Global well-posedness for the Schrödinger equation coupled to a nonlinear oscillator

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Abstract

The Schrödinger equation with the nonlinearity concentrated at a single point proves to be an interesting and important model for the analysis of long-time behavior of solutions, such as the asymptotic stability of solitary waves and properties of weak global attractors. In this note, we prove global well-posedness of this system in the energy space H^1.

1 Introduction and main results

We are going to prove the well-posedness in H^1 for the nonlinear Schrödinger equation with the nonlinearity concentrated at a single point:

$$i\dot{\psi}(x,t) = -\psi''(x,t) - \delta(x)F(\psi(0,t)), \qquad x \in \mathbb{R}, \tag{1.1}$$

where the dots and the primes stand for the partial derivatives in t and x, respectively. The equation describes the Schrödinger field coupled to a nonlinear oscillator. This equation is a convenient playground for developing the tools for the analysis of long-time behavior of solutions to $U(1)$-invariant Hamiltonian systems with dispersion. The asymptotic stability of the solitary manifold for equation (1.1) has been considered in [BKKS07]. Here we complete this result, giving the proof of the global well-posedness of (1.1) in the energy space.

Let us mention that for the Klein-Gordon equation with the nonlinearity of the same type the global attraction was addressed in [KK06], [KK07].

We assume that

$$F(\psi) = -\nabla_\psi U(\psi), \qquad \psi \in \mathbb{C}, \tag{1.2}$$

for some real-valued potential $U \in C^2(\mathbb{C})$, where ∇_ψ is the real derivative with respect to $(\operatorname{Re}\psi, \operatorname{Im}\psi)$. Equation (1.1) is a Hamiltonian system with the Hamiltonian

$$\mathscr{H}(\psi) = \int_{\mathbb{R}} \frac{|\psi'(x)|^2}{2}\,dx + U(\psi(0)), \qquad \psi \in H^1 = H^1(\mathbb{R}). \tag{1.3}$$

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The Hamiltonian form of (1.1) is

$$\dot{\Psi} = JD\mathscr{H}(\Psi), \tag{1.4}$$

where

$$\Psi = \begin{bmatrix} \operatorname{Re}\psi \\ \operatorname{Im}\psi \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{1.5}$$

and $D\mathscr{H}$ is the Fréchet derivative on the Hilbert space H^1. The value of the Hamiltonian functional is conserved for classical finite energy solutions of (1.1). We assume that equation (1.1) possesses $U(1)$-symmetry, thus requiring that

$$U(\psi) = u(|\psi|^2), \qquad \psi \in \mathbb{C}. \tag{1.6}$$

It then follows that $F(0) = 0$ and $F(e^{is}\psi) = e^{is}F(\psi)$ for $\psi \in \mathbb{C}$, $s \in \mathbb{R}$, and that

$$F(\psi) = a(|\psi|^2)\psi, \qquad \psi \in \mathbb{C}, \qquad \text{where} \quad a(\cdot) = 2u'(\cdot) \in \mathbb{R}. \tag{1.7}$$

This symmetry implies that $e^{i\theta}\psi(x,t)$ is a solution to (1.1) if $\psi(x,t)$ is. According to the Nöther theorem, the $U(1)$-invariance leads (formally) to the conservation of the charge, given by the functional

$$Q(\psi) = \frac{1}{2}\int_{\mathbb{R}} |\psi|^2\,dx. \tag{1.8}$$

We also assume that $U(\psi)$ is such that

$$U(z) \ge A - B|z|^2 \qquad \text{with some} \quad A \in \mathbb{R}, \quad B > 0. \tag{1.9}$$

We will show that equation (1.1) is globally well-posed in H^1. We will consider the solutions of class $\psi \in C_b(\mathbb{R}\times\mathbb{R})$. All the derivatives in equation (1.1) are understood in the sense of distributions.

Theorem 1.1 (Global well-posedness)**.** *Let the conditions (1.2), (1.6) and (1.9) hold with $U \in C^2(\mathbb{C})$. Then*

(*i*) *For any $\phi \in H^1(\mathbb{R})$, the equation for (1.1) with the initial data $\psi|_{t=0} = \phi$ has a unique solution $\psi \in C(\mathbb{R}, H^1(\mathbb{R}))$.*

(*ii*) *The values of the charge and energy functionals are conserved:*

$$Q(\psi(t)) = Q(\phi), \qquad \mathscr{H}(\psi(t)) = \mathscr{H}(\phi), \qquad t \in \mathbb{R}. \tag{1.10}$$

(*iii*) *There exists $\Lambda(\phi) > 0$ such that the following a priori bound holds:*

$$\sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1} \le \Lambda(\phi) < \infty. \tag{1.11}$$

(*iv*) *The map $\mathbf{U}:\ \psi(0) \mapsto \psi$ is continuous from H^1 to $L^\infty([0,T], H^1(\mathbb{R}))$, for any $T > 0$.*

Theorem 1.2. *Under conditions of Theorem 1.1, $\psi \in C^{(1/4)}(\mathbb{R}\times\mathbb{R})$.*

Let us give the outline of the proof. We need a small preparation first: We show that, without loss of generality, it suffices to prove the theorem assuming that U is uniformly bounded together with its derivatives. Indeed, the a priori bounds on the L^∞-norm of ψ imply that the nonlinearity $F(z)$ may be modified for large values of $|z|$. Then we will prove the existence and uniqueness of the solution $\psi \in C_b(\mathbb{R}\times[0,\tau])$, for some $\tau > 0$. This is accomplished in Section 2.

