Lectures on Global Attractors of Hamilton Nonlinear Wave Equations

A.I.Komech [1]
Department of Mechanics-Mathematics
Moscow State University
Moscow 119992, Russia

Preface

This is a survey on recent results on global attractors of the Hamilton nonlinear wave equations in an infinite space. We propose a unifying conjecture concerning the global attractors and long-time asymptotics of finite energy solutions of a class of nonlinear wave equations (Klein-Gordon, Maxwell, Schrödinger, Dirac, coupled systems, etc) with a Lie symmetry group:

For a generic equation with a Lie symmetry group each finite energy solution converges, in the long-time limit, to the sum of finite combination of solitary waves and a dispesive wave.

The conjecture was proved in the last decade for a list of equations: for 1D nonlinear wave and Klein-Gordon equations, for nonlinear systems of 3D wave, Klein-Gordon and Maxwell equations coupled to a classical particle, Maxwell-Landau-Lifschitz-Gilbert Equations. The equations correspond to the following four symmetry groups: i) trivial group $\{e\}$, ii) translation group $T = \mathbb{R}^d$, iii) rotation group $U(1)$, and iv) rotation group $SO(3)$.

We formulate our results which justify the conjecture for model equations with the four symmetry groups, and describe our numerical experiments for the Lorentz-invariant equations. We explain main ideas of proofs and discuss open problems. Main role in the proofs plays an analysis of energy radiation to infinity.

The investigation is inspired by mathematical problems of Quantum Mechanics: Bohr's Transitions to Quantum Stationary States and de-Broglie's Wave-Particle Duality. The conjecture is still open problem for coupled Maxwell-Dirac and Maxwell-Schrödinger Equations. The equations correspond to the translation, rotation and the Lorentz symmetry groups.

[1]Supported partly by the Faculty of Mathematics of the Vienna University, Max-Planck Institute for Mathematics in the Sciences (Leipzig), Project ONDES (INRIA, Rocquencourt), and Wolfgang Pauli Institute.

Contents

0 Introduction

0.1 Physical Motivations and Suggestions

Quantum Mechanics inspires the investigation of the global attractors of nonlinear hyperbolic equations. This is also suggested by the Heisenberg Program [26]. We keep in mind the following fundamental quantum phenomena:

I. Transitions between Quantum Stationary States or "quantum jumps" predicted by N.Bohr in 1913:

$$|E_-\rangle \;\mapsto\; |E_+\rangle \tag{0.1}$$

where $|E_\pm\rangle$ stands for Quantum Stationary State with the energy $E_\pm$.
II. Wave-Particle Duality predicted by L. de Broglie in 1922: diffraction of electrons discovered experimentally by C.Davisson and L.Germer in 1927, etc.
III. Gell-Mann – Ne'eman classification of the elementary particles, [19].

On the other hand, since 1925, basic quantum phenomena are described by hyperbolic partial differential equations, like the Schrödinger, Klein-Gordon, Dirac, Yang-Mills Eqns, etc, for a wave function $\psi(x,t)$, [5, 6, 43, 63]. In particular,

- Schrödinger has identified Quantum Stationary States with the wave functions $\psi(x)e^{i\omega t}$.
- Elementary Particles seem to correspond to the "solitary waves" $\psi(x-vt)e^{i\Phi(x,t)}$.

The identifications suggest the following mathematical conjectures:

I. The transitions (0.1) can be treated mathematically as the long-time asymptotics

$$\psi(x,t)\sim\psi_\pm(x)e^{i\omega_\pm t},\qquad t\to\pm\infty, \tag{0.2}$$

where the limit wave functions $\psi_\pm(x)e^{i\omega_\pm t}$ correspond to the stationary states $|E_\pm\rangle$.
II. The Wave-Particle Duality can be treated mathematically as the soliton-type asymptotics

$$\psi(x,t)\sim\sum_{k=1}^{N_\pm}\psi_\pm^k(x-v_\pm^k t)e^{i\Phi_\pm^k(x,t)},\qquad t\to\pm\infty. \tag{0.3}$$

The asymptotics (0.2) would mean that the set of all Quantum Stationary States is the *point global attractor* of the dynamical equations. The attraction might clarify Schrödinger's identification of the Quantum Stationary States with eigenfunctions. The asymptotics (0.3) claim an inherent mechanism of the "reduction of wave packets" in the Davisson-Germer experiment and clarify the description of the electron beam by plane waves. This description plays the key role in quantum mechanical scattering problems.

In particular, it is instructive to explain the distinguished role of the exponential function $e^{i\omega t}$ and travelling waves appearing in (0.2) and (0.3). We suggest that the role is provided by the symmetry of corresponding dynamical equations: the (global) gauge-invariance w.r.t. the group $U(1)$ for the asymptotics (0.2) and translation-invariance for (0.3). The suggestion is inspired by the fact that the function $e^{i\omega t}$ is a one-parametric subgroup of the corresponding symmetry group $U(1)$, and the traveling waves also correspond to one-parametric subgroup of translations.
III. A natural extension of the asymptotics (0.2) to the equations with a higher symmetry group G would look

$$\psi(\cdot,t)\sim e^{i\Omega_\pm t}\psi_\pm(\cdot),\qquad t\to\pm\infty, \tag{0.4}$$

where $i\Omega_\pm$ is (a representation of) an element of the Lie algebra $\mathbf{G}$ of the group G. For example, $\Omega_\pm$ is an Hermitian $N\times N$ matrix if $G=U(N)$. The matrix $\Omega_\pm$ and the functions $\psi_\pm$

give a solution to the corresponding *eigenmatrix stationary problem* (cf. (0.13) below). This correspondence is confirmed by the Gell-Mann – Ne'eman parallelism between the classification of the elementary particles and Lie algebras, [19], since the elementary particles appear to be the quantum stationary states.

We will justify the asymptotics of type (0.2) and (0.3 for model equations. Main role in the proofs plays an analysis of energy radiation to infinity.

0.2 Minimal Global Point Attractors

We discuss *minimal global point attractors* for Hamilton nonlinear wave equations in the entire space $\mathbb{R}^d$, $d \geq 1$. For example, consider nonlinear Klein-Gordon equations of type

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - m^2\psi(x,t) + f(x,\psi(x,t)), \qquad x \in \mathbb{R}^d. \tag{0.5}$$

We consider vector-valued solutions with the values $\psi \in \mathbb{R}^M$ or $\psi \in \mathbb{C}^N$ and identify $\mathbb{C}^N \equiv \mathbb{R}^{2N}$, $M, N \geq 1$. We can write the equation as the dynamical system

$$\dot{Y}(t) = \mathbf{F}(Y(t)), \quad t \in \mathbb{R}, \tag{0.6}$$

where $Y(t) = Y(x,t) := (\psi(x,t), \dot{\psi}(x,t))$. We will introduce a metric space $\mathcal{E}_F$, which is the space of finite energy states of the equation, and construct the corresponding dynamical group $W(t) : Y(0) \mapsto Y(t)$.

Definition 0.1. *(cf. [1, 25, 69]) A subset $\mathcal{A} \subset \mathcal{E}_F$ is a* **minimal global point attractor** *of the group $W(t)$ if*
i) For any $Y \in \mathcal{E}_F$ the convergence holds

$$W(t)Y \xrightarrow{\mathcal{E}_F} \mathcal{A}, \quad t \to \pm\infty. \tag{0.7}$$

ii) The subset is invariant, i.e. $W(t)\mathcal{A} = \mathcal{A}$, $t \in \mathbb{R}$.
iii) $\mathcal{A}$ is minimal set with the properties i) and ii).

By definition, (0.7) means that

$$\rho(W(t)Y, \mathcal{A}) := \inf_{X \in \mathcal{A}} \rho(W(t)Y, X) \to 0, \quad t \to \pm\infty, \tag{0.8}$$

where ρ stands for the metric in $\mathcal{E}_F$.

0.3 G-Invariant Equations

Let G be a *Lie group* and

$$g \mapsto g_* \tag{0.9}$$

a representation of G by the (linear or nonlinear) automorphisms of the phase space $\mathcal{E}_F$. Denote by $\mathbf{g} \mapsto \mathbf{g}_*$ the corresponding representation of the *Lie algebra* $\mathbf{G} = T_e G$. Namely, $\mathbf{g} \mapsto \mathbf{g}_*$ is the differential of the map (0.9) at $g = e$.

Definition 0.2. *(cf. [23]) i) The equation (0.6) (and (0.5)) is G-invariant with respect to the representation (0.9) if for any solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$, the trajectory $g_* Y(t)$ is also a solution.*
ii) Solitary Wave Solution of the equation (0.6) is any solution of the form (cf. (0.4))

$$Y(t) = e^{\mathbf{g}_* t} Y_{\mathbf{g}}, \tag{0.10}$$

where $Y_{\mathbf{g}} \in \mathcal{E}_F$ and $\mathbf{g}_*$ is the representation of an element $\mathbf{g}$ of the corresponding Lie algebra $\mathbf{G}$.

Remark 0.3. i) For *linear* representations (0.9), the equation (0.6) is G-invariant if *formally*

$$\mathbf{F}(g_*Y) = g_*\mathbf{F}(Y), \qquad g \in G. \tag{0.11}$$

ii) $e^{\mathbf{g}_*t}$ is the representation of the one-parametric subgroup $e^{\mathbf{g}t}$ of G. Hence, the solitary wave solution (0.10) can be written as

$$Y(t) = (e^{\mathbf{g}t})_* Y_{\mathbf{g}}, \tag{0.12}$$

iii) The amplitude $Y_{\mathbf{g}}$ of the solitary wave (0.10) satisfies formally the *stationary equation*

$$\mathbf{g}_* Y_{\mathbf{g}} = \mathbf{F}(Y_{\mathbf{g}}). \tag{0.13}$$

0.4 Examples of Symmetry Groups and Solitary Waves

A Every equation (0.5) is invariant w.r.t. trivial symmetry group $G = \{e\}$ with the identity representation $e_*\psi := \psi$. Then the Lie algebra $\mathbf{G} = \{0\}$ and the solitary waves are the *static stationary solutions* $Y(x,t) \equiv S(x)$. The stationary equation (0.13) becomes

$$0 = \mathbf{F}(S). \tag{0.14}$$

B The symmetry group $G = T := \mathbb{R}^d$ with the translation representation

$$a_*\psi(x) := \psi(x-a), \quad a \in \mathbb{R}^d.$$

It corresponds to translation-invariant equations (0.5) with $f(x,\psi) \equiv f(\psi)$. Then the Lie algebra $\mathbf{G} = \mathbb{R}^d$ and the solitary waves are *solitons (or traveling wave) solutions* $Y(x,t) \equiv Y_v(x - vt)$. The stationary equation (0.13) becomes

$$-v \cdot \nabla Y_v = \mathbf{F}(Y_v). \tag{0.15}$$

C The rotation symmetry group $G = U(1) := \{z \in \mathbb{C} : |z| = 1\}$ with the representation

$$z_*\psi(x) := z\psi(x), \quad |z| = 1.$$

It corresponds to 'phase-invariant' equations (0.5) with $f(x,\psi) \equiv a(x,|\psi|)\psi$. Then the Lie algebra $\mathbf{G} = \mathbb{R}$ and the solitary waves have the form $Y(x,t) \equiv e^{i\omega t}Y_\omega(x)$ like the Schrödinger *Quantum Stationary States*. The stationary equation (0.13) becomes the *nonlinear eigenvalue problem*

$$i\omega Y_\omega = \mathbf{F}(Y_\omega). \tag{0.16}$$

Next example with the symmetry group $U(1)$: the coupled Maxwell-Dirac Eqns

$$\left\{\begin{array}{l} \displaystyle\sum_{\alpha=0}^{3} \gamma^\alpha (i\hbar\partial_\alpha - \frac{e}{c}[A_\alpha(x) + A_\alpha^{ext}(\mathbf{x})])\psi(x) = mc\psi(x) \\ \\ \Box A_\alpha(x) = e\psi(x)\cdot\gamma^\alpha\gamma^0\overline{\psi(x)}, \ \alpha = 0,...,3 \end{array}\right| \quad x \in \mathbb{R}^4 \tag{0.17}$$

with an external static Maxwell field $A_\alpha^{ext}(\mathbf{x})$, $\mathbf{x} := (x_1, x_2, x_3)$. The Eqns are $U(1)$-invariant with respect to the representation (*global gauge group*)

$$z_*(A(\mathbf{x}), \psi(\mathbf{x})) = (A(\mathbf{x}), z\psi(\mathbf{x})), \quad |z| = 1. \tag{0.18}$$

The corresponding solitary waves read

$$(A(\mathbf{x}), e^{i\omega t}\psi(\mathbf{x})). \tag{0.19}$$

D The product symmetry group $G = T \times U(1)$ with the product representation

$$(a, z)_* \psi(x) := z\psi(x - a), \quad a \in \mathbb{R}^d, \quad |z| = 1.$$

It corresponds to equations (0.5) with $f(x, \psi) \equiv a(|\psi|)\psi$. Then the Lie algebra $\mathbf{G} = \mathbb{R}^d \times \mathbb{R}$ and the solitary waves have the form $Y(x, t) \equiv e^{i\omega t} Y_{v,\omega}(x - vt)$. The stationary equation (0.13) becomes

$$(-v \cdot \nabla + i\omega) Y_{v,\omega} = \mathbf{F}(Y_{v,\omega}). \tag{0.20}$$

Another example with the symmetry group $T \times U(1)$: the coupled Maxwell-Dirac Eqns

$$\left\{ \begin{array}{l} \displaystyle\sum_{\alpha=0}^{3} \gamma^\alpha (i\hbar\partial_\alpha - \frac{e}{c} A_\alpha(x))\psi(x) = mc\psi(x) \\ \\ \Box A_\alpha(x) = e\psi(x) \cdot \gamma^\alpha \gamma^0 \overline{\psi(x)}, \;\; \alpha = 0, ..., 3 \end{array} \right| \quad x \in \mathbb{R}^4 \tag{0.21}$$

without an external Maxwell field. The Eqns are $U(1)$-invariant with respect to the representation

$$(\mathbf{a}, z)_* (A(\mathbf{x}), \psi(\mathbf{x})) = (A(\mathbf{x} - \mathbf{a}), z\psi(\mathbf{x} - \mathbf{a})), \quad \mathbf{a} \in \mathbb{R}^3, \quad |z| = 1. \tag{0.22}$$

The corresponding Solitary Waves read

$$(A(\mathbf{x} - \mathbf{v}t), e^{i\omega t} \psi(\mathbf{x} - \mathbf{v}t)). \tag{0.23}$$

E The rotation symmetry group $G = SO(3)$ with the representation

$$R_* \psi(x) := R\psi(R^{-1}x), \quad R \in SO(3).$$

It corresponds to 'rotation-invariant' equations (0.5) with $f(x, \psi) \equiv a(|x|, |\psi|)\psi$ where $\psi \in \mathbb{R}^3$. Then the Lie algebra $\mathbf{G}$ is the Lie algebra of antisymmetric 3×3-matrices Ω, $\Omega' = -\Omega$, and the solitary waves have the form $Y(x, t) \equiv e^{\Omega t} Y_\Omega(e^{-\Omega t} x)$. The stationary equation (0.13) becomes the *nonlinear eigenmatrix problem*

$$\Omega Y_\Omega - \nabla Y_\Omega \cdot \Omega x = \mathbf{F}(Y_\Omega). \tag{0.24}$$

Remarks 0.4. *i) The existence of the solitary waves (0.20) is proved by Beresticky and Lions, [3], for a wide class of Eqns (0.5) with* $f(x, \psi) \equiv a(|\psi|)\psi$.
ii) The existence of the solitary waves (0.23) for the Eqns (0.21) is proved by Esteban, Georgiev and Séré, [14].

