Russian Journal of Mathematical Physics, Vol. 10, No. 4, 2003, pp. 399–410.

On the Convergence to a Statistical Equilibrium for the Dirac Equation

T. V. Dudnikova*, A. I. Komech, and N. J. Mauser*****

**Keldysh Institute of Applied Mathematics RAS*
Moscow 125047, Russia
E-mail: dudnik@elsite.ru, dudnik@mat.univie.ac.at
***, *** Wolfgang-Pauli Institute c/o Institute of Mathematics, Vienna University*
1090 Vienna, Austria
E-mail: komech@mat.univie.ac.at; mauser@courant.nyu.edu

Received September 28, 2003

Abstract. We consider the Dirac equation in $\mathbb{R}^3$ with constant coefficients and study the distribution μ_t of the random solution at time $t \in \mathbb{R}$. It is assumed that the initial measure μ_0 has zero mean, a translation-invariant covariance, and finite mean charge density. We also assume that μ_0 satisfies a mixing condition of Rosenblatt type or Ibragimov–Linnik type. The main result is the convergence of μ_t to a Gaussian measure as $t \to \infty$. The proof uses the study of long-time asymptotics of the solution and S. N. Bernstein's "room-corridors" method.

1. INTRODUCTION

This paper can be regarded as a continuation of our papers [1–5] concerning the analysis of long-time convergence to an equilibrium distribution for hyperbolic partial differential equations and harmonic crystals. Here we develop the analysis for the Dirac equation

$$\begin{cases} \dot{\psi}(x,t) = [-\alpha \cdot \nabla - i\beta m]\, \psi(x,t), & x \in \mathbb{R}^3 \\ \psi(x,0) = \psi_0(x), \end{cases} \tag{1.1}$$

where $\nabla = (\partial_1, \partial_2, \partial_3)$, $\partial_k = \partial/\partial x_k$, $m > 0$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, and α_k and β are 4×4 Dirac matrices (see (2.12), (2.13)). The solution $\psi(x,t)$ takes the values in $\mathbb{C}^4$ for $(x,t) \in \mathbb{R}^4$.

It is assumed that the initial data $\psi_0(x)$ are given by a random element of the function space $\mathcal{H} \equiv H^0_{\rm loc}(\mathbb{R}^3)$ of states with finite local energy, see Definition 2.1 below. The distribution of ψ_0 is a zero-mean probability measure μ_0 satisfying some additional assumptions, see Conditions **S1-S3** below. Denote by μ_t, $t \in \mathbb{R}$, the measure on $\mathcal{H}$ giving the distribution of the random solution $\psi(t)$ of problem (1.1). We identify the complex and real spaces $\mathbb{C}^4 \equiv \mathbb{R}^8$, and $\otimes$ stands for the tensor product of real vectors. The correlation functions of the initial measure are supposed to be translation-invariant,

$$Q_0(x,y) := E\Big(\psi_0(x) \otimes \psi_0(y)\Big) = q_0(x-y), \qquad x, y \in \mathbb{R}^3. \tag{1.2}$$

We also assume that the initial mean charge density is finite,

$$e_0 := E|\psi_0(x)|^2 = \operatorname{tr} q_0(0) < \infty, \qquad x \in \mathbb{R}^3. \tag{1.3}$$

Finally, assume that the measure μ_0 satisfies a mixing condition of Rosenblatt type or Ibragimov–Linnik type, which means that

$$\psi_0(x) \quad \text{and} \quad \psi_0(y) \quad \text{are asymptotically independent as} \quad |x-y| \to \infty. \tag{1.4}$$

*,**Supported partly by the research grants of DFG (436 RUS 113/615/0-1) and of RFBR (01-01-04002) and by the Austrian Science Foundation (FWF) Project (P16105-N05).

**Supported partly by the Austrian Science Foundation (FWF) START Project (Y-137-TEC).

Our main result gives the (weak) convergence of μ_t to a limit measure μ_∞,

$$\mu_t \rightharpoonup \mu_\infty, \qquad t \to \infty, \tag{1.5}$$

which is an equilibrium Gaussian measure on $\mathcal{H}$. A similar convergence holds as $t \to -\infty$ because our system is time-reversible. Explicit formulas (2.17) for the correlation functions of μ_∞ are given.

To prove the convergence (1.5), we follow the strategy of [1–5]. There are three steps.

I. The family of measures μ_t, $t \geqslant 0$, is weakly compact in an appropriate Fréchet space.
II. The correlation functions converge to a limit,

$$Q_t(x,y) \equiv \int \psi(x) \otimes \psi(y)\, \mu_t\,(d\psi) \to Q_\infty(x,y), \qquad t \to \infty. \tag{1.6}$$

III. The characteristic functionals converge to a Gaussian functional,

$$\hat{\mu}_t(\phi) := \int e^{i\langle \psi, \phi\rangle} \mu_t(d\psi) \to \exp\left\{-\frac{1}{2}\mathcal{Q}_\infty(\phi,\phi)\right\}, \qquad t \to \infty. \tag{1.7}$$

Here $\langle \cdot\,, \cdot\rangle$ stands for a real scalar product in L^2, $\mathcal{Q}_\infty$ for a quadratic form with the integral kernel $Q_\infty(x,y)$, and ϕ for an arbitrary element of the dual space.

Property I follows from the Prokhorov compactness theorem by a method used in [13]. Namely, we first establish a uniform bound for the mean local charge with respect to the measure μ_t, $t \geqslant 0$. Then the Prokhorov condition follows from the Sobolev embedding theorem by Chebyshev's inequality. Property II is derived by studying oscillatory integrals arising in the Fourier transform. However, the Fourier transform by itself is insufficient to prove Property III. We derive it by using an explicit representation of the solution in the coordinate space with the help of the Bernstein "room-corridor" technique by a method of [1–5]. The method gives a representation of the solution as a sum of weakly dependent random variables. Then (1.5) follows from the Ibragimov–Linnik central limit theorem under a Lindeberg-type condition. We sketch the proofs by using the technique of [1].

