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Scattering asymptotics for a charged particle coupled to the Maxwell field

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We establish long time soliton asymptotics for the nonlinear system of Maxwell equations coupled to a charged particle. The coupled system has a six-dimensional manifold of soliton solutions. We show that in the long time approximation, any solution, with an initial state close to the solitary manifold, is a sum of a soliton and a dispersive wave which is a solution of the free Maxwell equations. It is assumed that the charge density satisfies the Wiener condition. The proof further develops the general strategy based on the symplectic projection in Hilbert space onto the solitary manifold, modulation equations for the parameters of the projection, and decay of the transversal component. © *2011 American Institute of Physics*. [doi:10.1063/1.3567957]

I. INTRODUCTION

Our paper deals with an old and important problem of mathematical physics, namely, the problem of particle-field interaction. The equations of motion of a charged particle in external electromagnetic fields were introduced by Lorentz in 1892,[1] though for the first time it was written down by Maxwell in one of his investigations in the 1860s. On the other hand, formulas for the electromagnetic field generated by a moving charge were obtained by Liénard and Wiechert independently in 1898, respectively, in 1900. Thus the problem of the interaction of a charge with its self-generated field arises. The Liénard–Wiechert potentials imply that the field generated by an accelerated charge transports energy to infinity, hence the acceleration should tend to zero as $t \to \infty$. This *radiative decay* is known since Abraham[2] and is claimed in most of manuals on electrodynamics. However, it was proven only fairly recently in Refs. 3 and 4 for the model of the scalar field coupled to extended charge, and in Refs. 5 and 6 for the Maxwell field coupled to extended charge, as introduced by Abraham. The corresponding scalar or Maxwell fields converge to the static solutions in the models with an external confining potentials,[3,6] or to the solitons (travelling wave solutions) in the translation invariant models.[4,5] Here we refine the asymptotics[5] for the Maxwell–Lorentz equations identifying the outgoing dispersive wave and the rate of the convergence for initial states close to a soliton.

It is convenient to write the equations of motion in Hamiltonian form. The dynamical variables come then in canonically conjugate pairs. They are the position, q, of the particle, together with its momentum P, and the transverse vector potential, A, together with the transverse electric field E. We refer to Ref. 7, Chap. 13 for details. In these variables the Hamiltonian function reads

$$\mathcal{H}(E, A, q, P) = \frac{1}{2}\langle E, E\rangle + \frac{1}{2}\langle \nabla A, \nabla A\rangle + \left[1 + (P - A_\rho(q))^2\right]^{1/2}, \tag{1.1}$$

on the subspace defined by the transversality conditions:

$$\nabla \cdot E(x) = 0, \quad \nabla \cdot A(x) = 0. \tag{1.2}$$

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Here $\langle\cdot,\cdot\rangle$ denotes the L^2-scalar product and, written in components,

$$\langle E, E\rangle = \sum_{j=1}^{3}\langle E_j, E_j\rangle, \quad \langle\nabla A, \nabla A\rangle = \sum_{i,j=1}^{3}\langle\nabla_i A_j, \nabla_i A_j\rangle.$$

The function ρ is the charge density and A_ρ the convolution,

$$A_\rho(x) = \int d^3x'\rho(x'-x)A(x').$$

The canonical equations of motion follow then as

$$\dot{E}(x,t) = -\Delta A(x,t) - \Pi_s(\rho(x-q(t))\dot{q}(t)), \quad \dot{A}(x,t) = -E(x,t), \tag{1.3}$$

$$\dot{q}(t) = \frac{P(t) - A_\rho(q(t))}{\left[1 + \left(P(t) - A_\rho(q(t))\right)^2\right]^{1/2}}, \quad \dot{P}(t) = [\nabla(\dot{q}(t)\cdot A)]_\rho(q(t), t) \tag{1.4}$$

with $t \in \mathbb{R}$; $x, q, P \in \mathbb{R}^3$. Here and below all derivatives are understood in the sense of distributions. The operator $\hat{\Pi}_s$ is the projection onto the space of solenoidal (divergence-free) vector fields, which in Fourier space reads:

$$\hat{\Pi}_s(k)\, a = a - \frac{a\cdot k}{k^2}k.$$

It is easily checked that the transversality condition is preserved in time. We use units such that the velocity of light $c = 1$, $\varepsilon_0 = 1$, and the mechanical mass of the charge $m = 1$.

Let us write the system (1.2)–(1.4) as

$$\dot{Y}(t) = F(Y(t)), t \in \mathbb{R}, \tag{1.5}$$

where $Y(t) = (E(x,t), A(x,t), q(t), P(t))$ and the phase space is defined through $\mathcal{H} < \infty$. Below we always deal with column vectors but often write them as row vectors. The system (1.2)–(1.4) admits special solutions where the charge travels with constant velocity. In analogy with travelling solutions of nonlinear wave equation we call them *solitons*. Explicitly they are given by

$$Y_{a,v}(t) = (E_v(x - vt - a), A_v(x - vt - a), vt + a, P_v), \; P_v = p_v + \langle\rho, A_v\rangle, \tag{1.6}$$

for all $a, v \in \mathbb{R}^3$ with $|v| < 1$, where $E_v = \Pi_s E^v$, $A_v = \Pi_s A^v$, and E^v, A^v, p_v are given by

$$\left.\begin{array}{l} E^v(x) = -\nabla\phi_v(x) + v\cdot\nabla A^v(x), \quad A^v(x) = v\phi_v(x), \\ \phi_v(x) = \dfrac{\gamma}{4\pi}\displaystyle\int \frac{\rho(y)d^3y}{|\gamma(y-x)_\parallel + (y-x)_\perp|}, \quad p_v = \gamma v \end{array}\right| . \tag{1.7}$$

Here $\gamma = 1/\sqrt{1-v^2}$ and $x = x_\parallel + x_\perp$ with $x_\parallel$ the component parallel and $x_\perp$ the component orthogonal to v. The formulas (1.7) follow resolving the stationary equations which read

$$\left.\begin{array}{ll} E_v(x) = v\cdot\nabla A_v(x), & v\cdot\nabla E_v(x) = \Delta A_v(x) + \Pi_s(\rho(x)v), \\ v = \dfrac{P_v - \langle\rho, A_v\rangle}{\left[1 + (P_v - \langle\rho, A_v\rangle)^2\right]^{1/2}}, & 0 = \displaystyle\int \rho(x)v\cdot\nabla A_v(x)d^3x \end{array}\right| . \tag{1.8}$$

The states $S_{a,v} = Y_{a,v}(0)$ form the solitary manifold,

$$\mathcal{S} = \{S_{a,v} : a, v \in \mathbb{R}^3, |v| < 1\}. \tag{1.9}$$

For general initial data, one expects that for large times the solution splits up into two parts: one piece consists of a soliton with a definite velocity and the second piece are scattered fields escaping to infinity. In fact, this will be our main result. If the initial data are close to the solitary manifold, then we will prove that for large t,

$$(E(x,t), A(x,t)) \sim (E_{v_\pm}(x - v_\pm t - a_\pm), A_{v_\pm}(x - v_\pm t - a_\pm)) + W^0(t)\mathbf{\Psi}_\pm, \quad t \to \pm\infty. \tag{1.10}$$

Here $W^0(t)$ is the dynamical group of the free wave equation [Eqs. (1.3) with $\rho = 0$ and (1.2)], $\boldsymbol{\Psi}_\pm$ are the corresponding *asymptotic scattered fields*, and the remainder converges to zero *in the global energy norm*, i.e., in the norm of the space $\mathcal{F} := H^0_s(\mathbb{R}^3) \oplus \dot{H}^1_s(\mathbb{R}^3)$, see Sec. II. For the particle trajectory we prove that

$$\dot{q}(t) \to v_\pm, \quad q(t) \sim v_\pm t + a_\pm, \quad t \to \pm\infty. \tag{1.11}$$

The results are established under the following conditions on the charge distribution: ρ is a real-valued function of the Sobolev class $H^2(\mathbb{R}^3)$, compactly supported, and spherically symmetric, i.e.,

$$\rho, \nabla\rho, \nabla\nabla\rho \in L^2(\mathbb{R}^3), \qquad \rho(x) = 0 \ \text{ for } \ |x| \geq R_\rho, \qquad \rho(x) = \rho_1(|x|). \tag{1.12}$$

An essential point of our asymptotic analysis is the Wiener condition:

$$\hat{\rho}(k) = (2\pi)^{-3/2} \int e^{ikx} \rho(x) d^3x \neq 0 \ \text{ for all } \ k \in \mathbb{R}^3 \setminus \{0\}\,. \tag{1.13}$$

The Wiener condition was noted already in the previous works.[4–6] It expresses that all modes of the Maxwell field are coupled to the particle.

There is no restriction on $\int |\rho(x)| d^3x$. However if $\int \rho(x) d^3x \neq 0$, then the soliton fields have a slow decay at infinity, namely, $A_v(x) \sim |x|^{-1}$ and $E_v(x) \sim |x|^{-2}$. With our methods such a decay seems to be difficult to control and we have to impose the condition of vanishing the momenta of ρ up to the fourth order:

$$\int x^\alpha \rho(x) d^3x = 0, \quad |\alpha| \leq 4. \tag{1.14}$$

In particular, the total charge $\int \rho(x) d^3x$ equals zero (neutrality of the particle). Equivalently, $\hat{\rho}$ has a fifth order zero at $k = 0$,

$$\hat{\rho}^{(\alpha)}(0) = 0, \quad |\alpha| \leq 4. \tag{1.15}$$

We believe (1.14) to be a technical condition. Physically, one expects (1.10) to hold even without imposing charge neutrality and it is of interest to extend our proof in this direction.

Let us briefly comment on earlier works. The first mathematical investigation is the contribution of Bambusi and Galgani.[8] They consider a nonrelativistic kinetic energy for the charge and prove orbital stability of the solitons without Wiener condition. The asymptotics of type (1.10) for the fields alone, without $q, \dot{q}$ were proved under the Wiener condition for charged particle coupled to scalar or Maxwell field with a potential in Refs. 3 and 6 and for the translation invariant systems without potential in Refs. 4 and 5. However, the asymptotics were proved only in the local energy seminorms and did not involve the dispersive term.

Full asymptotics (1.10), (1.11) were established under the weak coupling condition $\|\rho\|_{L^2} \ll 1$ for translation invariant Maxwell–Lorentz system in Ref. 9 and for Maxwell–Lorentz system with a rotating particle in Ref. 10. In the present paper we establish the full asymptotics (1.10), (1.11) without the weak coupling condition under the Wiener condition (1.13).

Long time asymptotics of type (1.10) also appear in nonlinear wave equations, such as the Korteweg-deVries (Refs. 11 and 12) and the $U(1)$-invariant nonlinear Schrödinger equations.[13–21] In these equations there are no particle degrees of freedom and the solitons (1.6) correspond to the solitary wave solutions travelling at constant velocity.

Let us comment on basic peculiarity of our problem. Namely, the asymptotics (1.10), (1.11) mean the asymptotic stability of the solitary manifold $\mathcal{S}$ in the dynamics (1.3)–(1.4). However, the dynamics *along* the solitary manifold is unstable, and this is the main difficulty in the proofs. Namely, for two soliton solutions with close but different velocities v_1 and v_2 and close initial positions q_1^0 and q_2^0 one has

$$q_1(t) - q_2(t) = q_1^0 - q_2^0 + (v_1 - v_2)t \to \infty \ \text{ as } \ t \to \infty.$$

Moreover, the fields $(E_1(x,t), A_1(x,t))$ and $(E_2(x,t), A_2(x,t))$ being close at $t = 0$ do not remain so as $t \to \infty$, since they are centered at $q_1(t)$ and $q_2(t)$, although their difference remains bounded.

The nonlinear instability corresponds to the fact that tangent vectors $\partial_{a_j} S_{a,v}$ and $\partial_{v_j} S_{a,v}$, $j = 1, 2, 3$ to the solitary manifold are the zero eigenvectors and root vectors for the generator of the linearized equation. Respectively, the linearized equation admits linear in t secular solutions, see (8.6). The existence of these runaway solutions prohibits the direct application of the Liapunov strategy and requires significant modification of the classical stability theory.

Our approach relies on and further develops the general strategy introduced in the cited papers in the context of the $U(1)$-invariant Schrödinger equation. The approach uses (i) symplectic projection of the dynamics in the Hilbert phase space onto the symplectic orthogonal directions to the solitary manifold to kill the runaway secular solutions, (ii) the modulation equations for the motion along the solitary manifold, and (iii) freezing of the dynamics in the nonautonomous linearized equation. See more details in Introduction[22] where the general strategy has been developed for the case of the Klein–Gordon equation. The Maxwell–Lorentz equations (1.3)–(1.4) differ significantly from the Klein–Gordon case because of slow Coulombic decay of the solitons and presence of the embedded eigenvalue in the continuous spectrum of the linearized equation (see the comments below).

Developing the general strategy for the Maxwell–Lorentz equations (1.3)–(1.4), we obtain our main result in Secs. III–IX and Appendix A of the paper. The main novelty in our case is thorough establishing the appropriate decay of the linearized dynamics in Secs. X–XIII and Appendixes B and C:

I. We do not postulate any spectral properties of the linearized equation, calculating all the properties from the Wiener condition (1.13). Namely, we show that (i) the full zero spectral space of the linearized equation is spanned by the tangent vectors, and moreover, (ii) there are no others (nonzero) discrete eigenvalues (see Lemmas 12.5, 12.6 and Proposition 11.1).

II. Using these spectral properties, we prove that the linearized equation is stable in the *symplectic orthogonal complement* to the tangent space $\mathcal{T}_S$ spanned by the tangent vectors $\partial_{a_j} S_{a,v}$ and $\partial_{v_j} S_{a,v}$, $j = 1, 2, 3$. We exactly calculate in Lemma 12.6 the corresponding symplectic orthogonality conditions for initial data of the linearized dynamics.

III. One of the main peculiarities of the Maxwell–Lorentz equations is the presence of embedded eigenvalue $\lambda = 0$ in the continuous spectrum $\sigma_c = \mathbb{R}$ of the linearized equation. This situation never happens in all previous works on the asymptotic stability of the solitary waves for the Schrödinger and Klein–Gordon equations. Thus, the symplectic orthogonality condition is imposed now at the interior point of the continuous spectrum in contrast to all previous works in the field. Respectively, the integrand at this point in the spectral representation of the solution is not smooth even if the symplectic orthogonality condition holds. Hence, the integration by parts in this spectral representation, as in the case of the Schrödinger and Klein–Gordon equation, is impossible. For the proof of the decay in this new situation, we transform the spectral representation in the proofs of Propositions 12.2 and 12.4, and develop new more subtle technique of convolutions.

Our paper is organized as follows. In Sec. II, we formulate the main result. In Sec. III, we introduce the symplectic projection onto the solitary manifold. The linearized equation is defined and studied in Secs. IV–V. In Sec. VI, we split the dynamics in two components: along the solitary manifold and in transversal directions. In Sec. VII, we justify the slow motion of the longitudinal component, and in Sec. VIII the decay of the transversal component assuming the corresponding decay in the linearized dynamics, which is proved in Secs. X–XIII. In Sec. IX, we prove the main result. In Appendixes A, B, and C we collect routine calculations.

II. MAIN RESULTS

A. Existence of dynamics

Let us introduce a phase space for the system (1.2)–(1.4) and state the existence of dynamics. Set $H^0 = L^2(\mathbb{R}^3, \mathbb{R}^3)$, $\dot{H}^1$ is the closure of $C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ with respect to the norm $\|A\|_1 = |\nabla A| = \|\nabla A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}$. Let H_s^0, $\dot{H}_s^1$ be the subspaces constituted by solenoidal vector fields, namely, the closure in H^0, $\dot{H}^1$, respectively, of C_0^∞ vector fields with vanishing divergence. Define the phase space

$$\mathcal{E} = H_s^0 \oplus \dot{H}_s^1 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3, \quad Y = (E, A, q, P), \quad \|Y\|_{\mathcal{E}} = |E| + \|A\|_1 + |q| + |P|.$$

Let us define the corresponding space for fields alone:

$$\mathcal{F} = H_s^0 \oplus \dot{H}_s^1, \quad \|(E, A)\|_{\mathcal{F}} = |E| + \|A\|_1.$$

We write the Cauchy problem for the system (1.2)–(1.4) as

$$\dot{Y} = F(Y(t)), \quad t \in \mathbb{R}; \quad Y(0) = Y^0. \tag{2.1}$$

Proposition 2.1 (Ref. 5): Let (1.12) holds, let $Y^0 = (E^0, A^0, q^0, P^0) \in \mathcal{E}$. *Then*
(i) there exists a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *to the Cauchy problem (2.1).*
(ii) The energy conserves,

$$H(Y(t)) = H(Y^0), \quad t \in \mathbb{R}.$$

(iii) The estimate holds,

$$|\dot{q}(t)| \le \overline{v} < 1, \quad t \in \mathbb{R}. \tag{2.2}$$

B. The main result

To state our main result we have to introduce the following weighted Sobolev spaces. Let $H^0_{s,\alpha}$, $H^1_{s,\alpha}$ be the subspaces of H^0_s, respectively, $\dot{H}^1_s$ consisting of all the fields E, respectively, A with the finite norms:

$$\|E\|_{0,\alpha} = |(1+|x|)^\alpha E|, \quad \|A\|_{1,\alpha} = \|A\|_{0,\alpha} + |(1+|x|)^\alpha \nabla A|.$$

Let us define

$$\mathcal{E}_\alpha = H^0_{s,\alpha+1} \oplus H^1_{s,\alpha} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3, \quad \|Y\|_\alpha = \|E\|_{0,\alpha+1} + \|A\|_{1,\alpha} + |q| + |P|, \quad Y \in \mathcal{E}_\alpha.$$

For the fields we set

$$\mathcal{F}_\alpha = H^0_{s,\alpha+1} \oplus H^1_{s,\alpha}, \quad \|(E_s, A)\|_\alpha = \|E\|_{0,\alpha+1} + \|A\|_{1,\alpha}.$$

Definition 2.2: A soliton state is $S(\sigma) := (E_v(x-b), A_v(x-b), b, P_v)$, *where* $\sigma := (b, v)$ *with* $b, v \in \mathbb{R}^3$ *and* $|v| < 1$.

