On nonlinear wave equations with parabolic potentials

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Abstract. We introduce a new class of piece-wise quadratic potentials for nonlinear wave equations with a kink solutions. The potentials allow an exact description of the spectral properties for the linearized equation at the kink. This description is necessary for the study of the stability properties of the kinks.

In particular, we construct examples of the potentials of Ginzburg–Landau type providing the asymptotic stability of the kinks [6] and [7].

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1. Introduction

Last two decades there was an outstanding activity in the field of asymptotic stability of solitary waves for nonlinear Schrödinger equations [1], [2], [3], [11], [12], [13], [15], [16], [17], and [18], nonlinear Klein-Gordon equations [5] and [14], relativistic Ginzburg–Landau equations [6] and [7], and other Hamiltonian PDEs [8] and [10]. All these results rely on different assumptions on the spectral properties of the corresponding linearized dynamics. On the other hand, the examples were mostly unknown. Here we construct a model nonlinear wave equations, providing various spectral properties: different number of the eigenvalues, absence of the resonances, Fermi golden rule.

In particular, we construct the examples of relativistic Ginzburg–Landau equations providing all properties assumed in [6] and [7]. The properties imply the asymptotic stability of kinks for real solutions to 1D nonlinear Ginzburg–Landau equations

$$\ddot{\psi}(x, t) = \psi''(x, t) + F(\psi(x, t)), \quad x \in \mathbb{R},$$

where $F(\psi) = -U'(\psi)$.

We assume the following conditions.

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Condition U1. For some $K > 3$ and $m > 0$ the potential $U(\psi)$ is smooth even function satisfying

$$(\psi) > 0, \quad \psi \neq a.$$ \hspace{1cm} (\psi) > 0, \quad \psi \neq a.$$

$$U(\psi) = \frac{m^2}{2} (\psi \mp a)^2 + \mathcal{O}(\psi \mp a^K), \quad \psi \to \pm a.$$ $U(\psi) = \frac{m^2}{2} (\psi \mp a)^2 + \mathcal{O}(\psi \mp a^K), \quad \psi \to \pm a.$$

(see. Figure 1).

![Potential of Ginzburg–Landau type.](image)

Figure 1. Potential of Ginzburg–Landau type.

The corresponding stationary equation reads

$$s''(x) - U'(s(x)) = 0, \quad x \in \mathbb{R}. \quad (1.2)$$

Constant stationary solutions are: $\psi(x) \equiv 0$ and $\psi(x) \equiv \pm a$. There are also the “kinks”, i.e. nonconstant finite energy solutions $s(x)$ to (1.2) such that

$$s(x) \to \pm a, \quad x \to \pm \infty.$$ $s(x) \to \pm a, \quad x \to \pm \infty.$$

Condition U1 implies that $(s(x) \mp a)'' \sim m^2 (s(x) \mp a)$ for $x \to \pm \infty$, hence

$$|s(x) \mp a| \sim C e^{-m|x|}, \quad x \to \pm \infty. \quad (1.3)$$ $|s(x) \mp a| \sim C e^{-m|x|}, \quad x \to \pm \infty. \quad (1.3)$

(see. Figure 2).

Due to relativistic invariance of equation (1.1) the moving kinks

$$s_{q,v}(x,t) = s(\kappa(x - vt - q)), \quad q, v \in \mathbb{R}, |v| < 1, \kappa = 1/\sqrt{1 - v^2}$$ $s_{q,v}(x,t) = s(\kappa(x - vt - q)), \quad q, v \in \mathbb{R}, |v| < 1, \kappa = 1/\sqrt{1 - v^2}$

also are the solutions to (1.1). Let us linearize equation (1.1) at the kink $s(x)$. Substituting $\psi(x,t) = s(x) + \varphi(x,t)$, we obtain formally

$$\ddot{\varphi}(x,t) = -\nabla \varphi(x,t) + \mathcal{O}(|\varphi(x,t)|^2),$$ \ddot{\varphi}(x,t) = -\nabla \varphi(x,t) + \mathcal{O}(|\varphi(x,t)|^2),
where $H$ is the Schrödinger operator

$$
H \overset{def}{=} -\frac{d^2}{dx^2} + m^2 + W(x)
$$

with the potential

$$
W(x) = -F'(s(x)) - m^2 = U''(s(x)) - m^2.
$$

Condition U1 and the asymptotics (1.3) imply that

$$
|W(x)| = \mathcal{O}(|s(x) + a|^{K-1}) \sim Ce^{-(K-1)m|x|}, \quad x \to \pm \infty.
$$

The next properties of $H$ hold true.

**H1.** The continuous spectrum of $H$ is $\sigma_c = [m^2, \infty)$.

**H2.** The point $\lambda_0 = 0$ belongs to the discrete spectrum, and corresponding eigenfunction is $s'(x)$.

**H3.** Since $s'(x) > 0$, the point $\lambda_0 = 0$ is the groundstate, and all remaining discrete spectrum is contained in $(0, m^2]$.