0.5 On the Structure of Minimal Global Point Attractor

We will discuss the following general conjecture **G** concerning the structure of the attractors of a G-invariant equation with a fixed symmetry Lie group G:

$\mathcal{G}$ *For a generic G-invariant equation, the minimal global point attractor is the set*

$$\mathcal{A} = \{Y_{\mathbf{g}} \in \mathcal{E}_F : \mathbf{g} \in \mathbf{G} \;\text{ and }\; e^{\mathbf{g}_* t} Y \;\text{ is the Solitary Wave Solution }\}. \tag{0.25}$$

Here the expression *for generic G-invariant equation* means *for almost all G-invariant equations (0.6)*, i.e. *for almost all dynamical systems (0.6) (or the functions $f(x, \psi)$) satisfying the identity (0.11).*

We justify this general conjecture for a list of model equations with the Lie groups and their representations from previous section. Then the general conjecture $\mathcal{G}$ reads as follows

A For the trivial symmetry group $\{e\}$: the minimal global point attractor *generically* is the set of all static stationary solutions, i.e.

$$\mathcal{A} = \{Y(\cdot) \in \mathcal{E}_F : Y(x) \text{ is a static solution}\}. \tag{0.26}$$

B For the translation symmetry group $T = \mathbb{R}^d$: the minimal global point attractor *generically* is the set of all soliton solutions, i.e.

$$\mathcal{A} = \{Y_v(\cdot) \in \mathcal{E}_F : v \in \mathbb{R}^d \text{ and } Y_v(x - vt) \text{ is a soliton solution}\}. \tag{0.27}$$

C For the rotation symmetry group $U(1)$: the minimal global point attractor *generically* is the set

$$\mathcal{A} = \{Y_\omega(\cdot) \in \mathcal{E}_F : \omega \in \mathbb{R} \text{ and } e^{i\omega t} Y_\omega(x) \text{ is a solution}\}. \tag{0.28}$$

D For the product symmetry group $T \times U(1)$: the minimal global point attractor *generically* is the set

$$\mathcal{A} = \{Y_{v,\omega}(\cdot) \in \mathcal{E}_F : v \in \mathbb{R}^d, \omega \in \mathbb{R} \text{ and } e^{i\omega t} Y_{v,\omega}(x - vt) \text{ is a solution}\}. \tag{0.29}$$

E For rotation symmetry group $SO(3)$: the minimal global point attractor *generically* is the set

$$\mathcal{A} = \{Y_\Omega(\cdot) \in \mathcal{E}_F : \Omega' = -\Omega, \text{ and } e^{\Omega t} Y_\Omega(e^{-\Omega t} x) \text{ is a solution}\}. \tag{0.30}$$

Remark 0.5. It is instructive to stress that the word *generically* means *for generic equations with the corresponding fixed symmetry group.* For example: the trivial group $\{e\}$ is a subgroup of $U(1)$, hence each $U(1)$-invariant equation is also $\{e\}$-invariant. Therefore, the set of all $U(1)$-invariant equations is a subset of all ($\{e\}$-invariant) equations. This would contradict the different forms of the global attractors (0.28) and (0.26) if one omits the word *generic*. However, the $U(1)$-invariant equations constitute an *exceptional class* among all ($\{e\}$-invariant) equations. Therefore, one could expect much more sophisticate long-time behavior of the solutions to $U(1)$-invariant equations, hence different form of the global attractor.

Remark 0.6. *For the coupled Maxwell-Dirac Eqns (0.17), the convergence to the global attractor (0.28) means that, roughly speaking,*

$$(A(\mathbf{x}, t), \psi(\mathbf{x}, t)) \sim (A_\pm(\mathbf{x}), e^{i\omega_\pm t} \psi_\pm(\mathbf{x})), \qquad t \to \pm\infty, \tag{0.31}$$

that would clarify the Schrödinger identification of the Quantum Stationary States with the "eigenfunctions" $e^{i\omega t}\psi(x)$.

0.6 Asymptotics in Local and Global Norms

We suggest that the convergence to an attractor, (0.7), holds in *local energy seminorms* that defines the corresponding metric in (0.8). The attraction in a *global energy norm* generally is impossible because of energy conservation.

We also suggest long-time "scattering asymptotics" in the *global energy norm.*

A For the trivial symmetry group the suggested scattering asymptotics read

$$Y(x, t) \approx Y_\pm(x) + W_0(t)\Psi_\pm, \qquad t \to \pm\infty, \tag{0.32}$$

where $W_0(t)$ is the dynamical group of the *free* Klein-Gordon equation (0.5) with $f(x, \psi) \equiv 0$, and $\Psi_\pm$ are the *asymptotic scattering states.*

B For the translation-invariant equations (0.5) the scattering asymptotics read like (0.3):

$$Y(x, t) \approx \sum_{k=1}^{N_\pm} Y_\pm^k(x - v_\pm^k t) + W_0(t)\Psi_\pm, \qquad t \to \pm\infty, \tag{0.33}$$

C For the $U(1)$-invariant equations (0.5) the scattering asymptotics read

$$Y(x,t) \approx e^{i\omega_\pm t} Y_\pm(x) + W_0(t)\Psi_\pm, \quad t \to \pm\infty. \tag{0.34}$$

D Corresponding extension to the case **D** reads

$$Y(x,t) \approx \sum_{k=1}^{N_\pm} e^{i\omega_\pm^k t} Y_\pm^k(x - v_\pm^k t) + W_0(t)\Psi_\pm, \quad t \to \pm\infty. \tag{0.35}$$

E For the $SO(3)$-invariant equations (0.5) the scattering asymptotics read

$$Y(x,t) \approx e^{\Omega_\pm t} Y_\pm(e^{-\Omega_\pm t} x) + W_0(t)\Psi_\pm, \quad t \to \pm\infty. \tag{0.36}$$

0.7 Known Results

We will refer the attraction to the set (0.26) as to "attraction of type **A**", etc.

Attraction to Static Stationary States

The attraction to the static stationary states has been established initially in the theory of attractors of dissipative systems: Navier-Stokes, diffusion-reaction, damped wave equation, etc, by Babin and Vishik, Foias, Hale, Henry, Temam and others, [1, 16, 25, 27, 69]. The attraction holds then in the global energy norm, however only for $t \to +\infty$.

For the Hamilton equations, first results on the attraction of type **A** and the scattering asymptotics (0.32) with $Y_\pm = 0$ have been obtained in linear scattering theory by Morawetz, Lax and Phillips, Vainberg and others, [21, 52, 55, 60, 71]. The results were extended to nonlinear scattering theory by Segal, Strauss, Morawetz, Glassey, Klainerman, Hörmander and others, [10, 20, 22, 29, 34, 56, 59, 60, 64, 67, 68]. All the results concern the case of the global attractor which consists of one point which is zero solution, i.e. $\mathcal{A} = \{0\}$. The attraction to the zero solution in the local energy seminorms, is equivalent to the **local energy decay**, and (0.32) reduces to the dispersive wave: $Y(x,t) \approx W_0(t)\Psi_\pm$.

The attraction of type **A** to a nontrivial global attractor $\mathcal{A} \neq \{0\}$ has been established
i) by Komech, [36]-[38], for the 1D equations (0.5) with $m = 0$ and different classes of space-localized nonlinear terms, i.e. $f(x,\psi) = 0$, $|x| > a$, (see Sections 2 and 3 of Chapter I), and
ii) by Komech, Spohn and Kunze, [48], for the nonlinear system of 3D wave equation coupled to a classical particle (see Section 4 of Chapter 1). The corresponding global point attractors can contain an arbitrary finite or infinite number of isolated points, as well as continuous finite-dimensional components. The system is an analog of the coupled Maxwell-Lorentz equations of Classical Electrodynamics with the Abraham model of the *extended electron* (Eqns (5.34) of Chapter 1). For the coupled Maxwell-Lorentz equations the results have been proved by Komech and Spohn, [47].

Komech and Vainberg have established the Liapunov-type criterion for the asymptotic stability of stationary states of general nonlinear Klein-Gordon equations, [49].

Attraction to Solitary Waves

The attraction of type **B** and soliton-type asymptotics of type (0.33) have been discovered initially for the *integrable equations*: KdV, sine-Gordon, cubic Schrödinger, etc (see [57] for the survey of the results). The results have been extended by Imaikin, Komech, Kunze and Spohn, [33, 44, 46], to the (nonintegrable) 3D translation-invariant nonlinear system studied in [48] (see Chapter 2). The generalization of the results to the Maxwell-Lorentz resp. Klein-Gordon equations is done in by Imaikin, Komech, Spohn, [30], resp. Imaikin, Komech, Markowich,

[32]. Bensoussan, Iliine and Komech have extended the asymptotics to the relativistic-invariant nonlinear 1D equations (0.5) with $f(x,\psi)=\sum_k F_k\delta(\psi-\psi_k)$ and $m=0$, [2].

Numerous numerical experiments suggest that the asymptotics (0.33) and (0.35) hold for "any" 1D relativistic-invariant equations (0.5) with a positive Hamiltonian (see Chapter 4). However, the proof is still an open problem. Numerical experiments were made by Collino, Fouquet, Rhaouti and Vacus (Project ONDES, INRIA), by Radvogin (M.Keldysh Institute of Applied Mathematics, RAS) and Vinnichenko, [45].

The first results on the attraction of type **C** have been established by Soffer and Weinstein, [65, 66], for $U(1)$-invariant 3D nonlinear Schrödinger equation (see also [58]).

Buslaev, Perelman and Sulem have established the attraction of type **D** and the asymptotics (0.35) with $N_\pm=1$ for translation-invariant $U(1)$-invariant 1D nonlinear Schrödinger equations, [7, 8, 9]. The results [8] are extended by Cuccagna to the dimension $n\geq 3$, [12].

All the results [7, 8, 9, 12, 58, 65, 66] concern initial states which are *sufficiently close to the attractor*. The *global* attraction **C** for *all finite energy states* is established for the first time by Komech, [42], for the nonlinear $U(1)$-invariant 1D equations (0.5) with $m>0$ and $f(x,\psi)=\delta(x)F(\psi)$ (see Chapter 3).

The first result on the attraction of type **E** and the scattering asymptotics (0.36) have been established by Imaikin, Komech and Spohn, [31], for the coupled Maxwell-Lorentz equations with a spinning charge.

Adiabatic Effective Dynamics

An *adiabatic effective dynamics* is established by Komech, Kunze and Spohn for the solitons of 3D wave equation or Maxwell field coupled to a classical particle, in a *slowly varying external potential* , [44, 50]. The effective dynamics explains the increment of the mass of the particle caused by its interaction with the field. In [31], the effective dynamics is extended to the Maxwell field coupled to the spinning charge. The effective dynamics is extended by Fröhlich, Gustafson, Jonsson, Sigal, Tsai, and Yau, to the solitons of a nonlinear Schrödinger and Hartree equations, [17, 18]. An extension to relativistic-invariant equations is still an open problem. On the other hand, the Einstein mass-energy identity are proved by Dudnikova, Komech and Spohn, [13], for general relativistic-invariant nonlinear Klein-Gordon equations (0.5).

0.8 The Attraction by Energy Radiation to Infinity

For dissipative systems (Navier-Stokes, diffusion-reaction, damped wave equation, etc) the attraction is provided by energy dissipation. For the Hamilton equations, the energy dissipation is absent, and the attraction is provided by radiation of energy to infinity. The radiation plays the role of a dissipation.

A general strategy of the proof is to deduce that all omega-limiting points of a trajectory belong to an attractor. The deduction relies on the finiteness of the total amount of the radiated energy. For 1D equations, the deduction of the asymptotics **A** uses either integral representation of the solutions (as in [36, 37]) or more advanced technique of a "nonlinear Goursat problem" [38]. For 3D equations the deduction of the asymptotics **A** and **B** in [48, 46, 47] relies on the Wiener Tauberian theorem.

The proof of the asymptotics **C** in [42] relies on the spectral analysis of energy radiation to infinity. The radiation is provided by two different mechanisms: dispersive and nonlinear. The dispersive mechanism is responsible for the radiation of the harmonics with the frequencies $|\omega|>m$ corresponding to the continuous spectrum of the lineare part of the generator (the RHS of (0.5) with $d=1$). The nonlinear mechanism is responsible for the energy flow from the 'spectral gap' $|\omega|\leq m$ to the continuous spectrum $|\omega|>m$. The flow is provided by the polynomial character of the nonlinear term which multiply the frequencies. The multiplication follows from Titchmarsh Convolution Theorem [70] (see [28, Thm 4.3.3] for highly simplified proof and multidimensional generalization). We sketch the application of the Titchmarsh theorem in Chapter III.

The proof of the asymptotics (0.33), in [30]-[33] and [44], rely on an integral inequality perturbative technique developed in [58, 65, 66].

The proofs of the results [7, 8, 9, 12, 58, 65, 66] are based on new techniques which includes the linearization of the dynamical equation near the solitary manifold. Main difficulty of the analisys is provided by two facts:
i) The linearized equation is non-autonomous.
ii) The 'friezed' linearized equation is unstable.
The instability is provided by the discrete spectrum which is always nonempty: this is the automatic consequence of the invariance of the equation. Namely, the tangent vectors to the solitary manifold correspond to the zero point of the discrete spectrum. The authors develop new method of the 'simplectic projection' which kills the tangent vectors. Then the 'orthogonal' component decays since it corresponds to the continuous spectrum of the linearized equation. The presence of the nonzero eigenvalues is first handled in [8, 9]. Then the decay of the orthogonal component in the linear approximation fails. In this case the decay of the orthogonal component is provided by the nonlinear radiative mechanism, like [42]. Namely, the nonlinearity translates the discrete eigenmodes to the continuous spectrum which results in the energy radiation. The translation is provided by the condition of the 'Fermi Golden Rule' type.

0.9 Open Problems

• The proving of the scattering asymptotics (0.32) - (0.36) for the nonlinear Klein-Gordon equation (0.5) with $n > 1$ or with $n = 1$ and $f(x, \psi) \equiv f(\psi)$ are still open problems.
• The scattering asymptotics (0.32) and (0.34) are not proved yet for any nontrivial example with the nonzero limit states $Y_\pm(x)$.
• The extension of adiabatic effective dynamics [17, 18, 44, 50] to the solitons of *relativistic* nonlinear Klein-Gordon equations.
• The extension of the results on global attractors, scattering asymptotics and adiabatic effective dynamics to the nonlinear Dirac and Yang-Mills equations.
• The asymptotics (0.34) resp. (0.35) are not proved yet for the coupled nonlinear Maxwell-Dirac Equations (0.17) resp. (0.21). The same questions are open for the Maxwell-Schrödinger equations [24].