Obviously, the soliton solution admits the representation $S(\sigma(t))$, where

$$\sigma(t) = (b(t), v(t)) = (vt + a, v). \tag{2.3}$$

Definition 2.3: A solitary manifold is the set $\mathcal{S} := \{S(b, v) : b \in \mathbb{R}^3, |v| < 1\}$.

By (1.7) and the condition (1.14) we obtain that

$$A_v(y) = \mathcal{O}(|y|^{-6}), \quad E_v(y) = \mathcal{O}(|y|^{-7}), \quad |y| \to \infty.$$

Thus,

$$E_v \in H^0_{s,\alpha} \ \text{ for } \ \alpha < 11/2, \quad A_v \in \dot{H}^1_{s,\alpha} \ \text{ for } \ \alpha < 9/2,$$

and we have for the soliton states:

$$S(\sigma) \in \mathcal{E}_\alpha, \quad \text{for } \ \alpha < 9/2. \tag{2.4}$$

The main result of our paper is the following theorem.

Theorem 2.4: *Let the condition (1.12), Wiener condition (1.13), and the condition (1.14) hold, let* $\beta = 4 + \delta$, $0 < \delta < 1/2$. *Suppose that the initial state* $Y^0 \in \mathcal{E}_\beta$ *and is sufficiently close to the* *solitary manifold:*

$$Y^0 = S_{a_0,v_0} + Z_0, \quad d_\beta := \|Z_0\|_\beta \ll 1. \tag{2.5}$$

Let $Y(t) \in C(\mathbb{R}, \mathcal{E})$ be the solution to the Cauchy problem (2.1). Then the asymptotics hold for $t \to \pm\infty$,

$$\dot q(t) = v_\pm + \mathcal{O}(|t|^{-1-\delta}), \quad q(t) = v_\pm t + a_\pm + \mathcal{O}(|t|^{-2\delta}), \tag{2.6}$$

$$(E(x,t), A(x,t)) = (E_{v_\pm}(x - v_\pm t - a_\pm), A_{v_\pm}(x - v_\pm t - a_\pm)) + W^0(t)\mathbf{\Psi}_\pm + r_\pm(x,t) \tag{2.7}$$

with

$$\|r_\pm(t)\|_{\mathcal{F}} = \mathcal{O}(|t|^{-\delta}). \tag{2.8}$$

It suffices to prove the asymptotics (2.6), (2.7) for $t \to +\infty$ since the system (1.2)–(1.4) is time reversible.

III. SYMPLECTIC PROJECTION

A. Symplectic structure

The system (1.2) to (1.4) reads as the Hamiltonian system,

$$\dot Y = J\mathcal{DH}(Y), \quad J := \begin{pmatrix} 0 & E_3 & 0 & 0 \\ -E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_3 \\ 0 & 0 & -E_3 & 0 \end{pmatrix}, \quad Y = (E, A, q, P) \in \mathcal{E}, \tag{3.1}$$

where $\mathcal{DH}$ is the Fréchet derivative of the Hamilton functional (1.1), E_3 is the 3×3 identity matrix. Let us identify the tangent space to $\mathcal{E}$, at every point, with $\mathcal{E}$. Consider the symplectic form Ω defined on $\mathcal{E}$ by

$$\Omega = \int dE(x) \wedge dA(x)\,dx + dq \wedge dP, \ \text{ i.e. } \Omega(Y_1, Y_2) = \int (E_1 \cdot A_2 - E_2 \cdot A_1)dx + q_1 \cdot P_2 - q_2 \cdot P_1, \tag{3.2}$$

for $Y_k = (E_k, A_k, q_k, P_k) \in \mathcal{E}$, $k = 1, 2$ if the integral converges.

Definition 3.1: (i) $Y_1 \nmid Y_2$ means that $Y_1 \in \mathcal{E}$ is symplectic orthogonal to $Y_2 \in \mathcal{E}$, i.e., $\Omega(Y_1, Y_2) = 0$.

(ii) A projection operator $\mathbf{P} : \mathcal{E} \to \mathcal{E}$ is called symplectic orthogonal if $Y_1 \nmid Y_2$ for $Y_1 \in \operatorname{Ker}\mathbf{P}$ and $Y_2 \in \operatorname{Im}\mathbf{P}$.

B. Symplectic projection onto solitary manifold

Let us consider the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ to the manifold $\mathcal{S}$ at a point $S(\sigma)$. The vectors $\tau_j := \partial_{\sigma_j} S(\sigma)$, where $\partial_{\sigma_j} := \partial_{b_j}$ and $\partial_{\sigma_{j+3}} := \partial_{v_j}$ with $j = 1, 2, 3$, form a basis in $\mathcal{T}_\sigma \mathcal{S}$. In detail,

$$\left.\begin{array}{l} \tau_j = \tau_j(v) := \partial_{b_j} S(\sigma) = (-\partial_j E_v(y),\ -\partial_j A_v(y),\ e_j,\ \ 0) \\ \tau_{j+3} = \tau_{j+3}(v) := \partial_{v_j} S(\sigma) = \ \ (\partial_{v_j} E_v(y),\ \ \partial_{v_j} A_v(y),\ 0\ ,\partial_{v_j} P_v) \end{array}\right| \quad j = 1, 2, 3, \tag{3.3}$$

where $y := x - b$ is the *moving coordinate frame*, $e_1 = (1, 0, 0)$, etc. Let us stress that the functions τ_j will be considered always as the functions of y, not of x.

By (2.4) we have for the tangent vectors:

$$\tau_j(v) \in \mathcal{E}_\alpha, \quad \text{for } \alpha < 9/2, \quad j = 1, \ldots, 6. \tag{3.4}$$

Lemma 3.2: The matrix with the elements $\Omega(\tau_l(v), \tau_j(v))$ is nondegenerate for $|v| < 1$.

The proof is made by a straightforward computation, see Appendix A.

Let us show that in a small neighborhood of the soliton manifold $\mathcal{S}$ a "symplectic orthogonal projection" onto $\mathcal{S}$ is well defined. Introduce the translations $T_a : (\psi(\cdot), \pi(\cdot), q, p) \mapsto (\psi(\cdot - a), \pi(\cdot - a), q + a, p)$, $a \in \mathbb{R}^3$. The manifold $\mathcal{S}$ is invariant with respect to the translations.

Definition 3.3: Put $v(Y) := P/\sqrt{1 + P^2}$, *where* $P \in \mathbb{R}^3$ *is the last component of the vector* Y.

Lemma 3.4: Let (1.12) hold, $-9/2 < \alpha$ *and* $\overline{v} < 1$. *Then*

(i) there exists a neighborhood $\mathcal{O}_\alpha(\mathcal{S})$ *of* $\mathcal{S}$ *in* $\mathcal{E}_\alpha$ *and a map* $\mathbf{\Pi} : \mathcal{O}_\alpha(\mathcal{S}) \to \mathcal{S}$ *such that* $\mathbf{\Pi}$ *is uniformly continuous on* $\mathcal{O}_\alpha(\mathcal{S}) \cap \{Y \in \mathcal{E}_\alpha : v(Y) \le \overline{v}\}$ *in the metric of* $\mathcal{E}_\alpha$,

$$\mathbf{\Pi} Y = Y \quad \textit{for} \quad Y \in \mathcal{S}, \quad \textit{and} \quad Y - S \nmid \mathcal{T}_S\mathcal{S}, \quad \text{where} \quad S = \mathbf{\Pi} Y. \tag{3.5}$$

(ii) $\mathcal{O}_\alpha(\mathcal{S})$ is invariant with respect to the translations T_a, and

$$\mathbf{\Pi} T_a Y = T_a \mathbf{\Pi} Y, \quad \textit{for} \quad Y \in \mathcal{O}_\alpha(\mathcal{S}) \quad \textit{and} \quad a \in \mathbb{R}^3. \tag{3.6}$$

(iii) For any $\overline{v} < 1$ *there exists a* $\tilde{v} < 1$ *s.t.* $|v(\mathbf{\Pi} Y)| < \tilde{v}$ *when* $|v(Y)| < \overline{v}$.
(iv) For any $\tilde{v} < 1$ *there exists an* $r_\alpha(\tilde{v}) > 0$ *s.t.* $S(\sigma) + Z \in \mathcal{O}_\alpha(\mathcal{S})$ *if* $|v(S(\sigma))| < \tilde{v}$ *and* $\|Z\|_\alpha < r_\alpha(\tilde{v})$.
The proof is similar to that of Lemma 3.4 in Ref. 22.
We will call $\mathbf{\Pi}$ the *symplectic orthogonal projection* onto $\mathcal{S}$.

Corollary 3.5: The condition (2.5) implies that $Y_0 = S + Z_0$, *where* $S = S(\sigma_0) = \mathbf{\Pi} Y_0$, *and*

$$\|Z_0\|_\beta \ll 1. \tag{3.7}$$

IV. LINEARIZATION ON THE SOLITARY MANIFOLD

Let us consider a solution to the system (1.2)–(1.4), and split it as the sum,

$$Y(t) = S(\sigma(t)) + Z(t), \tag{4.1}$$

where $\sigma(t) = (b(t), v(t)) \in \mathbb{R}^3 \times \{|v| < 1\}$ is an arbitrary smooth function of $t \in \mathbb{R}$. In detail, denote $Y = (E, A, q, P)$ and $Z = (e, a, r, \pi)$. Then (4.1) means that

$$\left.\begin{array}{ll} E(x,t) = E_{v(t)}(x - b(t)) + e(x - b(t), t), & q(t) = b(t) + r(t) \\ A(x,t) = A_{v(t)}(x - b(t)) + a(x - b(t), t), & P(t) = P_{v(t)} + \pi(t) \end{array}\right| . \tag{4.2}$$

Let us substitute (4.2) to (1.2)–(1.4) and linearize the equations in Z. Later we will choose $S(\sigma(t)) = \mathbf{\Pi} Y(t)$, i.e., $Z(t)$ is symplectic orthogonal to $\mathcal{T}_{S(\sigma(t))}\mathcal{S}$. However, this orthogonality condition is not needed for the formal process of linearization. The orthogonality condition will be important in Secs. VI–VII, where we derive "modulation equations" for the parameters $\sigma(t)$.

Let us proceed to linearization. Setting $y = x - b(t)$ which is the *moving coordinate frame*, we obtain from (4.2) and (1.3)–(1.4) that

$$\dot{E} = \dot{v} \cdot \nabla_v E_{v(t)}(y) – \dot{b} \cdot \nabla E_{v(t)}(y) + \dot{e}(y,t) – \dot{b} \cdot \nabla e(y,t) = -\Delta(A_v(y) + a(y,t)) - \Pi_s(\rho(y - r)\dot{q}), \tag{4.3}$$

$$\dot{A} = \dot{v} \cdot \nabla_v A_{v(t)}(y) - \dot{b} \cdot \nabla A_{v(t)}(y) + \dot{a}(y,t) - \dot{b} \cdot \nabla a(y,t) = -E_{v(t)}(y) - e(y,t), \tag{4.4}$$

$$\dot{q} = \dot{b} + \dot{r} = \frac{P_{v(t)} + \pi - \langle \rho(y - r), A_{v(t)}(y) + a(y,t) \rangle}{(1 + (P_{v(t)} + \pi - \langle \rho(y - r), A_{v(t)}(y) + a(y,t) \rangle)^2)^{1/2}}, \tag{4.5}$$

$$\dot P=\dot v\cdot\nabla_v P_{v(t)}+\dot\pi=\langle\rho(y-r),\nabla(\dot q\cdot(A_{v(t)}(y)+a(y,t))\rangle. \tag{4.6}$$

Step (i): First we linearize Eq. (4.5). Note that

$$\rho(y-r)=\rho(y)-r\cdot\nabla\rho(y)+N_2(r), \tag{4.7}$$

where

$$\|N_2(r)\|_{0,\alpha}\le C_\alpha(\overline r)r^2, \tag{4.8}$$

uniformly in $|r|\le\overline r$ for any fixed $\overline r$, for an arbitrary $\alpha>0$. Then (let us write v instead of $v(t)$ and omit the other arguments for simplicity),

$$\langle\rho(y-r),A_v+a\rangle=\langle\rho,A_v\rangle+\langle\rho,a\rangle-\langle r\cdot\nabla\rho,A_v\rangle+N_2'=\langle\rho,A_v\rangle+\langle\rho,a\rangle+N_2', \tag{4.9}$$

where $N_2'(r,a)=-\langle r\cdot\nabla\rho,a\rangle+\langle N_2,A_v+a\rangle$. Here we use the equality $\langle r\cdot\nabla\rho,A_v\rangle=0$ which holds, since A_v is even and $\nabla\rho$ is odd. Further, since $P_v-\langle\rho,A_v\rangle=p_v$ by (1.6), we get $P_v+\pi-\langle\rho(y-r),A_v+a\rangle=P_v+\pi-\langle\rho,A_v\rangle-\langle\rho,a\rangle-N_2'=p_v+\pi-\langle\rho,a\rangle-N_2'=p_v+s$, where $s:=\pi-\langle\rho,a\rangle-N_2'$. Applying Taylor expansion we obtain

$$(1+(p_v+s)^2)^{-1/2}=\frac{1}{(1+p_v^2)^{1/2}}-\frac{p_v\cdot s}{(1+p_v^2)^{3/2}}+N_3'=\frac{1}{(1+p_v^2)^{1/2}}-\frac{v\cdot s}{1+p_v^2}+N_3',$$

since $p_v/(1+p_v^2)^{1/2}=v$. Finally,

$$\frac{P_v+\pi-\langle\rho(y-r),A_v+a\rangle}{(1+(P_v+\pi-\langle\rho(y-r),A_v(y)+a\rangle)^2)^{1/2}}=\frac{p_v+s}{(1+(p_v+s)^2)^{1/2}}$$

$$=v-\frac{(v,s)v}{(1+(p_v)^2)^{1/2}}+\frac{s}{(1+(p_v)^2)^{1/2}}+N_3''=v+\nu(s-(v\cdot s)v)+N_3'',$$

where $\nu=(1-v^2)^{1/2}=(1+p_v^2)^{-1/2}$. Insert the expression for s, then Eq. (4.5) becomes

$$\dot r=v-\dot b+B_v(\pi-\langle\rho,a\rangle)+N_3, \tag{4.10}$$

where $B_v:=\nu(E-v\otimes v)$, and

$$|N_3(Z)|\le C(\tilde v)\|Z\|^2_{-\alpha}, \tag{4.11}$$

uniformly in $|v|\le\tilde v<1$, for an arbitrary $\alpha>0$.

Step (ii): Next we linearize Eq. (4.3). By (4.7) and (4.10) we obtain

$$\rho(y-r)\dot q=\rho v+\rho B_v(\pi-\langle\rho,a\rangle)-r\cdot\nabla\rho v+N_1'.$$

Substitute to Eq. (4.3) and take (1.8) into account, then we get

$$\dot e=\dot b\cdot\nabla e-\Delta a+(\dot b-v)\cdot\nabla E_v-\dot v\cdot\nabla_v E_v-\Pi_s(\rho B_v(\pi-\langle\rho,a\rangle)-r\cdot\nabla\rho v)+N_1, \tag{4.12}$$

where for N_1 the same bound holds,

$$\|N_1(Z)\|\le C(\tilde v)\|Z\|^2_{-\alpha},\ \ \forall\alpha>0. \tag{4.13}$$

Step (iii): Further, by (1.8) Eq. (4.4) becomes

$$\dot a=-e+\dot b\cdot\nabla a+(\dot b-v)\cdot\nabla A_v-\dot v\cdot\nabla_v A_v. \tag{4.14}$$

Step (iv): Let us proceed to Eq. (4.6). We have

$$\dot P=\dot v\cdot\nabla_v P_v+\dot\pi=\langle\rho(y-r),\nabla(\dot q\cdot(A_v+a))\rangle$$

$$=\ \langle\rho-r\cdot\nabla\rho+N_2,\nabla((v+B_v(\pi-\langle\rho,a\rangle)+N_3)\cdot(A_v+a))\rangle=\langle\rho,v\cdot\nabla a\rangle$$
$$-\langle r\cdot\nabla\rho,\nabla(v\cdot A_v)\rangle+N_4,$$

since $\langle\rho,v\cdot\nabla A_v\rangle=0$ and $\langle\rho,B_v(\pi-\langle\rho,a\rangle)\cdot\nabla A_v\rangle=0$. Finally, the equation becomes

$$\dot\pi=\langle\rho,\nabla(v\cdot a)\rangle-\langle r\cdot\nabla\rho,\nabla(v\cdot A_v)\rangle-\dot v\cdot\nabla_v P_v+N_4, \tag{4.15}$$

where for $N_4(v, Z)$ the estimate such as (4.11) holds. We write Eqs. (4.10), (4.12)–(4.15) as

$$\dot{Z}(t) = A(t)Z(t) + T(t) + N(t), \quad t \in \mathbb{R}. \tag{4.16}$$

Here the operator $A(t)$ depends on $\sigma(t) = (b(t), v(t))$. We will use the parameters $v = v(t)$ and $w := \dot{b}(t)$. Then $A(t) = A_{v,w}$ can be written in the form:

$$A_{v,w}\begin{pmatrix} e \\ a \\ r \\ \pi \end{pmatrix} := \begin{pmatrix} w\cdot\nabla & -\Delta + \Pi_s(\rho B_v\langle\rho,\cdot\rangle) & \Pi_s(\cdot\nabla\rho v) & -\Pi_s(\rho B_v\cdot) \\ -1 & w\cdot\nabla & 0 & 0 \\ 0 & -B_v\langle\rho,\cdot\rangle & 0 & B_v \\ 0 & \langle\rho,\nabla(v\cdot)\rangle & -\langle\cdot\nabla\rho,\nabla(v\cdot A_v)\rangle & 0 \end{pmatrix}\begin{pmatrix} e \\ a \\ r \\ \pi \end{pmatrix}. \tag{4.17}$$

Furthermore, $T(t)$ and $N(t)$ in (4.16) stand for

$$T(t) = T_{v,w} = \begin{pmatrix} (w-v)\cdot\nabla E_v - \dot{v}\cdot\nabla_v E_v \\ (w-v)\cdot\nabla A_v - \dot{v}\cdot\nabla_v A_v \\ v - w \\ -\dot{v}\cdot\nabla_v P_v \end{pmatrix}, \quad N(t) = N(v,Z) = \begin{pmatrix} N_1(v,Z) \\ 0 \\ N_3(v,Z) \\ N_4(v,Z) \end{pmatrix}, \tag{4.18}$$

where $v = v(t)$, $w = w(t)$, and $Z = Z(t)$. The estimates (4.8), (4.11), and (4.13) imply the following.