To establish an asymptotic stability of the kinks $s_{q,v}(x, t)$ one need certain spectral properties of $H$ (cf. [6] and [7]).

**Condition U2.** The edge point $\lambda = m^2$ of the continuous spectrum is neither eigenvalue nor resonance.
Condition U3. The discrete spectrum of $H$ consists of two points: $\lambda_0 = 0$ and $\lambda_1 \in (0, m^2)$ satisfying
\[ 4\lambda_1 > m^2. \] (1.4)

We assume also a non-degeneracy condition known as “Fermi golden rule” meaning the strong coupling of the nonlinear term to the continuous spectrum. This coupling provides the energy radiation to infinity (cf. condition (10.0.11) in [2] and condition (1.11) in [7]).

Condition U4. The following inequality holds:
\[ \int \varphi_{4\lambda_1}(x) F''(s(x)) \varphi_{4\lambda_1}^2(x) \, dx \neq 0, \]
where $\varphi_{4\lambda_1}$ is the nonzero odd solution to $H \varphi_{4\lambda_1} = 4\lambda_1 \varphi_{4\lambda_1}$.

Note that the known quartic double well Ginzburg–Landau potential $U_{GL}(\psi) = (\psi^2 - a^2)^2 / (4a^2)$ satisfies condition U1 with $m^2 = 2$ and $K = 3$ as well as conditions U3 and U4. However, there exist the resonance for the corresponding operator $H$ at the edge point $\lambda = m^2$. Hence, the asymptotic stability of the kinks for $U_{GL}$ is the open problem.

Let us note that the result [4] concerns the wave front solution $\psi(x_1 - vt)$ to 3D wave with the potential $U_{GL}$. The solution has an infinite energy, so it is not a soliton, and its asymptotic stability is provided by the strong dispersion properties of the 3D case. The 1D case requires different arguments [6] and [7].

Our main result is the following theorem.

Theorem 1.1. There exist potentials $U(\psi)$ satisfying conditions U1–U4.

2. Piece wise parabolic potentials

As a first step, we will consider the class of the potentials which are piece-wise second order polynomials.

\[ U_0(\psi) = \begin{cases} \frac{1}{2} - \frac{b}{2} \psi^2, & |\psi| \leq \gamma, \\ \frac{d}{2}(\psi^2 - 1)^2, & \pm \psi \geq \gamma, \end{cases} \] (2.1)

with some constants $b, d > 0$ and $0 < \gamma < 1$. Let us find the parameters $b = b(\gamma)$ and $d = d(\gamma)$ providing $U_0(\psi) \in C^1(\mathbb{R})$. We have

\[ U_0(\gamma) = \frac{1}{2} - \frac{b}{2} \gamma^2 = \frac{d}{2}(\gamma - 1)^2 \quad \text{and} \quad U_0'(\gamma) = -b \gamma = d(\gamma - 1). \]
Solving the equations, we obtain
\[ b = \frac{1}{\gamma} \quad \text{and} \quad d = \frac{1}{1 - \gamma}, \quad 0 < \gamma < 1. \tag{2.2} \]

Then the functions \( U_0''(\psi) \) are piece-wise constant with the jumps at the points \( \psi = \pm \gamma \). Thus, the potentials \( U_0 \in C^4(\mathbb{R}) \) form one-dimensional manifold parametrized by \( \gamma \in (0, 1) \).

2.1. Kink. Let us solve the equation of type (1.2) for the kink in the case of potential (2.1):
\[ s_0''(x) - U_0'(s_0(x)) = 0, \quad x \in \mathbb{R}. \tag{2.3} \]
We search an odd solution to
\[ s_0''(x) = \begin{cases} -bs_0(x), & 0 < s_0(x) \leq \gamma, \\ d(s_0(x) - 1), & s_0(x) > \gamma. \end{cases} \]
We have
\[ s_0(x) = \begin{cases} C \sin \sqrt{b}x, & 0 < x \leq q, \\ Ae^{-\sqrt{d}x} + 1, & x > q, \end{cases} \]
where \( C > \gamma, A < 0, q = \frac{1}{\sqrt{b}} \arcsin \frac{\gamma}{C} \). Equating the values of \( s_0(x) \) and its left and right derivatives at \( x = q \) we obtain
\[ \begin{cases} Ae^{-\sqrt{d}q} + 1 = C \sin \sqrt{b}q = \gamma, \\ -\sqrt{d}Ae^{-\sqrt{d}q} = \sqrt{b}C \cos \sqrt{b}q. \end{cases} \tag{2.4} \]
The first line of (2.4) implies \( Ae^{-\sqrt{d}q} = \gamma - 1 \). Hence the second line of (2.4) becomes
\[ \sqrt{d}(1 - \gamma) = \sqrt{b}C \cos \sqrt{b}q. \]
The both side of the last equality is positive. Hence it is equivalent to
\[ d(1 - \gamma)^2 = b(C^2 - \gamma^2). \]
Substituting (2.2) we obtain \( 1 - \gamma = C^2/\gamma - \gamma \). Then
\[ C = \sqrt{\gamma}, \quad A = (\gamma - 1)e^{\sqrt{\gamma}/(1-\gamma)\arcsin \sqrt{\gamma}} \]
and
\[ q = \sqrt{\gamma} \arcsin \sqrt{\gamma}. \tag{2.5} \]
2.2. Linearized equation. Let us linearize equation (1.1) with $F(\psi) = F_0(\psi) = -U_0(\psi)$ at the kink $s_0(x)$ splitting the solution as the sum

$$\psi(t) = s_0 + \varphi(t).$$ (2.6)