0.10 Plan of the Exposition

In Chapter 1 we consider the attraction of type **A** to static stationary states. We start in Section 1 with an analysis of well known examples of linear wave equations in the dimensions $d = 1$ and $d = 3$. In Section 2 we formulate our result for 1D equations (0.5) with $m = 0$ and $f(x, \psi) = 0$, $|x| > a$, [38]. In Section 3 we give the detailed proof for the case $f(x, \psi) = \delta(x)F(\psi)$ considered in [36]. Finally, in Section 4, we sketch the proof of our result [48] on attraction of type **A** for 3D nonlinear system of wave equation coupled to a classical particle in an external confining potential.

In Chapter 2 we sketch the proof of our result [30] on the scattering asymptotics (0.33) for the 3D nonlinear system of wave equation coupled to a classical particle without an external potential. In that case the system is translation invariant and admits the soliton solutions moving with an arbitrary speed $v \in \mathbb{R}^3$, $|v| < 1$.

In Chapter 3 we sketch the proof of our result [42] on the attraction of type **C** for 1D equations (0.5) with $f(x, \psi) = \delta(x)F(\psi)$ and $m > 0$.

In Chapter 4 we describe our numerical observations of the scattering asymptotics (0.33) and (0.35) for 1D nonlinear Lorentz-invariant equations, [45]. We also describe the adiabatic effective dynamics of the solitons in slowly varying potentials.

Acknowledgments I thank H.Brezis, I.M.Gelfand, P.Joly, J.Lebowitz, E.Lieb, I.M.Sigal, A.Shnirelman, M.Shubin, H.Spohn, M.I.Vishik and E.Zeidler for fruitful discussions. I thank very much A.Vinnichenko who made the numerical simualtion of the soliton asymptotics for Chapter 4. I am indebted to Mechanical-Mathematical Department of Moscow State University for the support of my research. Finally, I thank the Faculty of Mathematics of Vienna University, Max-Planck Institute for Mathematics in the Sciences (Leipzig), Project ONDES (INRIA, Rocquencourt), and Wolfgang Pauli Institute for their hospitality.

A.Komech

Moscow, 12.10.2005

Chapter 1

Trivial Symmetry Group: Attraction to Stationary States

1 Example I: Linear d'Alembert Equation

Let us consider the one-dimensional linear wave equation

$$\ddot{u}(x,t) = u''(x,t), \quad (x,t) \in \mathbb{R}^2. \tag{1.1}$$

For concreteness, we look for real solutions in the sense of distributions. It is well known that the general solution is the sum

$$u(x,t) = f(x-t) + g(x+t), \tag{1.2}$$

where f and g are some distributions of one variable. We will consider the Cauchy problem with initial conditions

$$u|_{t=0} = u_0(x);\ \dot{u}|_{t=0} = v_0(x), \qquad x \in \mathbb{R}. \tag{1.3}$$

Then the solution is unique and given by the d'Alembert formula

$$u(x,t) = \frac{u_0(x+t) + u_0(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)dy. \tag{1.4}$$

Hence, the derivative in time is

$$\dot{u}(x,t) = \frac{u_0'(x+t) - u_0'(x-t)}{2} + \frac{v_0(x+t) + v_0(x-t)}{2}, \tag{1.5}$$

1.1 Energy Conservation

Let us assume that $u_0', v_0 \in L^2 := L^2(\mathbb{R})$. Then the energy is conserved,

$$E(t) := \frac{1}{2}\int_{\mathbb{R}} [|\dot{u}(x,t)|^2 + |u'(x,t)|^2]dx = \text{const}, \qquad t \in \mathbb{R}. \tag{1.6}$$

Formal proof Differentiating and substituting $\ddot{u} = u''$, we get

$$\dot{E}(t) = \int_{-\infty}^{+\infty} [\dot{u}\ddot{u} + u'\dot{u}']dx = \int_{-\infty}^{+\infty} [\dot{u}u'' + u'\dot{u}']dx = [\dot{u}u']|_{-\infty}^{+\infty} = 0 \tag{1.7}$$

by partial integration. ■

1.2 Existence of Dynamics

The Cauchy problem (1.1), (1.3) can be written as

$$\dot{Y}(t) = AY(t), \quad t \in \mathbb{R}; \quad Y(0) = Y_0, \tag{1.8}$$

where $Y(t) := (u(\cdot,t), \dot{u}(\cdot,t))$, $Y_0 = (u_0, v_0)$ and

$$A := \begin{pmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{pmatrix}.$$

Denote by $\|\cdot\|$ resp. $\|\cdot\|_R$ the norm in the Hilbert space $L^2 := L^2(\mathbb{R})$ resp. $L^2(-R,R)$. Let us introduce the phase space for the dynamical system (1.8).

Definition 1.1. *i)* $\mathcal{E}$ *is the Hilbert space* $\{Y(x) = (u(x), v(x)) : u \in C(\mathbb{R}), \quad u', v \in L^2\}$ *with the* **global energy norm**

$$\|Y\|_{\mathcal{E}} = \|u'\| + |u(0)| + \|v\|. \tag{1.9}$$

where u' *means the derivative in the sense of distributions.*
ii) $\mathcal{E}_F$ *is the linear space* $\mathcal{E}$ *endowed with the topology defined by the* **local energy seminorms**

$$\|Y\|_{\mathcal{E},R} = \|u'\|_R + |u(0)| + \|v\|_R. \tag{1.10}$$

iii) $Y_n \xrightarrow{\mathcal{E}_F} Y$ *iff* $\|Y_n - Y\|_{\mathcal{E},R} \to 0$, $\forall R > 0$.

This convergence is equivalent to the convergence w.r.t. the metric

$$\rho(X,Y) = \sum_{R=1}^{\infty} 2^{-R} \frac{\|X-Y\|_{\mathcal{E},R}}{1 + \|X-Y\|_{\mathcal{E},R}}, \qquad X, Y \in \mathcal{E}.$$

The next proposition is trivial:

Proposition 1.2. *i) For all* $Y_0 \in \mathcal{E}$ *there exists a unique solution* $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *to the Cauchy problem (1.8).*
ii) The map $W_0(t) : Y(0) \mapsto Y(t)$ *is continuous in* $\mathcal{E}$ *and* $\mathcal{E}_F$.
iii) The energy is conserved, (1.6).
iv) The d'Alembert representation (1.2) holds, where $f', g' \in L^2$.

Proof All the statements follow directly from (1.4) and (1.5). ■

1.3 Stationary States

Let us determine all finite energy stationary states for (1.1), i.e. the set $\mathcal{S} := \{Y \in \mathcal{E} : AY = 0\}$.

Exercise 1.3. *Check that* $\mathcal{S} = \{(c, 0) : c \in \mathbb{R}\}$.

1.4 Attractor of Finite Energy States

First let us consider the attractor of all finite energy states.

Proposition 1.4. *For every initial state* $Y_0 = (u_0, v_0) \in \mathcal{E}$ *the corresponding solution* $Y(t) := W_0(t)Y_0$ *converges to* $\mathcal{S}$ *in the* **local energy seminorms**:

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}, \qquad t \to \pm\infty. \tag{1.11}$$

Proof We have to show that $\rho(Y(t),\mathcal{S}) := \inf_{S\in\mathcal{S}} \rho(Y(t),S) \to 0$. Equivalently, there exists a trajectory $S(t) \in \mathcal{S}$ such that $\|Y(t) - S(t)\|_{\mathcal{E},R} \to 0$, $t \to \pm\infty$ for any $R > 0$. Let us set $S(t) := (u(0,t), 0)$. Then, according to the definition (1.10),

$$\|Y(t)-S(t)\|_{\mathcal{E},R} = \|(u(x,t)-u(0,t))'\|_R + \|\dot u(x,t)-0\|_R = \|u'(\cdot,t)\|_R + \|\dot u(\cdot,t)\|_R. \tag{1.12}$$

The d'Alembert representation (1.2) implies that the right hand side of (1.12) is majorized by $C(\|f'(\cdot - t)\|_R + \|g'(\cdot + t)\|_R)$. However,

$$\|f'(\cdot - t)\|_R^2 = \int_{-R}^{R} |f'(x-t)|^2 dx = \int_{-R-t}^{R-t} |f'(y)|^2 dy \to 0, \qquad t \to \pm\infty$$

by Proposition 1.2 *iv)*. A similar argument holds for $\|g'(\cdot + t)\|_R$. ■

Corollary 1.5. *Proposition 1.4 implies that the set $\mathcal{S}$ is the point attractor of the group $W_0(t)$ in the space $\mathcal{E}_F$, i.e. $\mathcal{A} = \mathcal{S}$.*

1.5 Space-Localized States

Now consider the initial states $(u_0, v_0) \in \mathcal{E}$ with space-localized energy density $|v_0(x)|^2 + |u_0'(x)|^2$, i.e.

$$v_0(x) = u_0'(x) = 0, \qquad |x| > a, \tag{1.13}$$

where $a < \infty$. Then

$$u_0(x) = \text{const} =: u_\pm, \qquad \pm x > a. \tag{1.14}$$

In this case Proposition 1.4 can be improved: the trajectory converges to some limit points $S_\pm \in \mathcal{S}$ which depend on the trajectory (see Fig. 1.1).

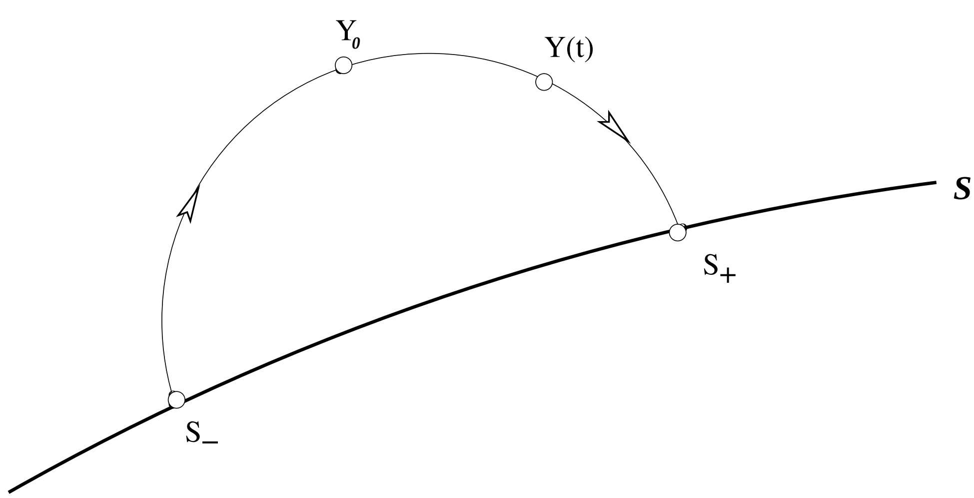

Figure 1.1: Global attraction.

Proposition 1.6. *Let us assume that $Y_0 = (u_0, v_0) \in \mathcal{E}$ and (1.13) holds. Then*

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm \in \mathcal{S}, \qquad t \to \pm\infty. \tag{1.15}$$

Proof The limit points $S_\pm = (c_\pm, 0)$ can be calculated from the d'Alembert formula (1.4):

$$c_+ = \frac{u_+ + u_-}{2} + \frac{1}{2}\int_{-\infty}^{+\infty} v_0(y)dy, \qquad c_- = \frac{u_+ + u_-}{2} - \frac{1}{2}\int_{-\infty}^{+\infty} v_0(y)dy. \tag{1.16}$$

The convergence to $S_\pm$ can be checked as in the previous proposition. ■

Remarks 1.7. *i) The asymptotics (1.15) give a mathematical model of Bohr's transitions to quantum stationary states, (0.1),*

$$\begin{array}{ccc} S_- & - - - - - - - - - > & S_+ \\ t = -\infty & & t = +\infty, \end{array} \tag{1.17}$$

see Fig. 1.1.
ii) The energy of the limit state $S_\pm = (c_\pm, 0)$ is zero which is less or equal to the nonnegative energy of the solution:

$$\mathcal{H}(S_\pm) \le \mathcal{H}(Y(t)) \equiv \mathcal{H}(Y_0), \quad t \in \mathbb{R} \tag{1.18}$$

iii) For any two stationary states $S_\pm \in \mathcal{S}$ there exists a solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ such that (1.15) holds. This follows from the formulae (1.16).

1.6 Drift Along the Attractor

Let us note that for all finite energy solutions the limit points $S_\pm$ generally do not exist. This means that the trajectory approaches the attractor and drifts along it.

Example 1.8. Consider the function $u(x,t) = \sin\log(1 + |x - t|)$:
i) u is a solution to the d'Alembert equation (1.1);
ii) $\dot{u}(\cdot,t), u'(\cdot,t) \in L^2$, since $\dot{u}, u' \sim \dfrac{1}{1+|x-t|}$, hence $Y(t) := (u(\cdot,t), \dot{u}(\cdot,t)) \in C(\mathbb{R}, \mathcal{E})$ and the convergence (1.11) holds by Proposition 1.4.
iii) The limit points $S_\pm$ for the trajectory $Y(t) := (u(\cdot,t), \dot{u}(\cdot,t))$ do not exist in the space $\mathcal{E}_F$ since the solution visits any neighborhood of the stationary solutions $S(x) \equiv \pm 1$ infinitely many times.
iv) The rate of the drift of the function $u(x,t)$, in any fixed interval $-R < x < R$, decays to zero in the long-time limit.

The drift is shown in Figures 1.2 and 1.3:
• Fig. 1.2 demonstrates the traveling wave $\sin\log(1+|x-t|)$ and its form on the interval $(-R, R)$ for the moments 0, t and $2t$.
• Fig. 1.3 demonstrates the oscillations of the wave on the interval $(-R, R)$ (left) and the drift in the phase space $\mathcal{E}$ (right).

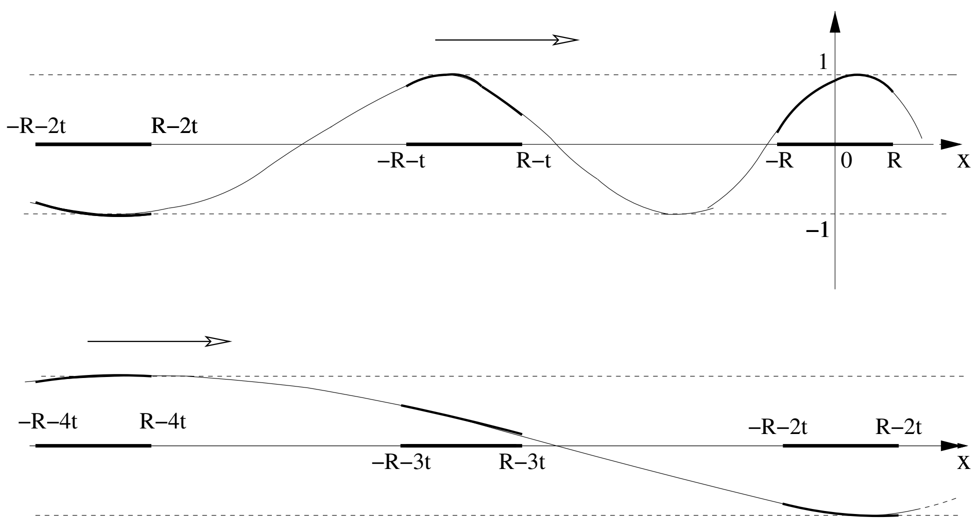

Figure 1.2: Traveling wave.