Lemma 4.1: For any $\alpha > 0$,

$$\|N(v,Z)\|_\alpha \le C(\tilde{v})\|Z\|_{-\alpha}^2, \tag{4.19}$$

uniformly in v *and* Z *with* $\|Z\|_{-\alpha} \le r_{-\alpha}(\tilde{v})$ *and* $|v| < \tilde{v} < 1$.

Remarks 4.2: (i) The term $A(t)Z(t)$ in the right-hand side of Eq. (4.16) is linear in $Z(t)$, and $N(t)$ is a *high order term* in $Z(t)$.

(ii) Formulas (3.3) and (4.18) imply:

$$T(t) = -\sum_{l=1}^{3}[(w-v)_l\tau_l + \dot{v}_l\tau_{l+3}], \tag{4.20}$$

and hence $T(t) \in \mathcal{T}_{S(\sigma(t))}\mathcal{S}$, $t \in \mathbb{R}$. The term $T(t)$ vanishes if $S(\sigma(t))$ is a soliton solution since in this case $\dot{v} = 0$ and $w = \dot{b} = v$. Otherwise $T(t)$ is a *zero order term* which does not vanish although $S(\sigma(t))$ belongs to the solitary manifold. In our context we will show that $T(t)$ rapidly decays as $t \to \infty$ [see (8.2) below].

V. THE LINEARIZED EQUATION

Here we study some properties of the operator (4.17). First, let us compute the action of $A_{v,w}$ on the tangent vectors τ_j to the solitary manifold $\mathcal{S}$.

Lemma 5.1: The operator $A_{v,w}$ acts on the tangent vectors $\tau_j(v)$ to the solitary manifold as follows,

$$A_{v,w}[\tau_j(v)] = (w-v)\cdot\nabla\tau_j(v), \quad A_{v,w}[\tau_{j+3}(v)] = (w-v)\cdot\nabla\tau_{j+3}(v) + \tau_j(v), \quad j = 1,2,3. \tag{5.1}$$

Proof: To get (5.1), differentiate the stationary equations (1.8) in x_j and v_j, cf. Ref. 22. □

Consider the linear equation

$$\dot{X}(t) = A_{v,w}X(t), t \in \mathbb{R} \tag{5.2}$$

with an arbitrary fixed v such that $|v| < 1$ and $w \in \mathbb{R}^3$. Let us define the space:

$$\mathcal{E}^+ = H_s^1 \oplus \dot{H}_s^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3.$$

Lemma 5.2: (i) For any v, $|v| < 1$, $w \in \mathbb{R}^3$ Eq. (5.2) formally can be written as the Hamiltonian system [cf. (3.1)],

$$\dot{X}(t) = JD\mathcal{H}_{v,w}(X(t)), \quad t \in \mathbb{R}, \tag{5.3}$$

where $D\mathcal{H}_v$ is the Fréchet derivative of the Hamilton functional:

$$\mathcal{H}_{v,w}(X) = \frac{1}{2}\int \Big[|e|^2 + |\nabla a|^2\Big]dy + \int a(w\cdot\nabla)edy + \frac{1}{2}(B_v\langle\rho, a\rangle)\cdot\langle\rho, a\rangle$$

$$+\frac{1}{2}\pi\cdot B_v\pi + \langle r\cdot\nabla\rho v, a\rangle - \langle\rho B_v\pi, a\rangle + \frac{1}{2}\langle r\cdot\nabla\rho, v\cdot(\nabla A_v)\rangle, \quad X = (e, a, r, \pi) \in \mathcal{E}. \tag{5.4}$$

(ii) *Energy conservation law holds for the solutions $X(t) \in C(\mathbb{R}, \mathcal{E})$,*

$$\mathcal{H}_{v,w}(X(t)) = \text{const}, t \in \mathbb{R}. \tag{5.5}$$

(iii) *The skew-symmetry relation holds,*

$$\Omega(A_{v,w}X_1, X_2) = -\Omega(X_1, A_{v,w}X_2), \quad X_1 \in \mathcal{E}, \quad X_2 \in \mathcal{E}^+. \tag{5.6}$$

The proof is similar to that in Ref. 22. We will apply Lemma 5.2 mainly to the operator $A_{v,v}$ corresponding to $w = v$. In that case the linearized equation has the following additional essential features.

Lemma 5.3: Let us assume that $w = v$ and $|v| < 1$. Then

(i) *the tangent vectors $\tau_j(v)$ with $j = 1, 2, 3$ are eigenvectors, and $\tau_{j+3}(v)$ are root vectors of the operator $A_{v,v}$, corresponding to zero eigenvalue, i.e.,*

$$A_{v,v}[\tau_j(v)] = 0, \quad A_{v,v}[\tau_{j+3}(v)] = \tau_j(v), \quad j = 1, 2, 3. \tag{5.7}$$

(ii) *The Hamilton function (5.4) is positive definite,*

$$\mathcal{H}_{v,v}(X) \geq 0. \tag{5.8}$$

Proof: The first statement follows from (5.1). To prove the second statement note that for $X = (e, a, r, \pi) \in \mathcal{E}$ one has

$$\begin{aligned}\mathcal{H}_{v,v}(X) &= \frac{1}{2}\int\Big[|e|^2 + |\nabla a|^2\Big]dy + \int a(v\cdot\nabla)edy + \frac{1}{2}(B_v\langle\rho, a\rangle)\cdot\langle\rho, a\rangle\\ &\quad+\frac{1}{2}\pi\cdot B_v\pi + \langle r\cdot\nabla\rho v, a\rangle - \langle\rho B_v\pi, a\rangle + \frac{1}{2}\langle r\cdot\nabla\rho, v\cdot(\nabla A_v)\rangle\\ &= \frac{1}{2}(B_v(\pi - \langle\rho, a\rangle))\cdot(\pi - \langle\rho, a\rangle) + \frac{1}{2}(\langle e, e\rangle + \langle(v\cdot\nabla)a, (v\cdot\nabla)a\rangle - \langle e, (v\cdot\nabla)a\rangle)\\ &\quad+\frac{1}{2}(\langle(-\Delta + (v\cdot\nabla)^2)a, a\rangle + \langle(r\cdot\nabla)\rho v, a\rangle + \langle(r\cdot\nabla)\rho, v\cdot(\nabla A_v)\rangle).\end{aligned}$$

Here the first line is clearly non-negative, since B_v is non-negative definite. The last line in Fourier space by (A3) equals,

$$\frac{1}{2}\int\left((k^2 - (kv)^2)|\hat{a}|^2 - 2i(kr)\hat{\rho}(v\cdot\overline{\hat{a}}) + \frac{(kr)^2|\hat{\rho}|^2v^2}{k^2 - (kv)^2}\right)dk.$$

The integrand is non-negative, since $|\text{Re}\,[i(kr)\hat{\rho}(v\cdot\overline{\hat{a}})]| \leq |(kr)||\hat{\rho}||v||\hat{a}|$. □

Remark 5.4: For a soliton solution of the system (1.2)–(1.4) we have $\dot{b} = v$, $\dot{v} = 0$, and hence $T(t) \equiv 0$. Thus, Eq. (5.2) is the linearization of the system (1.2)–(1.4) on a soliton solution. In fact, we do not linearize (1.2)–(1.4) on a soliton solution but on a trajectory $S(\sigma(t))$ with $\sigma(t)$ being nonlinear in t. We will show later that $T(t)$ is quadratic in $Z(t)$ if we choose $S(\sigma(t))$ to be the symplectic orthogonal projection of $Y(t)$. Then (5.2) is again the linearization of (1.2)–(1.4).

VI. SYMPLECTIC DECOMPOSITION OF THE DYNAMICS

Here we decompose the dynamics in two components: along the manifold $\mathcal{S}$ and in transversal directions. Equation (4.16) is obtained without any assumption on $\sigma(t)$ in (4.1). We are going to choose $S(\sigma(t)) := \mathbf{\Pi} Y(t)$ but then we need to know that

$$Y(t) \in \mathcal{O}_\alpha(\mathcal{S}), \quad t \in \mathbb{R}, \tag{6.1}$$

with some $\mathcal{O}_\alpha(\mathcal{S})$ defined in Lemma (3.5). It is true for $t = 0$ and $\alpha = \beta$ by our main assumption (2.5) with sufficiently small $d_\beta > 0$. Then $S(\sigma(0)) = \mathbf{\Pi} Y(0)$ and $Z(0) = Y(0) - S(\sigma(0))$ are well defined. We will prove below that (6.1) holds with $\alpha = -\beta$ if d_β is sufficiently small. First, the *a priori* estimate (2.2) together with Lemma 3.4 (iii) imply that $\mathbf{\Pi} Y(t) = S(\sigma(t))$ with $\sigma(t) = (b(t), v(t))$, and

$$|v(t)| \le \tilde{v} < 1, \quad t \in \mathbb{R}, \tag{6.2}$$

if $Y(t) \in \mathcal{O}_{-\beta}(\mathcal{S})$. Denote by $r_{-\beta}(\tilde{v})$ the positive number from Lemma 3.4 (iv) which corresponds to $\alpha = -\beta$. Then $S(\sigma) + Z \in \mathcal{O}_{-\beta}(\mathcal{S})$ if $\sigma = (b, v)$ with $|v| < \tilde{v}$ and $\|Z\|_{-\beta} < r_{-\beta}(\tilde{v})$. Note that (2.2) implies $\|Z(0)\|_{-\beta} < r_{-\beta}(\tilde{v})$ if d_β is sufficiently small. Therefore, $S(\sigma(t)) = \mathbf{\Pi} Y(t)$ and $Z(t) = Y(t) - S(\sigma(t))$ are well defined for $t \ge 0$ so small that $\|Z(t)\|_{-\beta} < r_{-\beta}(\tilde{v})$. This is formalized by the following standard definition.

Definition 6.1: t_* is the "exit time,"

$$t_* = \sup\{t > 0 : \|Z(s)\|_{-\beta} < r_{-\beta}(\tilde{v}), \quad 0 \le s \le t\}, \quad Z(s) = Y(s) - S(\sigma(s)). \tag{6.3}$$

One of our main goals is to prove that $t_* = \infty$ if d_β is sufficiently small. This would follow if we show that

$$\|Z(t)\|_{-\beta} < r_{-\beta}(\tilde{v})/2, \quad 0 \le t < t_*. \tag{6.4}$$

Note that

$$|r(t)| \le \bar{r} := r_{-\beta}(\tilde{v}), \quad 0 \le t < t_*. \tag{6.5}$$

Now $N(t)$ in (4.16) satisfies, by (4.19) with $\alpha = -\beta$, the following estimate,

$$\|N(t)\|_\beta \le C_\beta(\tilde{v}) \|Z(t)\|^2_{-\beta}, \quad 0 \le t < t_*. \tag{6.6}$$

VII. LONGITUDINAL DYNAMICS: MODULATION EQUATIONS

From now on we fix the decomposition $Y(t) = S(\sigma(t)) + Z(t)$ for $0 < t < t_*$ by setting $S(\sigma(t)) = \mathbf{\Pi} Y(t)$, which is equivalent to the symplectic orthogonality condition of type (3.5),

$$Z(t) \nmid \mathcal{T}_{S(\sigma(t))}\mathcal{S}, \quad 0 \le t < t_*. \tag{7.1}$$

This allows us to simplify drastically the asymptotic analysis of the dynamical equations (4.16) for the transversal component $Z(t)$. As the first step, we derive the longitudinal dynamics, i.e., the "modulation equations" for the parameters $\sigma(t)$. Let us derive a system of ordinary differential equations for the vector $\sigma(t)$. For this purpose, let us write (7.1) in the form,

$$\Omega(Z(t), \tau_j(t)) = 0, \quad j = 1, \ldots, 6, \quad 0 \le t < t_*, \tag{7.2}$$

where the vectors $\tau_j(t) = \tau_j(\sigma(t))$ span the tangent space $\mathcal{T}_{S(\sigma(t))}\mathcal{S}$. Note that $\sigma(t) = (b(t), v(t))$, where

$$|v(t)| \le \tilde{v} < 1, \quad 0 \le t < t_*, \tag{7.3}$$

by Lemma 3.4 (iii). It would be convenient for us to use some other parameters (c, v) instead of $\sigma = (b, v)$, where $c(t) = b(t) - \int_0^t v(\tau)d\tau$ and

$$\dot{c}(t) = \dot{b}(t) - v(t) = w(t) - v(t), \quad 0 \le t < t_*. \tag{7.4}$$

We do not need an explicit form of the equations for (c, v) but the following statement.

Lemma 7.1: (cf. Ref. 22, Lemma 6.2:): Let $Y(t)$ be a solution to the Cauchy problem (2.1), and (4.1), (7.2) hold. Then $(c(t), v(t))$ satisfies the equation

$$\begin{pmatrix} \dot{c}(t) \\ \dot{v}(t) \end{pmatrix} = \mathcal{N}(\sigma(t), Z(t)), \quad 0 \le t < t_*, \tag{7.5}$$

where

$$\mathcal{N}(\sigma, Z) = \mathcal{O}(\|Z\|^2_{-\beta}), \tag{7.6}$$

uniformly in $\sigma \in \{(b, v) : |v| \le \tilde{v}\}$.

Proof: We differentiate (7.2) in t and take Eq. (4.16) into account. Then [see details of computation in Ref. 22, Lemma 6.2] we obtain, in the vector form [Ref. 22, (6.18)]

$$0 = \Omega(v)\begin{pmatrix} \dot{c} \\ \dot{v} \end{pmatrix} + \mathcal{M}_0(\sigma, Z)\begin{pmatrix} \dot{c} \\ \dot{v} \end{pmatrix} + \mathcal{N}_0(\sigma, Z), \qquad \mathcal{N}_{0j}(\sigma, Z) = \Omega(N, \tau_j). \tag{7.7}$$

Here the matrix $\Omega(v)$ has the matrix elements $\Omega(\tau_l, \tau_j)$ and hence is invertible by Lemma 3.2. The 6×6 matrix $\mathcal{M}_0(\sigma, Z)$ has the matrix elements $\sim \|Z\|_{-\beta}$ and hence we can resolve Eq. (7.7) with respect to $(\dot{c}, \dot{v})$. Then (7.6) follows from Lemma 4.1 with $\alpha = \beta$, since $\mathcal{N}_0 = \mathcal{O}(\|Z\|^2_{-\beta})$. □

Remark 7.2: Equations (7.5), (7.6) imply that the soliton parameters $c(t)$ and $v(t)$ are *adiabatic invariants* [see Ref. 23].

VIII. DECAY FOR THE TRANSVERSAL DYNAMICS

Here we prove the following time decay of the transversal component $Z(t)$.

Proposition 8.1: Let all conditions of Theorem 2.4 hold. Then $t_* = \infty$, and

$$\|Z(t)\|_{-\beta} \le \frac{C(\rho, \tilde{v}, d_\beta)}{(1 + |t|)^{1+\delta}}, \quad t \ge 0. \tag{8.1}$$

In next section, we will show that our main Theorem 2.4 can be derived from the transversal decay (8.1). We will derive this decay from Eq. (4.16) for the transversal component $Z(t)$. This equation can be specified using Lemma 7.1. Namely, by (4.20) and (7.4),

$$T(t) = -\sum_{l=1}^{3}[\dot{c}_l \tau_l + \dot{v}_l \tau_{l+3}].$$

Then Lemma 7.1 implies that

$$\|T(t)\|_\beta \le C(\tilde{v})\|Z(t)\|^2_{-\beta}, \quad 0 \le t < t_*. \tag{8.2}$$

Note that the norm $\|T(t)\|_\beta$ is well defined by the condition (1.14). Thus, in (4.16) we should combine the terms $T(t)$ and $N(t)$ and obtain

$$\dot{Z}(t) = A(t)Z(t) + \tilde{N}(t), \quad 0 \le t < t_*, \tag{8.3}$$

where $A(t) = A_{v(t),w(t)}$, and $\tilde{N}(t) := T(t) + N(t)$. By (8.2) and (6.6) we have

$$\|\tilde{N}(t)\|_\beta \le C\|Z(t)\|^2_{-\beta}, \quad 0 \le t < t_*. \tag{8.4}$$

In all remaining part of our paper we will analyze mainly the *basic equation* (8.3) to establish the decay (8.1). We are going to derive the decay using the bound (8.4) and the orthogonality condition (7.1).

Let us comment on two main difficulties in proving (8.1). The difficulties are common for the problems studied in Refs. 14 and 24. First, the linear part of the equation is nonautonomous, hence we cannot apply directly the known methods of scattering theory. Similarly to the approach of Refs. 14 and 24, we reduce the problem to the analysis of the *frozen* linear equation,

$$\dot{X}(t) = A_1 X(t), \quad t \in \mathbb{R}, \tag{8.5}$$

where A_1 is the operator A_{v_1,v_1} defined in (4.17) with $v_1 = v(t_1)$ and a fixed $t_1 \in [0, t_*)$. Then we estimate the error by the method of majorants.