Substituting (2.6) to (1.1), we obtain

$$\ddot{\varphi}(x, t) = \varphi''(x, t) - U_0'(s_0(x) + \varphi(x, t)) + U_0'(s_0(x)).$$ (2.7)

By (2.1) we can write equations (2.7) as

$$\ddot{\varphi}(t) = -H_0\varphi(t) + N(\varphi(t)), \quad t \in \mathbb{R},$$

where

$$H_0 = -\frac{d^2}{dx^2} + W_0(x), \quad W_0(x) = U_0''(s_0(x)) = \begin{cases} -b, & |x| \leq q, \\ d, & |x| > q, \end{cases}$$ (2.8)

(see Figure 3) and

$$N(\varphi(t)) = -U_0'(s_0 + \varphi(t)) + U_0'(s_0) + U_0''(s_0).$$

The continuous spectrum of $H_0$ coincides with $[d, \infty)$. The point $\lambda_0 = 0$ is the groundstate since it corresponds to the even positive eigenfunction $\varphi_0(x) = s_0'(x)$:

$$H_0\varphi_0 = -s_0'''(x) + U_0''(s_0(x))s_0'(x) = 0,$$

which follows by differentiation of (2.3). Therefore, the discrete spectrum of $H_0$ belongs to $[0, d]$, and the next eigenfunction $\varphi_1(x)$ should be odd.
2.3. Odd eigenfunctions. Given an eigenvalue \( \lambda \), the corresponding eigenfunction \( \varphi(x) \) should satisfy the equation

\[
\begin{cases} 
-\varphi''(x) - b\varphi(x) = \lambda\varphi(x), & |x| \leq q, \\
-\varphi''(x) + d\varphi(x) = \lambda\varphi(x), & |x| > q.
\end{cases}
\]

(Equations 2.9) imply that the odd eigenfunctions have the form

\[
\varphi(x) = \begin{cases} 
B \sin \beta x, & |x| \leq q, \\
A \operatorname{sgn} x e^{-\alpha |x|}, & |x| > q.
\end{cases}
\]

(2.10)

where \( \alpha = \sqrt{d - \lambda} > 0 \) and \( \beta = \sqrt{b + \lambda} > 0 \). Equating the values of the eigenfunction and its left and right derivatives at \( x = q \), we obtain

\[
\begin{cases} 
Ae^{-\alpha q} = B \sin \beta q, \\
-A\alpha e^{-\alpha q} = B\beta \cos \beta q,
\end{cases}
\]

(2.11)

where \( a \) and \( b \) are related as follows

\[ \alpha^2 + \beta^2 = b + d. \]

System (2.11) admits nonzero solutions only if its determinant vanishes, that is

\[ -\alpha = \beta \cot \beta q. \]

At last, multiplying by \( q \), and denoting \( \xi = \beta q \) and \( \eta = \alpha q \), we obtain the system of equations

\[ -\eta = \xi \cot \xi, \quad \xi^2 + \eta^2 = R^2, \]

(2.12)

where \( R = q \sqrt{b + d} \) is the radius of the circle. Substituting \( b, d \) and \( q \) from (2.2) and (2.5) respectively, we obtain

\[
R = q \sqrt{\frac{1}{\gamma} + \frac{1}{1 - \gamma}} = \frac{q}{\sqrt{\gamma(1 - \gamma)}} = \arcsin \sqrt{\gamma}.
\]

Finally, the solutions to (2.12) can be found graphically (see Figure 4). Taking into account that \( \eta > 0 \), we obtain that

\[
R \in \left(0, \frac{\pi}{2}\right]: \text{system (2.12) has no solution}; \quad R \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]: \text{system (2.12) has one solution}; \quad R \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]: \text{system (2.12) has two solutions}; \quad R \in \left[\frac{5\pi}{2}, \frac{7\pi}{2}\right]: \text{system (2.12) has no solution}.
\]

(2.13)
Let us note that $R(0) = 0$ and $R(1) = \infty$, and the radius $R(\gamma)$ is monotone increasing on $[0, 1]$. Denote by $\gamma_k, k \in \mathbb{N}$ the solution to the equation

$$\arcsin \frac{\sqrt{\gamma_k}}{\sqrt{1 - \gamma_k}} = \frac{k \pi}{2}, \quad k \in \mathbb{N}. \tag{2.14}$$

