$. We conclude from (6.11) that $\phi \in H^1(\mathbb{R})$, finishing the proof of Proposition 7.4. Note that $\omega = \pm m$ could only correspond to the zero solution (see Remark 2.9). □

According to Remark 6.5, Proposition 7.4 completes the proof of Theorem 2.10.

8 Multifrequency solitary waves

We will show that when the assumptions of Theorem 2.10 are not satisfied, then the attractor could be more complicated because the equation admits multifrequency solitary wave solutions.

8.1 Wide gaps

Let us consider equation (2.1) with $N = 2$, under Assumptions 2.1 and 2.2.

Proposition 8.1. *If the Assumption 2.3 is violated, then the conclusion of Theorem 2.10 may no longer be correct.*

Proof. We will show that if $L := X_2 - X_1$ is sufficiently large, then one can take $F_1(\psi)$ and $F_2(\psi)$ satisfying Assumptions 2.1 and 2.2 such that the global attractor of the equation contains the multifrequency solutions which do not converge to solitary waves of the form (2.14). For our convenience, we assume that $X_1 = 0$, $X_2 = L$. We consider the model (2.1) with the nonlinearity

$$F_1(\psi) = F_2(\psi) = F(\psi), \qquad \text{where} \quad F(\psi) = \alpha\psi + \beta|\psi|^2\psi, \qquad \alpha,\, \beta \in \mathbb{R}. \tag{8.1}$$

In terms of the condition (2.17), $p_1 = p_2 = 2$. We take L to be large enough:

$$L > \frac{\pi}{2^{3/2}m}. \tag{8.2}$$

Consider the function

$$\psi(x,t) = A(e^{-\kappa(\omega)|x|} + e^{-\kappa(\omega)|x-L|})\sin(\omega t) + B\chi_{[0,L]}(x)\sin(k(3\omega)x)\,\sin(3\omega t), \qquad A,\, B \in \mathbb{C}. \tag{8.3}$$

Then $\psi(x,t)$ solves (2.1) for x away from the points X_J. We require that

$$k(3\omega) = \frac{\pi}{L}, \tag{8.4}$$

so that $\psi(x,t)$ is continuous in $x \in \mathbb{R}$ and symmetric with respect to $x = L/2$:

$$\psi(x,t) = \psi(\frac{L}{2} - x, t), \qquad x \in \mathbb{R}.$$

We need $|\omega| < m$ to have $\kappa(\omega) > 0$, and $3|\omega| > m$ to have $k(3\omega) \in \mathbb{R}$. We take $\omega > 0$, and thus $m < 3\omega < 3m$. By (8.4), this means that we need

$$m < \sqrt{\frac{\pi^2}{L^2} + m^2} < 3m.$$

The second inequality is satisfied by (8.2).

Due to the symmetry of $\psi(x,t)$ with respect to $x = L/2$, the jump condition (7.9) both at $x = 0$ and at $x = L$ takes the following identical form:

$$2A\kappa(\omega)\sin\omega t - Bk(3\omega)\sin 3\omega t = F\big(A(1 + e^{-\kappa(\omega)L})\sin(\omega t)\big). \tag{8.5}$$

Using the identity

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta, \tag{8.6}$$

we see that

$$F(A(1+e^{-\kappa(\omega)L})\sin\omega t) = \Big(\alpha A(1+e^{-\kappa(\omega)L}) + \frac{3}{4}\beta|A|^2A(1+e^{-\kappa(\omega)L})^3\Big)\sin(\omega t) - \frac{1}{4}\beta|A|^2A(1+e^{-\kappa(\omega)L})^3\sin(3\omega t). \tag{8.7}$$

Collecting in (8.5) the terms at $\sin\omega t$ and at $\sin 3\omega t$, we obtain the following system:

$$\begin{cases} 2A\kappa(\omega) = \alpha A(1+e^{-\kappa(\omega)L}) + \frac{3}{4}\beta|A|^2 A(1+e^{-\kappa(\omega)L})^3, \\ Bk(3\omega) = \frac{1}{4}\beta|A|^2 A(1+e^{-\kappa(\omega)L})^3. \end{cases} \tag{8.8}$$