Second, even for the frozen equation (8.5), the decay of type (8.1) for all solutions does not hold without the orthogonality condition of type (7.1). Namely, by (5.7) Eq. (8.5) admits the *secular solutions,*

$$X(t) = \sum_1^3 C_j \tau_j(v_1) + \sum_1^3 D_j[\tau_j(v_1)t + \tau_{j+3}(v_1)], \tag{8.6}$$

which arise also by differentiation of the soliton (1.6) in the parameters a and v_1 in the moving coordinate $y = x - v_1 t$. Hence, we have to take into account the orthogonality condition (7.1) in order to avoid the secular solutions. For this purpose we will apply the corresponding symplectic orthogonal projection, which kills the "runaway solutions" (8.6).

Definition 8.2: (i) Denote by $\mathbf{\Pi}_v$, $|v| < 1$, *the symplectic orthogonal projection of* $\mathcal{E}$ *onto the tangent space* $\mathcal{T}_{S(\sigma)}\mathcal{S}$, *and* $\boldsymbol{P}_v = \mathbf{I} - \mathbf{\Pi}_v$.

(ii) Denote by $\mathcal{Z}_v = \boldsymbol{P}_v\mathcal{E}$ *the space symplectic orthogonal to* $\mathcal{T}_{S(\sigma)}\mathcal{S}$ *with* $\sigma = (b, v)$ *(for an arbitrary* $b \in \mathbb{R}$*).*

Note that by the linearity,

$$\mathbf{\Pi}_v Z = \sum \mathbf{\Pi}_{jl}(v)\tau_j(v)\Omega(\tau_l(v), Z), \quad Z \in \mathcal{E}, \tag{8.7}$$

with some smooth coefficients $\mathbf{\Pi}_{jl}(v)$. Hence, the projector $\mathbf{\Pi}_v$, in the variable $y = x - b$, does not depend on b, and this explains the choice of the subindex in $\mathbf{\Pi}_v$ and $\boldsymbol{P}_v$.

Now we have the symplectic orthogonal decomposition

$$\mathcal{E}_\beta = \mathcal{T}_{S(\sigma)}\mathcal{S} + \mathcal{Z}_v, \quad \sigma = (b, v), \tag{8.8}$$

and the symplectic orthogonality (7.1) can be written in the following equivalent forms:

$$\mathbf{\Pi}_{v(t)} Z(t) = 0, \quad \boldsymbol{P}_{v(t)} Z(t) = Z(t), \quad 0 \le t < t_*. \tag{8.9}$$

Remark 8.3: The tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ is invariant under the operator $A_{v,v}$ by Lemma 5.3 (i), hence the space $\mathcal{Z}_v$ is also invariant by (5.6): $A_{v,v}Z \in \mathcal{Z}_v$ for *sufficiently smooth* $Z \in \mathcal{Z}_v$.

The following proposition is one of the main ingredients for proving (8.1). Let us consider the Cauchy problem for Eq. (8.5) with $A_1 = A_{v_1,v_1}$ for a fixed v_1, $|v_1| < 1$. Recall that $\beta = 4 + \delta$, $0 < \delta < 1/2$.

Proposition 8.4: Let the Wiener condition (1.13) and the condition (1.14) hold, $|v_1| \le \tilde{v} < 1$, *and* $X_0 \in \mathcal{E}$. *Then*

(i) *Equation (8.5), with* $A_1 = A_{v_1,v_1}$, *admits the unique solution* $e^{A_1 t}X_0 := X(t) \in C_b(\mathbb{R}, \mathcal{E})$ *with the initial condition* $X(0) = X_0$.

(ii) *For* $X_0 \in \mathcal{Z}_{v_1} \cap \mathcal{E}_\beta$, *the the following decay holds,*

$$\|e^{A_1 t}X_0\|_{-2-\delta} \le \frac{C(\tilde{v})}{(1 + |t|)^{1+\delta}}\|X_0\|_\beta, \quad t \in \mathbb{R}. \tag{8.10}$$

Part (i) follows by standard arguments using the positivity (5.8) of the Hamilton functional. Part (ii) will be proved in Secs. X–XIII developing general strategy.[22] Namely, Eq. (8.5) is a system of four equations involving field components, E and A as well as vector components, r and π. We apply Fourier–Laplace transform, express the field components in terms of the vector components from the first two equations and substitute to the third and the fourth equations. Then we obtain a closed system for the vector components alone and prove their decay. Finally, for the field components we come to a wave equation with a right-hand side which has the established decay. This implies the corresponding decay for the field components.

A. Frozen form of transversal dynamics

Now let us fix an arbitrary $t_1 \in [0, t_*)$, and rewrite Eq. (8.3) in a "frozen form,"

$$\dot{Z}(t) = A_1 Z(t) + (A(t) - A_1)Z(t) + \tilde{N}(t), \qquad 0 \le t < t_*, \tag{8.11}$$

where $A_1 = A_{v(t_1),v(t_1)}$ and

$$A(t)-A_1=\begin{pmatrix} [w-v_1]\cdot\nabla & \Pi_s(\rho(B_v - B_{v_1})\langle\rho,\cdot\rangle) & \Pi_s(\cdot\nabla\rho(v-v_1)) & -\Pi_s(\rho(B_v - B_{v_1})\cdot) \\ 0 & [w-v_1]\cdot\nabla & 0 & 0 \\ 0 & -(B_v - B_{v_1})\langle\rho,\cdot\rangle & 0 & B_v - B_{v_1} \\ 0 & \langle\rho,(v-v_1)\nabla\cdot\rangle & -\langle\cdot\nabla\rho,(v\nabla A_v - v_1\nabla A_{v_1})\rangle & 0 \end{pmatrix}, \tag{8.12}$$

where $w = w(t)$, $v = v(t)$, $v_1 = v(t_1)$. The next trick is important since it allows us to kill the "bad terms" $[w(t)-v(t_1)]\cdot\nabla$ in the operator $A(t) - A_1$.

Definition 8.5: Let us change the variables $(y, t) \mapsto (y_1, t) = (y + d_1(t), t)$, *where*

$$d_1(t) := \int_{t_1}^{t} (w(s) - v(t_1))ds, \quad 0 \le t \le t_1. \tag{8.13}$$

Next define

$$\begin{aligned} Z_1(t) &= (e_1(y_1, t), a_1(y_1, t), r(t), \pi(t)) := (e(y, t), a(y, t), r(t), \pi(t)) \\ &= (e(y_1 - d_1(t), t), a(y_1 - d_1(t), t), r(t), \pi(t)). \end{aligned} \tag{8.14}$$

Then we obtain the final form of the "frozen equation" for the transversal dynamics

$$\dot{Z}_1(t) = A_1 Z_1(t) + B_1(t)Z_1(t) + N_1(t), \ \ 0 \le t \le t_1, \tag{8.15}$$

where $N_1(t) = \tilde{N}(t)$ is expressed in terms of $y = y_1 - d_1(t)$, and

$$B_1(t)=\begin{pmatrix} 0 & \Pi_s(\rho(B_{v(t)} - B_{v(t_1)})\langle\rho,\cdot\rangle) & \Pi_s(\cdot\nabla\rho(v(t)-v(t_1))) & -\Pi_s(\rho(B_{v(t)} - B_{v(t_1)})\cdot) \\ 0 & 0 & 0 & 0 \\ 0 & -(B_{v(t)} - B_{v(t_1)})\langle\rho,\cdot\rangle & 0 & B_{v(t)} - B_{v(t_1)} \\ 0 & \langle\rho,(v(t)-v(t_1))\cdot\nabla\cdot\rangle & -\langle\cdot\nabla\rho,(v(t)\nabla A_{v(t)} - v(t_1)\nabla A_{v(t_1)})\rangle & 0 \end{pmatrix}.$$

Let us derive appropriate bounds for the "remainder terms" $B_1(t)Z_1(t)$ and $N_1(t)$ in (8.15).

Lemma 8.6 (Ref. 22, Corollaries 7.3 and 7.4): The following bounds hold:

$$\|N_1(t)\|_\beta \le \|Z_1(t)\|^2_{-\beta}(1 + |d_1(t)|)^{3\beta}, \quad 0 \le t \le t_1, \tag{8.16}$$

$$\|B_1(t)Z_1(t)\|_\beta \le C\|Z_1(t)\|_{-\beta} \int_t^{t_1} (1 + |d_1(\tau)|)^{2\beta}\|Z_1(\tau)\|^2_{-\beta} d\tau, \quad 0 \le t \le t_1. \tag{8.17}$$

B. Integral inequality

Equation (8.15) can be written in the integral form:

$$Z_1(t) = e^{A_1 t} Z_1(0) + \int_0^t e^{A_1(t-s)}[B_1 Z_1(s) + N_1(s)]ds, \ \ 0 \le t \le t_1. \tag{8.18}$$

We apply the symplectic orthogonal projection $\boldsymbol{P}_1 := \boldsymbol{P}_{v(t_1)}$ to both sides, and get

$$\boldsymbol{P}_1 Z_1(t) = e^{A_1 t}\boldsymbol{P}_1 Z_1(0) + \int_0^t e^{A_1(t-s)}\boldsymbol{P}_1[B_1 Z_1(s) + N_1(s)]ds.$$

We have used here that $\boldsymbol{P}_1$ commutes with the group $e^{A_1 t}$ since the space $\mathcal{Z}_1 := \boldsymbol{P}_1\mathcal{E}$ is invariant with respect to $e^{A_1 t}$, see Remark 8.3. Applying (8.10) we obtain that

$$\|\boldsymbol{P}_1 Z_1(t)\|_{-2-\delta} \le \frac{C}{(1+t)^{1+\delta}}\|\boldsymbol{P}_1 Z_1(0)\|_\beta + C\int_0^t \frac{1}{(1+|t-s|)^{1+\delta}}\|\boldsymbol{P}_1[B_1 Z_1(s) + N_1(s)]\|_\beta ds. \tag{8.19}$$

The operator $\boldsymbol{P}_1 = \mathbf{I} - \mathbf{\Pi}_1$ is continuous in $\mathcal{E}_\beta$ by (8.7). Hence, from (8.16), (8.17), and (8.19) we obtain that

$$\|\boldsymbol{P}_1 Z_1(t)\|_{-2-\delta} \le \frac{C}{(1+t)^{1+\delta}}\|Z_1(0)\|_\beta$$
$$+C(\overline{d}_1)\int_0^t \frac{1}{(1+|t-s|)^{1+\delta}}\left[\|Z_1(s)\|_{-\beta}\int_s^{t_1}\|Z_1(\tau)\|^2_{-\beta}d\tau + \|Z_1(s)\|^2_{-\beta}\right]ds, \quad 0 \le t \le t_1, \tag{8.20}$$

where $\overline{d}_1 := \sup_{0\le t\le t_1}|d_1(t)|$. Since $\|Z_1(t)\|_{\pm\beta} \le C(\overline{d}_1)\|Z(t)\|_{\pm\beta}$, we can rewrite (8.20) as

$$\|\boldsymbol{P}_1 Z_1(t)\|_{-2-\delta} \le \frac{C(\overline{d}_1)}{(1+t)^{1+\delta}}\|Z(0)\|_\beta$$
$$+C(\overline{d}_1)\int_0^t \frac{1}{(1+|t-s|)^{1+\delta}}\left[\|Z(s)\|_{-\beta}\int_s^{t_1}\|Z(\tau)\|^2_{-\beta}d\tau + \|Z(s)\|^2_{-\beta}\right]ds, \quad 0 \le t \le t_1. \tag{8.21}$$

Let us introduce the *majorant,*

$$m(t) := \sup_{s\in[0,t]}(1+s)^{1+\delta}\|Z(s)\|_{-\beta}, \qquad t \in [0, t_*). \tag{8.22}$$

To estimate $d_1(t)$ by $m(t_1)$ we note that

$$w(s) - v(t_1) = w(s) - v(s) + v(s) - v(t_1) = \dot{c}(s) + \int_s^{t_1}\dot{v}(\tau)d\tau, \tag{8.23}$$

by (7.4). Hence, (8.13), Lemma 7.1 and Definition (8.22) imply:

$$|d_1(t)| = |\int_{t_1}^t (w(s) - v(t_1))ds| \le \int_t^{t_1}\left(|\dot{c}(s)| + \int_s^{t_1}|\dot{v}(\tau)|d\tau\right)ds$$
$$\le Cm^2(t_1)\int_t^{t_1}\left(\frac{1}{(1+s)^{2+2\delta}} + \int_s^{t_1}\frac{d\tau}{(1+\tau)^{2+2\delta}}\right)ds \le Cm^2(t_1), \qquad 0 \le t \le t_1. \tag{8.24}$$

We can replace in (8.21) the constants $C(\overline{d}_1)$ by C if $m(t_1)$ is bounded for $t_1 \ge 0$. In order to do this replacement, we reduce the exit time. Let us denote by ε a fixed positive number which we will specify below.

Definition 8.7: t'_* is the exit time,

$$t'_* = \sup\{t \in [0, t_*) : m(s) \le \varepsilon, \quad 0 \le s \le t\}. \tag{8.25}$$

Now (8.21) implies that for $t_1 < t'_*$,

$$\|\boldsymbol{P}_1 Z_1(t)\|_{-2-\delta} \le \frac{C}{(1+t)^{1+\delta}}\|Z(0)\|_\beta$$
$$+C\int_0^t \frac{1}{(1+|t-s|)^{1+\delta}}\left[\|Z(s)\|_{-\beta}\int_s^{t_1}\|Z(\tau)\|^2_{-\beta}d\tau + \|Z(s)\|^2_{-\beta}\right]ds, \quad 0 \le t \le t_1, \tag{8.26}$$

C. Symplectic orthogonality

The following important bound (8.27) allows us to change the norm of $\boldsymbol{P}_1 Z_1(t)$ in the left hand side of (8.26) by the norm of $Z(t)$.

Lemma 8.8 (cf. Ref. 25, Lemma 10.2): For sufficiently small $\varepsilon > 0$, we have for $t_1 < t_*'$,

$$\|Z(t)\|_{-2-\delta} \le C\|\boldsymbol{P}_1 Z_1(t)\|_{-2-\delta}, \quad 0 \le t \le t_1, \tag{8.27}$$

where C depends only on ρ and $\overline{v}$.

D. Decay of transversal component

Here we complete the proof of Proposition 8.1.

Step (i) We fix ε, $0 < \varepsilon < r_{-\beta}(\tilde{v})$, and $t_*' = t_*'(\varepsilon)$ for which Lemma 8.8 holds. Then the bound of type (8.26) holds with $\|\boldsymbol{P}_1 Z_1(t)\|_{-2-\delta}$ in the left-hand side replaced by $\|Z(t)\|_{-\beta}$:

$$\|Z(t)\|_{-\beta} \le \|Z(t)\|_{-2-\delta} \le C\|\boldsymbol{P}_1 Z_1(t)\|_{-2-\delta} \le \frac{C}{(1+t)^{1+\delta}}\|Z(0)\|_\beta$$
$$+ C\int_0^t \frac{1}{(1+|t-s|)^{1+\delta}}\left[\|Z(s)\|_{-\beta}\int_s^{t_1}\|Z(\tau)\|_{-\beta}^2 d\tau + \|Z(s)\|_{-\beta}^2\right] ds, \quad 0 \le t \le t_1, \tag{8.28}$$

for $t_1 < t_*'$. This implies an integral inequality for the majorant $m(t)$ introduced by (8.22). Namely, multiplying both sides of (8.28) by $(1+t)^{1+\delta}$, and taking the supremum in $t \in [0, t_1]$, we get

$$m(t_1) \le C\|Z(0)\|_\beta + C\sup_{t\in[0,t_1]}\int_0^t \frac{(1+t)^{1+\delta}}{(1+|t-s|)^{1+\delta}}\left[\frac{m(s)}{(1+s)^{1+\delta}}\int_s^{t_1}\frac{m^2(\tau)d\tau}{(1+\tau)^{2+2\delta}} + \frac{m^2(s)}{(1+s)^{2+2\delta}}\right] ds,$$

for $t_1 \le t_*'$. Taking into account that $m(t)$ is a monotone increasing function, we get

$$m(t_1) \le C\|Z(0)\|_\beta + C[m^3(t_1) + m^2(t_1)]I(t_1), \quad t_1 \le t_*', \tag{8.29}$$

where

$$I(t_1) = \sup_{t\in[0,t_1]}\int_0^t \frac{(1+t)^{1+\delta}}{(1+|t-s|)^{1+\delta}}\left[\frac{1}{(1+s)^{1+\delta}}\int_s^{t_1}\frac{d\tau}{(1+\tau)^{2+2\delta}} + \frac{1}{(1+s)^{2+2\delta}}\right] ds \le \overline{I} < \infty, t_1 \ge 0.$$

Therefore, (8.29) becomes

$$m(t_1) \le C\|Z(0)\|_\beta + C\overline{I}[m^3(t_1) + m^2(t_1)], \quad t_1 < t_*'. \tag{8.30}$$

This inequality implies that $m(t_1)$ is bounded for $t_1 < t_*'$, and moreover,

$$m(t_1) \le C_1\|Z(0)\|_\beta, \quad t_1 < t_*', \tag{8.31}$$

since $m(0) = \|Z(0)\|_\beta$ is sufficiently small by (3.7).

Step (ii) The constant C_1 in the estimate (8.31) does not depend on t_* and t_*' by Lemma 8.8. We choose d_β in (2.5) so small that $\|Z(0)\|_\beta < \varepsilon/(2C_1)$. It is possible due to (3.7). Then the estimate (8.31) implies that $t_*' = t_*$ and therefore (8.31) holds for all $t_1 < t_*$. Then the bound (8.24) holds for all $t < t_*$. Therefore, (8.31) holds for all $t_1 < t_*$ and (6.4) holds as well. Finally, this implies that $t_* = \infty$, hence also $t_*' = \infty$ and (8.31) holds for all $t_1 > 0$ if d_β is small enough. □

The transversal decay (8.1) is proved.