In Section 3, we construct approximate solutions $\psi_\varepsilon \in C_b(\mathbb{R}, H^1(\mathbb{R}))$ that are solutions to a regularized problem (the δ-function substituted by its smooth approximations ρ_ε, $\varepsilon > 0$). On one hand, the approximate solutions have their energy and charge conserved. On the other hand, we will show in Section 4 that the approximate solutions converge to $\psi(x,t)$ uniformly for $|x| \le R$, $0 \le t \le \tau$.

In Section 5, we use the uniform convergence of approximate solutions to conclude that $\psi \in L^\infty([0,\tau], H^1(\mathbb{R}))$ and moreover that ψ could be extended to all $t \ge 0$. Then we show that the energy and the charge are conserved. We will use these conservations to extend the solution $\psi(x,t)$ for $t \in \mathbb{R}$. Then we prove that $\psi \in C(\mathbb{R}, H^1(\mathbb{R}))$.

In Section 6, we study the Hölder continuity in time, showing that $\psi \in C^{(1/4)}(\mathbb{R}\times\mathbb{R})$.

2 Local well-posedness in C_b

Lemma 2.1. *A priori bound (1.11) follows from (1.9) and the energy and charge conservation (1.10).*

Proof. Let $A \in \mathbb{R}$, $B > 0$ be constants from (1.9), and let $\psi \in H^1(\mathbb{R})$. To estimate $\|\psi\|_{H^1}$ in terms of the values of $Q(\psi)$ and $\mathscr{H}(\psi)$, we need to control the possibly negative contribution of $U(\psi)$ into $\mathscr{H}(\psi)$. We achieve this control by using the inequality

$$B|\psi(0)|^2 \le B\Big[\int_{\mathbb{R}} \hat\psi(k)\frac{dk}{2\pi}\Big]^2 \le B\int_{\mathbb{R}} \big(B^2+\tfrac{k^2}{4}\big)|\hat\psi(k)|^2\frac{dk}{2\pi}\cdot\int_R \frac{dk}{2\pi\big(B^2+\frac{k^2}{4}\big)} = B^2\|\psi\|_{L^2}^2 + \tfrac14\|\psi'\|_{L^2}^2. \qquad (2.1)$$

This allows us to write

$$\mathscr{H}(\psi) \ge \tfrac12\|\psi'\|_{L^2}^2 + A - B|\psi(0)|^2 \ge \tfrac14\|\psi'\|_{L^2}^2 + A - B^2\|\psi\|_{L^2}^2 = \tfrac14\|\psi\|_{H^1}^2 + A - (B^2+\tfrac14)\|\psi\|_{L^2}^2. \qquad (2.2)$$

The first inequality follows from (1.9), while the second one holds due to the bound (2.1). We rewrite (2.2) as the bound on $\|\psi\|_{H^1}^2$:

$$\|\psi\|_{H^1}^2 \le (8B^2+2)Q(\psi) + 4\mathscr{H}(\psi) - 4A. \qquad (2.3)$$

When we take into account the energy and charge conservation (1.10), the inequality (2.3) leads to the bound (1.11) with

$$\Lambda(\phi) = \sqrt{(8B^2+2)Q(\phi)+4\mathscr{H}(\phi)-4A}. \qquad (2.4)$$

□

Lemma 2.2. *Let us assume that Theorem 1.1 is true for the nonlinearities U that satisfy the following additional condition:*

$$\textit{For}\quad k=0,1,2\quad \textit{there exist}\quad U_k<\infty\quad \textit{so that}\quad \sup_{z\in\mathbb{C}}|\nabla^k U(z)| \le U_k. \qquad (2.5)$$

Then Theorem 1.1 is also true without this additional condition.

Proof. Fix a nonlinearity U that does not necessarily satisfy (2.5). For a particular initial data $\phi \in H^1(\mathbb{R})$ in Theorem 1.1, we choose $\widetilde U(z) \in C^2(\mathbb{C})$ so that $\widetilde U(z) = \widetilde U(|z|)$ for $z \in \mathbb{C}$ and $\widetilde U(z) = U(z)$ for $|z| \le \Lambda(\phi)$, where $\Lambda(\phi)$ is defined by (2.4). We can choose $\widetilde U$ so that it satisfies (1.9) with the same A, B as U does, and also satisfies the uniform bounds

$$\sup_{z\in\mathbb{C}}|\nabla^k\widetilde U(z)| < \infty, \qquad k=0,1,2.$$

By the assumption of the Lemma, Theorem 1.1 is true for the nonlinearity $\widetilde F = -\nabla\widetilde U$ instead of $F = -\nabla U$. Hence, there is a unique solution $\psi(x,t) \in L^\infty(\mathbb{R}, H^1)\cap C_b(\mathbb{R}\times\mathbb{R})$ to the equation

$$i\dot\psi(x,t) = -\psi''(x,t) - \delta(x)\widetilde F(\psi(0,t)),$$

with $\psi|_{t=0} = \phi$. By Lemma 2.1, ψ satisfies the a priori bound (1.11) with $\Lambda(\phi)$ defined by (2.4). This bound implies that $|\psi(0,t)| \le \Lambda(\phi)$ for $t \in \mathbb{R}$. Therefore, $\widetilde{F}(\psi(0,t)) = F(\psi(0,t))$ for $t \in \mathbb{R}$, and $\psi(x,t)$ is also a solution to (1.1) with the nonlinearity $F = -\nabla U$. $\square$

From now on, we shall assume in the proof of Theorem 1.1 that the bounds (2.5) hold true.

