On Asymptotic Completeness of Scattering in the Nonlinear Lamb System, II

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Abstract

We establish the asymptotic completeness in the nonlinear Lamb system for hyperbolic stationary states. For the proof we construct a trajectory of a reduced equation (which is a nonlinear nonautonomous ODE) converging to a hyperbolic stationary point using the Inverse Function Theorem in a Banach space. We give the counterexamples showing nonexistence of such trajectories for nonhyperbolic stationary points.

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1 Introduction

In this paper we consider the asymptotic completeness of scattering in the nonlinear Lamb system for the case of zero oscillator mass $m = 0$. This system describes the string coupled to the n-dimensional nonlinear oscillator with the force function $F(y)$, $y \in \mathbb{R}^n$,

$$(1.1) \qquad \left\{ \begin{array}{rcl} \ddot{u}(x,t) & = & u''(x,t), \quad x \in \mathbb{R} \setminus \{0\}, \\ 0 & = & F(y(t)) + u'(0+,\, t) - u'(0-,\, t); \quad y(t) := u(0,t), \end{array} \right.$$

where $\dot{u} := \dfrac{\partial u}{\partial t}$, $u' := \dfrac{\partial u}{\partial x}$. The solution $u(x,t)$ takes the values in $\mathbb{R}^n$ with $n \geq 1$.

The system (1.1) has been introduced originally by H. Lamb [8] in the linear case when $F(y) = -\omega^2 y$ and $n = 1$. The Lamb system with general nonlinear function $F(y)$ and the oscillator of mass $m \geq 0$ has been considered in [3] where the questions of irreversibility and nonrecurrence were discussed. The system was studied further in [4] where the global attraction to stationary states has been established for the first time, and in [2] where metastable regimes were studied for the stochastic Lamb system. The scattering asymptotics with a diverging free wave were established in [6].

In present paper we continue the study of the asymptotic completeness in the nonlinear scattering for the Lamb system. The case $n = 1$ was studied in [7] for hyperbolic stationary states under condition $F'(0) \neq 0$. We prove the asymptotic completeness for all $n > 1$ in the hyperbolic case.

The asymptotics completeness for nonlinear wave equations was considered in [11]) for small initial states. We prove the asymptotic completeness without the smallness assumption.

The paper is organized as follows. In Section 2 we introduce basic notations, and we recall some statements and constructions from [4, 5, 7, 9]. In Section 3 we reduce the asymptotic completeness to the existence of incoming trajectory of a reduced ODE. In Section 4 we prove the existence of the incoming trajectory for small perturbations. First, we prove this for linear F and then for nonlinear F using the Inversion Function Theorem. In Section 5 we extend the results of Section 4 to arbitrary perturbations without the smallness assumption. First, the solution is constructed for large t and then is continued back using a priory estimate.

2 Scattering asymptotics for the Lamb system

We consider the Cauchy problem for the system (1.1) with the initial conditions

$$(2.1) \qquad u|_{t=0} = u_0(x);\ \dot{u}|_{t=0} = v_0(x).$$

Denote by $\|\cdot\|_{L^2}$ the norm in the Hilbert space $L^2(\mathbb{R}, \mathbb{R}^n)$.

Definition 2.1. *The phase space $\mathcal{E}$ of finite energy states for the system (1.1) is the Hilbert space of the pars $(u(x), v(x)) \in H^1(\mathbb{R}, \mathbb{R}^n) \oplus L^2(\mathbb{R}, \mathbb{R}^n)$ with $u'(x) \in L^2(\mathbb{R}, \mathbb{R}^n)$ and the global energy norm*

$$\|(u, v)\|_{\mathcal{E}} = \|u'\| + |u(0)| + \|v\|.$$

The Cauchy problem (1.1), (2.1) can be written in the form

$$\dot{Y}(t) = \mathbf{F}(Y(t)), \quad t \in \mathbb{R}; \quad Y(0) = Y_0 \tag{2.2}$$

where $Y(t) := (u(\cdot,t), \dot{u}(\cdot,t))$, and $Y_0 = (u_0, v_0)$ is the initial date. In [5, 9], the scattering asymptotics have been proved,

$$Y(t) \sim S_\pm + W(t)\Psi_\pm, \qquad t \to \pm\infty \tag{2.3}$$

where $S_\pm$ are the limit stationary states, $W(t)$ is the dynamical group of the free wave equation, and $\Psi_\pm \in \mathcal{E}$ are the corresponding *asymptotic states.*