IX. SOLITON ASYMPTOTICS

Here we prove our main Theorem 2.4 relying on the decay (8.1). First we will prove the asymptotics (2.6) for the vector components, and afterward the asymptotics (2.7) for the fields.

Asymptotics for the vector components: From (4.2) we have $\dot{q} = \dot{b} + \dot{r}$, and from (8.3), (8.4), (4.17) it follows that $\dot{r} = -B_{v(t)}\langle\rho, a\rangle + B_{v(t)}\pi + \mathcal{O}(\|Z\|_{-\beta}^2)$. Recall that $\beta = 4+\delta$, $0 < \delta < 1/2$. Thus,

$$\dot{q} = \dot{b} + \dot{r} = v(t) + \dot{c}(t) - B_{v(t)}\langle\rho, a\rangle + B_{v(t)}\pi + \mathcal{O}(\|Z\|_{-\beta}^2). \tag{9.1}$$

Equation (7.5) and the estimates (7.6), (8.1) imply that

$$|\dot{c}(t)| + |\dot{v}(t)| \le \frac{C_1(\rho, \overline{v}, d_\beta)}{(1+t)^{2+2\delta}}, \quad t \ge 0. \tag{9.2}$$

Therefore, $c(t) = c_+ + \mathcal{O}(t^{-1-2\delta})$ and $v(t) = v_+ + \mathcal{O}(t^{-1-2\delta})$, $t \to \infty$. Since $\|a\|_{-2-\delta}$ and $|\pi|$ decay, such as $(1+t)^{-1-\delta}$, the estimate (8.1), and (9.2), (9.1) imply that

$$\dot{q}(t) = v_+ + \mathcal{O}(t^{-1-\delta}). \tag{9.3}$$

Similarly,

$$b(t) = c(t) + \int_0^t v(s)ds = v_+ t + a_+ + \mathcal{O}(t^{-2\delta}), \tag{9.4}$$

hence the second part of (2.6) follows:

$$q(t) = b(t) + r(t) = v_+ t + a_+ + \mathcal{O}(t^{-2\delta}), \tag{9.5}$$

since $r(t) = \mathcal{O}(t^{-1-\delta})$ by (8.1).

Asymptotics for the fields: We apply the approach developed in Ref. 26, see also Refs. 27 and 28. For the field part of the solution, $F(t) = (E(x,t), A(x,t))$ let us define the accompanying soliton field as $F_{\mathrm{v(t)}}(t) = (E_{\mathrm{v(t)}}(x - q(t)), A_{\mathrm{v(t)}}(x - q(t)))$, where $\mathrm{v}(t) := \dot{q}(t)$. Then for the difference $Z(t) = F(t) - F_{\mathrm{v(t)}}(t)$, we obtain easily the equation,[28] Eq. (2.5),

$$\dot{Z}(t) = \mathcal{A}Z(t) - \dot{\mathrm{v}} \cdot \nabla_{\mathrm{v}} F_{\mathrm{v}(t)}(t), \quad \mathcal{A}(E, A) = (-\Delta A, -E).$$

Then

$$Z(t) = W^0(t)Z(0) - \int_0^t W^0(t-s)[\dot{\mathrm{v}}(s) \cdot \nabla_{\mathrm{v}} F_{\mathrm{v}(s)}(s)]ds. \tag{9.6}$$

To obtain the asymptotics (2.7) it suffices to prove that $Z(t) = W^0(t)\Psi_+ + r_+(t)$ with some $\Psi_+ \in \mathcal{F}$ and $\|r_+(t)\|_{\mathcal{F}} = \mathcal{O}(t^{-\delta})$. This is equivalent to

$$W^0(-t)Z(t) = \mathbf{\Psi}_+ + r'_+(t), \tag{9.7}$$

where $\|r'_+(t)\|_{\mathcal{F}} = \mathcal{O}(t^{-\delta})$ since $W^0(t)$ is a unitary group in the Sobolev space $\mathcal{F}$ by the energy conservation for the free wave equation. Finally, (9.7) holds since (9.6) implies that

$$W^0(-t)Z(t) = Z(0) + \int_0^t W^0(-s)R(s)ds, \quad R(s) = \dot{\mathrm{v}}(s) \cdot \nabla_{\mathrm{v}} F_{\mathrm{v}(s)}(s), \tag{9.8}$$

where the integral in the right-hand side of (9.8) converges in the Hilbert space $\mathcal{F}$ with the rate $\mathcal{O}(t^{-\delta})$. The latter holds since $\|W^0(-s)R(s)\|_{\mathcal{F}} = \mathcal{O}(s^{-1-\delta})$ by the unitarity of $W^0(-s)$ and the decay rate $\|R(s)\|_{\mathcal{F}} = \mathcal{O}(s^{-1-\delta})$, which follows from the asymptotics for the vector components. More precisely, differentiating the first equation (1.4) in t and using the asymptotics (9.3), (8.1) we obtain an estimate for $\dot{\mathrm{v}}(t) = \ddot{q}(t)$ providing the mentioned decay rate of $R(s)$. □

X. SOLVING THE LINEARIZED EQUATION

In Secs. X–XIII, we prove Proposition 8.4 in order to complete the proof of the main result. First, let us make a change of variables in Eq. (8.5) to simplify its structure. Equation (8.5) reads

$$\begin{aligned} \dot{e} &= v \cdot \nabla e - \Delta a + \Pi_s(r \cdot \nabla\rho v - \rho B_v(\pi - \langle\rho, a\rangle)), & \dot{a} &= -e + v \cdot \nabla a, \\ \dot{r} &= \qquad\qquad B_v(\pi - \langle\rho, a\rangle), & \dot{\pi} &= \langle\rho, \nabla(v \cdot a)\rangle - \langle r \cdot \nabla\rho, \nabla(v \cdot A_v)\rangle. \end{aligned} \tag{10.1}$$

Put $\varphi = \pi - \langle\rho, a\rangle$. Then $\pi = \varphi + \langle\rho, a\rangle$. If we prove a decay of φ and a, then π has the corresponding decay as well. Further, $\dot{\varphi} = \dot{\pi} - \langle\rho, \dot{a}\rangle = \dot{\pi} - \langle\rho, -e + v \cdot \nabla a\rangle = \langle\rho, e\rangle + \langle\rho, \nabla(v \cdot a)\rangle - (v \cdot$

$\nabla)a\rangle - \langle r\cdot\nabla\rho, \nabla(v\cdot A_v)\rangle$ by the last equation of (10.1). Thus, the system (10.1) is equivalent to the following system:

$$\begin{aligned} \dot e &= v\cdot\nabla e - \Delta a + \Pi_s(r\cdot\nabla\rho v - \rho B_v\varphi), \quad \dot a = -e + v\cdot\nabla a,\\ \dot r &= \qquad B_v\varphi, \qquad \dot\varphi = \langle\rho, e\rangle + \langle\rho, v\wedge(\nabla\wedge a)\rangle - \langle r\cdot\nabla\rho, \nabla(v\cdot A_v)\rangle. \end{aligned} \tag{10.2}$$

For the last equation we have applied the identity $\nabla(v\cdot a) - (v\cdot\nabla)a = v\wedge(\nabla\wedge a)$. Denote by the same letter A the operator,

$$A\begin{pmatrix} e\\ a\\ r\\ \varphi\end{pmatrix} := \begin{pmatrix} v\cdot\nabla e - \Delta a + \Pi_s(r\cdot\nabla\rho v - \rho B_v\varphi)\\ -e + v\cdot\nabla a\\ B_v\varphi,\\ \langle\rho, e\rangle + \langle\rho, v\wedge(\nabla\wedge a)\rangle - \langle r\cdot\nabla\rho, \nabla(v\cdot A_v)\rangle\end{pmatrix}. \tag{10.3}$$

Below we prove the decay for the solution $X = (e, a, r, \varphi)$ to the equation,

$$\dot X(t) = AX(t). \tag{10.4}$$

So, now we construct and study the resolvent of A.

Let us apply the Laplace transform,

$$\Lambda X = \tilde X(\lambda) = \int_0^\infty e^{-\lambda t}X(t)dt, \quad \mathrm{Re}\,\lambda > 0, \tag{10.5}$$

to (8.5). The integral converges in $\mathcal{E}$, since $\|X(t)\|_{\mathcal{E}}$ is bounded by Proposition 8.4, (i). The analyticity of $\tilde X(\lambda)$ and Paley–Wiener arguments should provide the existence of a $\mathcal{E}$-valued distribution $X(t) = (\Psi(t), \Pi(t), Q(t), P(t))$, $t\in\mathbb{R}$, with a support in $[0,\infty)$. Formally,

$$\Lambda^{-1}\tilde X = X(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}\tilde X(i\omega + 0)d\omega, \quad t\in\mathbb{R}. \tag{10.6}$$

To prove the decay (8.10), we have to study the smoothness of $\tilde X(i\omega+0)$ at $\omega\in\mathbb{R}$. After the Laplace transform Eq. (8.5) becomes:

$$\lambda\tilde X(\lambda) = A\tilde X(\lambda) + X_0, \quad \mathrm{Re}\,\lambda > 0. \tag{10.7}$$

To justify the representation (10.6), we construct the resolvent as a bounded operator in $\mathcal{E}$ for $\mathrm{Re}\,\lambda > 0$. We shall write $(e(y), a(y), r, \varphi)$ instead of $(\tilde e(y,\lambda), \tilde a(y,\lambda), \tilde r(\lambda), \tilde\varphi(\lambda))$ to simplify the notations. Then (10.7) reads:

$$\left.\begin{array}{l} v\cdot\nabla e - \Delta a + \Pi_s(r\cdot\nabla\rho v - \rho B_v\varphi) - \lambda e = -e_0, \qquad -e + v\cdot\nabla a - \lambda a = -a_0\\[2ex] B_v\varphi - \lambda r = -r_0, \qquad \langle\rho, e\rangle + \langle\rho, v\wedge(\nabla\wedge a)\rangle - \langle r\cdot\nabla\rho, \nabla(v\cdot A_v)\rangle - \lambda\varphi = -\varphi_0 \end{array}\right|. \tag{10.8}$$

Step (i): Let us consider the first two equations. After Fourier transform they become,

$$-i(kv)\hat e + k^2\hat a - \hat\Pi_s(i(kr)\hat\rho v + \hat\rho B_v\varphi) - \lambda\hat e = -\hat e_0, \quad -\hat e - i(kv)\hat a - \lambda\hat a = -\hat a_0. \tag{10.9}$$

From the last equation we have $\hat e = -(\lambda + i(kv))\hat a + \hat a_0$. Substitute to the first equation of (10.9) and obtain

$$\hat a = \frac{1}{\hat D}((\lambda + ikv)\hat a_0 - \hat e_0 + \hat\Pi), \quad \hat\Pi := \hat\rho\hat\Pi_s(i(kr)v + B_v\varphi)), \tag{10.10}$$

where

$$\hat D = \hat D(\lambda) = k^2 + (\lambda + ikv)^2. \tag{10.11}$$

It is easy to see that $\hat D(\lambda)\neq 0$ for $\mathrm{Re}\,\lambda > 0$. Finally,

$$\hat e = \frac{k^2\hat a_0 + (\lambda + ikv)\hat e_0 - (\lambda + ikv)\hat\Pi}{\hat D}. \tag{10.12}$$

Step (ii): Let us proceed to the fourth equations of (10.8). The equation reads:

$$\langle \rho, e\rangle + \langle \rho, v \wedge (\nabla \wedge a)\rangle - \langle r \cdot \nabla \rho, \nabla(v \cdot A_v)\rangle - \lambda\varphi = -\varphi_0.$$

From now on we use the system of coordinates in x-space in which $v = (|v|, 0, 0)$, hence $vk = |v|k_1$. By (10.12) and a straightforward computation we obtain

$$\langle \rho, e\rangle = \Phi - C_1 r + F_1\varphi,$$

where

$$\Phi = \Phi(\lambda, e_0, a_0) := \int (k^2\hat{a}_0 + (\lambda + ikv)\hat{e}_0)\overline{\hat{\rho}}/\hat{D}\,dk \tag{10.13}$$

and C_1, F_1 are the following diagonal 3×3-matrices:

$$C_1(\lambda) = \begin{pmatrix} c_{11}(\lambda) & 0 & 0 \\ 0 & c_{12}(\lambda) & 0 \\ 0 & 0 & c_{13}(\lambda)\end{pmatrix}, \quad F_1(\lambda) = \begin{pmatrix} f_{11}(\lambda) & 0 & 0 \\ 0 & f_{12}(\lambda) & 0 \\ 0 & 0 & f_{13}(\lambda)\end{pmatrix}, \tag{10.14}$$

$$c_{11}(\lambda) = i|v|\int \frac{k_1(\lambda + ik_1|v|)|\hat{\rho}|^2}{\hat{D}(\lambda)}(1 - \frac{k_1^2}{k^2})dk,\;\; c_{1j}(\lambda) = -i|v|\int \frac{k_1k_j^2(\lambda + ik_1|v|)|\hat{\rho}|^2}{k^2\hat{D}(\lambda)}dk,\;\; j = 2, 3, \tag{10.15}$$

$$f_{11}(\lambda) = \nu^3\int \frac{(\lambda + ik_1|v|)|\hat{\rho}|^2}{\hat{D}(\lambda)}(\frac{k_1^2}{k^2} - 1)dk,\;\; f_{1j}(\lambda) = \nu\int \frac{(\lambda + ik_1|v|)|\hat{\rho}|^2}{\hat{D}(\lambda)}(\frac{k_j^2}{k^2} - 1)dk,\;\; j = 2, 3, \tag{10.16}$$

recall that $\nu = \sqrt{1 - v^2}$. By the change of variables $k_2 \mapsto k_3$, we obtain that $c_{12} = c_{13}$. Moreover, $c_{11} + c_{12} + c_{13} = 0$ and thus, $c_{11} = -c_{12} - c_{13} = -2c_{12}$. The matrix $C_1(\lambda)$ simplifies to

$$C_1(\lambda) = \begin{pmatrix} c_1(\lambda) & 0 & 0 \\ 0 & c_{12}(\lambda) & 0 \\ 0 & 0 & c_{12}(\lambda)\end{pmatrix}, \; c_1(\lambda) := -2c_{12}(\lambda),\; c_{12}(\lambda)$$
$$= -\frac{i|v|}{2}\int \frac{k_1(\lambda + ik_1|v|)|\hat{\rho}|^2}{\hat{D}(\lambda)}(1 - \frac{k_1^2}{k^2})dk. \tag{10.17}$$

Similarly, $f_{12} = f_{13}$ and

$$f_{12}(\lambda) = \nu\int \frac{(\lambda + ik_1|v|)|\hat{\rho}|^2}{\hat{D}(\lambda)}(\frac{k_2^2 + k_3^2}{2k^2} - 1)dk = \nu\int \frac{(\lambda + ik_1|v|)|\hat{\rho}|^2}{\hat{D}(\lambda)}(\frac{k^2 - k_1^2}{2k^2} - 1)dk$$
$$= -\frac{\nu}{2}\int \frac{(\lambda + ik_1|v|)|\hat{\rho}|^2}{\hat{D}(\lambda)}(1 + \frac{k_1^2}{k^2})dk. \tag{10.18}$$

Further, $\langle \rho, v \wedge (\nabla \wedge a)\rangle = \Psi - C_2 r + F_2\varphi$, where

$$\Psi = \Psi(\lambda, e_0, a_0) := \int dk\, v \wedge (-ik \wedge \frac{(\lambda + ikv)\hat{a}_0 - \hat{e}_0}{\hat{D}(\lambda)})\overline{\hat{\rho}} \tag{10.19}$$

and

$$C_2(\lambda) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22}(\lambda) & 0 \\ 0 & 0 & c_{23}(\lambda)\end{pmatrix}, \quad F_2(\lambda) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & f_{22}(\lambda) & 0 \\ 0 & 0 & f_{23}(\lambda)\end{pmatrix}, \tag{10.20}$$

where

$$c_{2j}(\lambda) = -v^2\int dk\frac{|\hat{\rho}|^2k_j^2}{\hat{D}(\lambda)},\quad f_{2j}(\lambda) = i\nu|v|\int dk\frac{|\hat{\rho}|^2k_1}{\hat{D}(\lambda)},\quad j = 2, 3. \tag{10.21}$$

Remark 10.1: Note that

$$\Phi(\lambda) = \Lambda\langle W^1(t)(e_0, a_0), \rho\rangle, \tag{10.22}$$

where $W^1(t)$ is the first component of the dynamical group $W(t)$ defined below by (13.5). Similarly,

$$\Psi(\lambda) = \Lambda\langle v \wedge (\nabla \wedge W^2(t)(e_0, a_0)), \rho\rangle, \tag{10.23}$$

where $W^2(t)$ is the second component of the same dynamical group.
Further, $c_{22} = c_{23}$, $f_{22} = f_{23}$, and

$$c_{22}(\lambda) = -\frac{v^2}{2}\int dk \frac{|\hat\rho|^2(k_2^2 + k_3^2)}{\hat D(\lambda)} = -\frac{v^2}{2}\int dk \frac{|\hat\rho|^2(k^2 - k_1^2)}{\hat D(\lambda)}. \tag{10.24}$$

At last, $\langle r \cdot \nabla\rho, \nabla(v \cdot A_v)\rangle = Gr$, where

$$G = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix}, \quad g_j = v^2\int \frac{(k^2 - k_1^2)k_j^2|\hat\rho|^2}{k^2(k^2 - k_1^2 v^2)} dk, \quad j = 1, 2, 3. \tag{10.25}$$

Again,

$$g_2 = g_3 \text{ and we set } g := g_2 = g_3. \tag{10.26}$$

Put

$$C(\lambda) = C_1(\lambda) + C_2(\lambda), \quad F(\lambda) = F_1(\lambda) + F_2(\lambda). \tag{10.27}$$

In detail, by (10.14), (10.17), and (10.20),

$$C(\lambda) = \begin{pmatrix} c_1(\lambda) & 0 & 0 \\ 0 & c(\lambda) & 0 \\ 0 & 0 & c(\lambda) \end{pmatrix}, \quad c(\lambda) := c_{12}(\lambda) + c_{22}(\lambda), \tag{10.28}$$

$$F(\lambda) = \begin{pmatrix} f_1(\lambda) & 0 & 0 \\ 0 & f(\lambda) & 0 \\ 0 & 0 & f(\lambda) \end{pmatrix}, \quad f_1(\lambda) := f_{11}(\lambda), \quad f(\lambda) := f_{12}(\lambda) + f_{22}(\lambda). \tag{10.29}$$

Finally, the fourth equation becomes $(C(\lambda) + G)r + (\lambda E - F(\lambda))\varphi = \varphi_0 + \Phi(\lambda) + \Psi(\lambda)$. We write this equation and the third equation of (10.8) together in the form:

$$M(\lambda)\begin{pmatrix} r \\ \varphi \end{pmatrix} = \begin{pmatrix} r_0 \\ \varphi_0 + \Phi(\lambda) + \Psi(\lambda) \end{pmatrix}, \quad \text{where } M(\lambda) = \begin{pmatrix} \lambda E & -B_v \\ C(\lambda) + G & \lambda E - F(\lambda) \end{pmatrix}. \tag{10.30}$$

Assume for a moment that the matrix $M(\lambda)$ is invertible for $\operatorname{Re}\lambda > 0$ (see below). Then

$$\begin{pmatrix} r \\ \varphi \end{pmatrix} = M^{-1}(\lambda)\begin{pmatrix} r_0 \\ \varphi_0 + \Phi(\lambda) + \Psi(\lambda) \end{pmatrix}, \quad \operatorname{Re}\lambda > 0. \tag{10.31}$$

Formulas (10.10), (10.12), and (10.31) give the expression of the resolvent $R(\lambda) = (A - \lambda)^{-1}$, $\operatorname{Re}\lambda > 0$, in Fourier representation.