Lemma 2.3. *(i) Let $\phi \in H^1 := H^1(\mathbb{R})$. There exists $\tau > 0$ that depends only on U_2 in (2.5) so that there is a unique solution $\psi \in C_b(\mathbb{R} \times [0,\tau])$ to equation (1.1) with the initial data $\psi|_{t=0} = \phi$.*

(ii) The map $\phi \mapsto \psi$ is continuous from H^1 to $C_b(\mathbb{R} \times [0,\tau])$.

Proof. Let us denote the dynamical group for the free Schrödinger equation by

$$W_t\phi(x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{i\frac{|x-y|^2}{2t}} \phi(y)\,dy, \qquad x \in \mathbb{R}. \tag{2.6}$$

For its Fourier transform, we have:

$$\mathscr{F}_{x\to k}[W_t\phi(x)](k) = e^{ik^2 t}\hat{\phi}(k), \qquad k \in \mathbb{R}. \tag{2.7}$$

Then the solution ψ to (1.1) with the initial data $\psi|_{t=0} = \phi$ admits the Duhamel representation

$$\psi(x,t) = W_t\phi(x) = W_t\phi(x) + Z\psi(x,t), \tag{2.8}$$

where

$$Z\psi(x,t) = -\int_0^t W_s\delta(x)F(\psi(0,t-s))\,ds = -\int_0^t \frac{e^{i\frac{x^2}{2s}}}{\sqrt{2\pi s}} F(\psi(0,t-s))\,ds. \tag{2.9}$$

The Fourier representation (2.7) implies that $W_t\phi(x) \in C_b(\mathbb{R}, H^1) \subset C_b(\mathbb{R} \times \mathbb{R})$. Further, we compute for ψ_1, $\psi_2 \in C_b(\mathbb{R} \times [0,\tau])$:

$$|Z\psi_2(x,t) - Z\psi_1(x,t)| \le \int\limits_0^t \frac{|F(\psi_2(0,t-s)) - F(\psi_1(0,t-s))|}{\sqrt{2\pi s}}\,ds \le U_2\sqrt{t} \sup_{0\le s\le t} |\psi_2(s) - \psi_1(s)|,$$

where we used (2.5) with $k = 2$. For definiteness, we set

$$\tau = \frac{1}{4U_2^2}. \tag{2.10}$$

Then the map $\psi \mapsto W_t\phi + Z\psi$ is contracting in the space $C_b(\mathbb{R} \times [0,\tau])$. It follows that equation (2.8) admits a unique solution $\psi \in C_b(\mathbb{R} \times [0,\tau])$, proving the first part of the theorem. The second part of the theorem also follows by contraction. $\square$

3 Regularized equation

We proved that there is a unique solution $\psi(x,t) \in C([0,\tau] \times \mathbb{R})$. Now we are going to prove that $\psi \in L^\infty(\mathbb{R}_+, H^1)$ and moreover that $\|\psi(t)\|_{H^1}$ is bounded uniformly in time.

Let us fix a family of functions $\rho_\varepsilon(x)$ approximating the Dirac δ-function. We pick $\rho_1(x) \in C_0^\infty[-1,1]$, nonnegative, and such that $\int_{\mathbb{R}} \rho_1(x)\,dx = 1$, and define

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho_1\left(\frac{x}{\varepsilon}\right), \qquad \varepsilon \in (0,1), \tag{3.1}$$

so that

$$\operatorname{supp}\rho_\varepsilon(x)\subseteq[-\varepsilon,\varepsilon],\qquad \rho_\varepsilon(x)\ge 0,\qquad \int_{\mathbb{R}}\rho_\varepsilon(x)\,dx=1.$$

Consider the smoothed equation with the "mean field interaction"

$$i\dot\psi(x,t)=-\Delta\psi(x,t)-\rho_\varepsilon(x)F(\langle\rho_\varepsilon,\psi(t)\rangle),\tag{3.2}$$

where

$$\langle\rho_\varepsilon,\psi(t)\rangle=\langle\rho_\varepsilon(\cdot),\psi(\cdot,t)\rangle=\int_{\mathbb{R}}\rho_\varepsilon(x)\psi(x,t)\,dx.$$

Clearly, equation (3.2) is the Hamiltonian equation, with the Hamilton functional

$$\mathscr{H}_\varepsilon(\psi)=\int\frac{|\nabla\psi|^2}{2}\,dx+U(\langle\rho_\varepsilon,\psi\rangle).\tag{3.3}$$

The Hamiltonian form of (3.2) is (cf. (1.4))

$$\dot\Psi_\varepsilon=JD\mathscr{H}_\varepsilon(\Psi_\varepsilon).\tag{3.4}$$

The solution ψ_ε to (3.2) with the initial data $\psi_\varepsilon|_{t=0}=\phi$ admits the Duhamel representation

$$\psi_\varepsilon(x,t)=\mathbf{W}_t\phi(x)+\mathbf{Z}_\varepsilon\psi_\varepsilon(x,t),\tag{3.5}$$

where

$$\mathbf{Z}_\varepsilon\psi_\varepsilon(x,t)=-\int_0^t\mathbf{W}_s\rho_\varepsilon(x)F(\langle\rho_\varepsilon,\psi_\varepsilon(t-s)\rangle)\,ds.\tag{3.6}$$

Lemma 3.1 (Local well-posedness). (i) *For any* $\varepsilon\in(0,1)$, *there exists* $\tau_\varepsilon>0$ *that depends on* ε *and on* U_2 *from (2.5) so that there is a unique solution* $\psi_\varepsilon\in C_b([0,\tau_\varepsilon],H^1)$ *to equation (3.2) with* $\psi_\varepsilon|_{t=0}=\phi$.