The asymptotics (2.3) hold in the norm of the Hilbert phase space $\mathcal{E}$ if the following limits exist:

$$u_0^+ := \lim_{x\to+\infty} u_0(x), \quad u_0^- := \lim_{x\to-\infty} u_0(x), \quad \overline{v}_0 := \int_{-\infty}^{\infty} v_0(y)dy. \tag{2.4}$$

We denote by $\mathcal{E}_\infty$ the subspace of $\mathcal{E}$ consisting of functions satisfying (2.4).

The stationary states $S(x) = (s(x), 0) \in \mathcal{E}$ for (2.2) are evidently determined with $s(x) \equiv s \in Z = \{z \in \mathbb{R}^n : \ F(z) = 0\}$. We denote by $\mathcal{S}$ the set of all stationary states of system (1.1). The following theorem is proved in [5, Theorem 4.5 ii) b)] and [9], Theorem 3.1. We assume that

$$F(u) = -\nabla V(u), \quad V(u) \in C^2(\mathbb{R}^n, \mathbb{R}), \quad \text{and} \quad V(u) \to +\infty, \quad |u| \to \infty. \tag{2.5}$$

Proposition 2.2. *Let the assumptions (2.5) and (2.4) hold, Z be a discrete subset in $\mathbb{R}^n$, and initial state $Y_0 \in \mathcal{E}_\infty$. Then*
i) For the corresponding solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to the Cauchy problem (2.2), the scattering asymptotics hold

$$Y(t) = S_+ + W(t)\Psi_+ + r_+(t), \qquad t \geq 0, \tag{2.6}$$

with some limit stationary state $S_+ \in \mathcal{S}$ and asymptotic state $\Psi_+ \in \mathcal{E}_\infty$. The remainder is small in the global energy norm:

$$\|r_+(t)\|_{\mathcal{E}} \to 0, \ \ t \to \infty.$$

We will call (S_+, Ψ_+) as the scattering data of the solution $Y(t)$. Our goal is to describe all *admissible pairs* $(S_+, \Psi_+) \in \mathcal{S} \times \mathcal{E}_\infty$ such that there exists $Y_0 \in \mathcal{E}_\infty$ satisfying (2.6).

Let us comment on previous results in this direction (see [7], Lemma 2.7, Lemma 5.1 and Theorem 6.1.)

A. Any asymptotic state $\Psi_+ = (\Psi_0, \Psi_1) \in \mathcal{E} \subset \mathcal{E}_\infty$ satisfies the identity

$$\Psi_0^+ + \Psi_0^- + \overline{\Psi}_1 = 0 \tag{2.7}$$

where $\Psi_0^+ = \lim\limits_{x\to+\infty} \Psi_0(x)$, $\Psi_0^- = \lim\limits_{x\to-\infty} \Psi_0(x)$, and $\overline{\Psi}_1 = \displaystyle\int_{-\infty}^{\infty} \Psi_1(y)dy$.

We denote by $\mathcal{E}_\infty^+$ the subspace of $\mathcal{E}_\infty$ consisting of functions satisfying (2.7).

B. A pair (S_+, Ψ_+) is admissible if $\Psi_+(x) \in \mathcal{E}_\infty^+$ satisfies the identity (2.7) and has a compact support.

C. For $n = 1$ any pair $(S_+, \Psi_+) \in \mathcal{S} \times \mathcal{E}_\infty^+$ with $S_+ = (s_+, 0)$ is admissible if $F'(s_+) \neq 0$.