The paper is organized as follows. The main result is stated in Section 2. The compactness (Property I) is established in Section 3, the convergence relation (1.6) in Section 4, and the convergence relation (1.7) in Section 5.

2. MAIN RESULTS

Let us describe our results more precisely.

2.1. *Notation*

We assume that the initial date ψ_0 in (1.1) is a complex-valued vector function belonging to the phase space $\mathcal{H}$.

Definition 2.1. Denote by $\mathcal{H} \equiv H^0_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^4)$ the Fréchet space of complex-valued functions $\psi(x)$ for which the local (charge) seminorms are finite,

$$\|\psi\|^2_{0,R} \equiv \int_{|x|<R} |\psi(x)|^2\, dx < \infty, \qquad R > 0. \tag{2.1}$$

Proposition 2.2. (i) *For any $\psi_0 \in \mathcal{H}$, there exists a unique solution $\psi(\cdot, t) \in C(\mathbb{R}, \mathcal{H})$ of the Cauchy problem* (1.1).

(ii) *For any $t \in \mathbb{R}$, the operator $U(t)\colon \psi_0 \mapsto \psi(\cdot, t)$ is continuous on $\mathcal{H}$.*

Proposition 2.2 follows from [10, Ths. V.3.1 and V.3.2] because the speed of propagation for Eq. (1.1) is finite. Let us choose a function $\zeta(x) \in C_0^\infty(\mathbb{R}^3)$ such that $\zeta(0) \neq 0$. Denote by $H^s_{\mathrm{loc}}(\mathbb{R}^3)$,

$s \in \mathbb{R}$, the local Sobolev spaces, i.e., the Fréchet spaces of distributions $u \in D'(\mathbb{R}^3)$ with the finite seminorms

$$\|u\|_{s,R} := \Big\|\Lambda^s\Big(\zeta(x/R)u\Big)\Big\|_{L^2(\mathbb{R}^3)}, \tag{2.2}$$

where $\Lambda^s v := F^{-1}_{k\to x}(\langle k\rangle^s \hat{v}(k))$, $\langle k\rangle := \sqrt{|k|^2+1}$, and $\hat{v} := Fv$ is the Fourier transform of a tempered distribution v. For $\phi \in C_0^\infty(\mathbb{R}^3)$ write

$$F\phi(k) = \int e^{ik\cdot x}\phi(x)\,dx.$$

Note that the space $H^s_{\rm loc}(\mathbb{R}^3)$ for $s=0$ agrees with Definition 2.1.

Definition 2.3. For $s \in \mathbb{R}$ write, $\mathcal{H}^s \equiv H^s_{\rm loc}(\mathbb{R}^3)$.

Using the standard technique of pseudodifferential operators and Sobolev's embedding theorem (see, e.g., [8]), one can prove that $\mathcal{H} = \mathcal{H}^0 \subset \mathcal{H}^{-\varepsilon}$ for every $\varepsilon > 0$, and the embedding is compact.

2.2. *Random Solution. Convergence to an Equilibrium*

Let (Ω, Σ, P) be a probability space with expectation E and let $\mathcal{B}(\mathcal{H})$ be the Borel σ-algebra of $\mathcal{H}$. Assume that $\psi_0 = \psi_0(\omega,\cdot)$ in (1.1) is a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. In other words, $(\omega, x) \mapsto \psi_0(\omega, x)$ is a measurable mapping $\Omega \times \mathbb{R}^3 \to \mathbb{C}^4$ with respect to the (completed) σ-algebras $\Sigma \times \mathcal{B}(\mathbb{R}^3)$ and $\mathcal{B}(\mathbb{C}^4)$. Then, by virtue of Proposition 2.2, $\psi(t) = U(t)\psi_0$ is again a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. Denote by $\mu_0(d\psi_0)$ the Borel probability measure on $\mathcal{H}$ giving the distribution of ψ_0. Without loss of generality, we can assume that $(\Omega, \Sigma, P) = (\mathcal{H}, \mathcal{B}(\mathcal{H}), \mu_0)$ and $\psi_0(\omega, x) = \omega(x)$ for $\mu_0(d\omega)\times dx$-almost all points $(\omega, x) \in \mathcal{H}\times\mathbb{R}^3$.

Definition 2.4. Let μ_t be the probability measure on $\mathcal{H}$ giving the distribution of $Y(t)$,

$$\mu_t(B) = \mu_0(U(-t)B), \qquad B \in \mathcal{B}(\mathcal{H}), \quad t \in \mathbb{R}. \tag{2.3}$$

Our main objective is to derive the weak convergence of the measures μ_t in the Fréchet space $\mathcal{H}^{-\varepsilon}$ for each $\varepsilon > 0$,

$$\mu_t \overset{\mathcal{H}^{-\varepsilon}}{\rightharpoondown} \mu_\infty \quad \text{as} \quad t \to \infty, \tag{2.4}$$

where μ_∞ is some Borel probability measure on the space $\mathcal{H}$. This means the convergence

$$\int f(\psi)\mu_t(d\psi) \to \int f(\psi)\mu_\infty(d\psi), \qquad t\to\infty, \tag{2.5}$$

for any bounded continuous functional $f(\psi)$ on $\mathcal{H}^{-\varepsilon}$.