Numerical calculations give

$$\gamma_1 \approx 0.64643, \quad \gamma_2 \approx 0.8579, \quad \gamma_3 \approx 0.92472, \quad \gamma_4 \approx 0.95359 \ldots \tag{2.15}$$

Further, (2.13) implies that

\[
\begin{align*}
\gamma &\in (0, \gamma_1]: \text{no nonzero odd eigenfunctions}, \\
\gamma &\in (\gamma_1, \gamma_3]: \text{one linearly independent odd eigenfunction}, \\
\gamma &\in (\gamma_3, \gamma_5]: \text{two linearly independent odd eigenfunctions}, \\
\end{align*}
\]
In particular, for $\gamma \in (\gamma_1, \gamma_3]$ we have one odd eigenfunction and the corresponding eigenvalue $\lambda_1 \in (0, d)$ reads

$$\lambda_1 = \lambda_1(\gamma) = \beta^2 - b = \frac{\xi^2}{q^2} - b$$

$$= \frac{1}{\gamma} \left( \frac{\xi^2}{\arcsin^2 \sqrt{\gamma}} - 1 \right) = \frac{1}{\gamma} \left( \frac{\sin^2 \xi}{1 - \gamma} - 1 \right),$$

where $\xi$ is the solution to

$$\frac{\xi^2}{\sin^2 \xi} = \frac{\arcsin^2 \sqrt{\gamma}}{1 - \gamma}. \quad (2.17)$$

### 2.4. Even eigenfunctions.

Equations (2.9) imply that the even eigenfunctions have the form

$$\varphi(x) = \begin{cases} B \cos \beta x, & |x| \leq q, \\ A e^{-\alpha |x|}, & |x| > q. \end{cases} \quad (2.18)$$

where $\alpha = \sqrt{d - \lambda} > 0, \beta = \sqrt{b + \lambda} > 0$. Equating the values of the eigenfunction and its left and right derivatives at $x = q$, we obtain

$$\begin{cases} Ae^{-\alpha q} = B \cos \beta q, \\ A\alpha e^{-\alpha q} = B\beta \sin \beta q. \end{cases} \quad (2.19)$$

The system admits nonzero solutions if and only if its determinant vanishes:

$$\alpha = \beta \tan \beta q.$$  

Similarly (2.12), we obtain the following equations for $\xi = \beta q$ and $\eta = \alpha q$:

$$\eta = \xi \tan \xi, \quad \xi^2 + \eta^2 = R^2, \quad (2.19)$$

where $R = \frac{\arcsin \sqrt{\gamma}}{\sqrt{1 - \gamma}}$. The solutions can also be found graphically (see Figure 5).
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Figure 5. Graphical solution of system (2.19).

We have

\[
R \in (0, \pi]: \text{system (2.19) has one solution;} \\
R \in (\pi, 2\pi]: \text{system (2.19) has two solutions;}
\]

\[
(2.20)
\]

Note that for any \( \gamma \in (0, 1) \) equation (2.19) has the solution \( \xi = \arcsin \sqrt{\gamma} \in (0, \pi/2) \). The solution corresponds to eigenvalue \( \lambda = 0 \) and the first even eigenfunction. Moreover, (2.20) implies that

\[
\gamma \in (0, \gamma_2]: \text{one linearly independent even eigenfunction,}
\gamma \in (\gamma_2, \gamma_4]: \text{two linearly independent even eigenfunctions,}
\]

where \( \gamma_i \) are defined in (2.14).

**Conclusion.** 1) There is one eigenvalue \( \lambda_0 = 0 \) for \( \gamma \in (0, \gamma_1] \). 2) There are two eigenvalues \( \lambda_0 = 0 \) and \( 0 < \lambda_1 < d \) for \( \gamma \in (\gamma_1, \gamma_2] \). Etc. (See Figure 6.)
3. Spectral conditions

We deduce Theorem 1.1 in Section 4 below from the following proposition.

**Proposition 3.1.** For any \( \gamma \in (\gamma_1, \gamma_2) \) the piecewise parabolic potentials \( U_0 \), defined in (2.1), satisfy conditions U1–U3 except for the smoothness condition at the points \( \psi = \pm \gamma \). Condition U4 holds for any \( \gamma \in (\gamma_1, \gamma_2) \) except for one point \( \gamma^* \).

**Proof.** Step i) Obviously, for \( U_0(\psi) \) condition U1 with \( a = 1 \), \( m^2 = d \), and any integer \( K \geq 3 \) holds except the smoothness at the points \( \psi = \pm \gamma \).

Consider condition U2. Note that the solutions to (2.12) or (2.19) with \( \eta = 0 \) and \( R = k\pi/2, \ k \in \mathbb{N} \) correspond to \( \alpha = 0 \) i.e. \( \lambda = d \). Then the functions (2.10) or (2.18) with \( A \neq 0 \) are a nonzero constant for \( |x| \geq \gamma \). Hence, the functions are the resonances corresponding to the edge point \( \lambda = d \) of the continuous spectrum. Thus, the resonances exist only for the discrete set of parameters \( \gamma_k, \ k \in \mathbb{N} \), defined in (2.14). Evidently, the set has just one limit point 1. Hence, condition U2 holds for \( \gamma \in (0, 1) \setminus \{ \bigcup_{k \in \mathbb{N}} \gamma_k \} \).