Moreover, the solution is not necessarily bounded:

Example 1.9. Consider the function $u(x,t) = \log(1 + |x - t|)$:
i) u is a solution to the d'Alembert equation (1.1);
ii) $\dot{u}(\cdot,t), u'(\cdot,t) \in L^2$, since $\dot{u}, u' \sim \dfrac{1}{1+|x-t|}$, hence $Y(t) := (u(\cdot,t), \dot{u}(\cdot,t)) \in C(\mathbb{R}, \mathcal{E})$ and the convergence (1.11) holds by Proposition 1.4.
iii) The limit points $S_\pm$ for the trajectory $Y(t) := (u(\cdot,t), \dot{u}(\cdot,t))$ do not exist in the space $\mathcal{E}_F$ since $u(x,t) \to \infty$, $t \to \infty$ for each fixed $x \in \mathbb{R}$.
iv) The rate of the drift of the function $u(x,t)$, in any fixed interval $-R < x < R$, decays to zero in the long-time limit.

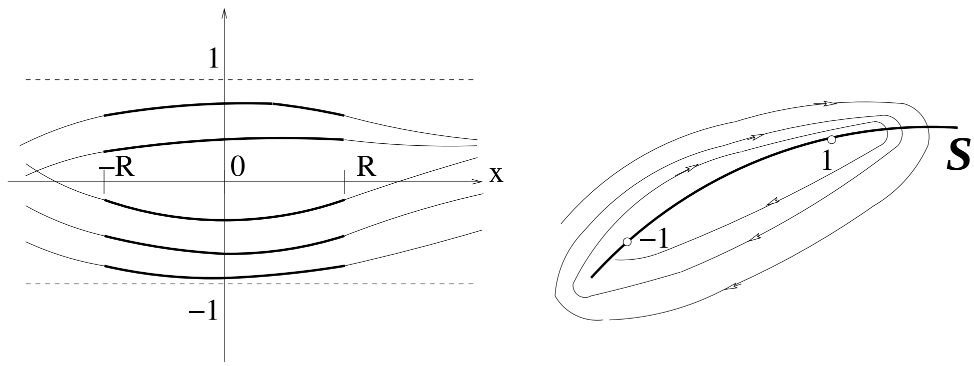

Figure 1.3: Drift along the attractor.

2 Example II: Three-Dimensional Wave Equation

Now let us consider the three-dimensional acoustic equation

$$\ddot{u}(x,t) = \Delta u(x,t), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}. \tag{2.1}$$

We will consider the Cauchy problem with initial conditions

$$u|_{t=0} = u_0(x);\ \dot{u}|_{t=0} = v_0(x), \qquad x \in \mathbb{R}^3. \tag{2.2}$$

2.1 Energy Conservation

Let us assume that $\nabla u_0, v_0 \in L^2 := L^2(\mathbb{R}^3)$. Then the energy is conserved,

$$E(t) := \frac{1}{2}\int\limits_{\mathbb{R}^3} [|\dot{u}(x,t)|^2 + |\nabla u(x,t)|^2]dx = \text{const}, \qquad t \in \mathbb{R}. \tag{2.3}$$

Formal proof Differentiating and substituting $\ddot{u} = \Delta u$, we get

$$\dot{E}(t) = \int\limits_{\mathbb{R}^3} [\dot{u}\ddot{u} + \nabla u \cdot \nabla \dot{u}]dx = \int\limits_{\mathbb{R}^3} [\dot{u}\Delta u + \nabla u \cdot \nabla \dot{u}]dx = 0 \tag{2.4}$$

by partial integration. ■

Existence of Dynamics

The Cauchy problem (2.1), (2.2) can be written as

$$\dot{Y}(t) = AY(t), \quad t \in \mathbb{R}; \quad Y(0) = Y_0, \tag{2.5}$$

where $Y(t) := (u(\cdot,t), \dot{u}(\cdot,t))$, $Y_0 = (u_0, v_0)$ and

$$A := \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}.$$

Denote by $\|\cdot\|$ resp. $\|\cdot\|_R$ the norm in the Hilbert space $L^2 := L^2(\mathbb{R}^3)$ resp. $L^2(B_R)$, where B_R is the ball $|x| < R$. Let us introduce the phase space for the dynamical system (2.5).

Definition 2.1. *i)* $\mathcal{E}$ *is the Hilbert space which is the completion of* $\{Y(x) = (u(x), v(x)) : u, v \in C_0^\infty(\mathbb{R}^3)\}$ *in the* **global energy norm**

$$\|Y\|_{\mathcal{E}} = \|\nabla u\| + \|v\|. \tag{2.6}$$

ii) $\mathcal{E}_F$ *is the linear space* $\mathcal{E}$ *endowed with the topology defined by the* **local energy seminorms**

$$\|Y\|_{\mathcal{E},R} = \|\nabla u\|_R + \|u\|_R + \|v\|_R. \tag{2.7}$$

We will consider the solutions $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to (2.1) in the sense of tempered distributions of $x \in \mathbb{R}^3$ and $|t| < T$ for any $T > 0$. The next proposition is trivial:

Proposition 2.2. *i) For all* $Y_0 \in \mathcal{E}$ *there exists unique solution* $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *to the Cauchy problem (2.5).*
ii) The map $W_0(t) : Y(0) \mapsto Y(t)$ *is continuous in* $\mathcal{E}$ *and* $\mathcal{E}_F$*.*
iii) The energy is conserved, (2.3).
iv) For $Y_0 = (u_0, v_0) \in \mathcal{E}$ *with* $u_0 \in C^1(\mathbb{R}^3)$ *and* $v_0 \in C(\mathbb{R}^3)$*, the solution is given by the Kirchhoff formula*

$$u(x,t) = \frac{1}{4\pi t} \int\limits_{|y-x|=|t|} v_0(y) dS(y) + \partial_t \Big(\frac{1}{4\pi t} \int\limits_{|y-x|=|t|} u_0(y) dS(y) \Big), \quad x \in \mathbb{R}^3, \ t \in \mathbb{R}, \tag{2.8}$$

where $dS(y)$ *is the Lebesgue measure on the sphere* $|y - x| = t$*.*

Proof All statements follows easily in the Fourier transform: formally $\hat{u}(k,t) := \int_{\mathbb{R}^3} e^{ikx} u(x,t) dx$, $k \in \mathbb{R}^3$. ■

Corollary 2.3. *The Kirchhoff formula implies* **Strong Huygen's Principle***:*
For given (x,t)*, the value of the solution* $u(x,t)$ *depends only on the initial data* $v_0(y)$ *on the sphere* $|y - x| = |t|$ *and on* $u_0(y)$ *in an arbitrarily small neighborhood of the sphere* $|y - x| = |t|$*.*

2.2 Attractor of Finite Energy States

First, let us determine all finite energy stationary states for (2.1), i.e. the set $\mathcal{S} := \{Y \in \mathcal{E} : AY = 0\}$.

Exercise 2.4. *Check that* $\mathcal{S} = \{0\}$*.*

Proposition 2.5. *Let us assume that the initial state* $Y_0 = (u_0, v_0) \in \mathcal{E}$ *and additionally,* $u_0 \in L^2(\mathbb{R}^3)$*. Then the corresponding solution* $Y(t) := W_0(t)Y_0$ *converge to the zero solution in the* **local energy seminorms***:*

$$Y(t) \xrightarrow{\mathcal{E}_F} 0, \qquad t \to \pm\infty. \tag{2.9}$$

In other words, the point attractor $\mathcal{A}$ of the group $W_0(t)$ is a point 0, so $\mathcal{A} = \mathcal{S}$.

Remark 2.6. *By definition, (2.9) means that*

$$\int\limits_{|x|<R} [|\nabla u(x,t)|^2 + |u(x,t)|^2 + |\dot{u}(x,t)|^2] dx \to 0, \qquad t \to \pm\infty, \tag{2.10}$$

for all finite energy solutions to the equation (2.1). Such asymptotics are well known in linear and nonlinear scattering theory as **local energy decay** *(see Introduction).*

Proof of Proposition 2.5 i) use the **energy inequality** for the solutions $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to (2.1):

$$\int\limits_{|x|<R} [|\dot{u}(x,t)|^2 + |\nabla u(x,t)|^2]dx \le \int\limits_{|x|<R+|t|} [|v_0(x)|^2 + |\nabla u_0(x)|^2]dx. \tag{2.11}$$

ii) By Strong Huygen's Principle, the solution $u(x,t)$, for $|x| < R$, depends on the initial data $u_0(y), v_0(y)$ only in the region $|t| - R < |y| < |t| + R$: it allows us to modify the initial data arbitrarily outside of the region, without changing the LHS of (2.11). In particular, set $u_0(y) = v_0(y) = 0$ for $|y| < |t| - R - 1$ and apply (2.11) to the solution $u_{R,t}(x,t)$ with the modified initial data. Then we can obtain that

$$\int\limits_{|x|<R} [|\dot{u}(x,t)|^2 + |\nabla u(x,t)|^2]dx \le \int\limits_{|t|-R-1<|x|<|t|+R} [|v_0(x)|^2 + |\nabla u_0(x)|^2 + C|u_0(x)|^2]dx \to 0, \quad t \to \pm\infty. \tag{2.12}$$

iii) It remains to prove that

$$\int\limits_{|x|<R} |u(x,t)|^2 dx \to 0, \qquad t \to \infty \tag{2.13}$$

(since the case $t \to -\infty$ is similar). This follows from the integral representation for the modified solution,

$$u(x,t) = \int_{t-2R-1}^{t} \dot{u}_{R,t}(x,s)ds, \qquad |x| < R \tag{2.14}$$

which holds since $u_{R,t}(x, t - 2R - 1) = 0$ for $|x| < R$, by Strong Huygen's Principle. Namely, (2.14) implies that

$$\|u(\cdot,t)\|_R \le \int_{t-2R-1}^{t} \|\dot{u}_{R,t}(\cdot,s)\|_R ds, \tag{2.15}$$

and we have, uniformly in $s \in (t - 2R - 1, t)$,

$$\|\dot{u}_{R,t}(\cdot,s)\|_R^2 \le \int\limits_{|s|-R-1<|x|<|s|+R} [|v_0(x)|^2 + |\nabla u_0(x)|^2 + C|u_0(x)|^2]dx \to 0, \quad t \to \infty. \tag{2.16}$$

similarly to (2.12).

2.3 Space-Localized States

Now consider space-localized initial states $(u_0, v_0) \in \mathcal{E}$ i.e.

$$u_0(x) = v_0(x) = 0, \qquad |x| > a, \tag{2.17}$$

where $a < \infty$. In this case Proposition 2.5 can be improved:

$$u(x,t) = 0, \quad |x| < R, \quad |t| > a + R. \tag{2.18}$$

by Strong Huygen's Principle. In particular, the asymptotics (2.9) and (2.10) obviously hold.

3 String with Space-Localized Nonlinear Interaction

We consider certain classes of one-dimensional equations of the type

$$\ddot{u}(x,t) = u''(x,t) + f(x, u(x,t)), \quad x \in \mathbb{R}. \tag{3.1}$$

All derivatives here and below are understood in the sense of distributions, $u(x,t) \in \mathbb{R}^d$ with $d \geq 1$ and $f(x,u) = -\nabla_u V(x,u)$, where $V(x,u)$ is a real potential. Then (3.1) is a formally Hamilton system with the Hamilton functional

$$\mathcal{H}(u, \dot{u}) = \int \Big(\frac{|\dot{u}|^2}{2} + \frac{|u'|^2}{2} + V(x,u)\Big) dx. \tag{3.2}$$

Our results concern the equations (3.1) with the *space-localized nonlinear interaction*:

$$f(x,u) = 0, \qquad |x| > \text{const}. \tag{3.3}$$

Let us consider the Cauchy problem for the equation (3.1) with the initial conditions

$$u|_{t=0} = u_0(x); \; \dot{u}|_{t=0} = v_0(x), \qquad x \in \mathbb{R}. \tag{3.4}$$

We write the Cauchy problem, similarly to (0.6), in the form

$$\dot{Y}(t) = \mathbf{F}(Y(t)), \quad t \in \mathbb{R}; \quad Y(0) = Y_0, \tag{3.5}$$

where $\mathbf{F}((u(x), v(x))) := (v(x), u''(x) + f(x, u(x)))$ and $Y_0 := (u_0, v_0)$. We consider the general case of the potential $V(x,u)$ satisfying the following assumptions

$$V(\cdot,\cdot) \in C^2(\mathbb{R} \times \mathbb{R}^d), \tag{3.6}$$

$$V(x,u) = 0, \qquad |x| \geq a, \quad u \in \mathbb{R}^d, \tag{3.7}$$

$$\inf_{\mathbb{R}\times\mathbb{R}^d} V(x,u) > -\infty, \tag{3.8}$$

$$\max_{x\in\mathbb{R}} V(x,u) \to \infty, \qquad |u| \to \infty. \tag{3.9}$$

3.1 Phase Space and Dynamics

Denote by $\|\cdot\|$ resp. $\|\cdot\|_R$ the norm in $L^2 := L^2(\mathbb{R})$ resp. $L^2(-R,R)$.

Definition 3.1. *i) Phase Space of the system (3.5) is the Hilbert space* $\mathcal{E} := \{Y(x) = (u(x), v(x)) : u \in C(\mathbb{R}), \; u', v \in L^2\}$, *with the* **global energy norm**

$$\|Y\|_{\mathcal{E}} = \|u'\| + |u(0)| + \|v\|. \tag{3.10}$$

ii) $\mathcal{E}_F$ *is the linear space* $\mathcal{E}$ *endowed with* **local energy seminorms**

$$\|Y\|_{\mathcal{E},R} = \|u'\|_R + |u(0)| + \|v\|_R, \; R > 0. \tag{3.11}$$

By definiton, the convergence $Y_n \xrightarrow{\mathcal{E}_F} Y$ *means that* $\|Y_n - Y\|_{\mathcal{E},R} \to 0$, $\forall R > 0$.