Set $\mathcal{R}\psi \equiv (\operatorname{Re}\psi, \operatorname{Im}\psi) = \{\operatorname{Re}\psi_1,\dots,\operatorname{Re}\psi_4, \operatorname{Im}\psi_1,\dots,\operatorname{Im}\psi_4\}$ for $\psi = (\psi_1,\dots,\psi_4) \in \mathbb{C}^4$ and denote by $\mathcal{R}^j\psi$ the jth component of the vector $\mathcal{R}\psi$, $j = 1,\dots,8$. The brackets $(\cdot,\cdot)$ mean the inner product in the real Hilbert spaces $L^2 \equiv L^2(\mathbb{R}^3)$, in $L^2\otimes\mathbb{R}^N$, or in some their extensions. For $\psi(x), \phi(x) \in L^2(\mathbb{R}^3, \mathbb{C}^4)$, write

$$\langle\psi,\phi\rangle := (\mathcal{R}\psi, \mathcal{R}\phi) = \sum_{j=1}^{8}(\mathcal{R}^j\psi, \mathcal{R}^j\phi). \tag{2.6}$$

Definition 2.5. The correlation functions of the measure μ_t are defined by

$$Q_t^{ij}(x,y) \equiv E\Big(\mathcal{R}^i\psi(x)\mathcal{R}^j\psi(y)\Big) \quad \text{for almost all} \quad x,y\in\mathbb{R}^3, \quad i,j=1,\dots,8, \tag{2.7}$$

provided that the expectations on the right-hand side are finite.

Denote by D the space of complex-valued functions in $C_0^\infty(\mathbb{R}^3)$ and write $\mathcal{D} := [D]^4$. For a Borel probability measure μ on $\mathcal{H}$, denote by $\hat\mu$ the characteristic functional (the Fourier transform)

$$\hat\mu(\phi) \equiv \int \exp(i\langle\psi,\phi\rangle)\,\mu(d\psi), \qquad \phi\in\mathcal{D} \quad (\text{see} \ \ (2.6)).$$

A measure μ is said to be *Gaussian* (with zero expectation) if its characteristic functional is of the form

$$\hat\mu(\phi) = \exp\left\{-\frac{1}{2}\mathcal{Q}(\phi,\phi)\right\}, \qquad \phi\in\mathcal{D},$$

where $\mathcal{Q}$ is a real nonnegative quadratic form on $\mathcal{D}$. A measure μ is said to be *translation-invariant* if

$$\mu(T_hB) = \mu(B), \qquad B\in\mathcal{B}(\mathcal{H}), \quad h\in\mathbb{R}^3,$$

where $T_h\psi(x) = \psi(x-h)$, $x\in\mathbb{R}^3$.

2.3. *Mixing Condition*

Let $O(r)$ be the set of all pairs of open bounded subsets $\mathcal{A},\mathcal{B}\subset\mathbb{R}^3$ at the distance not less than r, $\operatorname{dist}(\mathcal{A},\mathcal{B}) \geqslant r$, and let $\sigma(\mathcal{A})$ be the σ-algebra in $\mathcal{H}$ generated by the linear functionals $\psi\mapsto\langle\psi,\phi\rangle$, where $\phi\in\mathcal{D}$ with $\operatorname{supp}\phi\subset\mathcal{A}$. Define the Ibragimov–Linnik mixing coefficient of a probability measure μ_0 on $\mathcal{H}$ by the rule (cf. [9, Def. 17.2.2])

$$\varphi(r) \equiv \sup_{(\mathcal{A},\mathcal{B})\in O(r)} \ \sup_{\substack{A\in\sigma(\mathcal{A}),\, B\in\sigma(\mathcal{B})\\ \mu_0(B)>0}} \frac{|\mu_0(A\cap B)-\mu_0(A)\mu_0(B)|}{\mu_0(B)}. \tag{2.8}$$

Definition 2.6. We say that the measure μ_0 satisfies the *strong uniform Ibragimov–Linnik mixing condition* if

$$\varphi(r)\to 0 \quad \text{as} \quad r\to\infty. \tag{2.9}$$

We specify the rate of decay of φ below (see Condition **S3**).

2.4. *Main Assumptions and Results*

We assume that the measure μ_0 has the following properties **S0–S3**.

S0. μ_0 has zero expectation value,

$$E\psi_0(x)\equiv 0, \qquad x\in\mathbb{R}^3.$$

S1. μ_0 has translation-invariant correlation functions,

$$Q_0^{ij}(x,y) \equiv E\Big(\mathcal{R}^i\psi(x)\mathcal{R}^j\psi(y)\Big) = q_0^{ij}(x-y) \quad \text{for almost all} \quad x,y\in\mathbb{R}^3, \quad i,j=1,\dots,8. \tag{2.10}$$

S2. μ_0 has finite mean charge density, i.e., Eq. (1.3) holds.

S3. μ_0 satisfies the strong uniform Ibragimov–Linnik mixing condition with

$$\int_0^\infty r^2\varphi^{1/2}(r)\,dr < \infty. \tag{2.11}$$

The standard form of the Dirac matrices α_k and β (in 2×2 blocks) is

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad (k=1,2,3), \tag{2.12}$$

where I stands for the unit matrix and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.13}$$

Introduce the following 8×8 real valued matrices (in 4×4 blocks)

$$\Lambda_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix}, \ \Lambda_2 = \begin{pmatrix} 0 & i\alpha_2 \\ -i\alpha_2 & 0 \end{pmatrix}, \ \Lambda_3 = \begin{pmatrix} \alpha_3 & 0 \\ 0 & \alpha_3 \end{pmatrix}, \ \Lambda_0 = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}. \tag{2.14}$$

Note that by (2.12) and (2.13) we have

$$i\alpha_2 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad \text{where} \quad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Moreover, $\Lambda_k^T = \Lambda_k$, $k=1,2,3$, $\Lambda_0^T = -\Lambda_0$. Write

$$\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3), \quad P(\nabla) = \Lambda\cdot\nabla + m\Lambda_0. \tag{2.15}$$