Further, the operator $D(\lambda)$ defined in Fourier space as multiplication by the symbol (10.11) is invertible in $L^2(\mathbb{R}^3)$ for $\operatorname{Re}\lambda > 0$ and its fundamental solution $g_\lambda(y)$ exponentially decays as $|y| \to \infty$.

Lemma 10.2: (i) *The distribution $g_\lambda(\cdot)$ admits an analytic continuation in λ from the domain* $\operatorname{Re}\lambda > 0$ *to the entire complex plane* $\mathbb{C}$.

(ii) *The matrix function $M(\lambda)$ ($M^{-1}(\lambda)$) admits an analytic (respectively, meromorphic) continuation in the parameter λ from the domain* $\operatorname{Re}\lambda > 0$ *to the entire complex plane.*

Proof: The fundamental solution $g_\lambda(y)$ is given by

$$g_\lambda(y) = \frac{e^{-\varkappa|\tilde y| - \varkappa_1\tilde y_1}}{4\pi|\tilde y|}, \quad \tilde y := (\gamma y_1, y_2, y_3), \tag{10.32}$$

where

$$\gamma := 1/\sqrt{1-v^2}, \quad \varkappa = \gamma\lambda, \quad \varkappa_1 := |v|\varkappa. \tag{10.33}$$

Thus, the statement (i) follows from the formulas (10.33) and (10.32). To prove the statement (ii) we first need to show, according to (10.30), that the matrices $C(\lambda)$ and $F(\lambda)$ admit an analytic continuation to the entire complex plane. Consider the matrix $C(\lambda)$, for $F(\lambda)$ the argument is similar. The analytic continuation of $C(\lambda)$ then exists by the expression of type (B3) for the entries of the matrix $C(\lambda)$, and the statement (i) of the present lemma since the function $\rho(x)$ is compactly supported by (1.12). The inverse matrix $M^{-1}(\lambda)$ is then meromorphic, since it is well defined for large $|\lambda|$ by (B6)–(B8) and Corollary B.3. □

XI. REGULARITY IN CONTINUOUS SPECTRUM

By Lemma 10.2, the limit matrix

$$M(i\omega) := M(i\omega+0) = \begin{pmatrix} i\omega E & -B_v \\ C(i\omega+0)+G & i\omega E - F(i\omega+0)\end{pmatrix}, \quad \omega \in \mathbb{R}, \tag{11.1}$$

exists, and its entries are analytic functions of $\omega \in \mathbb{R}$. Recall that the point $\lambda = 0$ belongs to the discrete spectrum of the operator A by Lemma 5.3 (i), hence $M(i\omega + 0)$ (probably) is also not invertible at $\omega = 0$.

Proposition 11.1: The matrix $M^{-1}(i\omega)$ is analytic in $\omega \in \mathbb{R} \setminus \{0\}$.

Proof: It suffices to prove that the limit matrix $M(i\omega) := M(i\omega + 0)$ is invertible for $\omega \neq 0$, $\omega \in \mathbb{R}$ if ρ satisfies the Wiener condition (1.13), and $|v| < 1$. Since $v = (|v|, 0, 0)$, the matrix B_v is also diagonal:

$$B_v := \nu(E - v\otimes v) = \begin{pmatrix} \nu^3 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu \end{pmatrix}. \tag{11.2}$$

By (10.30), (10.26), (10.28), (10.29), (11.2), for $\omega \in \mathbb{R}$,

$$\det M(i\omega) = \det \begin{pmatrix} i\omega E & -B_v \\ C(i\omega)+G & i\omega E - F(i\omega)\end{pmatrix} = d_1 d^2,$$

where

$$d_1 = -\omega^2 - i\omega f_1(i\omega) + \nu^3(c_1(i\omega) + g_1), \tag{11.3}$$

$$d = -\omega^2 - i\omega f(i\omega) + \nu(c(i\omega) + g). \tag{11.4}$$

The formula for the determinant is obvious since all of the matrices C, F, G, and B_v are diagonal. Then for $|\omega| > 0$ the invertibility of $M(i\omega)$ follows from (11.3), (11.4) by the following lemma.

Lemma 11.2: If (1.13) holds and $|\omega| > 0$, then the imaginary parts of d_1 and d are positive: $\mathrm{Im}\, d_1 > 0$, $\mathrm{Im}\, d > 0$.

Proof: Let $\omega > 0$, the case $\omega < 0$ is similar. Note that $\mathrm{Im}\, d_1 = -\omega \mathrm{Re}\, f_1(i\omega+0) + \nu^3 \mathrm{Im}\, c_1(i\omega + 0)$. For $\varepsilon > 0$ we have

$$f_1(i\omega+\varepsilon) = \nu^3 \int \frac{(i\omega+\varepsilon+ik_1|v|)|\hat\rho(k)|^2}{\hat D(i\omega+\varepsilon, k)}\left(\frac{k_1^2}{k^2} - 1\right)dk. \tag{11.5}$$

By Sokhotsky–Plemelj formula for C^1-functions, [Ref. 29, Chap. VII, formula (58)],

$$\mathrm{Re}\, f_1(i\omega+0) = \pi\nu^3 \int_{T_\omega} \frac{(\omega+k_1|v|)|\hat\rho(k)|^2}{|\nabla \hat D(i\omega, k)|}\left(\frac{k_1^2}{k^2} - 1\right)dS, \tag{11.6}$$

where

$$T_\omega = \{k : k^2 - (\omega + k_1|v|)^2 = 0\}$$

is the ellipsoid on which $\hat{D}(i\omega, k) = 0$. Similarly,

$$\text{Re } c_1(i\omega + 0) = \pi|v| \int_{T_\omega} \frac{k_1(\omega + k_1|v|)|\hat{\rho}(k)|^2}{|\nabla\hat{D}(i\omega, k)|}(1 - \frac{k_1^2}{k^2})dS. \tag{11.7}$$

Then

$$\text{Im}\, d_1 = v^3\pi \int_{T_\omega} \frac{(\omega + k_1|v|)^2|\hat{\rho}(k)|^2}{|\nabla\hat{D}(i\omega, k)|}(1 - \frac{k_1^2}{k^2})dS > 0,$$

by the Wiener condition (1.13). Further,

$$\text{Im}\, d = -\omega\text{Re } f(i\omega + 0) + v\text{Im } c(i\omega + 0)$$

$$= -\omega\text{Re } f_{12}(i\omega + 0) + v\text{Im } c_{12}(i\omega + 0) - \omega\text{Re } f_{22}(i\omega + 0) + v\text{Im } c_{22}(i\omega + 0).$$

By (10.18) we have

$$f_{12}(i\omega + \varepsilon) = -\frac{v}{2} \int \frac{(i\omega + \varepsilon + ik_1|v|)|\hat{\rho}(k)|^2}{\hat{D}(i\omega + \varepsilon, k)}(1 + \frac{k_1^2}{k^2})dk.$$

Then

$$\text{Re } f_{12}(i\omega + 0) = -\frac{\pi v}{2} \int_{T_\omega} \frac{(\omega + k_1|v|)|\hat{\rho}(k)|^2}{|\nabla\hat{D}(i\omega, k)|}(1 + \frac{k_1^2}{k^2})dS$$

$$= -\frac{\pi v}{2} \int_{T_\omega} \frac{(\omega + k_1|v|)|\hat{\rho}(k)|^2}{|\nabla\hat{D}(i\omega, k)|} \frac{(\omega + k_1|v|)^2 + k_1^2}{k^2} dS,$$

since $k^2 = (\omega + k_1|v|)^2$ on T_ω. By (10.17) we obtain that

$$c_{12}(i\omega + \varepsilon) = \frac{|v|}{2} \int \frac{k_1(\omega - i\varepsilon + k_1|v|)|\hat{\rho}(k)|^2}{\hat{D}(i\omega + \varepsilon, k)}(1 - \frac{k_1^2}{k^2})dk.$$

Then

$$\text{Im } c_{12}(i\omega + 0) = -\frac{\pi|v|}{2} \int_{T_\omega} \frac{k_1(\omega + k_1|v|)|\hat{\rho}(k)|^2}{|\nabla\hat{D}(i\omega, k)|}(1 - \frac{k_1^2}{k^2})dS$$

$$= -\frac{\pi|v|}{2} \int_{T_\omega} \frac{k_1(\omega + k_1|v|)|\hat{\rho}(k)|^2}{|\nabla\hat{D}(i\omega, k)|} \frac{(\omega + k_1|v|)^2 - k_1^2}{k^2} dS$$

and

$$-\omega\text{Re } f_{12} + v\text{Im } c_{12} = \frac{v\pi}{2} \int_{T_\omega} \frac{|\hat{\rho}(k)|^2(\omega^2 + k_1^2(1 - v^2))}{|\nabla\hat{D}(i\omega, k)|} dk. \tag{11.8}$$

Further, by (10.21)

$$\text{Re } f_{22}(i\omega + 0) = \pi v|v| \int_{T_\omega} dS \frac{|\hat{\rho}(k)|^2 k_1}{|\nabla\hat{D}(i\omega, k)|},$$

by (10.24)

$$\text{Im } c_{22}(i\omega + 0) = \frac{\pi v^2}{2} \int_{T_\omega} dS \frac{|\hat{\rho}(k)|^2(k^2 - k_1^2)}{|\nabla\hat{D}(i\omega, k)|},$$

thus,

$$-\omega\text{Re } f_{22}(i\omega+0)+\nu\text{Im } c_{22}(i\omega+0)=\frac{\pi\nu}{2}\int_{T_\omega}dS\frac{|\hat\rho(k)|^2(v^2((\omega+k_1|v|)^2-k_1^2)-2\omega k_1|v|)}{|\nabla\hat D(i\omega,k)|}.$$

Finally, combining with (11.8) we obtain

$$\text{Im } \hat d=\frac{\pi\nu}{2}\int_{T_\omega}dS\frac{|\hat\rho(k)|^2((k_1(v^2-1)+\omega|v|)^2+\omega^2)}{|\nabla\hat D(i\omega,k)|}>0,$$

by the Wiener condition. This completes the proofs of the lemma and Proposition 11.1. □

Remark 11.3: The proof of Lemma 11.2 is the unique point in the paper where the Wiener condition is indispensable.

XII. TIME DECAY OF THE VECTOR COMPONENTS

Let us prove the decay (8.10) for the vector components $r(t)$ and $\phi(t)$ of the solution $e^{At}X_0$. Formula (10.31) expresses the Laplace transforms $\tilde r(\lambda)$, $\tilde\varphi(\lambda)$. Hence, the components are given by the Fourier integral:

$$\begin{pmatrix} r(t)\\ \varphi(t)\end{pmatrix}=\frac{1}{2\pi}\int e^{i\omega t}M^{-1}(i\omega)\begin{pmatrix} r_0\\ \varphi_0+\Phi(i\omega)+\Psi(i\omega)\end{pmatrix}d\omega. \tag{12.1}$$

Recall that in Proposition 8.4 we assume that

$$X_0\in\mathcal{Z}_v\cap\mathcal{E}_\beta,\quad \beta=4+\delta,\qquad 0<\delta<1/2. \tag{12.2}$$

Theorem 12.1: The functions $r(t)$, $\varphi(t)$ are continuous for $t\ge 0$, and

$$|r(t)|+|\varphi(t)|\le\frac{C(\rho,\tilde v,d_\beta)}{(1+|t|)^{1+\delta}}\|X_0\|_\beta,\quad t\ge 0. \tag{12.3}$$

Proof: Proposition 11.1 alone is not sufficient for the proof of the convergence and decay of the integral. Namely, we need an additional information about a regularity of the matrix $L(i\omega)$ and of $\Phi(i\omega)+\Psi(i\omega)$. Let us split the Fourier integral (12.1) into two terms using the partition of unity $\zeta_1(\omega)+\zeta_2(\omega)=1$, $\omega\in\mathbb{R}$:

$$\begin{pmatrix} r(t)\\ \varphi(t)\end{pmatrix}=\frac{1}{2\pi}\int e^{i\omega t}(\zeta_1(\omega)+\zeta_2(\omega))\begin{pmatrix} \tilde r(i\omega)\\ \tilde\varphi(i\omega)\end{pmatrix}d\omega$$

$$=\begin{pmatrix} r_1(t)\\ \varphi_1(t)\end{pmatrix}+\begin{pmatrix} r_2(t)\\ \varphi_2(t)\end{pmatrix}=I_1(t)+I_2(t), \tag{12.4}$$

where the functions $\zeta_k(\omega)\in C^\infty(\mathbb{R})$ are supported by

$$\text{supp}\,\zeta_1\subset\{\omega\in\mathbb{R}:|\omega|<r+1\},\quad \text{supp}\,\zeta_2\subset\{\omega\in\mathbb{R}:|\omega|>r\}, \tag{12.5}$$

where r is introduced below in Lemma 12.3. We prove the decay (12.3) for (r_1,φ_1) and (r_2,φ_2) in Propositions 12.4 and 12.2, respectively.

Proposition 12.2: The function $I_2(t)$ is continuous for $t\ge 0$ and

$$|I_2(t)|\le C(\rho,\tilde v)(1+|t|)^{-3-\delta}\|X_0\|_\beta. \tag{12.6}$$

Proof: First, we need the asymptotic behavior of $M^{-1}(\lambda)$ at infinity. Let us recall that $M^{-1}(\lambda)$ was originally defined for $\operatorname{Re}\lambda > 0$, but it admits a meromorphic continuation to the entire complex plane $\mathbb{C}$ (see Lemma 10.2).

Lemma 12.3: There exist a matrix R_0 and a matrix-function $R_1(\omega)$, such that

$$M^{-1}(i\omega) = \frac{R_0}{\omega} + R_1(\omega), |\omega| > r > 0, \omega \in \mathbb{R}, \tag{12.7}$$

where, for every $k = 0, 1, 2, \ldots,$

$$|\partial_\omega^k R_1(\omega)| \leq \frac{C_k}{|\omega|^2}, \quad |\omega| > r > 0, \quad \omega \in \mathbb{R}, \tag{12.8}$$

r is sufficiently large.

Proof: The statement follows from the explicit formulas (B2), (B6)–(B8) for the inverse matrix $M^{-1}(i\omega)$ and from Lemma B.2. □

Further, (12.1) implies that

$$\begin{aligned} I_2(t) &= \frac{1}{2\pi}\int e^{i\omega t}\zeta_2(\omega)M^{-1}(i\omega)\left[\begin{pmatrix} r_0 \\ \varphi_0\end{pmatrix} + \begin{pmatrix} 0 \\ \Phi(i\omega)+\Psi(i\omega)\end{pmatrix}\right]d\omega \\ &= s(t)\begin{pmatrix} r_0 \\ \varphi_0\end{pmatrix} + s * \begin{pmatrix} 0 \\ f+\psi\end{pmatrix}, \end{aligned} \tag{12.9}$$

where [see (10.6)]

$$s(t) := \Lambda^{-1}\left[\zeta_2(\omega)M^{-1}(i\omega)\right]$$

and

$$f(t) := \Lambda^{-1}\Phi(i\omega) = \langle W^1(t)(e_0, a_0), \rho\rangle, \quad \psi(t) := \Lambda^{-1}\Psi(i\omega) = \langle v \wedge (\nabla \wedge W^1(t)(e_0, a_0)), \rho\rangle, \tag{12.10}$$

since Φ, Ψ are given by (10.13), (10.19), (10.22), (10.23). Recall that $(e_0, a_0, r_0, \pi_0) = X_0 \in \mathcal{F}_\beta$ with $\beta = 4 + \delta$, where $\delta > 0$ under conditions of Proposition 8.4 and Theorem 2.4. Hence, applying Lemma 13.2 (see below) with $\alpha = 4 + \delta$, we obtain that

$$|f(t)| + |\psi(t)| \leq C(\rho, \tilde{v})(1+t)^{-3-\delta}\|X_0\|_\beta. \tag{12.11}$$

On the other hand, (12.7)–(12.8) imply that

$$|s(t)| = \mathcal{O}(t^{-N}), \quad |t| \to \infty, \quad \forall N > 0.$$

Hence, all the terms in (12.9) are continuous for $t \geq 0$ and decay such as $Ct^{-3-\delta}\|X_0\|_\beta$. □

Now let us prove the decay for $I_1(t)$. In this case the proof will rely substantially on the symplectic orthogonality conditions. Namely, (12.2) implies that

$$\Omega(X_0, \tau_j) = 0, \quad j = 1, \ldots, 6. \tag{12.12}$$

Proposition 12.4: The function $I_1(t)$ is continuous for $t \geq 0$ and

$$|I_1(t)| \leq C(\rho, \tilde{v})(1+t)^{-1-\delta}\|X_0\|_\beta, \quad t \geq 0. \tag{12.13}$$

Proof: First, let us calculate the Fourier transforms $\tilde{r}_1(i\omega)$ and $\tilde{\varphi}_1(i\omega)$.