Step ii) For any \( \gamma \in (\gamma_1, \gamma_2) \) the operator \( H_0 \) defined in (2.8) has exactly two eigenvalues \( \lambda_0 = 0 \) and \( \lambda_1 = 2 \cos^2 \frac{\gamma}{2} \). For condition U3 it remains to verify (1.4) with \( m^2 = d \). Namely, due to (2.16) and (2.17) we must prove that for any \( \gamma \in (\gamma_1, \gamma_2) \) the following inequality holds:

\[
\frac{4}{\gamma} \left( \frac{\sin^2 \xi(\gamma)}{1 - \gamma} - 1 \right) > \frac{1}{1 - \gamma},
\]

where \( \xi(\gamma) \in (\pi/2, \pi) \) is the solution to (2.17). After the simple transformations we obtain

\[
4 \cos^2 \xi(\gamma) < 3\gamma,
\]

and

\[
\frac{\pi}{2} < \xi(\gamma) < \pi - \arccos \frac{\sqrt{3\gamma}}{2}.
\]

Since \( \frac{\xi}{\sin \xi} \) is monotonically increasing function for \( \xi \in (\pi/2, \pi) \), then

\[
\frac{\pi}{2} < \frac{\arcsin \sqrt{3\gamma}}{\sqrt{1 - \gamma}} < \frac{2(\pi - \arccos \frac{\sqrt{3\gamma}}{2})}{\sqrt{3 - 3\gamma}}.
\]
Finally, we obtain
\[ \gamma_1 < \gamma < \alpha, \]
where \( \alpha \) is the solution to
\[ \arcsin \sqrt{\alpha} = \frac{2(\pi - \arccos \sqrt{\frac{4\alpha}{3\alpha^2}})}{\sqrt{4 - 3\alpha^2}}. \]
Numerical calculation gives
\[ \alpha = 0.921485 > \gamma_2. \]
Therefore, condition \( \textbf{U3} \) holds for any \( \gamma \in (\gamma_1, \gamma_2) \).

Step iii) Finally, consider condition \( \textbf{U4} \) (Fermi golden rule). The condition can be rewritten as
\[ \int U_0'''(s_0(x))\varphi_{4\lambda_1}(x)\varphi_{4\lambda_1}^2(x)dx = \int \frac{d}{dx}U_0'''(s_0(x))\frac{\varphi_{4\lambda_1}(x)\varphi_{4\lambda_1}^2(x)}{s_0'(x)}dx \neq 0. \]
By (2.8) we have that \( U_0'''(s_0(x)) = W_0(x) \) is the piece wise constant function. Hence,
\[ \frac{d}{dx}U_0''(s_0(x)) = (b + d)\delta(x - q) - (b + d)\delta(x + q), \]
and (3.1) becomes
\[ \varphi_{4\lambda_1}(q)\varphi_{4\lambda_1}^2(q) \neq 0. \]
Formula (2.10) yields that \( \varphi_{\lambda_1}(q) = Ae^{-aq} \neq 0 \). Hence it is sufficient to verify that
\[ \varphi_{4\lambda_1}(q) \neq 0. \]
The eigenfunction \( \varphi_{4\lambda_1} \) satisfies the equations
\[ \begin{cases} -\varphi_{4\lambda_1}''(x) - b\varphi_{4\lambda_1}(x) = 4\lambda_1\varphi_{4\lambda_1}(x), & |x| \leq q, \\ -\varphi_{4\lambda_1}''(x) + d\varphi_{4\lambda_1}(x) = 4\lambda_1\varphi_{4\lambda_1}(x), & |x| > q. \end{cases} \]
For the odd solution to (3.2) we have
\[ \varphi_{4\lambda_1}(q) = \sin \beta q, \quad \beta = \sqrt{b + 4\lambda_1} > 0. \]
Therefore, \( \varphi_{4\lambda_1}(q) = 0 \) only if \( \beta q = k\pi, k \in \mathbb{N}, \) or
\[ \sqrt{1 + 4\gamma\lambda_1(\gamma)} \arcsin \sqrt{\gamma} = k\pi, \quad k \in \mathbb{N}, \]
where \( \lambda_1(\gamma) \) is defined in (2.16) and (2.17). Substituting \( \lambda_1(\gamma) \) into (3.3) we obtain from (2.16) and (2.17)
\[ \begin{align*} \frac{\arcsin \sqrt{\gamma}}{\sqrt{1 - \gamma}} & = \frac{4\sin^2 \xi - 3(1 - \gamma)}{4\sin^2 \frac{\xi}{2} - 3(1 - \gamma)} = k\pi, \\ \frac{\xi^2}{\sin^2 \frac{\xi}{2}} & = \frac{\arcsin \sqrt{\gamma}}{1 - \gamma}. \end{align*} \]
For $\gamma \in (\gamma_1, \gamma_2)$ the system has a solution only for $k = 1$ since

$$0 < \frac{\arcsin \sqrt{\gamma}}{\sqrt{1 - \gamma}} \sqrt{4 \sin^2 \xi - 3(1 - \gamma)} < 2\pi, \quad \gamma_1 < \gamma < \gamma_2.$$ 