This convergence is equivalent to the convergence w.r.t. the metric

$$\rho(Y,Z) = \sum_{R=1}^{\infty} 2^{-R} \frac{\|Y - Z\|_{\mathcal{E},R}}{1 + \|Y - Z\|_{\mathcal{E},R}}.$$

Proposition 3.2. *([38]) Let the assumptions (3.6)–(3.9) be fulfilled and* $d \geq 1$*. Then*
i) for every $Y_0 \in \mathcal{E}$ *the Cauchy problem (3.5) admits a unique solution* $Y(t) \in C(\mathbb{R}, \mathcal{E})$*.*
ii) The map $W(t) : Y(0) \mapsto Y(t)$ *is continuous in* $\mathcal{E}$ *and* $\mathcal{E}_F$*.*
iii) The energy is conserved,

$$E(t) := \mathcal{H}(Y(t)) = \text{const}, \qquad t \in \mathbb{R}. \tag{3.12}$$

iv) The a priori bound holds,

$$\sup_{t \in \mathbb{R}} \|Y(t)\|_{\mathcal{E}} < \infty. \tag{3.13}$$

3.2 Stationary States

We denote by $\mathcal{S}$ the set of all stationary states $S = (s(x), 0) \in \mathcal{E}$ for the equation (3.1), i.e. $\mathcal{S} := \{Y \in \mathcal{E} : \mathbf{F}(Y) = 0\}$. Let us denote $\mathcal{S}^h = \{S \in \mathcal{S} : \ \mathcal{H}(S) \leq h\}$ for $h \in \mathbb{R}$. Then (3.6)–(3.8) and (3.2) imply that $\mathcal{S}^h$ is a closed bounded set in $\mathcal{E}$, i.e.

$$\sup_{S \in \mathcal{S}^h} \|S\|_{\mathcal{E}} < \infty, \ \forall h \in \mathbb{R}. \tag{3.14}$$

Proposition 3.3. *Let the assumptions (3.6)–(3.9) be fulfilled,* $d = 1$ *and the function* $F(u)$ *is real analytic on* $\mathbb{R}$*. Then* $\mathcal{S}^h$ *is a finite set for every* $h \in \mathbb{R}$*.*

3.3 Main Results

The main result for the equation (3.1) means that the set $\mathcal{S}$ is the minimal global point attractor of the equation in the topology of the space $\mathcal{E}_F$. Let us denote by $W_0(t)$ the dynamical group of the 1D *free* wave equation.

Theorem 3.4. *([38]) Let the assumptions (3.6)–(3.9) hold and an initial state* $Y_0 \in \mathcal{E}$*. Then*
i) The corresponding solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *to the Cauchy problem (3.5) converges to the set* $\mathcal{S}$ *in the* **local energy seminorms***:*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}, \qquad t \to \pm\infty. \tag{3.15}$$

ii) Let us assume additionally that $d = 1$*, and the function* $F(u)$ *is real analytic on* $\mathbb{R}$*. Then*
a) There exist the limit stationary states $S_\pm \in \mathcal{S}$ *depending on the solution* $Y(t)$*:*

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm, \qquad t \to \pm\infty. \tag{3.16}$$

b) The **scattering asymptotics** *holds (cf. (0.33))*

$$Y(t) = S_\pm + W_0(t)\Psi_\pm + r_\pm(t) \tag{3.17}$$

with some scattering states $\Psi_\pm \in \mathcal{E}$*, and the remainder is small in the* **global energy norm***:*

$$\|r_\pm(t)\|_{\mathcal{E}} \to 0, \ \ t \to \pm\infty. \tag{3.18}$$

Remarks i) The statements i), ii) a) are proved in [38]. The statement ii) b) is a trivial consequence of ii) a).
ii) The weak convergence (3.16) and (3.6)–(3.9) imply that

$$\mathcal{H}(S_\pm) \leq \mathcal{H}(Y(t)) \equiv \mathcal{H}(Y_0), \ \ t \in \mathbb{R} \tag{3.19}$$

by the Fatou theorem.
iii) Theorem 3.4 is proved in [38] for $f(x, u) = \chi(x)F(u)$ for the simplicity of exposition. The

theorem can be extended easily to the general nonlinear term $f(x,u)$ satisfying the conditions (3.6)–(3.9) (cf. [37]).

Similar results are proved in [36] for the case $f(x,u) = \delta(x)F(u)$ and in [37] for finite sum $f(x,u) = \sum_1^N \delta(x - x_k)F_k(u)$. The proof of Theorem 3.4 is rather technical and is a natural development of the methods used in [36, 37]. We will explain basic ideas sketching the proof for the simplest case $f(x,u) = \delta(x)F(u)$ in the next section.

3.4 On Inelastic Scattering

The asymptotics (3.17) mean the *inelastic scattering* of the solitons by the nonlinear interaction: the *incident scattering data* (S_-, Ψ_-) at $t = -\infty$ pass to the *outgoing scattering data* (S_+, Ψ_+) at $t = +\infty$. This defines the *nonlinear scattering operator*

$$\mathbf{S} : (S_-, \Psi_-) \mapsto (S_+, \Psi_+). \tag{3.20}$$

However a justification of this definition is an open question. The *elastic scattering* corresponds to the case when $S_+ = S_-$.

4 String Coupled to a Nonlinear Oscillator

Here we prove Theorem 3.4 for the case $f(x,u) = \delta(x)F(u)$ studied in [36]. More generally, we will consider the coupled system

$$\left\{\begin{array}{rcl} \ddot u(x,t) &=& u''(x,t), \quad x \in \mathbb{R}\setminus\{0\}, \\ m\ddot y(t) &=& F(y(t)) + u'(0+,\,t) - u'(0-,\,t);\ y(t) \equiv u(0,t), \end{array}\right. \tag{4.1}$$

where $m \geq 0$. The solutions $u(x,t)$ take the values in $\mathbb{R}^d$ with $d \geq 1$. Physically, the problem (4.1) describes small crosswise oscillations of an infinite string stretched parallel to the $0x$-axis; a particle of mass $m \geq 0$ is attached to the string at the point $x = 0$; $F(y)$ is an external (nonlinear) force field perpendicular to $0x$, the field subjects the particle (see Fig. 1.4).

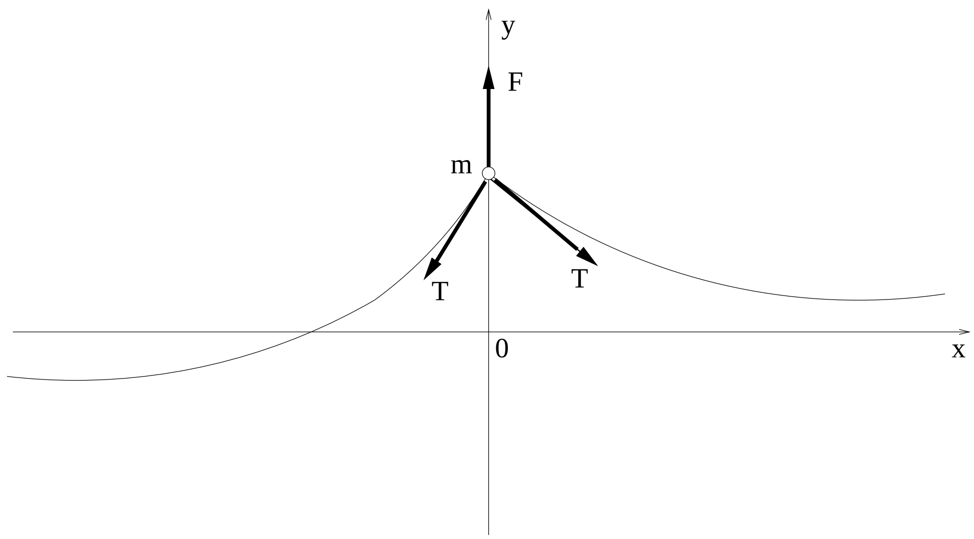

Figure 1.4: String with the oscillator.

The system (4.1) is formally equivalent to a one-dimensional nonlinear wave equation with the nonlinear term $\delta(x)F(u)$ concentrated at the single point $x = 0$ and with a mass m concentrated at the same point:

$$(1 + m\delta(x))\ddot u(x,\,t) = u''(x,\,t) + \delta(x)F(u(x,\,t)),\ (x,\,t) \in \mathbb{R}^2. \tag{4.2}$$

In the linear case when $F(y) = -\omega^2 y$, the system (4.1) has a single stationary finite energy solution $s(x) \equiv 0$. In this case the stabilization to zero was considered originally by H.Lamb [51].

Consider the Cauchy problem for the system (4.1) with the initial conditions

$$u|_{t=0} = u_0(x);\ \dot u|_{t=0} = v_0(x);\ \dot y|_{t=0} = p_0. \tag{4.3}$$

The last condition in (4.3) is necessary only if $m > 0$. We consider below the case $m > 0$ and sometimes we comment the case $m = 0$. Write the Cauchy problem (4.1), (4.3) in the form

$$\dot Y(t) = \mathbf{F}(Y(t)) \ \text{ for } \ t \in \mathbb{R}, \quad Y(0) = Y_0, \tag{4.4}$$

where $Y_0 = (u_0, v_0, p_0)$.

4.1 Phase Space and Dynamics

Let us introduce a phase space $\mathcal{E}$ of finite energy states for the system (4.1) with $m > 0$. Denote by $\|\cdot\|$ resp. $\|\cdot\|_R$ the norm in the Hilbert space $L^2 := L^2(\mathbb{R}, \mathbb{R}^d)$ resp. $L^2(-R, R; \mathbb{R}^d)$.

Definition 4.1. *i)* $\mathcal{E}$ *is the Hilbert space of the triples* $(u(x), v(x), p) \in C(\mathbb{R}, \mathbb{R}^d) \oplus L^2 \oplus \mathbb{R}^d$ *with* $u'(x) \in L^2$ *and the* **global energy norm**

$$\|(u, v, p)\|_{\mathcal{E}} = \|u'\| + |u(0)| + \|v\| + |p|. \tag{4.5}$$

ii) $\mathcal{E}_F$ *is the space* $\mathcal{E}$ *endowed with the topology defined by the* **local energy seminorms**

$$\|(u, v, p)\|_{\mathcal{E},R} \equiv \|u'\|_R + |u(0)| + \|v\|_R + |p|, \quad R > 0. \tag{4.6}$$

We assume that

$$F(u) \in C^1(\mathbb{R}^d, \mathbb{R}^d), \qquad F(u) = -\nabla V(u) \tag{4.7}$$

$$V(u) \to +\infty, \qquad |u| \to \infty. \tag{4.8}$$

Then the system (4.1) is formally Hamiltonian with the phase space $\mathcal{E}$ and the Hamilton functional

$$\mathcal{H}(u, v, p) = \frac{1}{2}\int [|v(x)|^2 + |u'(x)|^2]dx + m\frac{|p|^2}{2} + V(u(0)) \tag{4.9}$$

for $(u, v, p) \in \mathcal{E}$. We consider solutions $u(x,t)$ such that $Y(t) = (u(\cdot,t), \dot{u}(\cdot,t), \dot{y}(t)) \in C(\mathbb{R}, \mathcal{E})$, where $y(t) \equiv u(0,t)$.

Let us discuss the definition of the Cauchy problem (4.4) for the functions $Y(t) \in C(0,\infty; \mathcal{E})$. At first, $u(x,t) \in C(\mathbb{R}^2)$ due to $Y(t) \in C(0,\infty;\mathcal{E})$. Then the wave equation (1.1) is understood in the sense of distributions. This is equivalent to the d'Alembert decomposition

$$u(x,t) = f_\pm(x-t) + g_\pm(x+t), \quad \pm x > 0, \tag{4.10}$$

where $f_\pm, g_\pm \in C(\mathbb{R})$. Therefore,

$$\dot{u}(x,t) = -f'_\pm(x-t) + g'_\pm(x+t), \quad u'(x,t) = f'_\pm(x-t) + g'_\pm(x+t) \ \text{ for } \ \pm x > 0, \quad t \in \mathbb{R},$$

where all the derivatives are understood in the sense of distributions. The condition $Y(t) \in C(0,\infty;\mathcal{E})$ implies that

$$f'_\pm, g'_\pm \in L^2_{loc}(\mathbb{R}). \tag{4.11}$$

We now explain the second equation in (4.1).

Definition 4.2. *In the second equation of (4.1) put*

$$u'(0\pm, t) \equiv f'_\pm(-t) + g'_\pm(t) \in L^2_{loc}(\mathbb{R}), \tag{4.12}$$

while the derivative $\ddot{y}(t)$ *of* $y(t) \equiv u(0\pm,t) \in C(\mathbb{R})$ *(or of* $\dot{y}(t) \in L^2_{loc}(\mathbb{R})$ *by (4.11)) is understood in the sense of distributions.*

Note that the functions $f_\pm$ and $g_\pm$ in (4.10) are unique up to an additive constant. Hence definition (4.12) is unambiguous.

Proposition 4.3. *([36]) Let the conditions (4.7), (4.8) be fulfilled,* $m > 0$ *and* $d \geq 1$*. Then*
i) For every $Y_0 \in \mathcal{E}$ *the Cauchy problem (4.4) admits a unique solution* $Y(t) \in C(\mathbb{R}, \mathcal{E})$*.*
ii) The map $W(t) : Y_0 \mapsto Y(t)$ *is continuous in* $\mathcal{E}$ *and* $\mathcal{E}_F$*.*
iii) The energy is conserved,

$$\mathcal{H}(Y(t)) = \text{const}, \qquad t \in \mathbb{R}. \tag{4.13}$$

iv) The a priori bound holds,

$$\sup_{t\in\mathbb{R}} \|Y(t)\|_{\mathcal{E}} < \infty. \tag{4.14}$$

4.2 Stationary States

The stationary states $S = (s(x), 0, 0) \in \mathcal{E}$ for (4.4) are evidently determined. We define for every $c \in \mathbb{R}^d$ the constant function

$$s_c(x) = c \quad \text{for} \quad x \in \mathbb{R}. \tag{4.15}$$

Then the set $\mathcal{S}$ of all stationary states $S \in \mathcal{E}$ is given by

$$\mathcal{S} = \{S_z = (s_z(x), 0, 0) : \quad z \in Z\}, \tag{4.16}$$

where $Z = \{z \in \mathbb{R}^d : \ F(z) = 0\}$.

Definition 4.4. *The potential $V(u)$ is "non-degenerate", if the set Z is a discrete subset in $\mathbb{R}^d$.*

For $d = 1$ this implies that

$$F(u) \not\equiv 0 \text{ on every nonempty interval } c_1 < u < c_2. \tag{4.17}$$

4.3 Main Results

The main result means that the set $\mathcal{S}$ is the minimal global point attractor of the system (4.1) in the space $\mathcal{E}_F$. Let us denote by $\tilde{W}_0(t)(u, v, 0) := (W_0(t)(u, v), 0)$, where $W_0(t)$ is the dynamical group of free wave equation corresponding to $F(u) \equiv 0$.