(ii) *For each* $t\le\tau_\varepsilon$, *the map* $\mathbf{U}_\varepsilon(t):\phi=\psi_\varepsilon(0)\mapsto\psi_\varepsilon(t)$ *is continuous in* H^1.

(iii) *The values of the functionals* $\mathscr{H}_\varepsilon$ *and* Q *on solutions to (3.2) are conserved in time.*

Proof. (i) For $\psi_1,\ \psi_2\in C_b([0,\tau_\varepsilon],H^1)$, we compute:

$$\begin{aligned}&\|\mathbf{Z}_\varepsilon\psi_2(\cdot,t)-\mathbf{Z}_\varepsilon\psi_1(\cdot,t)\|_{H^1}\\&=\Big\|\int_0^t\mathbf{W}_s\rho_\varepsilon F(\langle\rho_\varepsilon,\psi_2(t-s)\rangle)-F(\langle\rho_\varepsilon,\psi_1(t-s)\rangle)\,ds\Big\|_{H^1}\\&\le\int_0^t\|\mathbf{W}_s\rho_\varepsilon\|_{H^1}\,|F(\langle\rho_\varepsilon,\psi_2(t-s)\rangle)-F(\langle\rho_\varepsilon,\psi_1(t-s)\rangle)|\,ds.\end{aligned}$$

The first factor under the integral sign is bounded uniformly for $0<s\le t$:

$$\|\mathbf{W}_s\rho_\varepsilon\|_{H^1_x}=\frac{1}{\sqrt{2\pi}}\left\|\sqrt{1+k^2}e^{ik^2s/2}\widehat{\rho}_\varepsilon(k)\right\|_{L^2_k}=\|\rho_\varepsilon\|_{H^1}.$$

Taking this into account, we get:

$$\begin{aligned}\|\mathbf{Z}_\varepsilon\psi_2(\cdot,t)-\mathbf{Z}_\varepsilon\psi_1(\cdot,t)\|_{H^1}&\le\|\rho_\varepsilon\|_{H^1}\int_0^t|F(\langle\rho_\varepsilon,\psi_2(t-s)\rangle)-F(\langle\rho_\varepsilon,\psi_1(t-s)\rangle)|\,ds\\&\le tU_2\|\rho_\varepsilon\|_{H^1}\sup_{s\in[0,t]}|\langle\rho_\varepsilon,\psi_2(s)-\psi_1(s)\rangle|.\end{aligned}$$

Therefore, the map $\psi\mapsto\mathbf{W}_t\phi+\mathbf{Z}_\varepsilon\psi$ is contracting if we choose, for definiteness,

$$\tau_\varepsilon=\frac{1}{4U_2\|\rho_\varepsilon\|_{H^1}}.\tag{3.7}$$

(*ii*) The continuity of the mapping $\mathbf{U}_\varepsilon(t)$ also follows from the contraction argument.

(*iii*) It suffices to prove the conservation of the values of $\mathscr{H}_\varepsilon(\psi_\varepsilon(t))$ and $Q(\psi_\varepsilon(t))$ for $\phi \in H^2 := H^2(\mathbb{R})$ since the functionals are continuous on H^1. For $\phi \in H^2$, the corresponding solution belongs to the space $C_b([0,\tau_\varepsilon],H^2)$ by the Duhamel representation (3.5). Then the energy and charge conservation follows by the Hamiltonian structure (3.4). Namely, the differentiation of the Hamilton functional gives by the chain rule,

$$\frac{d}{dt}\mathscr{H}_\varepsilon(\Psi_\varepsilon(t)) = \langle D\mathscr{H}_\varepsilon(\Psi_\varepsilon(t)), \dot\Psi_\varepsilon(t)\rangle = \langle D\mathscr{H}_\varepsilon(\Psi_\varepsilon(t)), JD\mathscr{H}_\varepsilon(\Psi_\varepsilon(t))\rangle = 0 \tag{3.8}$$

since the Fréchet derivative $D\mathscr{H}_\varepsilon(\Psi_\varepsilon(t)) = -\Delta\Psi_\varepsilon(\cdot,t) - \rho_\varepsilon(\cdot)F(\langle\rho_\varepsilon,\Psi_\varepsilon(t)\rangle)$ belongs to $L^2(\mathbb{R})$ for $t \in [0,\tau_\varepsilon]$. Similarly, the charge conservation follows by the differentiation,

$$\begin{aligned}\frac{d}{dt}Q(\Psi_\varepsilon(t)) &= \langle DQ(\Psi_\varepsilon(t)), \dot\Psi_\varepsilon(t)\rangle = \langle DQ(\Psi_\varepsilon(t)), JD\mathscr{H}_\varepsilon(\Psi_\varepsilon(t))\rangle \\ &= \langle \Psi_\varepsilon(x,t), J\Delta\Psi_\varepsilon(x,t)\rangle - \langle \Psi_\varepsilon(x,t), J\rho_\varepsilon(x)F(\langle\rho_\varepsilon,\Psi_\varepsilon(t)\rangle)\rangle.\end{aligned} \tag{3.9}$$