Denote

$$\theta = \arcsin \sqrt{\gamma} \in (\pi \sqrt{1 - \gamma_1}, \pi \sqrt{1 - \gamma_2}). \quad (3.5)$$

Then (3.4) with $k = 1$ is equivalent to

$$\begin{cases}
4\xi^2 - 3\theta^2 = \pi^2, \\
\sin \frac{\xi}{\theta} = \cos \frac{\theta}{\theta}.
\end{cases} \quad (3.6)$$

Let us prove that (3.6) has a unique solution. Consider two functions $\theta_1(\xi)$ and $\theta_2(\xi)$, where $\theta_1(\xi) \equiv \frac{1}{\sqrt{3}} \sqrt{4\xi^2 - \pi^2}$ and $\theta_2(\xi)$ is the solution of $\sin \frac{\xi}{\theta} = \cos \frac{\theta}{\theta}$.

The function $\theta_1(\xi)$ increases for $\xi(\gamma_1) < \xi < \xi(\gamma_2)$, and

$$\theta'_1(\xi) = \frac{1}{\sqrt{3}} \frac{4\xi}{\sqrt{4\xi^2 - \pi^2}} > \frac{1}{\sqrt{3}} \frac{4(\pi/2)}{\sqrt{4(3\pi/4)^2 - \pi^2}} = \frac{4}{\sqrt{15}} > 1 \quad (3.7)$$

for $\xi(\gamma_2) < \xi < \xi(\gamma_2)$, since $\xi(\gamma_1) = \pi/2$ and $\xi(\gamma_2) \sim 2.3137 < 3\pi/4$. On the other hand,

$$\theta'_2(\xi) = \frac{\sin \frac{\xi}{\theta} - \xi \cos \frac{\xi}{\theta}}{\cos \theta + \theta \sin \theta} \frac{\theta^2}{\xi^2} > 0, \quad \pi/2 < \xi < \xi(\gamma_2).$$

Moreover, by (3.5) and (3.6) we obtain

$$\theta'_2(\xi) = \theta \frac{\sin \frac{\xi}{\theta} - \cos \frac{\xi}{\theta}}{\cos \theta + \sin \theta} < \frac{\sin \frac{\xi}{\theta} - \cos \frac{\xi}{\theta}}{\sin \frac{\xi}{\theta} + \sin \theta} < 1, \quad \pi/2 < \xi < \xi(\gamma_2)$$

since $|\cos \xi| < |\cos \xi(\gamma_2)| < \sqrt{2}/2$, and $\sin \theta = \sqrt{\gamma} > \sqrt{\gamma_1} > \sqrt{2}/2$ by (2.15). Finally,

$$\theta_2(\pi/2) > \theta_1(\pi/2) = 0, \quad \theta_2(\xi(\gamma_2)) \sim 1.1843 < \theta_1(\xi(\gamma_2)) \sim 1.9616. \quad (3.8)$$

Therefore, (3.7) and (3.8) imply that $\theta_1(\xi) = \theta_2(\xi)$ for a single value $\xi(\gamma_*) \in (\pi/2, \xi(\gamma_2))$ (see Figure 7).
Numerical calculation gives \( \gamma_* \approx 0.7925 \). Hence, system (3.6) on the interval \((\gamma_1, \gamma_2)\) has the only solution \( \gamma = \gamma_* \). Thus, the Fermi golden rule holds for any \( \gamma \in (\gamma_1, \gamma_2) \) except for the one point \( \gamma_* \).

**Conclusion.** The potential \( U_0(\psi) \) satisfies conditions U1–U4 except for the smoothness at the points \( \psi = \pm \gamma \) for any \( \gamma \in (\gamma_1, \gamma_*) \cup (\gamma_*, \gamma_2) \).

4. **Smooth potentials**

We deduce Theorem 1.1 from Proposition 3.1 by an approximation of the potential \( U_0 \) with a smooth functions satisfying conditions U1–U4. Namely, let \( h(\psi) \in C_0^\infty(\mathbb{R}) \) be an even mollifying function with the following properties:

\[
h(\psi) \geq 0, \quad \text{supp } h \subset [-1, 1], \quad \int h(\psi) d\psi = 1.
\]

For \( \varepsilon \in (0, 1] \) we set

\[
\bar{U}_\varepsilon(\psi) \equiv \frac{1}{\varepsilon} \int h\left(\frac{\psi - \psi'}{\varepsilon}\right) U_0(\psi') d\psi'.
\]  

(4.1)