The similar results hold when $t \to -\infty$. Then the asymptotics (2.6) take the form

$$Y(t) = S_- + W(t)\Psi_- + r_-(t), \qquad t \leq 0,$$

where for $\Psi_- = (\Psi_0, \Psi_1)$. The relation (2.7) is changed to the following

$$\Psi_0^+ + \Psi_0^- - \overline{\Psi}_1 = 0.$$

In this paper we generalize the result **C** for an arbitrary $n > 1$ for hyperbolic stationary states. Let $\sigma(A)$ denote the spectrum of an $n \times n$ -matrix A.

Definition 2.3. *The stationary state $S_+ = (s_+, 0)$ of system (1.1) is hyperbolic if $\mathrm{Re}\,\lambda \neq 0$ for all $\lambda \in \sigma(F'(s_+))$.*

We will prove that a pair (S_+, Ψ_+) is admissible for $n > 1$ in the case of hyperbolic stationary state S_+, and arbitrary $\Psi_+ \in \mathcal{E}_\infty^+$.

3 Reduced equation

Let $Y(t) \in C(\mathbb{R}, \mathcal{E})$ be a solution to (2.2) with $Y_0 \in \mathcal{E}_\infty$. Let us set

$$W_+ Y_0 = (S_+, \Psi_+) \in \mathcal{S} \times \mathcal{E}_\infty^+ \tag{3.1}$$

where Ψ_+ is defined by (2.6), and $S_+ = (s_+, 0)$.

Definition 3.1. *The Lamb system (1.1) is asymptotically complete at a stationary state S_+ if for any $\Psi_+ \in \mathcal{E}_\infty^+$ there exists initial data $Y_0 \in \mathcal{E}_\infty$ such that (3.1) holds.*

For $\Psi_+ = (\Psi_0, \Psi_1) \in \mathcal{E}_\infty^+$, let us set

$$S(t) := \frac{\Psi_0(t) + \Psi_0(-t)}{2} + \frac{1}{2}\int_{-t}^{t} \Psi_1(y)dy, \quad t \in \mathbb{R}. \tag{3.2}$$

Lemma 3.2. *([7], Lemma 3.1) Let $Y(t) \in C(\mathbb{R}, \mathcal{E})$ be a solution of (2.2) with $Y(0) = Y_0 \in \mathcal{E}_\infty$, and (3.1) holds for some S_+ and $\Psi_+ \in \mathcal{E}_\infty^+$. Then $\dot{S} \in L^2(\mathbb{R}, \mathbb{R}^n)$, and*

$$\dot{y}(t) = -\frac{1}{2}F(y(t)) + \dot{S}(t), \quad t > 0\ ; \quad \dot{y} \in L^2(\mathbb{R}_+, \mathbb{R}^n)\ ; \quad y(t) \to s_+,\ t \to +\infty \tag{3.3}$$

where $y(t) := u(0, t)$.

We call the differential equation (3.3) the *inverse reduced equation*. It plays the crucial role in the proof of the asymptotics completeness. The following lemma, proved in [7], reduces the problem of asymptotic completeness to the construction of solutions to (3.3).

Lemma 3.3. *(see [7], Lemma 4.1) Let $(S_+, \Psi_+) \in \mathcal{S} \times \mathcal{E}_\infty^+$, and there exists a solution $y(t)$ to (3.3). Then there exists $Y_0 \in \mathcal{E}_\infty$ such that (3.1) holds.*

Lemma 3.2 implies the existence of solution to (3.3) for any $\Psi_+ \in \mathcal{E}_\infty^+$ if the system (2.2) is asymptotically complete at S_+. Conversely, Lemma 3.3 implies the asymptotic completeness when (3.3) has a solution for any $\Psi_+ \in \mathcal{E}_\infty^+$. Thus, (3.3) gives a characterization of admissible asymptotic states.

4 Incoming trajectories

In this section we prove the existence of a solution to (3.3) for small $\|\dot{S}\|_{L^2}$ in the case of hyperbolic stationary state $S_+ = (s_+, 0)$. We adapt to our case the methods [1], [12] of construction of stable and unstable invariant manifolds in the hyperbolic case. Namely, first we prove the existence for the linear $F(y)$ and then for the nonlinear $F(y)$ with small perturbations. We will extend these results to arbitrary perturbations $\dot{S} \in L^2(\mathbb{R}, \mathbb{R}^n)$ in the next section.