Lemma 12.5: The matrix $M^{-1}(i\omega)$ can be represented as follows:

$$M^{-1}(i\omega) = \begin{pmatrix} \frac{1}{\omega}\mathcal{L}_{11} & \frac{1}{\omega^2}\mathcal{L}_{12} \\ \mathcal{L}_{21} & \frac{1}{\omega}\mathcal{L}_{22}\end{pmatrix}, \tag{12.14}$$

where $\mathcal{L}_{ij}(\omega)$, $i, j = 1, 2$ *are smooth diagonal* 3×3*-matrices,* $\mathcal{L}_{ij}(\omega) \in C^\infty(-r-1, r+1)$. *Moreover,*

$$\mathcal{L}_{11} = i\mathcal{L}_{12}B_v^{-1} + i\mathcal{L}_3, \tag{12.15}$$

where $\mathcal{L}_3$ *is defined by (B21),* $\mathcal{L}_3$ *is a smooth diagonal* 3×3*-matrix,* $\mathcal{L}_3(\omega) \in C^\infty(-r-1, r+1)$.
For proof see Appendix B. Then the vector components are given by

$$\tilde{r}(i\omega) = \frac{1}{\omega}\mathcal{L}_{11}(\omega)r_0 + \frac{1}{\omega^2}\mathcal{L}_{12}(\omega)(\varphi_0 + \Phi(i\omega) + \Psi(i\omega)), \tag{12.16}$$

$$\tilde{\varphi}(i\omega) = \mathcal{L}_{21}(\omega)r_0 + \frac{1}{\omega}\mathcal{L}_{22}(\omega)(\varphi_0 + \Phi(i\omega) + \Psi(i\omega)). \tag{12.17}$$

Next we calculate the symplectic orthogonality conditions (12.12).

Lemma 12.6: The symplectic orthogonality conditions (12.12) read:

$$\varphi_0 + \Phi(0) + \Psi(0) = 0 \ \text{ and } \ (B_v^{-1} + \mathcal{L}_{12}^{-1}(0)\mathcal{L}_3(0))r_0 + \Phi'(0) + \Psi'(0) = 0. \tag{12.18}$$

For proof see Appendix C.

Now we can prove Proposition 12.4.

*Step (i):*Let us prove (12.13) for $\varphi_1(t)$ relying on the representation (12.17). Namely, (12.4) and (12.17) imply:

$$\varphi_1(t) = \Lambda^{-1}\zeta_1(\omega)\mathcal{L}_{21}(\omega)r_0 + \Lambda^{-1}\zeta_1(\omega)\mathcal{L}_{22}(\omega)\frac{\varphi_0 + \Phi(i\omega) + \Psi(i\omega)}{\omega} = \varphi_1'(t) + \varphi_1''(t).$$

The first term $\varphi_1'(t)$ decays such as $Ct^{-\infty}\|X_0\|_\beta$ by Lemma 12.5. The second term admits the convolution representation $\varphi_1''(t) = \Lambda^{-1}\zeta_1\mathcal{L}_{22} * g(t)$, where

$$g(t) := \Lambda^{-1}\frac{\varphi_0 + \Phi(i\omega) + \Psi(i\omega)}{\omega}.$$

Now we use the symplectic orthogonality conditions (12.18) and obtain

$$g(t) = \Lambda^{-1}\frac{\Phi(i\omega) + \Psi(i\omega) - \Phi(0) - \Psi(0)}{\omega} = i\int_\infty^t (f(s) + \psi(s))ds.$$

Finally, $g(t)$ decays such as $Ct^{-2-\delta}\|X_0\|_\beta$ for $t \geq 0$ by (12.11), hence $\varphi_1''(t)$ decays such as $Ct^{-2-\delta}\|X_0\|_\beta$ for $t \geq 0$.

Step (ii): Now let us prove (12.13) for $r_1(t)$. By (12.16), (12.15), and the symplectic orthogonality conditions (12.18),

$$\tilde{r}(i\omega) = \frac{1}{\omega}i(\mathcal{L}_{12}B_v^{-1} + \mathcal{L}_3)r_0 + \frac{1}{\omega^2}\mathcal{L}_{12}(\varphi_0 + \Phi(i\omega) + \Psi(i\omega))$$

$$= \frac{\mathcal{L}_{12}}{\omega}\left[i(B_v^{-1} + \mathcal{L}_{12}^{-1}\mathcal{L}_3)r_0 + \frac{\varphi_0 + \Phi(i\omega) + \Psi(i\omega)}{\omega}\right] = \frac{\mathcal{L}_{12}}{\omega}\left[i(B_v^{-1} + \mathcal{L}_{12}^{-1}\mathcal{L}_3)r_0 + \tilde{g}(\omega)\right]$$

$$= \mathcal{L}_{12}\frac{i(B_v^{-1} + \mathcal{L}_{12}^{-1}\mathcal{L}_3)r_0 + \tilde{g}(0) + \tilde{g}(\omega) - \tilde{g}(0)}{\omega} = \mathcal{L}_{12}\frac{\tilde{g}(\omega) - \tilde{g}(0)}{\omega},$$

since $i(B_v^{-1} + \mathcal{L}_{12}^{-1}\mathcal{L}_3(0))r_0 + \tilde{g}(0) = 0$ by the symplectic orthogonality conditions (12.18), because $\tilde{g}(0) = i(\Phi'(0) + \Psi'(0))$. Thus, $r_1(t) = \Lambda^{-1}\zeta_1(\omega)\mathcal{L}_{12} * h(t)$ by (12.4), where

$$h(t) := \Lambda^{-1}\frac{\tilde{g}(i\omega) - \tilde{g}(0)}{\omega} = i\int_\infty^t g(s)ds.$$

This integral decays such as $Ct^{-1-\delta}\|X_0\|_\beta$ for $t \geq 0$ by (12.11), hence $|r_1(t)| \leq Ct^{-1-\delta}\|X_0\|_\beta$ for $t \geq 0$. □

Now Theorem 12.1 is proved.

XIII. TIME DECAY OF FIELDS

Here we construct the field components $e(x,t), a(x,t)$ of the solution $X(t)$ and prove their decay corresponding to (8.10). Let us denote $F(t) = (e(\cdot,t), a(\cdot,t))$. We will construct the fields solving the first two equations of (10.4), where A is given by (10.3). These two equations have the form:

$$\dot{F}(t) = \begin{pmatrix} v\cdot\nabla & -\Delta \\ -1 & v\cdot\nabla \end{pmatrix} F + \begin{pmatrix} \Pi(t) \\ 0 \end{pmatrix}, \quad \Pi(t) := \Pi_s(r(t)\nabla\rho v - \rho B_v \varphi(t)). \tag{13.1}$$

By Theorem 12.1 we know that $r(t)$ and $\varphi(t)$ are continuous and

$$|r(t)| + |\varphi(t)| \leq \frac{C(\rho,\tilde{v})\|X_0\|_\beta}{(1+t)^{1+\delta}}, \quad t \geq 0. \tag{13.2}$$

Hence, Proposition 8.4 is reduced now to the following.

Proposition 13.1: (i) Let functions $r(t), \varphi(t) \in C([0,\infty);\mathbb{R}^3)$, and $F_0 \in \mathcal{F}$. Then Eq. (13.1) admits a unique solution $F(t) \in C[0,\infty;\mathcal{F})$ with the initial condition $F(0) = F_0$.

(ii) If $X_0 = (F_0; r_0, \varphi_0) \in \mathcal{E}_\beta$ and the decay (13.2) holds, the corresponding fields also decay uniformly in v:

$$\|F(t)\|_{-2-\delta} \leq \frac{C(\rho,\tilde{v})\|X_0\|_\beta}{(1+t)^{1+\delta}}, \quad t \geq 0, \tag{13.3}$$

for $|v| \leq \tilde{v}$ with any $\tilde{v} \in (0;1)$.

Proof: Both statements follow from the Duhamel representation,

$$F(t) = W(t)F_0 + \left[\int_0^t W(t-s)\begin{pmatrix} \Pi(s) \\ 0 \end{pmatrix} ds\right], \quad t \geq 0, \tag{13.4}$$

where $W(t)$ is the dynamical group of the modified wave equation,

$$\dot{F}(t) = \begin{pmatrix} v\cdot\nabla & -\Delta \\ -1 & v\cdot\nabla \end{pmatrix} F(t), \tag{13.5}$$

and from the following decay properties of the group $W(t)$.

Lemma 13.2: For $\tilde{v} < 1$ and $F_0 \in \mathcal{F}_\alpha$, $\alpha > 1$, the following decay holds,

$$\|W(t)F_0\|_{-\alpha} \leq \frac{C(\alpha,\tilde{v})}{(1+t)^{\alpha-1}}\|F_0\|_\alpha, \quad t \geq 0, \tag{13.6}$$

for the dynamical group $W(t)$ corresponding to the modified wave equation (13.5) with $|v| < \tilde{v}$.

Cf. the proof of (Ref. 22, Lemma 18.2). Proposition 8.4 is proved.

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APPENDIX A: COMPUTING SYMPLECTIC FORM

Here we compute the matrix elements $\Omega(\tau_j,\tau_l)$ of the matrix Ω and prove that the matrix is nondegenerate. For $j,l=1,2,3$ it follows from (3.3) and (3.2) that

$$\Omega(\tau_j,\tau_l)=\langle\partial_j E_v,\partial_l A_v\rangle-\langle\partial_j A_v,\partial_l E_v\rangle,\quad \Omega(\tau_{j+3},\tau_{l+3})=\langle\partial_{v_j}E_v,\partial_{v_l}A_v\rangle-\langle\partial_{v_j}A_v,\partial_{v_l}E_v\rangle, \tag{A1}$$

$$\Omega(\tau_j,\tau_{l+3})=-\langle\partial_j E_v,\partial_{v_l}A_v\rangle+\langle\partial_j A_v,\partial_{v_l}E_v\rangle+e_j\cdot\partial_{v_l}P_v. \tag{A2}$$

In Fourier representation the solitons read:

$$\hat{E}_v(k)=\frac{i(kv)\hat\rho}{\hat D_0}\left(\frac{(kv)}{k^2}k-v\right),\quad \hat{A}_v(k)=\frac{-\hat\rho}{\hat D_0}\left(\frac{(kv)}{k^2}k-v\right), \tag{A3}$$

$$P_v=p_v+\langle A_v,\rho\rangle=p_v+v\int\frac{|\hat\rho|^2dk}{\hat D_0}-\int\frac{|\hat\rho|^2dk}{k^2\hat D_0}(kv)k, \tag{A4}$$

where $\hat D_0:=k^2-(kv)^2$; $\hat D_0$ is non-negative and even in k. Differentiating in v we obtain for $j=1,2,3$:

$$\begin{aligned}\partial_{v_j}\hat E_v&=\frac{i\hat\rho}{\hat D_0}\left(\frac{2k_j(kv)}{\hat D_0}k-\frac{k_j(k^2+(kv)^2)}{\hat D_0}v-(kv)e_j\right),\\ \partial_{v_l}\hat A_v&=\frac{\hat\rho}{\hat D_0}\left(\frac{2k_l(kv)}{\hat D_0}v-\frac{k_l(k^2+(kv)^2)}{k^2\hat D_0}k+e_l\right),\end{aligned} \tag{A5}$$

$$\begin{aligned}\partial_{v_l}P_v&=\partial_{v_l}p_v+\langle\partial_{v_l}A_v,\rho\rangle\\ &=B_v^{-1}e_l+\int\frac{|\hat\rho|^2dk}{\hat D_0}e_l+2\int\frac{|\hat\rho|^2(kv)k_ldk}{\hat D_0^2}v-\int\frac{|\hat\rho|^2(k^2+(kv)^2)k_ldk}{k^2\hat D_0^2}k.\end{aligned} \tag{A6}$$

Then for $j,l=1,2,3$ we get from (A1) by the Parseval identity,

$$\langle\partial_jE_v,\partial_lA_v\rangle=-i\int\frac{k_jk_l(kv)|\hat\rho|^2}{\hat D_0^2}\left(\frac{(kv)}{k^2}k-v\right)^2dk=0,$$

since the integrand function is odd in k. Similarly, $\langle\partial_jA_v,\partial_lE_v\rangle=0$ and thus $\Omega(\tau_j,\tau_l)=0$. Further, by (A1),

$$\begin{aligned}\langle\partial_{v_j}E_v,\partial_{v_l}A_v\rangle=i\int\frac{|\hat\rho|^2}{\hat D_0^2}\Bigg(&\frac{4k_jk_l(kv)^3}{\hat D_0^2}+\frac{2k_jk_l(kv)}{\hat D_0}\\ &-\frac{2k_jk_l(kv)(k^2+(kv)^2)}{\hat D_0^2}-\frac{2k_jk_l(kv)(k^2+(kv)^2)v^2}{\hat D_0^2}-\frac{k_j(k^2+(kv)^2)v_l}{\hat D_0}\\ &+\frac{k_jk_l(kv)(k^2+(kv)^2)^2}{k^2\hat D_0^2}-\frac{2k_l(kv)^2v_j}{\hat D_0}-(kv)\delta_{jl}+\frac{k_jk_l(kv)(k^2+(kv)^2)}{k^2\hat D_0}\Bigg)dk=0,\end{aligned}$$

since the integrand function is odd in k. Note that the integral converges by the neutrality condition (1.14). Similarly, $\langle\partial_{v_j}A_v,\partial_{v_l}E_v\rangle=0$ and thus, $\Omega(\tau_{j+3},\tau_{l+3})=0$. Now let us compute $\Omega(\tau_j,\tau_{l+3})$. First,

$$-\langle\partial_jE_v,\partial_{v_l}A_v\rangle=\int\frac{k_j(kv)|\hat\rho|^2dk}{\hat D_0^2}\left(v_l+\frac{2k_l(kv)v^2}{\hat D_0}-\frac{k_l(kv)(k^2+(kv)^2)}{k^2\hat D_0}\right).$$

Second,

$$\langle \partial_j A_v, \partial_{v_l} E_v \rangle = \int \frac{k_j |\hat{\rho}|^2 dk}{\hat{D}_0^2} \left((kv) v_l + \frac{k_l (k^2 + (kv)^2) v^2}{\hat{D}_0} - \frac{2 k_l (kv)^2}{\hat{D}_0} \right).$$

And third,

$$e_j \cdot \partial_{v_l} P_v = e_j \cdot B_v^{-1} e_l + \int \frac{|\hat{\rho}|^2 dk}{\hat{D}_0} \delta_{jl} + 2 \int \frac{|\hat{\rho}|^2 (kv) k_l dk}{\hat{D}_0^2} v_j - \int \frac{|\hat{\rho}|^2 (k^2 + (kv)^2) k_j k_l dk}{k^2 \hat{D}_0^2}.$$

By a straightforward computation we obtain that the matrix $\Omega^+(v) = \|\Omega(\tau_j, \tau_{l+3})\||_{j,l=1,2,3}$ is positive definite and hence nondegenerate. Finally, the matrix,

$$\|\Omega(\tau_j, \tau_l)\||_{j,l=1,\ldots,6} = \begin{pmatrix} 0 & \Omega^+(v) \\ -\Omega^+(v) & 0 \end{pmatrix}, \tag{A7}$$

is also nondegenerate.

APPENDIX B: BOUNDS FOR THE MATRIX $M^{-1}(i\omega)$

Proposition B.1: The following bound holds:

$$\|M^{-1}(i\omega)\| = \mathcal{O}\left(\frac{1}{|\omega|}\right), \quad \omega \to \infty. \tag{B1}$$

Proof: Recall that

$$M(i\omega) = \begin{pmatrix} i\omega E & -B_v \\ C(i\omega) + G & i\omega E - F(i\omega) \end{pmatrix},$$

where

$$B_v = \begin{pmatrix} \nu^3 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad C(i\omega) = \begin{pmatrix} c_1(i\omega) & 0 & 0 \\ 0 & c(i\omega) & 0 \\ 0 & 0 & c(i\omega) \end{pmatrix},$$

$$G = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix}, \quad F(i\omega) = \begin{pmatrix} f_1(i\omega) & 0 & 0 \\ 0 & f(i\omega) & 0 \\ 0 & 0 & f(i\omega) \end{pmatrix}.$$

Here $\nu = \sqrt{1 - v^2}$ and $c_1, c; g_1, g; f_1, f$ are defined by (10.17), (10.25), (10.26), (10.28), (10.29), respectively. Further,

$$\det M(i\omega) = d_1 d^2, \quad d_1 = i\omega(i\omega - f_1) + \nu^3(c_1 + g_1), \quad d = i\omega(i\omega - f) + \nu(c + g). \tag{B2}$$

Lemma B.2: The functions $c_1(i\omega)$, $c(i\omega)$, $f_1(i\omega)$, $f(i\omega)$ are bounded for $\omega \in \mathbb{R}$ with large $|\omega|$.