Evidently, \( \bar{U}_\varepsilon(\psi) \geq 0 \) is a smooth, even function, symmetric with respect to the points \( \psi = \pm 1 \) in some neighborhoods of these points. In addition we have

\[
\bar{U}_\varepsilon(\psi) - U_0(\psi) = \begin{cases}
\mu_\varepsilon > 0, & |\psi| \geq \gamma + \varepsilon, \\
-\nu_\varepsilon < 0, & |\psi| \leq \gamma - \varepsilon,
\end{cases}
\]
where \( \mu_\varepsilon, \nu_\varepsilon = O(\varepsilon^2) \). Let us set

\[
U_\varepsilon(\psi) = \tilde{U}_\varepsilon(\psi) - \mu_\varepsilon. \tag{4.2}
\]

Then

\[
U_\varepsilon(\psi) = \begin{cases} 
U_0(\psi), & |\psi| \geq \gamma + \varepsilon, \\
U_0(\psi) - \mu_\varepsilon - \nu_\varepsilon, & |\psi| \leq \gamma - \varepsilon.
\end{cases} \tag{4.3}
\]

Obviously,

\[
\sup_{\psi \in \mathbb{R}} |U_\varepsilon(\psi) - U_0(\psi)| \leq C \varepsilon \tag{4.4}
\]

with some constant \( C \). Moreover,

\[
U''_\varepsilon(\psi) \leq 0 \text{ for } \psi \leq 0 \quad \text{and} \quad U''_\varepsilon(\psi) \geq 0 \text{ for } \psi \geq 0. \tag{4.5}
\]

The corresponding kink is an odd solution to the equation

\[
s''_\varepsilon(x) - U'_{\varepsilon}(s_\varepsilon(x)) = 0, \quad x \in \mathbb{R}. \tag{4.6}
\]

The equation can be integrated using the “energy conservation”

\[
\frac{|s'_\varepsilon(x)|^2}{2} - U_\varepsilon(s_\varepsilon(x)) = \text{const}, \quad x \in \mathbb{R}
\]

with \( \text{const} = 0 \):

\[
\int_0^{s_\varepsilon(x)} \frac{ds}{\sqrt{2U_\varepsilon(s)}} = x, \quad x \in \mathbb{R}. \tag{4.6}
\]

Hence, \( s_\varepsilon(x) \) is a monotone increasing function, and

\[
s_\varepsilon(x) \to \pm 1, \quad x \to \pm \infty.
\]

Moreover, (4.3), (4.4), and (4.6) imply that

\[
\sup_{x \in \mathbb{R}} |s_\varepsilon(x) - s_0(x)| \leq C_1 \varepsilon.
\]

Therefore,

\[
||s_\varepsilon(x)| - \gamma| \geq \varepsilon, \quad ||x| - q| \geq \delta,
\]

where

\[
\delta \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.7}
\]

Hence, for the linear potential \( W_\varepsilon(x) \triangleq U''_{\varepsilon}(s_\varepsilon(x)) \) we obtain

\[
W_\varepsilon(x) = W_0(x), \quad ||x| - q| \geq \delta. \tag{4.8}
\]

Further, (4.5) implies that

\[
W'_\varepsilon(x) \leq 0 \text{ for } x \leq 0 \quad \text{and} \quad W'_\varepsilon(x) \geq 0 \text{ for } x \geq 0.
\]
Therefore,
\begin{equation}
|W_\varepsilon(x) - W_0(x)| \leq b + d, \quad x \in \mathbb{R}.
\end{equation}
(see Figure 8).

\end{hlist}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{potential.png}
\caption{Potential $W_\varepsilon$.}
\end{figure}

As a result, denoting $w_\varepsilon(x) = W_\varepsilon(x) - W_0(x)$, we obtain
\begin{equation}
\|w_\varepsilon\|_{L^2(\mathbb{R})} \to 0, \quad \varepsilon \to 0
\end{equation}
by (4.7), (4.8), and (4.9).

**Lemma 4.1.** The eigenvalues of the Schrödinger operator
\begin{equation}
H_\varepsilon = -\frac{d^2}{dx^2} + W_\varepsilon(x)
\end{equation}
converge to the ones of $H_0$ as $\varepsilon \to 0$.

**Proof.** The eigenvalues of $H_0$ and $H_\varepsilon$ are the poles of the resolvents $R_0(\omega) = (H_0 - \omega)^{-1}$ and $R_\varepsilon(\omega) = (H_\varepsilon - \omega)^{-1}$ respectively. Hence, the lemma follows from (4.10) due to the relation
\begin{equation}
R_\varepsilon(\omega) = (H_0 - \omega + w_\varepsilon)^{-1} = R_0(\omega)(1 + w_\varepsilon R_0(\omega))^{-1}.
\end{equation}
\end{proof}

**Proof of Theorem 1.1.** Consider the potential $U(\psi) = U_\varepsilon(\psi)$ defined in eq. (4.1) and eq. (4.2). Let us prove that there exist $\varepsilon_0 > 0$ such that for any $\gamma \in (\gamma_1, \gamma_2) \setminus \gamma_*$, and $0 < \varepsilon < \varepsilon_0$ the potential $U_\varepsilon$ satisfies conditions U1–U4.
On nonlinear wave equations with parabolic potentials

Step i) Condition U1 with \( a = 1, m^2 = d, \) and any integer \( K \geq 3 \) obviously holds.