4.1 Linear equation

Let A be a linear operator $\mathbb{R}^n \to \mathbb{R}^n$ and

$$\mathrm{Re}\,\lambda \neq 0, \quad \lambda \in \sigma(A).$$

Then

$$\mathrm{Sp}\ A = \sigma_- \cup \sigma_+,$$

where $\mathrm{Re}\,\lambda < 0$ for all $\lambda \in \sigma_-$ and $\mathrm{Re}\,\lambda > 0$ for all $\lambda \in \sigma_+$. Let $\varepsilon > 0$ be such that

$$|\mathrm{Re}\,\lambda| > \varepsilon, \quad \lambda \in \sigma(A). \tag{4.1}$$

Denote by $P_\pm$ the projectors of $\mathbb{R}^n$ to the subspaces generated by the eigenvectors corresponding to $\sigma_\pm$ respectively. Then the operator A is decomposed

$$A = A_+ + A_-, \quad A_\pm = AP_\pm.$$

Definition 4.1. *Define the Banach space*

$$\mathcal{Y} := L^2 \cap C_b^0$$

where

$$L^2 := L^2(\mathbb{R}, \mathbb{R}^n)\ , C_b^0 := \{y \in C_b(\mathbb{R}, \mathbb{R}^n) : \ y(t) \to 0, \quad t \to \infty\},$$

and for $y \in \mathcal{Y}$

$$\|y\| := \|y\|_{L^2} + \|y\|_{C_b}.$$

Consider

$$\dot{y}(t) = Ay(t) + f(t), \quad t \in \mathbb{R}. \tag{4.2}$$

Lemma 4.2. *There exists a continuous linear operator $R : L^2 \to \mathcal{Y}$ such that for any $f \in L^2$ and $y \in \mathcal{Y}$ equation (4.2) is equivalent to*

$$y = Rf.$$

Proof. Let us introduce a fundamental solution of system (4.2)

$$E(t) := \begin{cases} e^{A_- t}, & t > 0, \\ -e^{A_+ t}, & t < 0. \end{cases}$$

By (4.1) we have

$$|E(t)| \leq Ce^{-\varepsilon|t|}, \quad t \in \mathbb{R}. \tag{4.3}$$

Let us check that

$$y = Rf := E * f. \tag{4.4}$$

is a solution to (4.2), belongs to $\mathcal{Y}$ and tends to 0, as $t \to \infty$.

i) Obviously, y satisfies (4.2). Now let us prove that $y \in C_b$. By (4.3) we have:

$$|y(t)| \quad = |(E * f)(t)| \le C \int\limits_{-\infty}^{\infty} e^{-\varepsilon|t-s|}|f(s)|ds \le C\|f\|_{L^2} \tag{4.5}$$

by the Cauchy-Schwartz inequality.

ii) Let us check that $y \in L^2$. Denote $M(t) := |f(t)|$. Passing to the Fourier transform $M \to \tilde{M}$ and using (4.3) and (4.4) we obtain

$$\|y\|_{L^2} \le \|e^{-\varepsilon|t|} * M(t)\|_{L^2} = \frac{1}{2\pi}\|\frac{2\varepsilon}{\omega^2+\varepsilon^2}\tilde{M}(\omega)\|_{L^2} \le C\|\tilde{M}\|_{L^2} = 2\pi C\|f\|_{L^2}.$$

iii) Finally, let us prove that $y(t) \to 0$, as $t \to \infty$. By (4.5) it suffices to check that

$$\int\limits_{-\infty}^{t/2} e^{-\varepsilon|t-s|}|f(s)|ds \to 0, \quad \int\limits_{t/2}^{\infty} e^{-\varepsilon|t-s|}|f(s)|ds \to 0, \quad t \to \infty.$$