For almost all $x, y \in \mathbb{R}^3$, introduce the matrix-valued function

$$Q_\infty(x,y) \equiv \Big(Q_\infty^{ij}(x,y)\Big)_{i,j=1,\dots,8} = \Big(q_\infty^{ij}(x-y)\Big)_{i,j=1,\dots,8}. \tag{2.16}$$

Here

$$\hat q_\infty(k) = \frac12 \hat q_0(k) + \frac12 \hat{\mathcal P}(k) P(-ik)\hat q_0(k) P^T(ik), \tag{2.17}$$

where $\hat{\mathcal P}(k) = 1/(k^2+m^2)$, and $\hat q_0(k)$ is the Fourier transform of the correlation matrix of the measure μ_0 (see (2.10)). Since $P^T(ik) = -P(-ik)$, we formally have

$$q_\infty(z) = \frac12 q_0(z) - \frac12 \mathcal P * P(\nabla) q_0(z) P(\overleftarrow{\nabla}), \tag{2.18}$$

where

$$\mathcal P(z) = e^{-m|z|}/(4\pi|z|)$$

is the fundamental solution for the operator $-\Delta + m^2$ and $*$ stands for the convolution of distributions. We show below that $\hat q_0 \in L^2 \equiv L^2(\mathbb{R}^3)$ (cf. (4.7)). Hence, $\hat q_\infty(k) \in L^2$ by (2.17), and the convolution in (2.18) also belongs to L^2.

Denote by $\mathcal Q_\infty$ a real quadratic form on L^2 defined by

$$\mathcal Q_\infty(\phi,\phi) \equiv (Q_\infty(x,y), \mathcal R\phi(x)\otimes\mathcal R\phi(y)) = \sum_{i,j=1}^{8} \int_{\mathbb{R}^3\times\mathbb{R}^3} Q_\infty^{ij}(x,y)\mathcal R^i\phi(x)\mathcal R^j\phi(y)\,dxdy.$$

The form $\mathcal Q_\infty$ is continuous on L^2 because $\hat q_\infty(k)$ is bounded by Corollary 4.3.

Theorem A. *Let $m > 0$, and let conditions* **S0–S3** *hold. Then*

(i) *the convergence in* (2.4) *holds for any $\varepsilon > 0$;*

(ii) *the limit measure μ_∞ is a Gaussian equilibrium measure on $\mathcal{H}$;*

(iii) *the characteristic functional of μ_∞ is of the form*

$$\hat{\mu}_\infty(\phi) = \exp\Big\{-\frac{1}{2}\mathcal{Q}_\infty(\phi,\phi)\Big\}, \qquad \phi \in \mathcal{D}.$$

Theorem A can be derived from Propositions 2.7 and 2.8 given below by using the same arguments as in [13, Theorem XII.5.2].

Proposition 2.7. *The family of measures $\{\mu_t, t \in \mathbb{R}\}$ is weakly compact in the space $\mathcal{H}^{-\varepsilon}$ for any $\varepsilon > 0$.*

Proposition 2.8. *For any $\phi \in \mathcal{D}$,*

$$\hat{\mu}_t(\phi) \equiv \int \exp\{i\langle\psi,\phi\rangle\}\,\mu_t(d\psi) = E\exp\{i\langle U(t)\psi,\phi\rangle\} \to \exp\Big\{-\frac{1}{2}\mathcal{Q}_\infty(\phi,\phi)\Big\}, \qquad t\to\infty. \tag{2.19}$$

Propositions 2.7 and 2.8 are proved in Sections 3 and 4–5, respectively.

2.5. *Remark on Various Mixing Conditions for the Initial Measure*

We use the *strong uniform* Ibragimov–Linnik mixing condition for the simplicity of our presentation. The *uniform* Rosenblatt mixing condition [12] with a higher degree (> 2) in the bound (1.3) is also sufficient. In this case we assume that there exists a δ, $\delta > 0$, such that

$$\sup_{x\in\mathbb{R}^3} E|\psi_0(x)|^{2+\delta} < \infty.$$

Then condition (2.11) requires the following modification:

$$\int_0^\infty r\alpha^p(r)dr < \infty, \qquad p = \min(\delta/(2+\delta), 1/2),$$

where $\alpha(r)$ is the Rosenblatt mixing coefficient defined as in (2.8), but without the denominator $\mu_0(B)$. The statements of Theorem A and their proofs remain essentially unchanged.

3. COMPACTNESS OF MEASURES

3.1. *Fundamental Solution of the Dirac Operator*

One can readily see that α_k and β are Hermitian symmetric matrices satisfying the anticommutation relations

$$\begin{cases} \alpha_k^* = \alpha_k, & \beta^* = \beta \\ \alpha_k\alpha_l + \alpha_l\alpha_k = 2\delta_{kl}I, & \alpha_k\beta + \beta\alpha_k = 0, \end{cases}$$

(δ_{kl} is Kronecker's delta). Therefore,

$$(\partial_t + \alpha\cdot\nabla + i\beta m)\,(\partial_t - \alpha\cdot\nabla - i\beta m) = (\partial_t^2 - \triangle + m^2)I.$$

Then we can construct a fundamental solution $\mathcal{E}(x,t)$ of the Dirac operator, i.e., a solution of the equation

$$(\partial_t + \alpha\cdot\nabla + i\beta m)\,\mathcal{E}(x,t) = \delta(x,t)I, \qquad \mathcal{E}(x,t) = 0 \quad \text{for} \quad t < 0,$$

of the form

$$\mathcal{E}(x,t) = (\partial_t - \alpha\cdot\nabla - i\beta m)\,E(x,t), \tag{3.1}$$

where $E(x,t) \equiv E_t(x)$ is a fundamental solution for the Klein–Gordon operator $(\partial_t^2 - \triangle + m^2)$, and E vanishes for $t < 0$.