Theorem 4.5. *([36]) Let all assumptions of Proposition 4.3 hold and an initial state $Y_0 \in \mathcal{E}$. Then*
i) The corresponding solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to the Cauchy problem (4.4) converges to the set $\mathcal{S}$ in the **local energy seminorms***:*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S} \quad as \quad t \to \pm\infty. \tag{4.18}$$

ii) Let additionally the set Z be a discrete subset in $\mathbb{R}^d$. Then
a) There exist the limit stationary states $S_\pm \in \mathcal{S}$ depending on the solution $Y(t)$:

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm, \qquad t \to \pm\infty. \tag{4.19}$$

b) Let additionally there exist the finite limits

$$\overline{u}_0 := \lim_{|x| \to \infty} u_0(x), \qquad I_0 := \int_{-\infty}^{\infty} v_0(y) dy. \tag{4.20}$$

Then the **scattering asymptotics** *hold*

$$Y(t) = S_\pm + \tilde{W}_0(t)\Psi_\pm + r_\pm(t) \tag{4.21}$$

with some scattering states $\Psi_\pm \in \mathcal{E}_0$, and the remainder is small in the **global energy norm***:*

$$\|r_\pm(t)\|_{\mathcal{E}} \to 0, \ \ t \to \pm\infty. \tag{4.22}$$

Remarks *i)* The statements i) and ii) a) are proved in [36]. The statement ii) b) is a trivial consequence of ii) a).
ii) The convergence (4.19) and (4.8), (4.9) imply (3.19) by Fatou theorem.
iii) The condition (4.8) describes an "open set" among all the potential functions $V(u)$. In this sense, the attraction (4.18) holds for "almost all" equations (4.1) which agrees with the conjecture **A** from Introduction.
iv) A similar theorem holds also in the case $m = 0$.

First we prove Proposition 4.3. We establish the uniqueness of the solution $Y(t) = (u(\cdot, t), \dot{u}(\cdot, t), \dot{y}(t)) \in C(\mathbb{R}, \mathcal{E})$ provided such a solution exists. At the same time, we get a method of constructing a solution. Thus, in fact, the existence will be proved as well.

4.4 Uniqueness of solution

The d'Alembert representation holds for $x > 0$ and $x < 0$:

$$u(x,t) = \begin{cases} f_+(x-t) + g_+(x+t), & x > 0, \\ f_-(x-t) + g_-(x+t), & x < 0. \end{cases} \tag{4.23}$$

Initial conditions (4.3) imply the well-known formulas for $f_\pm(z)$ and $g_\pm(z)$ in the region $\pm z > 0$:

$$\begin{cases} f_\pm(z) &= \dfrac{u_0(z)}{2} - \dfrac{1}{2}\displaystyle\int_0^z v_0(y)\,dy, \quad \pm z > 0, \\ g_\pm(z) &= \dfrac{u_0(z)}{2} + \dfrac{1}{2}\displaystyle\int_0^z v_0(y)\,dy, \quad \pm z > 0, \end{cases} \tag{4.24}$$

We omit here the constants of the integration since the d'Alembert waves $f_\pm$ and $g_\pm$ in (4.23) also are defined up to an additive constant. (4.24) implies

$$f'_\pm(z), g'_\pm(z) \in L^2(\mathbb{R}_\pm, \mathbb{R}^d) \tag{4.25}$$

since $(u_0, v_0, p_0) \in \mathcal{E}$. By (4.24), the usual d'Alembert formula is valid for $|x| \geq |t|$:

$$u(x,t) = \frac{u_0(x-t) + u_0(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)dy, \quad |x| \geq |t|. \tag{4.26}$$

Thus the solution $u(x,t)$ in the region $|x| \geq |t|$ is defined uniquely. It remains to prove the uniqueness in the region $|x| < |t|$.

4.5 Reduced equation for $t > 0$

Consider for example the case $t > 0$. In the region $|x| < t$; the unknown functions in (4.23) are $f_+(x-t)$ for $0 < x < t$ and $g_-(x+t)$ for $-t < x < 0$. Therefore, we have to find the functions $f_+(z)$ for $z < 0$ and $g_-(z)$ for $z > 0$.

To find these unknown functions, we derive a nonlinear ordinary differential equation for $y(t) := u(0,t)$. First, the initial conditions (4.3) imply that

$$y(0) = u_0(0), \quad \dot{y}(0) = p_0. \tag{4.27}$$

Lemma 4.6. *For every solution $Y(t) = (u(\cdot,t), \dot{u}(\cdot,t), \dot{y}(t)) \in C(\mathbb{R}, \mathcal{E})$, to the system (4.4) the function $y(t) := u(0,t)$ is a solution to the "reduced" equation*

$$m\ddot{y}(t) = F(y(t)) - 2\dot{y}(t) + 2\dot{w}_{in}(t), \quad a.a.\ t \geq 0, \tag{4.28}$$

where the function $w_{in}(t)$ is the sum of the incident waves at the point $x = 0$:

$$w_{in}(t) = g_+(t) + f_-(-t), \ t \in \mathbb{R}. \tag{4.29}$$

For this function we have

$$\dot{w}_{in}(t) \in L^2(0, \infty). \tag{4.30}$$

Proof First, the d'Alembert decomposition (4.23) implies that

$$y(t) := u(0,t) = f_+(-t) + g_+(t) = f_-(-t) + g_-(t), \quad t \geq 0. \tag{4.31}$$

Therefore, we can express the outgoing waves $f_+(x-t)$, $g_-(x+t)$ in terms of the incident waves $g_+(x+t)$, $f_-(x-t)$:

$$f_+(-t) = y(t) - g_+(t), \qquad g_-(t) = y(t) - f_-(-t), \qquad t \geq 0. \tag{4.32}$$

Substituting the expressions to (4.23), we get

$$u(x,t) = \left\{ \begin{array}{ll} y(t-x) + g_+(x+t) - g_+(t-x), & 0 < x < t \\ y(t+x) + f_-(x-t) - f_-(-x-t), & -t < x < 0 \end{array} \right| \qquad t \geq 0. \tag{4.33}$$

At last, substituting (4.33) to the second equation of the system (4.1), we get the formulas (4.28) and (4.29). Finally, (4.30) follows from (4.29) and (4.25). ■

4.6 A priori bounds

The reduced equation implies the basic a priori bounds.

Lemma 4.7. *Equation (4.28) with the initial conditions (4.27) admits a unique solution for all $t > 0$ and the a priori bound holds*

$$\sup_{t>0}|y(t)| + \sup_{t>0}|\dot{y}(t)| + \int_0^\infty |\dot{y}(t)|^2 dt \leq B < \infty, \tag{4.34}$$

where B is bounded for bounded $\|(u_0, v_0, p_0)\|_{\mathcal{E}}$.

Proof Let us get an a priori estimate for $y(t)$. To do this we multiply (4.28) by $\dot{y}(t)$. Then the identity $F(y) = -\nabla V(y)$ implies that

$$\frac{d}{dt}[m\frac{|\dot{y}(t)|^2}{2} + V(y(t))] = -2|\dot{y}(t)|^2 + 2\dot{w}_{in}(t)\dot{y}(t), \qquad a.a.\ t \in \mathbb{R}. \tag{4.35}$$

Majorizing the right hand side by the function $-|\dot{y}(t)|^2 + |\dot{w}(t)|^2$, we get after integration

$$m\frac{|\dot{y}(t)|^2}{2} + V(y(t)) + \int_0^t \dot{y}^2(s)ds \leq m\frac{|\dot{y}(0)|^2}{2} + V(y(0)) + \int_0^t |\dot{w}(s)|^2 ds, \qquad t > 0. \tag{4.36}$$

Hence (4.8) and (4.30) imply (4.34). ■

Corollary 4.8. *(4.34) and (4.32) imply that*

$$f'_+ \in L^2(\mathbb{R}_-), \qquad g'_- \in L^2(\mathbb{R}_+) \tag{4.37}$$

by (4.25).

4.7 Existence of solution

The existence of the local solution to the reduced equation (4.28) with the intial conditions (4.27) follows by the contraction mapping principle. Finally, the existence of the global solution follows from the a priori bounds (4.34). Then the formulas (4.33) give the solution to the Cauchy problem (4.4). Proposition 4.3 i) is proved. The a priori bounds (4.14) follow from (4.23) by (4.25) and (4.37). ■

4.8 Relaxation for the Reduced Equation

We will deduce Theorem 4.5 in the next section from the representation (4.33) and from the following lemma on relaxation for the reduced equation. Let us denote $\mathcal{Z} = \{(z,0) \in \mathbb{R}^{2d} : z \in Z\}$.

Lemma 4.9. *Let all assumptions of Theorem 4.5 hold. Then*
i) For every solution $y(t)$ to the equation (4.28)

$$(y(t), \dot{y}(t)) \to \mathcal{Z}, \qquad t \to \infty. \tag{4.38}$$

ii) Let, additionally, Z be a discrete subset in $\mathbb{R}^d$. Then there exists a $(z,0) \in \mathcal{Z}$ such that

$$(y(t), \dot{y}(t)) \to (z,0), \qquad t \to \infty. \tag{4.39}$$

Proof Obviously, *ii)* follows from *i)*. Let us check that *i)* follows from (4.34). Namely, (4.38) is equivalent to the system

$$\begin{aligned} y(t) &\to \mathcal{Z}, \quad t \to \infty, &(4.40)\\ \dot{y}(t) &\to 0, \quad t \to \infty. &(4.41)\end{aligned}$$

• First let us prove (4.41). Assume the contrary, that

$$|\dot{y}(t_k)| \geq \varepsilon > 0 \tag{4.42}$$

for a sequence $t_k \to \infty$. Integrating the equation (4.28), we get that

$$m(\dot{y}(t) - \dot{y}(s)) = \int_s^t F(y(\tau))d\tau - 2\int_s^t \dot{y}(\tau)d\tau + 2\int_s^t \dot{w}_{in}(\tau)d\tau, \qquad s,t \geq 0. \tag{4.43}$$

Let us estimate each of three integrals in the RHS. The first is $\mathcal{O}(|t-s|)$ since $y(\tau)$ is a bounded function by (4.34). Second and third are $\mathcal{O}(|t-s|^{1/2})$ by (4.34), (4.30) and the Cauchy-Schwartz inequality. Hence, (4.43) implies that $\dot{y}(t)$ is a Hölder function of degree 1/2, i.e.

$$|\dot{y}(t) - \dot{y}(s)| \leq C|t-s|^{1/2}, \qquad s,t \geq 0, \quad |t-s| \leq 1. \tag{4.44}$$

Therefore, $\displaystyle\int_0^\infty \dot{y}^2(t)dt = \infty$ by (4.42) which contradicts (4.34). ■

• Now we can prove (4.40). Again assume the contrary. Then

$$F(y(t_k)) \to \overline{F} \neq 0 \tag{4.45}$$

for a sequence $t_k \to \infty$ since $y(t)$ is a bounded function. Moreover, (4.41) implies the uniform convergence

$$F(y(\tau)) \to \overline{F}, \qquad |\tau - t_k| \leq T \tag{4.46}$$

for any $T > 0$. Now (4.43) and (4.41), (4.30) imply that

$$m(\dot{y}(t_k + T) - \dot{y}(t_k - T)) = 2T\overline{F} + o(1), \qquad t_k \to \infty, \tag{4.47}$$

which contradicts (4.41) since $T\overline{F} \neq 0$. ■

4.9 Examples

Let us illustrate Lemma 4.9 by certain typical examples. For simplicity let us assume that

$$u_0(x) = C_\pm, \quad v_0(x) = 0, \qquad \pm x > r_0 \tag{4.48}$$

with some $C_\pm \in \mathbb{R}$ and $r_0 \geq 0$. Then (4.29) implies that $\dot{w}(t) \equiv 0$ for $t > r_0$ and (4.28) is an autonomous equation for $t > r_0$. In the phase plane $(y, \dot{y})$ the orbits of the reduced equation (4.28) are determined by the following system:

$$\left\{ \begin{array}{rl} \dot{y}(t) & = v(t), \\ m\dot{v}(t) & = F(y(t)) - 2\nu v(t), \end{array} \right. \qquad t > r_0. \tag{4.49}$$

Here $\nu = 1$ for the system (4.1) and more generally $\nu = \sqrt{\mu T}$ where μ is a string density and T is its tension. Consider this system as a perturbation of the system with $\nu = 0$ corresponding to a *free* oscillator not coupled to a string,

$$\left\{ \begin{array}{rl} \dot{y} & = v, \\ m\dot{v} & = F(y). \end{array} \right. \tag{4.50}$$

Let us establish some simple relationships between phase portraits of these two systems.
A These system have the same stationary points.
B The vertical component $\dot{v}$ of the phase velocity vector of (4.49) is less then that of (4.50) if $v > 0$, and is greater if $v < 0$. The horizontal components of these vectors are equal.
C Hence the orbits of (4.49) intersect those of (4.50) from above in the halfplane $v > 0$ and from below in the halfplane $v < 0$.

Let us consider for instance a nondegenerate potential of Ginzburg-Landau type

$$V(y) = \frac{1}{4}(y^2 - 1)^2, \quad y \in \mathbb{R}. \tag{4.51}$$

It satisfies conditions (4.7), (4.8). Then the system (4.50) has the following orbits:

- closed curves corresponding to periodic solutions,
- two separatrices both leaving and entering the point $(0, 0)$,
- three stationary points: a saddle at the point $(0, 0)$ and two centers at the points $(\pm 1, 0)$, see Fig. 1.5.

Taking into account the property **C**, we see that for the system (4.49) with potential (4.51):

- the points $(\pm 1, 0)$ are stable foci for small $\nu > 0$ (stable nodes for large $\nu > 0$),
- the point $(0, 0)$ is a saddle, see Fig. 1.6.

4.10 Convergence to Global Attractor

Now we can prove Theorem 4.5 i).

A compact attracting set

At first we construct a compact attracting set $\mathcal{A}$ for the considered trajectory $Y(t)$.

Definition 4.10. $\mathcal{A} = \{S_c : \ c \in \mathbb{R}^d, \ \ |c| \leq B\}$, *where* S_c *is defined similarly to (4.16), and* B *is the bound (4.34).*

The set $\mathcal{A}$ is a compact subset in $\mathcal{E}_F$, since $\mathcal{A}$ is homeomorphic to a compact subset in $\mathbb{R}^d$.