The second limit follows from the Cauchy-Schwartz inequality since

$$\|f\|_{L^2(t/2,\infty)} \to 0, \quad t \to \infty.$$

It remains to prove the first limit. The limit holds since

$$\int\limits_{-\infty}^{t/2} e^{-\varepsilon|t-s|}|f(s)|ds \le Ce^{-\varepsilon t/2}\|f\|_{L^2} \to 0, \quad t \to \infty.$$

■

4.2 Nonlinear equation: Inverse Function Theorem

Let us consider the nonlinear equation:

$$\dot{y} = Ay + N(y) + f(t), \qquad t > 0 \tag{4.6}$$

where $f \in L^2$, and

$$N \in C^2(\mathbb{R}^n, \mathbb{R}^n), \quad N(0) = 0 \text{ and } \nabla N(0) = 0. \tag{4.7}$$

The function N may be considered as the functional map:

$$(\mathcal{N}(y))(t) := N(y(t)), \quad t \in \mathbb{R}.$$

Lemma 4.3. *i) The map $\mathcal{N}:\mathcal{Y}\to\mathcal{Y}$ is continuous.*

ii) There exists the Frechet derivative $\mathcal{N}'(y)\in\mathcal{L}(\mathcal{Y},\mathcal{Y})$ for $y\in\mathcal{Y}$.

iii) Moreover,

$$N'(0)=0.$$

iv)

$$N'\in C(\mathcal{Y},\mathscr{L}(\mathcal{Y},\mathcal{Y})).$$

Proof. i) Conditions (4.7) imply that for any $\delta>0$

$$|N(y)|\le C_\delta|y|,\quad |y|<\delta. \tag{4.8}$$

Hence $\mathcal{N}(y)\in\mathcal{Y}$ for $y\in\mathcal{Y}$. The Lagrange formula implies that the map $\mathcal{N}$ is continuous from $\mathcal{Y}$ to $\mathcal{Y}$ by (4.7).

ii) By (4.7) we have

$$N(y)-N(y_0)=N'(y_0)(y-y_0)+r(y,y_0),\quad |r(y,y_0)|\le\alpha(|y-y_0|)|y-y_0|,\quad |y|,|y_0|\le\delta.$$

for any $\delta>0$, where $\alpha(s)$ is a monotone increasing function of $s\ge0$ and

$$\alpha(s)\to0,\quad s\to0.$$

Hence,

$$|r(y(t),y_0(t))|\le\alpha(|y(t)-y_0(t)|)(|y(t)-y_0(t)|).$$

Therefore,

$$\|r\|_{C_b}\le\alpha(\|y-y_0\|_{C_b})\|y-y_0\|_{C_b},\quad \|r\|_{L^2}\le\alpha(\|y-y_0\|_{C_b})\|y-y_0\|_{L^2}.$$

Now

$$\mathcal{N}(y)-\mathcal{N}(y_0)=\mathcal{N}'(y_0)(y-y_0))+r(y,y_0),\quad \|r\|\le\alpha(\|y-y_0\|)\|y-y_0\|,\quad \|y\|,\|y_0\|\le\delta.$$

iii) Let us check that $\mathcal{N}'=0$. By (4.7) it suffices to prove that

$$\frac{\|\mathcal{N}(y)\|}{\|(y)\|}\to0,\quad \|y\|\to0.$$

This follows from the estimate

$$|N(y)|\le C_\delta|y|^2,\quad |y|<\delta$$

which holds by (4.7).

iv) We should prove that the map

$$y\to\mathcal{N}'(y)$$

is continuous: $\mathcal{Y}\to L(\mathcal{Y},\mathcal{Y})$, i.e.

$$\|\mathcal{N}'(y_1)-\mathcal{N}'(y_2)\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})}\to0,\quad \|y_1-y_2\|\to0. \tag{4.9}$$

Indeed, (4.9) means that

$$\sup_{\|y\|\le 1} \|[\mathcal{N}'(y_1) - \mathcal{N}'(y_2)]y\| \to 0, \quad \|y_1 - y_2\| \to 0. \tag{4.10}$$

Denote by $\mathcal{N}'_{ij}(y) := \frac{\partial}{\partial y_i} N_j(y)$. Then (4.10) is equivalent to

$$\|\frac{\partial}{\partial y_i} N_j(y_1(t)) - \frac{\partial}{\partial y_i} N_j(y_2(t))\| \to 0, \quad \|y_1 - y_2\| \to 0.$$

Finally, this follows from Lemma 4.3 i) and (4.7). ■.