Remark 3.1. The function $E_t(x)$ is given by

$$E_t(x) = F^{-1}_{k\to x} \frac{\sin \omega t}{\omega}, \qquad \omega \equiv \omega(k) \equiv \sqrt{|k|^2 + m^2}. \tag{3.2}$$

Then, by the Paley–Wiener theorem (see, e.g., [6, Th. II.2.5.1]), the function $E_t(\cdot)$ is supported by the ball $|x| \leqslant t$.

Denote by $U(t)$, $t \in \mathbb{R}$, the dynamical group for problem (1.1). Then $U(t)$ is a convolution operator given by

$$\psi(x,t) = U(t)\psi_0 = \mathcal{E}(\cdot,t) * \psi_0 = (\partial_t - \alpha \cdot \nabla - i\beta m)\, E_t(\cdot) * \psi_0. \tag{3.3}$$

The convolution exists because the distribution $\mathcal{E}(\cdot,t)$ is compactly supported by (3.1) and by Remark 3.1.

3.2. *Local Estimates*

Proposition 3.2. *For any $\psi_0 \in \mathcal{H}$ and $R > 0$,*

$$\|U(t)\psi_0\|_{0,R} \leqslant C\|\psi_0\|_{0,R+t}, \qquad t \in \mathbb{R}, \tag{3.4}$$

where $C < \infty$ does not depend on R and t.

Proof. In the Fourier transform, the solution $\psi(x,t)$ of the Cauchy problem (1.1) reads as

$$\hat\psi(k,t) = \hat{\mathcal{E}}(k,t)\hat\psi_0(k) = \Big[\cos\omega t - (\alpha\cdot(-ik) + i\beta m)\frac{\sin\omega t}{\omega}\Big]\hat\psi_0(k)$$

by (3.1) and (3.2). Then, for $\psi_0 \in L^2$,

$$\|\psi(\cdot,t)\|_{L^2} = \|\hat\psi(\cdot,t)\|_{L^2} \leqslant C\|\hat\psi_0(\cdot)\|_{L^2} = C\|\psi_0(\cdot)\|_{L^2}. \tag{3.5}$$

Let us consider $\psi_0 \in \mathcal{H}$. Introduce the function $\psi_0^*(x)$ equal to $\psi_0(x)$ for $|x| \leqslant R+t$ and to 0 otherwise. Denote by $\psi(x,t)$ (by $\psi^*(x,t)$) the solution of the Cauchy problem (1.1) with the initial data $\psi_0(x)$ ($\psi_0^*(x)$, respectively). Note that $\psi(x,t) = \psi^*(x,t)$ for $|x| \leqslant R$. Therefore, relation (3.5) implies

$$\|\psi(\cdot,t)\|_R = \|\psi^*(\cdot,t)\|_R \leqslant C\|\psi_0^*(\cdot)\|_{L^2} = C\|\psi_0(\cdot)\|_{R+t}.$$

3.3. *Proof of Compactness*

Proposition 2.7 follows from the estimate (3.3) below by using the Prokhorov theorem [13, Lemma II.3.1], as in the proof of [13, Th. XII.5.2].

Proposition 3.3. *Let the conditions of Theorem A hold. Then, for any positive R, there exists a constant $C(R) > 0$ such that*

$$\sup_{t\geqslant 0} E\|U(t)\psi_0\|^2_{0,R} \leqslant C(R) < \infty. \tag{3.6}$$

Proof. Let us write

$$e_t(x) := E|\psi(x,t)|^2, \qquad x \in \mathbb{R}^3. \tag{3.7}$$

The mathematical expectation is finite for almost every x by (3.4) and by the Fubini theorem. Moreover, $e_t(x) = e_t$ for almost every $x \in \mathbb{R}^3$ by Condition **S1**. Hence, it follows from the Fubini theorem, (3.4), and Condition **S2** that

$$E\|U(t)\psi_0\|^2_{0,R} \equiv e_t|B_R| \leqslant CE\|\psi_0\|^2_{0,R+|t|} \equiv Ce_0|B_{R+|t|}|, \qquad t \in \mathbb{R}. \tag{3.8}$$

Here B_R is the ball $|x| \leqslant R$ in $\mathbb{R}^3$, and $|B_R|$ is the volume of this ball. As $R \to \infty$, we see from (3.8) that $e_t \leqslant Ce_0$. Thus,

$$E\|U(t)\psi_0\|^2_{0,R} = e_t|B_R| \leqslant Ce_0|B_R| < \infty.$$

4. CONVERGENCE OF CORRELATION FUNCTIONS

We prove the convergence of the correlation functions for the measures μ_t. This implies Proposition 2.8 in the case of Gaussian measures μ_0. It follows from condition **S1** that

$$Q_t^{ij}(x,y) = q_t^{ij}(x-y), \qquad x,y \in \mathbb{R}^3, \tag{4.1}$$

for $i,j = 1,\dots,8$.

Proposition 4.1. *The correlation functions $q_t^{ij}(z)$, $i,j=1,\dots,8$, converge for any $z \in \mathbb{R}^3$,*

$$q_t^{ij}(z) \to q_\infty^{ij}(z), \qquad t \to \infty, \tag{4.2}$$

where the functions $q_\infty^{ij}(z)$ are defined in (2.17).