Here $\Psi_\varepsilon(x,t), J\Delta\Psi_\varepsilon(x,t)\rangle = \nabla\Psi_\varepsilon(x,t), J\nabla\Psi_\varepsilon(x,t)\rangle = 0$, and also

$$\begin{aligned}\langle \Psi_\varepsilon(x,t), J\rho_\varepsilon(x)F(\langle\rho_\varepsilon,\Psi_\varepsilon(t)\rangle)\rangle &= \int \Psi_\varepsilon(x,t)\cdot[J\rho_\varepsilon(x)F(\langle\rho_\varepsilon,\Psi_\varepsilon(t)\rangle)]\,dx \\ &= \langle\rho_\varepsilon,\Psi_\varepsilon(t)\rangle\cdot[JF(\langle\rho_\varepsilon,\Psi_\varepsilon(t)\rangle)] = 0.\end{aligned} \tag{3.10}$$

Here "$\cdot$" stands for the real scalar product in $\mathbb{R}^2$, and $Z\cdot[JF(Z)] = 0$ for $Z \in \mathbb{R}^2$ since $F(Z) = a(|Z|)Z$ with $a(|Z|) \in \mathbb{R}$ by (1.7).

□

Corollary 3.2 (Global well-posedness)**.** (*i*) *For any $\varepsilon > 0$, $\varepsilon \le 1$, there exists a unique solution $\psi_\varepsilon \in C(\mathbb{R},H^1)$ to equation (3.2) with $\psi_\varepsilon|_{t=0} = \phi$.*

The H^1-norm of ψ_ε is bounded uniformly in time:

$$\sup_{t\in\mathbb{R}} \|\psi_\varepsilon(t)\|_{H^1} \le \Lambda_\varepsilon(\phi), \qquad t \in \mathbb{R}, \tag{3.11}$$

where

$$\Lambda_\varepsilon(\phi) = \sqrt{(8B^2+2)Q(\phi) + 4\mathscr{H}_\varepsilon(\phi) - 4A}. \tag{3.12}$$

(*ii*) *For each $t \ge 0$, the map $\mathbf{U}_\varepsilon(t) : \psi_\varepsilon(0) \mapsto \psi_\varepsilon(t)$ is continuous in H^1.*

Proof. (*i*) The existence and uniqueness of the solution $\psi_\varepsilon \in C_b([0,\tau_\varepsilon],H^1)$ follow from Lemma 3.1 (*i*). The bound on the value of the H^1-norm of $\psi_\varepsilon(t)$ is obtained as in Lemma 2.1. Namely, noting that

$$U(\langle\rho,\psi_\varepsilon\rangle) \ge A - B\langle\rho,\psi_\varepsilon\rangle^2 \ge A - B\sup_{x\in\mathbb{R}}|\psi_\varepsilon|^2 \ge A - B^2\|\psi\|_{L^2}^2 - \frac{1}{4}\|\psi'\|_{L^2}^2$$

and using the energy and charge conservation proved in Lemma 3.1 (*iii*), we conclude that

$$(2B^2+\tfrac{1}{2})Q(\phi) + \mathscr{H}_\varepsilon(\phi) = (2B^2+\tfrac{1}{2})Q(\psi_\varepsilon) + \mathscr{H}_\varepsilon(\psi_\varepsilon) \ge A + \tfrac{1}{4}\|\psi_\varepsilon\|_{H^1}^2,$$

so that

$$\|\psi_\varepsilon\|_{H^1}^2 \le (8B^2+2)Q(\phi) + 4\mathscr{H}_\varepsilon(\phi) - 4A. \tag{3.13}$$

By (3.7), the time span τ_ε depends only on $\|\rho_\varepsilon\|_{H^1}$ and U_2. Hence, the bound (3.11) allows us to extend the solution to $t\in[\tau_\varepsilon,2\tau_\varepsilon]$. The bound (3.11) for $t\in[0,2\tau_\varepsilon]$ follows from (3.13) by the energy and charge conservation proved in Lemma 3.1 (*iii*). We conclude by induction that the solution exists and the bound (3.11) holds for all $t\in\mathbb{R}$.

(*ii*) The continuity of the mapping $\mathbf{U}_\varepsilon(t):\psi_\varepsilon(0)\mapsto\psi_\varepsilon(t)$ for all $t\ge 0$ follows from its continuity for small times by dividing the interval $[0,t]$ into small time intervals.

□

4 Convergence of regularized solutions

Lemma 4.1. *Let τ and $\psi\in C_b(\mathbb{R}\times[0,\tau])$ be as in Lemma 2.3, and let $\psi_\varepsilon\in C(\mathbb{R}_+,H^1)$ be as in Corollary 3.2. Then for any finite $R>0$*

$$\psi_\varepsilon(x,t)\underset{\varepsilon\to 0}{\rightrightarrows}\psi(x,t),\qquad |x|\le R,\quad 0\le t\le\tau. \tag{4.1}$$

Proof. We have

$$\psi_\varepsilon(x,t)=\boldsymbol{W}_t\phi(x)+\int_0^t\boldsymbol{W}_s\rho_\varepsilon(x)F(\langle\rho_\varepsilon,\psi_\varepsilon(t-s)\rangle)\,ds, \tag{4.2}$$

$$\psi(x,t)=\boldsymbol{W}_t\phi(x)+\int_0^t\boldsymbol{W}_s\delta(x)F(\psi(0,t-s))\,ds. \tag{4.3}$$