4.3 Incoming trajectory for small perturbations

By Lemma 1.1, equation (4.6) with $y \in \mathcal{Y}$ is equivalent to

$$\Phi(y) = Rf, \tag{4.11}$$

where

$$\Phi(y) := y - (RN)(y).$$

The map $RN : \mathcal{Y} \to \mathcal{Y}$ is continuous and admits the Fréchet differential $(RN)' \in C(\mathcal{Y}, \mathscr{L}(\mathcal{Y}, \mathcal{Y}))$ by Lemma 4.3, and

$$(RN)' = RN', \qquad (RN)'(0) = 0.$$

Therefore, the map Φ is continuous $\mathcal{Y} \to \mathcal{Y}$, $\Phi' \in C(\mathcal{Y}, \mathcal{L}(\mathcal{Y}, \mathcal{Y}))$, and $\Phi'(0) = I$, where I is the identity operator.

Theorem 4.4. *Let $f \in L^2$. There exist $\varepsilon > 0$, $C > 0$ such that equation (4.11) admits the unique solution $y \in \mathcal{Y}$ with $\|y\| < C$ for $\|f\|_{L^2} < \varepsilon$. This solution depends continuously on f.*

Proof. The map $\Phi : \mathcal{Y} \to \mathcal{Y}$ is continuously differentiable, $\Phi(0) = 0$ and $\Phi'(0) = I$. Hence, by the Inverse Function Theorem (Theorem 10.4, [10]) there exist $\varepsilon,\ C > 0$ such that for $\|Rf\| < \varepsilon$ there exists the unique $y \in \mathcal{Y}$ with $\|y\| < C$ satisfying (4.11) and depending continuously on Rf. It remains to note that R is continuous operator $L^2 \to \mathcal{Y}$ by Lemma 1.1. ■

5 Asymptotic completeness

In this section we prove asymptotic completeness for any hyperbolic stationary state. First, we construct the incoming trajectory for large t using Theorem 4.4, and afterwards we continue the trajectory backwards using a priory estimate.

Theorem 5.1. *Let conditions (2.5) hold and $F \in C^2$. Then system (1.1) is asymptotically complete at any hyperbolic stationary state.*

Proof. Let $S_+ = (s_+, 0)$ be a hyperbolic stationary state. We can consider $s_+ = 0$. According to Lemma 3.3, it suffices to prove that

i) for any $\Psi_+ = (\Psi_0, \Psi_1) \in \mathcal{E}^+_\infty$ there exists a trajectory $y(t)$ satisfying the differential equation (3.3) with S defined by (3.2) and $y \to 0$, as $t \to \infty$.

ii) $\dot{y} \in L^2$.

First, let us decompose the function F as

$$F(y) = Ay + N(y) \tag{5.1}$$

where $A := F'(0)$. Then A satisfies (4.9) since $(0,0)$ is the hyperbolic state and N satisfies (4.7), because $F \in C^2$. Then equation (3.3) can be written as

$$\dot{y} = -\frac{1}{2}A - \frac{1}{2}N + f(t) \tag{5.2}$$

where $f(t) := \frac{1}{2}\dot{S} \in L^2$ by Lemma 3.2. Now we are able to prove i) and ii).