Proof. Using the notation (2.14) and (2.15), by (3.3) we obtain

$$\mathcal{R}\psi(x,t) = \Big(\partial_t - P(\nabla)\Big) E_t * \mathcal{R}\psi_0.$$

Then, by (3.2) and (2.15), the Fourier transform of the solution of the Cauchy problem (1.1) becomes

$$\widehat{\mathcal{R}\psi}(k,t) = \hat{\mathcal{G}}_t(k)\widehat{\mathcal{R}\psi_0}(k), \quad \text{where} \quad \hat{\mathcal{G}}_t(k) := \cos\omega t - P(-ik)\frac{\sin\omega t}{\omega}. \tag{4.3}$$

The translation invariance condition (2.10) implies that

$$E(\widehat{\mathcal{R}\psi_0}(k) \otimes \widehat{\mathcal{R}\psi_0}(k')) = F_{x\to k, y\to k'} q_0(x-y) = (2\pi)^3 \delta(k+k')\hat{q}_0(k). \tag{4.4}$$

Further, (4.3) gives

$$E(\widehat{\mathcal{R}\psi}(k,t) \otimes \widehat{\mathcal{R}\psi}(k',t)) = (2\pi)^3\delta(k+k')\hat{\mathcal{G}}_t(k)\hat{q}_0(k)\hat{\mathcal{G}}_t^*(k). \tag{4.5}$$

Therefore, by the inverse Fourier transform we obtain

$$\begin{aligned} q_t(x-y) =& E(\mathcal{R}\psi(x,t) \otimes \mathcal{R}\psi(y,t)) = F^{-1}_{k\to(x-y)} \hat{\mathcal{G}}_t(k)\hat{q}_0(k)\hat{\mathcal{G}}_t^*(k) \\ =& (2\pi)^{-3} \int e^{-ik(x-y)} \left(\cos\omega t - P(-ik)\frac{\sin\omega t}{\omega}\right) \hat{q}_0(k) \left(\cos\omega t - P^T(ik)\frac{\sin\omega t}{\omega}\right) dk \\ =& (2\pi)^{-3} \int e^{-ik(x-y)} \Big[\frac{1+\cos 2\omega t}{2}\hat{q}_0(k) - \frac{\sin 2\omega t}{2\omega}\Big(\hat{q}_0(k)P^T(ik) + P(-ik)\hat{q}_0(k)\Big) \\ &+ \frac{1-\cos 2\omega t}{2\omega^2} P(-ik)\hat{q}_0(k)P^T(ik)\Big]\, dk. \end{aligned} \tag{4.6}$$

To prove (4.2), it remains to show that the oscillatory integrals in (4.6) converge to zero. Let us first analyze the entries of the matrix q_0^{ij}, $i,j=1,\dots,8$.

Lemma 4.2. *Let the assumptions of Theorem* A *hold. Then*

$$\hat{q}_0^{ij} \in L^1(\mathbb{R}^3) \quad \textit{for any} \quad i,j.$$

Proof. Let us first prove that

$$q_0^{ij}(z) \in L^p(\mathbb{R}^3), \qquad p \geqslant 1, \quad i,j = 1,\dots,8. \tag{4.7}$$

Conditions **S0**, **S2** and **S3** imply (cf. [9, Lemma 17.2.3]) that

$$|q_0^{ij}(z)| \leqslant Ce_0\varphi^{1/2}(|z|), \qquad z \in \mathbb{R}^3, \qquad i,j = 1,\dots,8. \tag{4.8}$$

The mixing coefficient φ is bounded, and hence relations (4.8) and (2.11) imply (4.7),

$$\int_{\mathbb{R}^3} |q_0^{ij}(z)|^p\,dz \leqslant Ce_0^p \int_{\mathbb{R}^3} \varphi^{p/2}(|z|)\,dz \leqslant C_1 \int_0^\infty r^2\varphi^{1/2}(r)\,dr < \infty.$$

By Bochner's theorem, $\hat{q}_0^{ij}$ is a nonnegative matrix-valued measure on $\mathbb{R}^3$, and condition **S2** implies that the total measure $\hat{q}_0(\mathbb{R}^3)$ is finite. On the other hand, relation (4.7) for $p=2$ gives $\hat{q}_0^{ij} \in L^2(\mathbb{R}^3)$. Hence, $\hat{q}_0^{ij} \in L^1(\mathbb{R}^3)$.

Let us apply this lemma to the oscillatory integrals entering (4.6). The convergence (4.2) follows from (4.6) by the Lebesgue–Riemann theorem. This completes the proof of Proposition 4.1.

Relation (4.7) with $p=1$ implies now that $\hat{q}_0(k)$ is bounded. Hence, the explicit formula (2.17) implies the following assertion.

Corollary 4.3. *All matrix elements $\hat{q}_\infty^{ij}(k)$, $i,j = 1,\dots,8$, are bounded.*

5. CONVERGENCE OF CHARACTERISTIC FUNCTIONALS

To prove Proposition 2.8 for the general case of a non-Gaussian measure μ_0, we develop a version of Bernstein's "room–corridor" method of [1–4]: (i) we use an integral representation for the solutions of (1.1), (ii) divide the region of the integration into "rooms" and "corridors" and (iii) evaluate their contribution. As the result, the value $\langle U(t)\psi_0, \phi\rangle$ for $\phi \in \mathcal{D}$ is represented as the sum of weakly dependent random variables. Then we apply Bernstein's "room–corridor" method and the Lindeberg central limit theorem.

(i) We first evaluate the inner product $\langle U(t)\psi_0, \phi\rangle$ in (2.19) by using duality arguments. For $t \in \mathbb{R}$, introduce "formal adjoint" operators $U'(t)$ from the space $\mathcal{D}$ to a suitable space of distributions. For example,

$$\langle \psi, U'(t)\phi\rangle = \langle U(t)\psi, \phi\rangle, \qquad \phi \in \mathcal{D}, \quad \psi \in \mathcal{H}. \tag{5.1}$$

Write $\phi(\cdot,t) = U'(t)\phi$. Then (5.1) can be represented as

$$\langle \psi(t), \phi\rangle = \langle \psi_0, \phi(\cdot,t)\rangle, \qquad t \in \mathbb{R}. \tag{5.2}$$

The adjoint groups admit a convenient description (see Lemma 5.1 for the group $U'(t)$).