Assuming that $A \neq 0$, we divide the first equation by A:

$$2\kappa(\omega) = \alpha(1+e^{-\kappa(\omega)L}) + \frac{3}{4}\beta|A|^2(1+e^{-\kappa(\omega)L})^3. \tag{8.9}$$

The condition for the existence of a solution $A \neq 0$ is

$$\Big(\frac{2\kappa(\omega)}{1+e^{-\kappa(\omega)L}} - \alpha\Big)\beta > 0. \tag{8.10}$$

Once we found A, the second equation in (8.8) can be used to express B in terms of A.

Remark 8.2. Condition (8.10) shows that we can choose $\beta < 0$ taking large $\alpha > 0$. The corresponding potential $U(\psi) = -\alpha|\psi|^2/2 - \beta|\psi|^4/4$ satisfies (2.13) and Assumptions 2.1 and 2.2.

□

8.2 Linear degeneration

Let us consider equation (2.1) with $N = 2$, under Assumptions 2.1 and 2.3.

Proposition 8.3. *If the Assumption 2.2 is violated, then the conclusion of Theorem 2.10 may no longer be correct.*

Proof. Again, we construct multifrequency solutions. Consider the equation

$$\ddot{\psi} = \psi'' - m^2\psi + \delta(x)F_1(\psi) + \delta(x-L)F_2(\psi), \tag{8.11}$$

where

$$F_1(\psi) = \alpha\psi + \beta|\psi|^2\psi, \qquad F_2(\psi) = \gamma\psi, \qquad \alpha,\ \beta,\ \gamma \in \mathbb{R}. \tag{8.12}$$

Note that the function F_2 is linear, failing to satisfy Assumption 2.2. The function

$$\psi(x,t) = \begin{cases} (A+B)e^{\kappa(\omega)x}\sin(\omega t), \qquad x \leq 0, \\ \big(Ae^{-\kappa(\omega)x} + Be^{\kappa(\omega)x}\big)\sin(\omega t) + C\sinh(\kappa(3\omega)x)\sin(3\omega t), \qquad x \in [0,L], \\ (Ae^{-\kappa(\omega)} + Be^{\kappa(\omega)(2L-x)})\sin(\omega t) + \frac{C}{\sinh(\kappa(3\omega)L)}e^{-\kappa(3\omega)(x-L)}\sin(3\omega t), \qquad x \geq L, \end{cases}$$

where $\omega \in (0, m/3)$, will be a solution if the jump conditions are satisfied at $x = 0$ and at $x = L$:

$$-\psi'(0+,t) + \psi'(0-,t) = \alpha\psi(0,t) + \beta\psi^3(0,t), \tag{8.13}$$

$$-\psi'(L+,t) + \psi'(L-,t) = \alpha\psi(L,t) + \beta\psi^3(L,t). \tag{8.14}$$

We use the identity

$$\alpha(A+B)\sin(\omega t) + \beta((A+B)\sin(\omega t))^3 = \Big(\alpha(A+B) + \beta\frac{3(A+B)^3}{4}\Big)\sin(\omega t) - \beta\frac{(A+B)^3}{4}\sin(3\omega t)$$

which follows from (8.6). Collecting the terms at $\sin(\omega t)$ and at $\sin(3\omega t)$, we write the condition (8.13) as the following system of equations:

$$2\kappa(\omega)A = \Big(\alpha(A+B) + \beta\frac{3(A+B)^3}{4}\Big), \tag{8.15}$$

$$-\kappa(3\omega)C = -\beta\frac{(A+B)^3}{4}. \tag{8.16}$$

Similarly, the condition (8.14) is equivalent to the following two equations:

$$2B\kappa(\omega)e^{\kappa(\omega)L} = \gamma(Ae^{-\kappa(\omega)L} + Be^{\kappa(\omega)L}), \tag{8.17}$$

$$\frac{\kappa(3\omega)C}{\sinh(\kappa(3\omega)L)} + \kappa(3\omega)C\cosh(\kappa(3\omega)L) = \gamma C\sinh(\kappa(3\omega)L). \tag{8.18}$$

Equations (8.15), (8.16), (8.17), and (8.18) could be satisfied for arbitrary $L > 0$. Namely, for any $\omega \in (0, m/3)$, one uses (8.18) to determine γ. For any $\beta \neq 0$, there is always a solution A, and B to the nonlinear system (8.15), (8.17). Finally, C is obtained from (8.16).