i) Let $T > 0$ be such that $\|f\|_{L^2(T,\infty)} < \varepsilon$ for $t \geq T$, where ε is chosen as in Theorem 4.4. Consider

$$f_1(t) := \begin{cases} f(t), & t \geq T \\ 0, & t < T. \end{cases}$$

By Theorem 4.4 there exists $y_1(t) \in \mathcal{Y}$ satisfying equation (5.2). Then y_1 satisfies the inverse reduced equation (3.3) for $t \geq T$. It remains to construct a solution y_2 to equation (5.2) or, equivalently, to (3.3) for $0 \leq t \leq T$ with the "initial condition" $y_2(T) = y_1(T)$. It suffices to prove a priori estimate. Multiplying equation (3.3) for y_2 by $2\dot{y}_2(t)$ and using (2.5), we obtain that

$$(\nabla V)(y_2(t))\dot{y}_2(t) = 2|\dot{y}_2(t)|^2 - 2f(t)\dot{y}_2(t), \quad 0 < t < T.$$

Integrating and using the initial condition, we obtain

$$V(y_1(T)) - V(y_2(t)) = 2\int_t^T |\dot{y}_2(\tau)|^2 d\tau - 2\int_t^T f(\tau)\dot{y}_2(\tau)d\tau, \quad 0 \leq t \leq T.$$

Using the Young inequality, we estimate the second term in the right hand side as

$$2\left|\int_t^T \dot{S}(\tau)\dot{y}_2(\tau)d\tau\right| \leq \int_t^T |\dot{S}(\tau)|^2 d\tau + \int_t^T |\dot{y}_2(\tau)|^2 d\tau.$$

Hence,

$$V(y_2(t)) + \int_t^T |\dot{y}_2(\tau)|^2 d\tau \leq V(y_2(T)) + \int_t^T |f(\tau)|^2 d\tau \leq B < \infty, \quad t \in [0,T] \tag{5.3}$$

since $f \in L^2$. Therefore, $y_2(t)$ is bounded for $t \in [0,T]$ by (2.5). Finally, defining

$$y(t) := \begin{cases} y_1(t), & t \geq T, \\ y_2(t), & t \in [0,T], \end{cases} \tag{5.4}$$

we obtain that y satisfies (3.3). Moreover, $y(t) \to 0$ by Def. (4.1) and the fact that $y_1 \in \mathcal{Y}$.

ii) Let us prove that $\dot{y} \in L^2$. First, $\dot{y}_1 \in L^2$ since y_1 satisfies (3.3), $\dot{S} \in L^2$,

$$|F(y)| \leq C_\delta |y| \quad |y| < \delta$$

by (4.8) and (5.1). Second, the function $\dot{y}_2 \in L^2$ by (5.3). So $\dot{y} \in L^2$ by (5.4). ■

6 Counterexamples

In this section we give two examples which show that the incoming solution may not exist for nonhyperbolic stationary state. This means that the system is not asymptotically complete in this state.

Example 6.1. *Let us consider equation (3.3) with F satisfying (2.5), and*

$$F(y) = 0, \quad |y| < 1.$$

then $s_+ = 0$ is the nonhyperbolic stationary point. Let us choose

$$f(t) = \frac{1}{1+|t|} \in L^2. \tag{6.1}$$

Let us prove that in this case a trajectory satisfying condition

$$y(t) \to 0, \quad t \to \infty. \tag{6.2}$$

does not exist. In fact, let y satisfy (3.3) with $s_+ = 0$. Then there exists $T > 0$ such that $|y(t)| < 1/2$ for $t > T$. Hence, we have $\dot{y} = 1/(1+t)$ for $t > T$, and therefore

$$y(t) = \ln(1+t) + C, \quad t > T,$$

which contradicts (6.2). ■

Example 6.2. *Let us consider equation (3.3) with F, satisfying (2.5), and*

$$F(y) = y^2, \quad |y| < 1.$$

Then $s_+ = 0$ is the nonhyperbolic stationary point. Let us choose $f(t)$ from (6.1) and prove that the trajectory satisfying condition (6.2) does not exist. In fact, let y satisfy (3.3) with $s_+ = 0$. Then there exists $T > 0$ such that $|y(t)| < 1/2$ for $t > T$. Hence, $\dot{y} = y^2 + 1/(1+t)$ for $t > T$, and therefore

$$\dot{y}(t) \geq 1/(1+t), \quad t > T.$$

Thus,

$$y(t) \geq \ln(1+t) + C, \quad t > T$$

which contradicts (6.2).

■

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