Taking the difference of these equations and regrouping the terms, we can write:

$$\begin{aligned}\psi_\varepsilon(x,t)-\psi(x,t)=&\int_0^t\boldsymbol{W}_s\rho_\varepsilon(x)\left(F(\langle\rho_\varepsilon,\psi_\varepsilon(t-s)\rangle)-F(\psi(0,t-s))\right)ds\\&+\int_0^t[\boldsymbol{W}_s\rho_\varepsilon(x)-\boldsymbol{W}_s\delta(x)]F(\psi(0,t-s))\,ds.\end{aligned} \tag{4.4}$$

Let us analyze the first term in the right-hand side of (4.4). It is bounded by

$$\begin{aligned}&\left|\int_0^t\frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}}\rho_\varepsilon(y)\,dy\,ds\right|\sup_{0\le s\le t}|F(\langle\rho_\varepsilon,\psi_\varepsilon(s)\rangle)-F(\psi(0,s))|\\&\le\left|\int_0^t\frac{ds}{\sqrt{2\pi s}}\right|U_2\sup_{|x|\le\varepsilon,0\le s\le t}|\psi_\varepsilon(x,s)-\psi(x,s)|\\&\le\sqrt{\frac{2t}{\pi}}U_2\sup_{|x|\le\varepsilon,0\le s\le t}|\psi_\varepsilon(x,s)-\psi(x,s)|\\&\le\frac12\sup_{|x|\le\varepsilon,0\le s\le t}|\psi_\varepsilon(x,s)-\psi(x,s)|,\end{aligned} \tag{4.5}$$

where in the last inequality we used (3.7). Setting $M_{R,\tau}=\sup_{|x|\le R,0\le t\le\tau}|\psi_\varepsilon(x,t)-\psi(x,t)|$, we can rewrite (4.4) as

$$M_{R,\tau}\le\frac12 M_{R,\tau}+\sup_{|x|\le R,0\le t\le\tau}\int_0^t[\boldsymbol{W}_s\rho_\varepsilon(x)-\boldsymbol{W}_s\delta(x)]F(\psi(0,t-s))\,ds.$$

Therefore,

$$M_{R,\tau}\le 2\sup_{|x|\le R,0\le t\le\tau}\int_0^t\int\frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}}[\rho_\varepsilon(y)-\delta(y)]\,dy\,F(\psi(0,t-s))ds. \tag{4.6}$$

We claim that the right-hand side tends to zero as $\varepsilon \to 0$. To prove this, we split the integral into two pieces:

$$I_1(\delta,\varepsilon) = \int_\delta^t \int \frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}} [\rho_\varepsilon(y) - \delta(y)]\, dy F(\psi(0,t-s))ds, \tag{4.7}$$

$$I_2(\delta,\varepsilon) = \int_0^\delta \int \frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}} [\rho_\varepsilon(y) - \delta(y)]\, dy F(\psi(0,t-s))ds, \tag{4.8}$$

where $\delta \in (0,t)$ is yet to be chosen. Let us analyze the term (4.7):

$$|I_1(\delta,\varepsilon)| \le CU_0 \sup_{s\ge\delta,|x|\le R} \left| \int_{|y|<\varepsilon} \frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}} [\rho_\varepsilon(y) - \delta(y)]\, dy \right|. \tag{4.9}$$

Since $s \ge \delta > 0$ and $|x| \le R$, the function $\frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}}$ is Lipschitz in $y \in [-\varepsilon,\varepsilon]$, uniformly in all the parameters. Therefore,

$$\int_{\mathbb{R}} \frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}} [\rho_\varepsilon(y) - \delta(y)]\, dy \to 0, \qquad \varepsilon \to 0, \tag{4.10}$$

uniformly in the parameters. We conclude that

$$\lim_{\varepsilon\to 0} I_1(\delta,\varepsilon) = 0, \tag{4.11}$$

for any fixed $\delta > 0$. We then bound (4.8) uniformly by

$$I_2(\delta,\varepsilon) \le CU_0 \int (\rho_\varepsilon(y) + \delta(y))\, dy \int_0^\delta \frac{ds}{\sqrt{s}} \le C\sqrt{\delta},$$

with C independent of ε. Now apparently the right-hand side of (4.6) tends to zero as $\varepsilon \to 0$. □

5 Well-posedness in energy space

Lemma 5.1 (Local well-posedness)**.** *There is a unique solution $\psi \in L^\infty([0,\tau], H^1(\mathbb{R})) \cap C_b(\mathbb{R}\times[0,\tau])$ to equation (1.1) with $\psi|_{t=0} = \phi$, where τ is as in (2.10).*

Proof. The unique solution $\psi \in C_b(\mathbb{R}\times[0,\tau])$ is constructed in Lemma 2.3. According to (3.11) and (4.1),

$$\|\psi(t)\|_{H^1} \le \liminf_{\varepsilon\to 0} \|\psi_\varepsilon(t)\|_{H^1} \le \Lambda(\phi), \qquad 0 \le t \le \tau. \tag{5.1}$$

□

Lemma 5.2. *The values of the functionals $\mathscr{H}$ and Q are conserved in time for $t \in [0,\tau]$.*