□

A Global well-posedness

Here we prove Theorem 2.4. We first need to adjust the nonlinearity F so that it becomes bounded, together with its derivatives. Define

$$\lambda_0 = \sqrt{\frac{\mathcal{H}(\psi_0, \pi_0) - \sum_J A_J}{m - \sum_J B_J}}, \tag{A.1}$$

where $(\psi_0, \pi_0) \in \mathscr{E}$ is the initial data from Theorem 2.4 and A_J, B_J are constants from (2.13). Then we may pick a modified potential function $\widetilde{U}_J \in C^2(\mathbb{C}, \mathbb{R})$, $\widetilde{U}_J(\psi) = \widetilde{U}_J(|\psi|)$, $j = 1,\ 2$, so that

$$\widetilde{U}_J(\psi) = U_J(\psi) \qquad \text{for } |\psi| \le \lambda_0, \quad \psi \in \mathbb{C}, \tag{A.2}$$

$\widetilde{U}_J(\psi)$ satisfy (2.13) with the same constants A_J, B_J as $U_J(\psi)$ do:

$$\widetilde{U}_J(\psi) \ge A_J - B_J|\psi|^2, \quad \text{for } \psi \in \mathbb{C}, \quad \text{where} \quad A_J \in \mathbb{R}, \quad B_J \ge 0, \quad 1 \le J \le N, \quad \sum_J B_J < m, \tag{A.3}$$

and so that $|\widetilde{U}_J(\psi)|$, $|\widetilde{U}_J'(\psi)|$, and $|\widetilde{U}_J''(\psi)|$ are bounded for $\psi \ge 0$. We define

$$\widetilde{F}_J(\psi) = -\nabla\widetilde{U}_J(\psi), \qquad \psi \in \mathbb{C}, \tag{A.4}$$

where ∇ denotes the gradient with respect to $\operatorname{Re}\psi$, $\operatorname{Im}\psi$; Then $\widetilde{F}_J(e^{is}\psi) = e^{is}\widetilde{F}_J(\psi)$ for any $\psi \in \mathbb{C}$, $s \in \mathbb{R}$.

We consider the Cauchy problem of type (2.1) with the modified nonlinearity,

$$\left\{\begin{array}{l} \ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \sum_J \delta(x - X_J)\widetilde{F}_J(\psi(X_J,t)), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ \psi|_{t=0} = \psi_0(x), \qquad \dot{\psi}|_{t=0} = \pi_0(x). \end{array}\right. \tag{A.5}$$

Equation (A.5) formally can be written as the following Hamiltonian system (Cf. (2.5)):

$$\dot{\Psi}(t) = \mathcal{J}\, D\widetilde{\mathcal{H}}(\Psi), \qquad \mathcal{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{A.6}$$

where $D\widetilde{\mathcal{H}}$ is the variational derivative of the Hamilton functional

$$\widetilde{\mathcal{H}}(\Psi) = \int\limits_{\mathbb{R}} \left(|\pi|^2 + |\nabla\psi|^2 + m^2|\psi|^2\right)\, dx + \sum_J \widetilde{U}_J(\psi(X_J,t)), \quad \Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix} \in \mathscr{E}, \tag{A.7}$$

which is Fréchet differentiable in the space $\mathscr{E} = H^1 \times L^2$. By the Sobolev embedding theorem, $L^\infty(\mathbb{R}) \subset H^1(\mathbb{R})$, and there is the following inequality:

$$\|\psi\|_{L^\infty}^2 \le \frac{1}{2m}(\|\psi'\|_{L^2}^2 + m^2\|\psi\|_{L^2}^2) \le \frac{1}{2m}\|\Psi\|_{\mathscr{E}}^2. \tag{A.8}$$