Lemma 4.11. *Let all the assumptions of Theorem 4.5 hold. Then* $Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{A}$ *as* $t \to \pm\infty$*.*

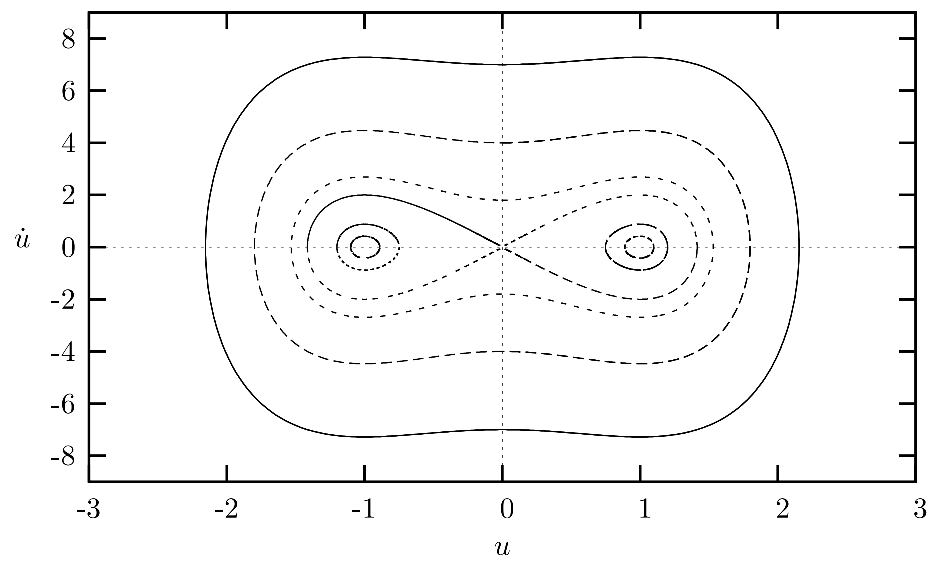

Figure 1.5: Hamilton system

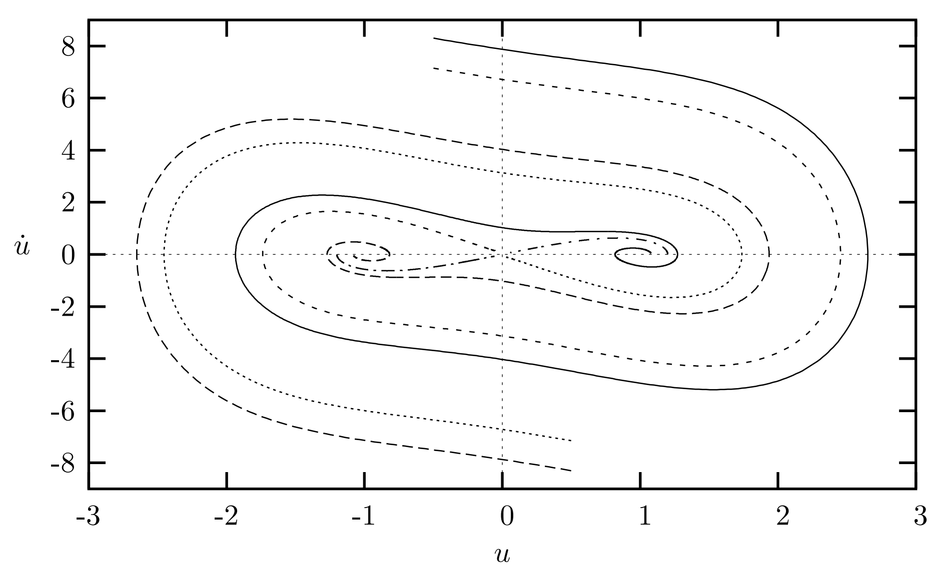

Figure 1.6: System with a friction

Proof According to (4.6), it suffices to verify that for every $R > 0$

$$\|Y(t) - S_{y(t)}\|_{\mathcal{E},R} = \|u'(\cdot,t)\|_R + \|\dot{u}(\cdot,t)\|_R + |\dot{y}(t)| \to 0 \text{ as } t \to \infty. \tag{4.52}$$

Here $\|...\|_R \to 0$ due to (4.33), (4.25) and (4.34). Therefore (4.41) completes the proof. ■

Omega-limiting set

Lemma 4.11 implies that the orbit $O(Y_0) := \{Y(t) \in \mathcal{E} : \ t \in \mathbb{R}\}$ is precompact in $\mathcal{E}_F$ since the set $\mathcal{A}$ is compact in $\mathcal{E}_F$. Therefore, the next lemma implies (4.18).

Definition 4.12. *Let us denote by $\Omega(Y_0)$ the omega-set of the trajectory $Y(t)$ in the topology of the space $\mathcal{E}_F$: $\overline{Y} \in \Omega(Y_0)$ if and only if $Y(t_k) \xrightarrow{\mathcal{E}_F} \overline{Y}$ for a sequence $t_k \to \pm\infty$.*

Lemma 4.13. *$\Omega(Y_0)$ is a subset of $\mathcal{S}$.*

Proof $\Omega(Y) \subset \mathcal{A}$, since $\mathcal{A}$ is an attracting set. Moreover, the set $\Omega(Y_0)$ is invariant with respect to the dynamical group $W(t)$, $t \in \mathbb{R}$, due to the continuity of $W(t)$ in $\mathcal{E}_F$. Hence, for every $\overline{Y} \in \Omega(Y_0)$ there exists a C^2-curve $t \mapsto c(t) \in \mathbb{R}^d$ such that $W(t)\overline{Y} = S_{c(t)} := (c(t), 0, 0)$. This means that $\dot{c}(t) = 0$, i.e. $c(t) \equiv c$, hence $W(t)\overline{Y} \equiv \overline{Y}$. Therefore, $\overline{Y} \in S$. ■

Now Theorem 4.5 i) is proved.

Remark 4.14. *The bound (4.34) is provided by the friction term in the reduced equation (4.28) for the nonlinear oscillator. The friction means the energy radiation by the oscillator, and the integral in (4.34) represents the energy radiated to infinity. Thus, our proof of Theorem 4.5 i) relies on the energy radiation to infinity.*

4.11 Dispersive Wave

Here we prove Theorem 4.5 ii). First, the attraction (4.18) (or (4.39)) implies the convergence (4.19) since the set $\mathcal{S}$, isomorphic to Z, is discrete.

It remains to prove the scattering asymptotics (4.21). For example, let us construct the dispersive wave $\widetilde{W}_0(t)\Psi_+ = (w_{\text{out}}(x,t), \dot{w}_{\text{out}}(x,t), 0)$, $t \geq 0$. Here $w(x,t)$ is a finite energy solution to the free d'Alembert equation. Let us set

$$w_{\text{out}}(x,t) = C_0 + f_+(x-t) + g_-(x+t), \tag{4.53}$$

where the constant C_0 will be chosen below. It remains to check (4.21) for $t \to \infty$. It suffices to verify the representation

$$(u(x,t), \dot{u}(x,t), \dot{y}(t)) = (s_+(x), 0, 0) + (w_{\text{out}}(x,t), \dot{w}_{\text{out}}(x,t), 0) + r_+(t), \quad t > 0, \tag{4.54}$$

where

$$\|r_+(t)\|_{\mathcal{E}} \to 0, \quad t \to +\infty. \tag{4.55}$$

By definition of the norm (4.5), this is equivalent to

$$\begin{aligned} &\|u'(\cdot,t) - w'_{\text{out}}(\cdot,t)\|_{L^2(\mathbb{R},\mathbb{R}^d)} + \quad |u(0,t) - s_+(0) - w_{\text{out}}(0,t)| \\ &\qquad + \quad \|\dot{u}(\cdot,t) - \dot{w}_{\text{out}}(\cdot,t)\|_{L^2(\mathbb{R},\mathbb{R}^d)} \to 0, \qquad t \to +\infty \end{aligned} \tag{4.56}$$

since $\dot{y}(t) \to 0$ by (4.41).

Step i) Let us start with the second term in the LHS of (4.56). Since $u(0,t) \to s_+(0)$ by (4.19), it suffices to prove that

$$w_{\text{out}}(0,t) = C_0 + f_+(-t) + g_-(t) \to 0, \quad t \to +\infty. \tag{4.57}$$

First, we have by (4.32) and (4.19) that

$$\lim_{t\to\infty} f_+(-t) = s_+(0) - \lim_{t\to+\infty} g_+(t); \quad \lim_{t\to+\infty} g_-(t) = s_+(0) - \lim_{t\to\infty} f_-(-t). \tag{4.58}$$

Second, (4.20) and (4.24) imply that

$$\lim_{t\to\infty} f_-(-t) = \frac{\overline{u}_0}{2} - \frac{1}{2}\int_0^{-\infty} v_0(y)dy, \qquad \lim_{t\to+\infty} g_+(t) = \frac{\overline{u}_0}{2} + \frac{1}{2}\int_0^{\infty} v_0(y)dy. \tag{4.59}$$

Substituting into (4.58), we obtain

$$\begin{cases} \lim\limits_{t\to\infty} f_+(-t) &= s_+(0) - \dfrac{\overline{u}_0}{2} - \dfrac{1}{2}\displaystyle\int_0^{\infty} v_0(y)dy, \\ \lim\limits_{t\to+\infty} g_-(t) &= s_+(0) - \dfrac{\overline{u}_0}{2} + \dfrac{1}{2}\displaystyle\int_0^{-\infty} v_0(y)dy \end{cases} \tag{4.60}$$

Now, the last identity from (4.20) implies (4.57) if we choose $C_0 := \overline{u}_0 + I_0 - 2s_+(0)$.

Step ii) Now, let us consider first term in the LHS of (4.56). It suffices to prove for example that

$$\|u'(\cdot,t) - w'_{\rm out}(\cdot,t)\|_{L^2(\mathbb{R}^+,\mathbb{R}^d)} \to 0, \qquad t\to\infty. \tag{4.61}$$

Using (4.53) and the d'Alembert representation (4.23) for $x > 0$, we get

$$u'(x,t) - w'_{\rm out}(x,t) = g'_+(x+t) - g'_-(x+t), \quad x \ge t. \tag{4.62}$$

by (4.23). Finally, (4.25) and (4.37) imply that

$$\begin{aligned} \|g'_+(x+t) - g'_-(x+t)\|^2_{L^2([t,+\infty)} &\le C\int_0^t \Big[|g'_+(x+t)|^2 + |g'_-(x+t)|^2\Big]dx \\ &= C\int_t^{2t} \Big[|g'_+(z)|^2 + |g'_-(z)|^2\Big]dz \to 0, \quad t\to\infty. \end{aligned} \tag{4.63}$$

Step iii) The third term in the LHS of (4.56) can be handled similarly. ■

4.12 Transitivity

Further, a question arises on a connection between the limit stationary states $S_\pm$ of solutions to the system (4.4) as $t\to\pm\infty$. Then next lemma means that the limit stationary states in (4.19) may be arbitrary.

Lemma 4.15. *Let us assume that $F(y)\in C(\mathbb{R}^d,\mathbb{R}^d)$ and $d = 1$. Then for every two stationary states $S_\pm \in \mathcal{S}$ there exists a solution $Y(t)\in C(\mathbb{R},\mathcal{E})$ to the system (4.4), intertwining $S_\pm$ in the sense (4.19).*

Remark Lemma 4.15 means that there is no exclusion principle in the system (4.1). This is the system with nontrivial "Bohr's transitions" between any distinct stationary states $S_+ \ne S_-$. Such transition is a purely nonlinear effect, which in general is impossible for linear autonomous Schrödinger or Dirac equations.

Proof Let us consider $S_\pm = (s_\pm(x),0,0)\in\mathcal{S}$ with $s_\pm(x)\equiv z_\pm \in Z$. It is possible to provide the transition $S_-\to S_+$ in different ways. We choose one of them, which is possibly the most obvious. Namely, we construct a solution $Y(t) = (u(\cdot,t),\dot u(\cdot,t),y(t))\in C(\mathbb{R},\mathcal{E})$ such that

$$y(t)\equiv u(0\pm,t) = \begin{cases} z_- & t\le -1, \\ z_+ & t\ge 1. \end{cases} \tag{4.64}$$

We extend $y(t)$ for $t \in (-1,1)$ arbitrarily so that $y \in C^2(\mathbb{R}, \mathbb{R}^d)$. Then we set $g_+ \equiv z_-$ and determine f_- by (4.28):

$$m\ddot{y}(t) = F(y(t)) + 2(f'_-(-t) - \dot{y}(t)), \quad t \in \mathbb{R}. \tag{4.65}$$

Then $f'_-(z) \in C(\mathbb{R}, \mathbb{R}^d)$ and

$$f'_-(-t) = 0, \qquad |t| \geq 1 \tag{4.66}$$

since $F(z_\pm) = 0$. To determine f_- uniquely, we may require that

$$f_-(-t) = z_- \qquad t \leq -1. \tag{4.67}$$

Then the reflected waves g_- and f_+ are determined by (4.31). Since $y(t)$, $f_-(-t)$, and $g_+(t)$ are constant for $|t| \geq 1$, $f_+(-t)$, $g_-(t)$ are also constant for $|t| \geq 1$. Then for $u(x,t)$ defined by (4.33), the function $Y(t) = (u(\cdot,t), \dot{u}(\cdot,t), \dot{u}(0,t)) \in C(\mathbb{R}, \mathcal{E})$ is a solution to (4.4), and (4.19) holds. ■

Remarks i) The constructed solution means that the oscillator is in the stationary point z_- for $t \leq -1$; then the wave $f_-(x-t)$ falls on the oscillator and takes it to the state z_+ by $t = 1$; moreover, for $t > -1$ it generates a pair of reflected waves: $g_-(x+t)$ for $x < 0$ and $f_+(x-t)$ for $x > 0$. The waves run in the strip $-1 < t - |x| < 1$.

ii) Physically, the inequality $z_+ \neq z_-$ means the capture of energy by the oscillator if $V(z_+) > V(z_-)$, or the emission of energy by the oscillator if $V(z_+) < V(z_-)$.

5 3D Nonlinear Wave-Particle System

We consider a real scalar field $\phi(x,t)$, $x\in\mathbb{R}^3$, coupled to a particle with a position $q(t)\in\mathbb{R}^3$:

$$
(5.1)\qquad \left\{\begin{array}{rcl}
\ddot\phi(x,t) &=& \Delta\phi(x,t)-\rho(x-q(t)),\ \ x\in\mathbb{R}^3\\[2ex]
\dfrac{d}{dt}\dfrac{\dot q(t)}{\sqrt{1-\dot q^2(t)}} &=& -\nabla V(q(t))-\displaystyle\int\nabla\phi(x,t)\rho(x-q(t))\,dx,
\end{array}\right.
$$

Denote the conjugate momenta by $\pi(x,t):=\dot\phi(x,t)$ and $p:=\dot q/\sqrt{1-\dot q^2}$. Then the system (5.1) reads as

$$
(5.2)\qquad \left\{\begin{array}{l}
\dot\phi(x,t)=\pi(x,t),\qquad \dot\pi(x,t)=\Delta\phi(x,t)-\rho(x-q(t)),\\[2ex]
\dot q(t)=\dfrac{p(t)}{\sqrt{1+p^2(t)}},\quad \dot p(t)=-\nabla V(q(t))-\displaystyle\int\nabla\phi(x,t)\rho(x-q(t))dx.
\end{array}\right.
$$

This is a Hamiltonian system with the Hamilton functional

$$
\begin{aligned}
H(\phi,\pi,q,p) \;&=\; \int\Big[\frac12|\pi(x)|^2+\frac12|\nabla\phi(x)|^2+\phi(x)\rho(x-q)\Big]dx\\
(5.3)\qquad &\qquad +\sqrt{1+p^2}+V(q).
\end{aligned}
$$

The interaction term of type $\phi(q)$ would result in an energy which is not bounded from below. Therefore we smoothen out the coupling by the real function $\rho(x)$, which is assumed to be radial and to have compact support. More precisely, we assume that

$$
(5.4)\qquad \rho,\nabla\rho\in L^2(\mathbb{R}^3),\qquad \rho(x)=0\ \text{ for }\ |x|\ge R_\rho.
$$

The system has been analyzed in [33, 46, 48]. It is an analog of the Abraham model of Classical Electrodynamics, with an *extended electron*, (5.34), studied in [30, 47]. In analogy to the Maxwell-Lorentz equations we call $\rho(x)$ the “charge distribution”.