Proof. The convergence (4.1) and the bounds (3.11) imply that

$$Q(\psi(t)) = \frac{1}{2}\|\psi(t)\|_{L^2}^2 \le \frac{1}{2}\lim_{\varepsilon\to 0}\|\psi_\varepsilon(t)\|_{L^2}^2 = Q(\phi), \tag{5.2}$$

where we used the conservation of Q for the approximate solutions ψ_ε (Lemma 3.1). The same argument applied to the initial data $\psi|_{t=t_0}$ with any $t_0 \in (0,\tau)$ and combined with the uniqueness of the solution, allows to conclude that $Q(\psi(t))$ is monotonically non-increasing when time changes from 0 to τ. Instead, solving the Schrödinger equation backwards in time and using the uniqueness of solution, we can as well conclude that $Q(\psi(t))$ is monotonically non-decreasing when time changes from 0 to τ. This proves that $Q(\psi(t)) = \mathrm{const}$ for $t \in [0,\tau]$.

To prove the conservation of $\mathscr{H}(\psi(t))$, we will need the relation

$$\lim_{\varepsilon\to 0} U(\langle\rho_\varepsilon,\psi_\varepsilon\rangle) = U(\psi(0,t)). \tag{5.3}$$

This relation follows from continuity of the potential U and from

$$\lim_{\varepsilon\to 0}\langle\rho_\varepsilon,\psi_\varepsilon(t)\rangle = \lim_{\varepsilon\to 0}\langle\rho_\varepsilon,(\psi_\varepsilon(t)-\psi(t))\rangle + \lim_{\varepsilon\to 0}\langle\rho_\varepsilon,\psi(t)\rangle = \psi(0,t), \tag{5.4}$$

where $\lim_{\varepsilon\to 0}\langle\rho_\varepsilon,(\psi_\varepsilon(t)-\psi(t))\rangle = 0$ since ψ_ε approaches ψ uniformly for $0 \le t \le \tau$ and $|x| \le R$ (including $x = 0$), while $\lim_{\varepsilon\to 0}\langle\rho_\varepsilon,\psi(t)\rangle = \psi(0,t)$ since ψ is continuous in x (due to the finiteness of H^1-norm of ψ that follows from (5.1)). We have:

$$\mathscr{H}(\psi(t)) = \frac{\|\nabla\psi(x,t)\|_{L^2}^2}{2} + U(\psi(0,t)) \le \lim_{\varepsilon\to 0}\left\{\frac{\|\nabla\psi_\varepsilon(x,t)\|_{L^2}^2}{2} + U(\langle\rho_\varepsilon,\psi_\varepsilon\rangle)\right\} = \mathscr{H}(\phi),$$

where we used the relation (5.3) and (4.1). We also used the conservation of the values of the functional $\mathscr{H}_\varepsilon$ for the approximate solutions ψ_ε (see Lemma 3.1). Proceeding just as with $Q(\psi(t))$ above, we conclude that $\mathscr{H}(\psi(t)) = \mathrm{const}$ for $0 \le t \le \tau$. □

Corollary 5.3 (Global well-posedness)**.** *There is a unique solution $\psi \in L^\infty(\mathbb{R}, H^1(\mathbb{R})) \cap C_b(\mathbb{R}\times\mathbb{R})$ to equation (1.1) with $\psi|_{t=0} = \phi$. The values of the functionals $\mathscr{H}$ and Q are conserved in time.*

Proof. The solution $\psi \in L^\infty([0,\tau], H^1)$ constructed in Lemma 5.1 exists for $0 \le t \le \tau$, where the time span τ defined in (2.10) depends only on U_2 from (2.5). Hence, the bound (1.11) at $t = \tau$ allows us to extend the solution ψ constructed in Lemma 5.1 to the time interval $[\tau, 2\tau]$. We proceed by induction. □

For the conclusion of Theorem 1.1, it remains to prove that $\psi \in C(\mathbb{R}, H^1(\mathbb{R}))$. This follows from the next two lemmas.

Lemma 5.4. $\psi \in C(\mathbb{R}, H^1_{weak}(\mathbb{R}))$.

Proof. Fix $f \in H^{-1}(\mathbb{R})$ and pick any $\delta > 0$. Since H^1 is dense in H^{-1}, there exists $g \in H^1(\mathbb{R})$ such that

$$\|f-g\|_{H^{-1}} < \frac{\delta}{4\Lambda(\phi)}, \tag{5.5}$$

where $\Lambda(\phi)$ given by (2.4) is the a priori bound on $\|\psi(t)\|_{H^1}$ proved in Lemma 2.1 on the grounds of the energy and the charge conservation for $\psi(t)$. Then

$$|\langle f,\psi(t)-\psi(t_0)\rangle| \le |\langle f-g,\psi(t)-\psi(t_0)\rangle| + |\langle g,\psi(t)-\psi(t_0)\rangle| \tag{5.6}$$

$$\le \|f-g\|_{H^{-1}}\big(\|\psi(t)\|_{H^1} + \|\psi(t_0)\|_{H^1}\big) + \|g\|_{H^1}\|\psi(t)-\psi(t_0)\|_{H^{-1}}. \tag{5.7}$$

By (5.5), the first term in the right-hand side of (5.7) is bounded by $\delta/2$. By Corollary 5.3, we have $\psi \in L^\infty(\mathbb{R}, H^1(\mathbb{R}))$, and equation (1.1) yields $\psi \in C(\mathbb{R}, H^{-1}(\mathbb{R}))$. Hence, the second term in the right-hand side of (5.7) becomes smaller than $\delta/2$ if t is sufficiently close to t_0. Since $\delta > 0$ was arbitrary, this proves that $\lim_{t\to t_0}\langle f,\psi(t)-\psi(t_0)\rangle = 0$. □

Proposition 5.5. $\psi \in C(\mathbb{R}, H^1(\mathbb{R}))$.