Thus, (A.3) leads to

$$\widetilde{U}_J(\psi(0)) \ge A_J - B_J\|\psi\|_{L^\infty}^2 \ge A_J - \frac{B_J}{2m}\|\Psi\|_{\mathscr{E}}^2. \tag{A.9}$$

Taking into account (A.7), we obtain the inequality

$$\|\Psi\|_{\mathscr{E}}^2 = 2\widetilde{\mathcal{H}}(\Psi) - 2\sum_J \widetilde{U}_J(\psi(X_J)) \le 2\widetilde{\mathcal{H}}(\Psi) - 2\sum_J A_J + \frac{\sum_J B_J}{m}\|\Psi\|_{\mathscr{E}}^2, \qquad \Psi \in \mathscr{E}. \tag{A.10}$$

It follows that

$$\|\Psi\|_{\mathscr{E}}^2 \le \frac{2m}{m - \sum_J B_J}\Big(\widetilde{\mathcal{H}}(\Psi) - \sum_J A_J\Big), \qquad \Psi \in \mathscr{E}. \tag{A.11}$$

Lemma A.1. (*i*) *There is the identity* $\widetilde{\mathcal{H}}(\Psi_0) = \mathcal{H}(\Psi_0)$.

(*ii*) *If* $\Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix} \in \mathscr{E}$ *satisfies* $\widetilde{\mathcal{H}}(\Psi) \le \widetilde{\mathcal{H}}(\Psi_0)$, *then* $\widetilde{U}_J(\psi(x)) = U_J(\psi(x))$ *for any* $x \in \mathbb{R}$.

Proof. According to (A.11), the Sobolev embedding (A.8), and the choice of λ_0 in (A.1),

$$\|\psi_0\|_{L^\infty}^2 \le \frac{1}{2m}\|\Psi_0\|_{\mathscr{E}}^2 \le \frac{\mathcal{H}(\Psi_0) - \sum_J A_J}{m - \sum_J B_J} = \lambda_0^2. \tag{A.12}$$

Thus, by (A.2), $\widetilde{U}(\psi_0(x)) = U(\psi_0(x))$ for all $x \in \mathbb{R}$. This proves (*i*).

By (A.8), the relation (A.11), the condition $\widetilde{\mathcal{H}}(\Psi) \le \widetilde{\mathcal{H}}(\Psi_0)$, and part (*i*) of the Lemma, we have:

$$\|\psi\|_{L^\infty}^2 \le \frac{1}{2m}\|\Psi\|_{\mathscr{E}}^2 \le \frac{\widetilde{\mathcal{H}}(\Psi) - \sum_J A_J}{m - \sum_J B_J} \le \frac{\widetilde{\mathcal{H}}(\Psi_0) - \sum_J A_J}{m - \sum B_J} = \frac{\mathcal{H}(\Psi_0) - \sum_J A_J}{m - \sum_J B_J} = \lambda_0^2.$$

Now the statement (*ii*) follows by (A.2). □

If $\Psi(t)$ solves (A.6), then $\widetilde{\mathcal{H}}(\Psi(t)) = \widetilde{\mathcal{H}}(\Psi_0)$, By Lemma A.1 (*ii*), $\widetilde{U}_J(\psi(x,t)) = U_J(\psi(x,t))$ for all $x \in \mathbb{R}$, $t \in \mathbb{R}$. Hence, $\widetilde{F}_J(\psi(x,t)) = F_J(\psi(x,t))$ for all $x \in \mathbb{R}$, $t \ge 0$, allowing us to conclude that $\psi(t)$ solves (2.1) as well as (A.5). The rest of the proof of Theorem 2.4 repeats the proof of a similar result for the case $N = 1$ [KK07, Theorem 2.3].

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