Lemma 5.1. *For $\phi \in \mathcal{D}$, the function $U'(t)\phi = \phi(x,t)$ is the solution of*

$$\dot{\phi}(x,t) = (\alpha\cdot\nabla + i\beta m)\phi(x,t), \qquad \phi(x,0) = \phi(x). \tag{5.3}$$

Proof. Differentiating (5.1) with respect to t for $\psi, \phi \in \mathcal{D}$, we obtain

$$\langle \psi, \dot{U}'(t)\phi\rangle = \langle \dot{U}(t)\psi, \phi\rangle. \tag{5.4}$$

The group $U(t)$ has the generator

$$\mathcal{A} = -\alpha\cdot\nabla - i\beta m.$$

Therefore, the generator of $U'(t)$ is the conjugate operator

$$\mathcal{A}' = \alpha\cdot\nabla + i\beta m. \tag{5.5}$$

Hence, relation (5.3) holds indeed with

$$\dot{\phi} = \mathcal{A}'\phi.$$

Remark 5.2. Comparing (5.3) and (1.1), we see that $\phi(x,t) = U'(t)\phi$ can be represented as a convolution (cf. (3.3)), namely,

$$\phi(\cdot,t) = \mathcal{R}_t * \phi, \qquad \mathcal{R}_t := (\partial_t + \alpha\cdot\nabla + im\beta)E_t. \tag{5.6}$$

(ii) Introduce a "room–corridor" partition of $\mathbb{R}^3$. For a given $t > 0$, choose $d_t \geqslant 1$ and $\rho_t > 0$ such that $\rho_t \sim t^{1-\delta}$ with some $\delta \in (0,1)$ and

$$d_t \sim t/\log t \quad \text{as} \quad t \to \infty.$$

Set $h_t = d_t + \rho_t$ and

$$a^j = jh_t, \quad b^j = a^j + d_t, \qquad j \in \mathbb{Z}. \tag{5.7}$$

We refer to the slabs

$$R_t^j = \{x \in \mathbb{R}^3 :\ a^j \leqslant x^3 \leqslant b^j\}$$

as "*rooms*" and to

$$C_t^j = \{x \in \mathbb{R}^3 :\ b^j \leqslant x^3 \leqslant a_{j+1}\}$$

as "*corridors.*" Here $x = (x^1, x^2, x^3)$. The symbol d_t stands for the *width* of a room and ρ_t for that of a corridor.

Denote by χ_r the indicator of the interval $[0, d_t]$ and by χ_c the indicator of $[d_t, h_t]$, which means that

$$\sum_{j\in\mathbb{Z}} (\chi_r(s - jh) + \chi_c(s - jh)) = 1 \quad \text{for (almost all)} \quad s \in \mathbb{R}.$$

The following decomposition holds:

$$\langle \psi_0, \phi(\cdot,t)\rangle = \sum_{j\in\mathbb{Z}} (\langle \psi_0, \chi_r^j \phi(\cdot,t)\rangle + \langle \psi_0, \chi_c^j \phi(\cdot,t)\rangle), \tag{5.8}$$

where $\chi_r^j := \chi_r(x^3 - jh)$ and $\chi_c^j := \chi_c(x^3 - jh)$. Consider the random variables r_t^j and c_t^j given by

$$r_t^j = \langle \psi_0, \chi_r^j \phi(\cdot,t)\rangle, \qquad c_t^j = \langle \psi_0, \chi_c^j \phi(\cdot,t)\rangle, \qquad j \in \mathbb{Z}. \tag{5.9}$$

Then (5.8) and (5.2) imply

$$\langle U(t)\psi_0, \phi\rangle = \sum_{j\in\mathbb{Z}} (r_t^j + c_t^j). \tag{5.10}$$

The series in (5.10) is in fact a finite sum. Indeed, for the support of ϕ we have

$$\operatorname{supp} \phi \subset B_{\overline{r}} \quad \text{for some} \quad \overline{r} > 0.$$

Then, by the convolution representation (5.6), the support of the function $\phi(\cdot,t)$ at $t > 0$ is a subset of an "inflated future cone"

$$\operatorname{supp} \phi \subset \{(x,t) \in \mathbb{R}^3 \times \mathbb{R}_+ :\ |x| \leqslant t + \overline{r}\}, \tag{5.11}$$

whereas $\mathcal{R}_t(x)$ is supported by the "future cone" $|x| \leqslant t$. The latter fact follows from (5.6) and from Remark 3.1. Finally, it follows from (5.9) that

$$r_t^j = c_t^j = 0 \qquad \text{for} \qquad jh_t + t < -\overline{r} \qquad \text{and for} \quad jh_t - t > \overline{r}. \tag{5.12}$$

Therefore, the series (5.10) becomes a sum,

$$\langle U(t)\psi_0, \phi\rangle = \sum_{-N_t}^{N_t} (r_t^j + c_t^j), \qquad N_t \sim \frac{t}{h_t}. \tag{5.13}$$

Lemma 5.3. *Let Conditions* **S0–S3** *hold. Then the following bounds hold for* $t > 1$:

$$E|r_t^j|^2 \leqslant C(\phi)\ d_t/t, \qquad E|c_t^j|^2 \leqslant C(\phi)\ \rho_t/t, \qquad j \in \mathbb{Z}. \tag{5.14}$$

Proof. We discuss the first bound in (5.14) only because the other can be proved in a similar way. Rewrite the left-hand side of (5.14) as the integral of correlation functions. We obtain

$$E|r_t^j|^2 = \langle \chi_r^j(x_3)\chi_r^j(y_3)q_0(x-y), \phi(x,t) \otimes \phi(y,t)\rangle. \tag{5.15}$$