Proof. Let us fix $t_0 \in \mathbb{R}$ and compute

$$\lim_{t\to t_0} \|\psi(t)-\psi(t_0)\|_{H^1}^2 = \lim_{t\to t_0} \left(\|\psi(t)\|_{H^1}^2 - 2\langle\psi(t),\psi(t_0)\rangle_{H^1} + \|\psi(t_0)\|_{H^1}^2\right). \tag{5.8}$$

The relation

$$\|\psi(t)\|_{H^1}^2 = 2\big(Q(\psi(t)) + H(\psi(t))\big) - 2U(\psi(0,t)),$$

together with the conservation of the energy and charge and the continuity of $\psi(0,t)$ for $t \in \mathbb{R}$ (see Corollary 5.3), shows that

$$\lim_{t\to t_0} \|\psi(t)\|_{H^1}^2 = \|\psi(t_0)\|_{H^1}^2.$$

By Lemma 5.4, $\lim_{t\to t_0}\langle\psi(t),\psi(t_0)\rangle_{H^1} = \langle\psi(t_0),\psi(t_0)\rangle_{H^1}$. This shows that the right-hand side of (5.8) is equal to zero. □

Now Theorem 1.1 is proved.

6 Hölder regularity of solution

In this section, we prove Theorem 1.2.

Lemma 6.1. *If $\phi \in H^1$, then $\boldsymbol{W}_{(\cdot)}\phi(x) \in C^{(1/4)}[0,\tau]$, uniformly in $x \in \mathbb{R}$.*

Proof. Let $t, t' \in [0,\tau]$. We have by the Cauchy-Schwarz inequality:

$$|\boldsymbol{W}_{t'}\phi(x) - \boldsymbol{W}_t\phi(x)| \le C\left|\int e^{-ikx}\left(e^{i\frac{t'k^2}{2}} - e^{i\frac{tk^2}{2}}\right)\hat\phi(k)\,dk\right|$$

$$\le C\int \min(1,|t'-t|k^2)|\hat\phi(k)|\,dk \le C\left[\int_{\mathbb{R}} \frac{\min(1,|t'-t|k^2)^2}{1+k^2}\,dk\right]^{\frac12}\|\phi\|_{H^1}.$$

We bound the last integral as follows:

$$\int_{\mathbb{R}} \frac{\min(1,|t'-t|k^2)^2}{1+k^2}\,dk \le \int\limits_{|k|<|t'-t|^{-\frac12}} \frac{|t'-t|^2k^4}{1+k^2}\,dk + \int\limits_{|k|>|t'-t|^{-\frac12}} \frac{dk}{1+k^2} \le \mathrm{const}\,|t'-t|^{\frac12}.$$

□

Lemma 6.2 (Regularity of $\psi(0,t)$)**.** *The unique solution $\psi \in C_b(\mathbb{R}\times[0,\tau])$ to equation (1.1) with the initial data $\psi|_{t=0} = \phi$ constructed in Lemma 2.3 satisfies*

$$\psi(0,\cdot) \in C^{(1/4)}[0,\tau].$$

Proof. Due to Lemma 6.1, it suffices to consider the regularity of $\mathrm{Z}\psi(0,t)$. For any $t, t' \in [0,\tau]$, $t' < t$, we have:

$$\mathrm{Z}\psi(0,t') - \mathrm{Z}\psi(0,t) = \int_0^t \left[\frac{F(\psi(0,s))}{\sqrt{2\pi(t'-s)}} - \frac{F(\psi(0,s))}{\sqrt{2\pi(t-s)}}\right]ds + \int_t^{t'} \frac{F(\psi(0,s))}{\sqrt{2\pi(t'-s)}}\,ds. \tag{6.1}$$

The first integral in the right-hand side of (6.1) is bounded by

$$C_1\int_0^t \left|\frac{1}{\sqrt{t'-s}} - \frac{1}{\sqrt{t-s}}\right| ds \le C_2|t'-t|^{1/2}.$$

The second integral in the right-hand side of (6.1) is also bounded by $C|t'-t|^{1/2}$. □

Lemma 6.3. *$\psi(x,\cdot)\in C^{(1/4)}(\mathbb{R})$, uniformly in $x\in\mathbb{R}$.*

Proof. We have the relation

$$\psi(x,t)=\boldsymbol{W}_{t-t_0}\psi(x,t_0)+\int_0^{t-t_0}\frac{e^{i\frac{x^2}{2s}}}{\sqrt{2\pi s}}F(\psi(0,t-s))\,ds. \tag{6.2}$$

By Lemma 6.1, the first term in the right-hand side of (6.2), considered as a function of time, belongs to $C^{(1/4)}(\mathbb{R})$ (uniformly in $x\in\mathbb{R}$). The second term in the right-hand side of (6.2) is bounded by const $|t-t_0|^{1/2}$. This proves that $\psi(x,\cdot)\in C^{(1/4)}(\mathbb{R})$, uniformly in x. □

It remains to mention that the Hölder continuity in x follows from the inclusion $H^1(\mathbb{R})\subset C^{(1/4)}(\mathbb{R})$. Theorem 1.2 is proved.

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