Step ii) For any \( \sigma \in \mathbb{R}, \) and \( s = 0, 1, 2, \ldots \) denote by \( \mathcal{H}_\sigma^s = \mathcal{H}_\sigma^s(\mathbb{R}) \) the weighted Sobolev spaces with the finite norms

\[
\| \psi \|_{\mathcal{H}_\sigma^s} = \sum_{k=0}^{s} \| (1 + |x|)^\sigma \psi^{(k)} \|_{L^2(\mathbb{R})} < \infty.
\]

By [9], Theorem 7.2, the absence of the resonance at the point \( \omega = d \) for the Schrödinger operator \( H \) is equivalent to the boundedness of the corresponding resolvent \( R(\omega) : \mathcal{H}_\sigma^0 \to \mathcal{H}_\sigma^{-2} \) at \( \omega = d \) for any \( \sigma > 1/2 \). Hence, the resolvent \( R_0(d) : \mathcal{H}_\sigma^0 \to \mathcal{H}_\sigma^{-2} \) is bounded by Proposition 3.1. Further, (4.8) and (4.9) imply

\[
\| w_{\varepsilon} \|_{\mathcal{H}_\sigma^0 \to \mathcal{H}_\sigma^0} \to 0, \quad \varepsilon \to 0
\]

Hence, for sufficiently small \( \varepsilon \) the operator \( R_\varepsilon(d) : \mathcal{H}_\sigma^0 \to \mathcal{H}_\sigma^{-2} \) is bounded by (4.11). Then condition U2 holds for \( U_\varepsilon \).

Step iii) Lemma 4.1 implies that for \( \gamma \in (\gamma_1, \gamma_2) \) and sufficiently small \( \varepsilon \) the operator \( H_\varepsilon \) has exactly two eigenvalues \( \lambda_0 = 0 \) and \( 0 < \lambda_1(\varepsilon) < d \). Moreover, \( \lambda_1(\varepsilon) \to \lambda_1(0) = \lambda_1 \) as \( \varepsilon \to 0 \) and then \( 4 \lambda_1(\varepsilon) > d \) for sufficiently small \( \varepsilon \). Hence, condition U3 holds.

Step iv) It remains to check condition U4. Consider arbitrary \( \gamma \in (\gamma_1, \gamma_2) \setminus \gamma_\ast \). Denote \( \varphi^{\varepsilon}_{\lambda_1(\varepsilon)} \) and \( \varphi^{\varepsilon}_{4\lambda_1(\varepsilon)} \) the corresponding odd eigenfunctions of \( H_\varepsilon \). Then we have

\[
\int U''_\varepsilon(s_{\varepsilon}(x))\varphi^{\varepsilon}_{4\lambda_1(\varepsilon)}(x)(\varphi^{\varepsilon}_{\lambda_1(\varepsilon)}(x))^2dx

= \int d_{x-q}^+ W_\varepsilon(x)\varphi^{\varepsilon}_{4\lambda_1(\varepsilon)}(x)(\varphi^{\varepsilon}_{\lambda_1(\varepsilon)}(x))^2 s'_\varepsilon(x)dx

= \sum_{\pm} d_{x-q}^\pm \frac{\varphi^{\varepsilon}_{4\lambda_1(\varepsilon)}(x-q+\delta)(\varphi^{\varepsilon}_{\lambda_1(\varepsilon)}(x+q+\delta))^2}{s'_\varepsilon(x-q+\delta)}

+ \sum_{\pm} b_{x-q}^\pm \frac{\varphi^{\varepsilon}_{4\lambda_1(\varepsilon)}(x-q-\delta)(\varphi^{\varepsilon}_{\lambda_1(\varepsilon)}(x+q-\delta))^2}{s'_\varepsilon(x-q-\delta)}

- \int d_{x-q}^+ W_\varepsilon(x)\varphi^{\varepsilon}_{4\lambda_1(\varepsilon)}(x)(\varphi^{\varepsilon}_{\lambda_1(\varepsilon)}(x))^2 s'_\varepsilon(x)dx

\xrightarrow{\varepsilon \to 0} 2(d + b)\frac{\varphi^{\varepsilon}_{4\lambda_1}(q)(\varphi^{\varepsilon}_{\lambda_1}(q))^2}{s'_\varepsilon(q)} = \int U'_0(s_0(x))\varphi^{\varepsilon}_{4\lambda_1}(x)(\varphi^{\varepsilon}_{\lambda_1}(x))^2dx \neq 0
\]

since \( \delta \to 0 \) as \( \varepsilon \to 0 \). Hence, U4 holds for sufficiently small \( \varepsilon \). \( \square \)
References


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