Proof: Let us consider only the first function $c_1(i\omega)$ in detail, for the rest three functions the argument is similar. By (10.17),

$$c_1(i\omega) = i|v| \int k_1 \frac{(i\omega + ik_1|v|)|\hat{\rho}|^2}{\hat{D}(i\omega)} (1 - \frac{k_1^2}{k^2}) dk$$

$$= i|v| \int k_1 \frac{(i\omega + ik_1|v|)|\hat{\rho}|^2}{\hat{D}(i\omega)} dk - i|v| \int k_1 \frac{(i\omega + ik_1|v|)|\hat{\rho}|^2}{\hat{D}(i\omega)} \frac{k_1^2}{k^2} dk =: \mathcal{C}_1(i\omega) + \mathcal{C}_2(i\omega).$$

Let us study $\mathcal{C}_1(i\omega)$, for $\mathcal{C}_2(i\omega)$ the argument is similar. In the x-space we obtain:

$$\mathcal{C}_1(i\omega) = |v| \langle -\partial_1 (i\omega - |v| \partial_1) \rho, D^{-1}(i\omega) \rho \rangle, \tag{B3}$$

where $D(i\omega) = -\Delta + (i\omega - v\nabla)^2$ is the differential operator with the symbol (10.11). Hence, $\mathcal{C}_1(i\omega)$ is bounded by the decay,

$$|\langle \partial_1^2 \rho, D^{-1}(i\omega)\rho\rangle| = \mathcal{O}(\frac{1}{|\omega|}), \quad |\omega| \to \infty, \tag{B4}$$

which follows from the Agmon estimates [Ref. 30, (A.2′)] (see also Appendix in Ref. 31). The Agmon estimates are applicable to the operator $D(i\omega)$ because of representation,

$$D(i\omega) = e^{-i\gamma v_1 x_1}[-(1-v_1^2)\partial_1^2 - \partial_2^2 - \partial_3^2 - \omega^2]e^{i\gamma v_1 x_1}, \tag{B5}$$

with $(1 - v_1^2)\gamma = \omega$ in the coordinates, where $v = (v_1, 0, 0)$. □

Corollary B.3: The determinants $d_1(i\omega)$ and $d(i\omega)$ are nonzero for $\omega \in \mathbb{R}$ with large $|\omega|$.

Further, the inverse matrix reads:

$$M^{-1}(i\omega) = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \tag{B6}$$

where

$$L_{11}(i\omega) = \begin{pmatrix} (i\omega - f_1)/d_1 & 0 & 0 \\ 0 & (i\omega - f)/d & 0 \\ 0 & 0 & (i\omega - f)/d \end{pmatrix}, \quad L_{12}(i\omega) = \begin{pmatrix} \nu^3/d_1 & 0 & 0 \\ 0 & \nu/d & 0 \\ 0 & 0 & \nu/d \end{pmatrix}, \tag{B7}$$

$$L_{21}(i\omega) = \begin{pmatrix} -(c_1 + g_1)/d_1 & 0 & 0 \\ 0 & -(c+g)/d & 0 \\ 0 & 0 & -(c+g)/d \end{pmatrix}, \quad L_{22}(i\omega) = \begin{pmatrix} i\omega/d_1 & 0 & 0 \\ 0 & i\omega/d & 0 \\ 0 & 0 & i\omega/d \end{pmatrix}. \tag{B8}$$

Now Proposition B.1 follows from (B2), (B6)–(B8), and Lemma B.2. □

1. Proof of Lemma 12.5

By Lemma 10.2 the functions $c_1(\lambda)$, $c(\lambda)$; $f_1(\lambda)$, $f(\lambda)$ are analytic in $\mathbb{C}$. Thus,

$$c_1(\lambda) = c_1(0) + c_1'(0)\lambda + \frac{c_1''(0)}{2}\lambda^2 + \dots, \quad c(\lambda) = c(0) + c'(0)\lambda + \frac{c''(0)}{2}\lambda^2 + \dots,$$
$$f_1(\lambda) = f_1(0) + f_1'(0)\lambda + \dots, \quad f(\lambda) = f(0) + f'(0)\lambda + \dots.$$

Below we write v instead of $|v|$ for simplicity of notations. By (10.16), (10.17), (10.29),

$$c_1'(\lambda) = iv\int dk|\hat\rho|^2 k_1 \left(1 - \frac{k_1^2}{k^2}\right)\frac{k^2 - (\lambda + ik_1 v)^2}{\hat D^2(\lambda)},$$

$$f_1'(\lambda) = \nu^3 \int dk|\hat\rho|^2 \left(\frac{k_1^2}{k^2} - 1\right)\frac{k^2 - (\lambda + ik_1 v)^2}{\hat D^2(\lambda)}. \tag{B9}$$

Further, $c'(\lambda) = (c_{12})'(\lambda) + (c_{22})'(\lambda)$, $f'(\lambda) = (f_{12})'(\lambda) + (f_{22})'(\lambda)$. By (10.16), (10.17), (10.21),

$$(c_{12})'(\lambda) = -\frac{iv}{2}\int dk|\hat\rho|^2 k_1(1 - \frac{k_1^2}{k^2})\frac{k^2 - (\lambda + ik_1 v)^2}{\hat D^2(\lambda)}, \quad (c_{22})'(\lambda) = v^2\int dk|\hat\rho|^2 \frac{(k^2 - k_1^2)(\lambda + ik_1 v)}{\hat D^2(\lambda)},$$
$$(f_{12})'(\lambda) = -\frac{\nu}{2}\int dk|\hat\rho|^2\left(1 + \frac{k_1^2}{k^2}\right)\frac{k^2 - (\lambda + ik_1 v)^2}{\hat D^2(\lambda)}, \quad (f_{22})'(\lambda) = -2i\nu v\int dk|\hat\rho|^2 k_1 \frac{(\lambda + ik_1 v)}{\hat D^2(\lambda)}. \tag{B10}$$

Further,

$$c_1''(\lambda) = -2iv \int dk |\hat{\rho}|^2 k_1 \left(1 - \frac{k_1^2}{k^2}\right) \frac{(\lambda + ik_1 v)(3k^2 - (\lambda + ik_1 v)^2)}{\hat{D}^3(\lambda)} \tag{B11}$$

and $c''(\lambda) = (c_{12})''(\lambda) + (c_{22})''(\lambda)$, where

$$(c_{12})''(\lambda) = iv \int dk |\hat{\rho}|^2 k_1 (1 - \frac{k_1^2}{k^2}) \frac{(\lambda + ik_1 v)(3k^2 - (\lambda + ik_1 v)^2)}{\hat{D}^3(\lambda)}, \tag{B12}$$

$$(c_{22})''(\lambda) = v^2 \int dk |\hat{\rho}|^2 \frac{(k^2 - k_1^2)(k^2 - 3(\lambda + ik_1 v)^2)}{\hat{D}^3(\lambda)}. \tag{B13}$$

Note that $c_1(0) = -g_1$, $c(0) = -g$, $c_1'(0) = c'(0) = 0$, then

$$c_1(\lambda) = -g_1 + \lambda^2 I_1(\lambda), \quad c(\lambda) = -g + \lambda^2 I(\lambda),$$

where the functions $I_1(\lambda)$, $I(\lambda)$ are analytic in $\mathbb{C}$ and $I_1(0) = c_1''(0)/2$, $I(0) = c''(0)/2$. By (B11)–(B13) we have

$$c_1''(0) = 2v^2 \int dk |\hat{\rho}|^2 k_1^2 \left(1 - \frac{k_1^2}{k^2}\right) \frac{3k^2 + (k_1 v)^2}{(k^2 - (k_1 v)^2)^3}, \tag{B14}$$

$$c''(0) = v^2 \int dk |\hat{\rho}|^2 \frac{(k^2 - k_1^2)(k^2(k^2 + 3(k_1 v)^2) - k_1^2(3k^2 + (k_1 v)^2))}{k^2(k^2 - (k_1 v)^2)^3}. \tag{B15}$$

Similarly, $f_1(0) = f(0) = 0$ and

$$f_1(\lambda) = \lambda J_1(\lambda), \quad f(\lambda) = \lambda J(\lambda),$$

where the functions $J_1(\lambda)$, $J(\lambda)$ are analytic in $\mathbb{C}$ and $J_1(0) = f_1'(0)$, $J(0) = f'(0)$. By (B10) we have

$$f_1'(0) = v^3 \int dk |\hat{\rho}|^2 (\frac{k_1^2}{k^2} - 1) \frac{k^2 + (k_1 v)^2}{(k^2 - (k_1 v)^2)^2}, \tag{B16}$$

$$f'(0) = \frac{v}{2} \int dk |\hat{\rho}|^2 \frac{(3k^2(k_1 v)^2 - k_1^2(k_1 v)^2 - k^2(k^2 + k_1^2))}{k^2(k^2 - (k_1 v)^2)^2}. \tag{B17}$$

We put $\lambda = i\omega$ and obtain

$$c_1(i\omega) + g_1 = -\omega^2 I_1(i\omega), \quad c(i\omega) + g = -\omega^2 I(i\omega), \quad f_1(i\omega) = i\omega J_1(i\omega), \quad f(i\omega) = i\omega J(i\omega). \tag{B18}$$

Then

$$d_1(i\omega) = -\omega^2(1 - J_1(i\omega) + v^3 I_1(i\omega)), \quad d(i\omega) = -\omega^2(1 - J(i\omega) + vI(i\omega)). \tag{B19}$$

Substitute (B18), (B19)–(B7), (B8) and obtain

$$L_{11}(i\omega) = \frac{1}{\omega}\mathcal{L}_{11}(i\omega), \quad L_{12}(i\omega) = \frac{1}{\omega^2}\mathcal{L}_{12}(i\omega), \quad L_{21}(i\omega) = \mathcal{L}_{21}(i\omega), \quad L_{22}(i\omega) = \frac{1}{\omega}\mathcal{L}_{22}(i\omega),$$

where (we omit the dependance on $i\omega$ for simplicity of notations)

$$\mathcal{L}_{11} = \begin{pmatrix} \frac{i(J_1 - 1)}{1 - J_1 + v^3 I_1} & 0 & 0 \\ 0 & \frac{i(J - 1)}{1 - J + vI} & 0 \\ 0 & 0 & \frac{i(J - 1)}{1 - J + vI} \end{pmatrix},$$

$$\mathcal{L}_{12} = \begin{pmatrix} \frac{-v^3}{1-J_1+v^3I_1} & 0 & 0 \\ 0 & \frac{-v}{1-J+vI} & 0 \\ 0 & 0 & \frac{-v}{1-J+vI} \end{pmatrix},$$

$$\mathcal{L}_{21} = \begin{pmatrix} \frac{-I_1}{1-J_1+v^3I_1} & 0 & 0 \\ 0 & \frac{-I}{1-J+vI} & 0 \\ 0 & 0 & \frac{-I}{1-J+vI} \end{pmatrix},$$

$$\mathcal{L}_{22} = \begin{pmatrix} \frac{-i}{1-J_1+v^3I_1} & 0 & 0 \\ 0 & \frac{-i}{1-J+vI} & 0 \\ 0 & 0 & \frac{-i}{1-J+vI} \end{pmatrix}.$$

Note that

$$\mathcal{L}_{11}(\omega) = i\mathcal{L}_{12}(\omega)B_v^{-1} + i\mathcal{L}_3, \tag{B20}$$

where

$$\mathcal{L}_3 = \begin{pmatrix} \frac{J_1}{1-J_1+v^3I_1} & 0 & 0 \\ 0 & \frac{J}{1-J+vI_1} & 0 \\ 0 & 0 & \frac{J}{1-J+vI_1} \end{pmatrix}. \tag{B21}$$

Finaly, we prove that the denominators of the matrix elements of each matrix $\mathcal{L}_{11}$ to $\mathcal{L}_{22}$ and $\mathcal{L}_3$ are nonzero at $\omega = 0$. Indeed, $-J_1(0) + v^3 I_1(0) > 0$, since $I_1(0) > 0$ and $J_1(0) < 0$ by (B14), (B16). Further, by a straightforward computation we obtain that

$$-J(0) + vI(0) = \frac{v}{2}\int dk \frac{|\hat{\rho}|^2(k^6(1+v^2) + k^4k_1^2(1+3v^4-8v^2) + k^2k_1^4v^2(3-v^2))}{k^2(k^2-(k_1v)^2)^3}.$$

It is easy to check that $k^6(1+v^2) + k^4k_1^2(1+3v^4-8v^2) + k^2k_1^4v^2(3-v^2) \geq 0$. This completes the proof of Lemma 12.5. □

APPENDIX C: SYMPLECTIC ORTHOGONALITY CONDITIONS

For $j = 1, 2, 3$ we have

$$0 = \Omega(Z_0, \tau_j) = -\langle e_0, \partial_j A_v\rangle + \langle a_0, \partial_j E_v\rangle - (\varphi_0 + \langle a_0, \rho\rangle)\cdot e_j$$

$$= -\int dk\ \hat{e}_0\ \overline{\frac{-ik_j(-\hat{\rho})}{\hat{D}}(\frac{kv}{k^2}k - v)} + \int dk\ \hat{a}_0\ \overline{\frac{-ik_j i(kv)\hat{\rho}}{\hat{D}}(\frac{kv}{k^2}k - v)} - (\varphi + \langle a_0, \rho\rangle)\cdot e_j.$$

Since $\hat{e}_0 \perp k$ and $\hat{a}_0 \perp k$, the condition simplifies to

$$-i\int dk\ \hat{e}_0\ \frac{k_j\overline{\hat{\rho}}}{\hat{D}}v - \int dk\ \hat{a}_0\ \frac{k_j(kv)\overline{\hat{\rho}}}{\hat{D}}v - \int dk\hat{a}_0\overline{\hat{\rho}}\cdot e_j - \varphi_0\cdot e_j = 0,$$

or, in the vector form,

$$\varphi_0 + \int dk\frac{\overline{\hat{\rho}}}{\hat{D}}[i(\hat{e}_0v)k + (\hat{a}_0v)(kv)k + k^2\hat{a}_0 - (kv)^2\hat{a}_0] = 0. \tag{C1}$$

On the other hand,

$$\varphi_0 + \Phi(0) = \varphi_0 + \int dk\frac{\overline{\hat{\rho}}}{\hat{D}}[i(kv)\hat{e}_0 + k^2\hat{a}_0]. \tag{C2}$$

Subtract (C1) from (C2) and obtain

$$\varphi_0 + \Phi(0) = -i \int dk \frac{\overline{\hat{\rho}}}{\hat{D}} [(\hat{e}_0 v)k - (kv)\hat{e}_0] - \int dk \frac{(kv)\overline{\hat{\rho}}}{\hat{D}} [(\hat{a}_0 v)k - (kv)\hat{a}_0]$$

$$= \int dk \frac{\overline{\hat{\rho}}}{\hat{D}} v \wedge ((-ik) \wedge \hat{e}_0) + \int dk \frac{\overline{\hat{\rho}}}{\hat{D}} (-ikv) v \wedge ((-ik) \wedge \hat{a}_0) = -\Psi(0).$$

Thus, (C1) reads $\varphi_0 + \Phi(0) + \Psi(0) = 0$.

Further, the symplectic orthogonality conditions $\Omega(Z_0, \tau_{j+3})$, $j = 1, 2, 3$ in the vector form read:

$$0 = \int dk \frac{\overline{\hat{\rho}}}{\hat{D}} \left[\frac{2(kv)(\hat{e}_0 v)}{\hat{D}} k + \hat{e}_0 \right] - i \int dk \frac{\overline{\hat{\rho}}}{\hat{D}} \left[\frac{k^2 + (kv)^2}{\hat{D}} (\hat{a}_0 v)k + (kv)\hat{a}_0 \right]$$

$$+ B_v^{-1} r_0 + \int dk \frac{|\hat{\rho}|^2}{\hat{D}} r_0 + 2 \int dk \frac{|\hat{\rho}|^2 (kv)(r_0 v)}{\hat{D}^2} k - \int dk \frac{|\hat{\rho}|^2 (k^2 + (kv)^2)(r_0 k)}{k^2 \hat{D}^2} k. \tag{C3}$$

The second line involving r_0, in the coordinate system, where $v = (v, 0, 0)$, simplifies to $B_v^{-1} r_0 +$

$$\begin{pmatrix} \int dk \frac{|\hat{\rho}|^2((k^2 - k_1^2)(k^2 + (k_v)^2))}{k^2 D^2} & 0 & 0 \\ 0 & \int dk \frac{|\hat{\rho}|^2 (k^2(k^2 + k_1^2) + k_1^2 (k_v)^2 - 3k^2 (k_1 v)^2)}{2k^2 D^2} & 0 \\ 0 & 0 & \int dk \frac{|\hat{\rho}|^2 (k^2(k^2 + k_1^2) + k_1^2 (k_v)^2 - 3k^2 (k_1 v)^2)}{2k^2 D^2} \end{pmatrix} r_0.$$

And this is exactly $(B_v^{-1} + \mathcal{L}_{12}\mathcal{L}_3(0)) r_0$, since

$$\mathcal{L}_{12}\mathcal{L}_3(0) = B_v^{-1} \begin{pmatrix} -J_1(0) & 0 & 0 \\ 0 & -J(0) & 0 \\ 0 & 0 & -J(0) \end{pmatrix},$$

where $J_1(0) = f_1'(0)$, $J(0) = f'(0)$, see (B16), (B17).

Finally, by (10.13), (10.19),

$$\Phi'(0) + \Psi'(0)$$

$$= \int dk \frac{\overline{\hat{\rho}} [D\hat{e}_0 - i(kv)D\hat{a}_0 + 2(kv)(v \cdot \hat{e}_0)k - i(k^2 + (kv)^2)(v \cdot \hat{a}_0)k]}{D^2},$$

which coincides with the first line of (C3) involving $\hat{e}_0$, $\hat{a}_0$.

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