The following uniform bound holds (cf. [11, Th. XI.17 (b)]):

$$\sup_{x\in\mathbb{R}^3} |\phi(x,t)| = \mathcal{O}(t^{-3/2}), \qquad t \to \infty. \tag{5.16}$$

In fact, (5.6) and (3.2) imply that the function $\phi(x,t)$ can be represented as the sum

$$\phi(x,t) = \sum_{\pm} \int_{\mathbb{R}^3} e^{-i(kx\pm\omega t)} a^{\pm}(\omega)\hat{\phi}(k)\,dk, \tag{5.17}$$

where $a^{\pm}(\omega)$ is a matrix whose entries are linear functions of ω or $1/\omega$. Let us prove the asymptotics (5.16) along each ray $x = vt + x_0$ with $|v| \leqslant 1$. The asymptotic relation thus obtained must hold uniformly in $x \in \mathbb{R}^3$ by (5.11). By (5.17) we have

$$\phi(vt + x_0, t) = \sum_{\pm} \int_{\mathbb{R}^3} e^{-i(kv\pm\omega)t - ikx_0} a^{\pm}(\omega)\hat{\phi}(k)\,dk. \tag{5.18}$$

This is a sum of oscillatory integrals with the phase functions

$$\phi_{\pm}(k) = kv \pm \omega(k).$$

Each function has two stationary points which are solutions of the equation

$$v = \mp\nabla\omega(k) \quad \text{if} \quad |v| < 1,$$

and has none if $|v| \geqslant 1$. The phase functions are nondegenerate, i.e.,

$$\det\left(\frac{\partial^2\phi_{\pm}(k)}{\partial k_i \partial k_j}\right)_{i,j=1}^{3} \neq 0, \qquad k \in \mathbb{R}^3. \tag{5.19}$$

Finally, $\hat{\phi}(k)$ is smooth and rapidly decays at infinity. Therefore,

$$\phi(vt + x_0, t) = \mathcal{O}(t^{-3/2})$$

according to the standard method of stationary phase, see [7].

According to (5.11) and (5.16), it follows from (5.15) that

$$E|r_t^j|^2 \leqslant Ct^{-3} \int_{|x|\leqslant t+\bar{r}} \chi_r^j(x^3)\|q_0(x-y)\|\,dxdy = Ct^{-3} \int_{|x|\leqslant t+\bar{r}} \chi_r^j(x^3)\,dx \int_{\mathbb{R}^3} \|q_0(z)\|dz, \tag{5.20}$$

where $\|q_0(z)\|$ stands for the norm of the matrix $\left(q_0^{ij}(z)\right)$. Therefore, relation (5.14) follows for $\|q_0(\cdot)\| \in L^1(\mathbb{R}^3)$ by (4.7).

Hence, the rest of the proof of Proposition 2.8 is the same as that in the case of the Klein–Gordon equation, [1, p. 20–25]. The proof of Theorem A is complete.

REFERENCES

1. Dudnikova, T. V., Komech, A. I., Kopylova, E. A., and Suhov, Yu. M., On Convergence to Equilibrium Distribution, I. The Klein–Gordon Equation with Mixing, *Comm. Math. Phys.*, 2002, vol. 225, no. 1, pp. 1–32.
2. Dudnikova, T. V., Komech, A. I., Ratanov, N. E., and Suhov, Yu. M., On Convergence to Equilibrium Distribution, II. The Wave Equation in Odd Dimensions, with Mixing, *J. Stat. Phys.*, 2002, vol. 108, no. 4, pp. 1219–1253.
3. Dudnikova, T. V., Komech, A. I., and Spohn, H., On a Two-Temperature Problem for Wave Equation, *Markov Processes and Related Fields*, 2002, vol. 8, pp. 43–80.
4. Dudnikova, T., Komech, A., and Spohn, H., On Convergence to Statistical Equilibrium for Harmonic Crystal, *J. Math. Phys.*, 2003, vol. 44, no. 6, pp. 2595–2620. ArXiv: math-ph/0210039.
5. Dudnikova, T., Komech, A., Mauser, N. J., On Two-Temperature Problem for Harmonic Crystals, *J. Stat. Phys.*, 2003. ArXiv: math-ph/0211017 (to appear).
6. Egorov, Yu. V., Komech, A. I., and Shubin, M. A., *Elements of the Modern Theory of Partial Differential Equations*, Springer, Berlin, 1999.
7. Fedoryuk, M. V., Method of Stationary Phase and Pseudodifferential Operators, *Uspekhi Mat. Nauk*, 1971, vol. 26, no. 1 (157), pp. 67–112.
8. Hörmander, L., *The Analysis of Linear Partial Differential Operators* III: *Pseudodifferential Operators*, Berlin–New York: Springer-Verlag, 1985.
9. Ibragimov, I. A. and Linnik, Yu. V., *Independent and Stationary Sequences of Random Variables*, Ed. by J. F. C. Kingman, Wolters-Noordhoff, Groningen, 1971.
10. Mikhailov, V. P., *Partial Differential Equations*, Mir, Moscow, 1978.
11. Reed, M. and Simon, B., *Methods of Modern Mathematical Physics* III: *Scattering Theory*, Academic Press, New York–London, 1979.
12. Rosenblatt, M. A., A Central Limit Theorem and a Strong Mixing Condition, *Proc. Nat. Acad. Sci. U. S. A.*, 1956, vol. 42, no. 1, pp. 43–47.
13. Vishik, M. I. and Fursikov, A. V., *Mathematical Problems of Statistical Hydromechanics*, Dordrecht: Kluwer Academic Publishers, 1988.