Let us consider the Cauchy problem for the system (5.2) with the initial conditions

$$
(5.5)\qquad \phi|_{t=0}=\phi_0(x),\quad \pi|_{t=0}=\pi_0(x),\quad q|_{t=0}=q_0,\quad p|_{t=0}=p_0.
$$

Denote by

$$
Y(t):=(\phi(\cdot,t),\pi(\cdot,t),q(t),p(t))),\qquad\qquad Y_0:=(\phi_0,\pi_0,q_0,p_0).
$$

Then the Cauchy problem (5.1), (5.5) reads

$$
(5.6)\qquad \dot Y(t)=\mathbf{F}(Y(t)),\quad t\in\mathbb{R};\ \ Y(0)=Y_0.
$$

5.1 Phase Space and Dynamics

Let us introduce the phase space for the dynamical system (5.6). Denote by $\|\cdot\|$ resp. $\|\cdot\|_R$ the norm in the Hilbert space $L^2:=L^2(\mathbb{R}^3)$ resp. $L^2(B_R)$, where B_R is the ball $|x|<R$. Let us denote by $\dot H^1$ the completion of the real space $C_0^\infty(\mathbb{R}^3)$ with norm $\|\nabla\phi(x)\|$. Equivalently, using the Sobolev embedding theorem, $\dot H^1=\{\phi(x)\in L^6(\mathbb{R}^3):\ \ |\nabla\phi(x)|\in L^2\}$ (see [53]).

Definition 5.1. *i)* $\mathcal{E} := H_1 \oplus L^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 = \{Y = (\phi(x), \pi(x), q, p)\}$ *is the Hilbert space with the* **global energy norm**

$$\| Y \|_{\mathcal{E}} := \|\nabla\phi\| + \|\pi\| + |q| + |p|.$$

ii) $\mathcal{E}_F$ *is the space* $\mathcal{E}$ *endowed with the topology defined by the* **local energy seminorms**

$$\|Y\|_{\mathcal{E},R} = \|\nabla\phi\|_R + \|\phi\|_R + \|\pi\|_R + |q| + |p|, \qquad R > 0.$$

Proposition 5.2. *i) For every* $Y_0 \in \mathcal{E}$ *the Cauchy problem (5.6) admits a unique solution* $Y(t) \in C(\mathbb{R}, \mathcal{E})$.
ii) The map $W(t) : Y_0 \to Y(t)$ *is continuous in* $\mathcal{E}$ *and* $\mathcal{E}_F$.
iii) The energy is conserved:

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y_0), \qquad t \in \mathbb{R}.$$

iv) The a priori estimate holds,

$$\sup_{t\in\mathbb{R}}(\|\nabla\phi(\cdot,t)\| + \|\pi(\cdot,t)\| + |p(t)|) < \infty, \tag{5.7}$$

$$\sup_{t\in\mathbb{R}} |\dot{q}(t)| \le \overline{v} < 1, \qquad \sup_{t\in\mathbb{R}}(|\ddot{q}(t)| + |\dddot{q}(t)|) < \infty. \tag{5.8}$$

Proof
Step I. Local Existence and Uniqueness The dynamical system (5.1) is a finite-dimensional perturbation of the free wave equation. Hence, it can be rewritten as an integral equation involving the unitary dynamical group of the free equation. Then the local existence and uniqueness follow by the contraction mapping principle.
Step II. Energy Conservation The energy conservation follows by a formal differentiation using the Hamilton structure:

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(Y(t)) &= \langle \mathcal{H}_\phi,\ \dot{\phi}\ \rangle + \langle \mathcal{H}_\pi,\ \dot{\pi}\ \rangle + \langle \mathcal{H}_q,\ \dot{q}\ \rangle + \langle \mathcal{H}_p,\ \dot{p}\ \rangle \\ &= \langle \mathcal{H}_\phi, \mathcal{H}_\pi\rangle - \langle \mathcal{H}_\pi, \mathcal{H}_\phi\rangle + \langle \mathcal{H}_q, \mathcal{H}_p\rangle - \langle \mathcal{H}_p, \mathcal{H}_q\rangle = 0. \end{aligned} \tag{5.9}$$

Step III. A Priori Estimates and Global Existence Next crucial point is the bound

$$\min \int \Big[\frac{1}{4}|\nabla\phi(x)|^2 + \phi(x)\rho(x-q)\Big]\, dx = -\frac{1}{(2\pi)^3}\int \frac{|\hat{\rho}(k)|^2}{|k|^2} d^3k = \langle \rho, \Delta^{-1}\rho\rangle > -\infty, \tag{5.10}$$

where $\hat{\rho}(k) := \displaystyle\int e^{ikx}\rho(x)dx$ is the Fourier transform of the charge density. The bound can be easily checked in the Fourier transform by the Parseval identity. Now the energy conservation implies that

$$\int \Big[\frac{1}{2}|\pi(x)|^2 + \frac{1}{4}|\nabla\phi(x)|^2\Big]\, dx + \sqrt{1+p^2} \le H(Y_0) + \langle \rho, \Delta^{-1}\rho\rangle. \tag{5.11}$$

This obviously implies (5.7). Hence, the global existence of the solution also follows. ∎

Remark 5.3. *The case of the point charge corresponds to* $\rho(x) = \delta(x)$, *hence* $\hat{\rho}(k) \equiv 1$. *Therefore, similarly to (5.10),*

$$\min \mathcal{H}(Y) = -\frac{1}{2(2\pi)^3}\int \frac{d^3k}{|k|^2} + 1 + \min V(x) = -\infty \tag{5.12}$$

which means the Ultraviolet Divergence. In this case the dynamics, probably, does not exist.

5.2 Attraction to Stationary States

Stationary States

Denote the Coulombic potential

$$\phi_q(x) = -\int \frac{\rho(y-q)dy}{4\pi|y-x|}. \tag{5.13}$$

Proposition 5.4. *The set of stationary states of (5.1) is equal to*

$$\mathcal{S} = \{(\phi, \pi, q, p) = (\phi_q, 0, q, 0) =: S_q| \quad q \in Z\}, \tag{5.14}$$

where $Z = \{q \in \mathbb{R}^3 : \nabla V(q) = 0\}$.

Proof The stationary problem (5.1) reads

$$\left\{\begin{array}{rcl} 0 & = & \Delta\phi(x,t) - \rho(x-q),\ x \in \mathbb{R}^3 \\ 0 & = & -\nabla V(q) - \int \nabla\phi(x,t)\rho(x-q)\,dx, \end{array}\right. \tag{5.15}$$

Now the formula (5.13) follows from first Eqn. In the second Eqn, the integral vanishes for $\phi = \phi_q$ since the integrand is antisymmetric w.r.t. reflection in q: the antistymmetry is obvious in the Fourier space. ■

Confining Potential

Let us assume that the potential $V(x)$ is confining, ik.e.

$$V(x) \to \infty, \qquad |x| \to \infty. \tag{5.16}$$

Then the energy conservation (5.9) together with a priori bound (5.7) imply that the particle trajectory is bounded, i.e.

$$\sup_{t\in\mathbb{R}} |q(t)| < \infty. \tag{5.17}$$

Wiener Condition

We introduce an important *Wiener condition* for the Fourier transform of the charge density $\rho(x)$:

$$\hat\rho(k) = \int e^{ikx}\rho(x)dx \neq 0, \qquad k \in \mathbb{R}^3. \tag{5.18}$$

The condition provides a *strong coupling* of the field and particle. Namely, the first equation of (5.1) reads, in the Fourier transform,

$$\ddot{\hat\phi}(k,t) = -|k|^2\hat\phi(k,t) - \hat\rho(k)e^{ikq(t)}. \tag{5.19}$$

If $\rho(k_0) = 0$, then the dynamics of $\hat\phi(k_0, t)$ and $q(t)$ are independent.

Decay of Initial Fields

We will need an additional decay and smoothness of the initial fields.

Definition 5.5. *The space* $\mathcal{E}^\sigma := \{Y = (\phi(x), q, \pi(x), p) \in \mathcal{E}\}$ *such that* $\phi(x) \in C^2$, $\pi(x) \in C^1$ *for* $|x| > \text{const}$, *and*

$$|\phi(x)| + |x|(|\nabla\phi(x)| + |\pi(x)|) + |x|^2(|\nabla\nabla\phi(x)| + |\nabla\pi(x)|) = \mathcal{O}(|x|^{-1/2-\sigma}), \quad |x| \to \infty. \tag{5.20}$$

Remark 5.6. *Note that the states* $Y \in \mathcal{E}^\sigma$ *have finite energy if* $\sigma > 0$, *and, generally, the infinite energy if* $\sigma = 0$.

Theorem 5.7. *([48])*
i) Let (5.18) hold, and $\sigma > 0$. *Then for any* $Y^0 \in \mathcal{E}^\sigma$, *the corresponding solution converges to the set* $\mathcal{S}$, *i.e.*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}, \qquad t \to \pm\infty. \tag{5.21}$$

ii) Let Z *be a discrete subset in* $\mathbb{R}^3$. *Then*

$$Y(t) \xrightarrow{\mathcal{E}_F} S^\pm \in \mathcal{S}, \qquad t \to \pm\infty. \tag{5.22}$$

Proof ii) follows from i). Main ingredient in the proof of i) is the following crucial lemma.

Lemma 5.8. *Let all conditions of Theorem 5.7 hold. Then the relaxation of the acceleration holds,*

$$\ddot{q}(t) \to 0, \qquad t \to \pm\infty. \tag{5.23}$$

Let us sketch the proof of the lemma. *Step I. Scattering of Energy to Infinity* We analyse the radiation to infinity and prove the bound

$$I := \int_0^\infty \Big(\int_{S_1} |R(\omega, t)|^2 d\omega\Big) dt < \infty, \tag{5.24}$$

where $S_1 := \{\omega \in \mathbb{R}^3 : \omega = 1\}$, and

$$R(\omega, t) = \int\limits_{\mathbb{R}^3} \frac{\rho(y - q(t + \omega\cdot y))\omega \cdot \ddot{q}(t + \omega \cdot y)}{(1 - \omega \cdot \dot{q}(t + \omega \cdot y))^2} dy \tag{5.25}$$

The integral I is a part of the total energy radiated to infinity, and $|R(\omega, t)|^2$ is the corresponding "power of radiation in the direction ω". The formula (5.25) follows by an asymptotic analysis of the Poynting vector for the energy flux,

$$S(x, t) = -\pi(x, t)\nabla\phi(x, t). \tag{5.26}$$

Namely, we substitute here the Liénard-Wiehert type expressions for $\pi(x, t)$ and $\nabla\phi(x, t)$ following from the Kirchhoff formula for the solution of the first Eqn of (5.1) (cf. (2.8)). Then we get the asymptotics

$$S(\omega r, r + t) \cdot \omega \sim \frac{|R(\omega, t)|^2}{r^2}, \qquad r \to \infty. \tag{5.27}$$

At last, the integral (5.24) is finite since the total energy is finite.

Remark 5.9. *i) The RHS of (5.25) is an integral over the 3D hyperplane in the Minkowski 4D space. The hyperlane appears as the limit of the conical surface in the Kirchhoff formula.*
ii) For small velocities, $|\dot{q}(t)| \ll 1$ *(i.e. in nonrelativistic approximation),* $|R(\omega, t)|^2$ *coincides with known Larmor-Lénard expression for the power of radiation, [62].*

Step II. Time Decay of the Radiation The bounds (5.8) imply that

$$\sup_{\omega\in S_1, t\in\mathbb{R}} \dot{R}(\omega,t) < \infty. \tag{5.28}$$

Hence, (5.24) implies that

$$R(\omega,t)\to 0, \qquad t\to\infty. \tag{5.29}$$

Step III. Convolution Representation Let us fix an $\omega\in S_1$, and set set $r(t)=\omega\cdot q(t)$, $s=\omega\cdot y$, and $\rho_1(s)=\int dq_1 dq_2 \rho(q_1,q_2,s)$. Furtheremore, introduce new "time variable" $\theta=\theta(\tau):=\tau-r(\tau)$. The change of variables $\tau\to\theta$ is diffeomorphism $\mathbb{R}\to\mathbb{R}$ since $|\dot r|\le|\dot q|\le\overline{v}<1$ by (5.8). Now (5.25) reads as a convolution

$$\begin{aligned} R(\omega,t) &= \int ds\,\rho_1(s-r(t+s))\,\frac{\ddot r(t+s)}{(1-\dot r(t+s))^2}\\ &= \int d\tau\,\rho_1(t-(\tau-r(\tau)))\,\frac{\ddot r(\tau)}{(1-\dot r(\tau))^2}\\ &= \int d\theta\,\rho_1(t-\theta)g_\omega(\theta)=\rho_1 * g_\omega(t), \end{aligned} \tag{5.30}$$

where

$$g_\omega(\theta):=\frac{\ddot r(\tau(\theta))}{(1-\dot r(\tau(\theta)))^3}.$$

Step IV. Wiener's Tauberian Theorem Now (5.29) reads

$$\rho_a * g_\omega(t)\to 0, \qquad t\to\infty. \tag{5.31}$$

The derivative $g'_\omega(\theta)$ is bounded, and $\hat\rho_1(k)=\hat\rho(k\omega)\neq 0$, $k\in\mathbb{R}$, by (5.18). Hence (5.31) implies by Pitt's extension to Wiener's Tauberian Theorem, cf. [61, Thm. 9.7(b)],

$$\lim_{\theta\to\infty} g_\omega(\theta)=0. \tag{5.32}$$

Since $\omega\in S_1$ is arbitrary and $\theta(t)\to\infty$ as $t\to\infty$, we have proved (5.23) for $t\to\infty$. For $t\to-\infty$ the lemma follows similarly. Lemma 5.8 is proved. ■

Remarks 5.10. *i) The spherical symmetry of ρ in (5.4) is not necessary.*
ii) We use the Wiener condition (5.18) in the application of the Wiener Tauberian Theorem. However, (5.18) is not necessary. Namely, (5.23) and Theorem 5.7 i) hold for $V(x)\equiv 0$ and $\|\rho\|\ll 1$ (see next section).
iii) The Wiener condition (5.18) describes an "open set" among all the functions $\rho(x)$ satisfying (5.4). In this sense, the asymptotics (5.21), (5.22) hold for "almost all" equations (5.1) which agrees with the conjecture $\mathcal{G}$ from Introduction.
iv) The condition (5.20) can be considerably weakened or modified. For instance, one may suppose that

$$|x|(|\nabla\phi(x)|+|\phi(x)|+|\pi(x)|)=\mathcal{O}(|x|^{-1/2-\sigma}), \quad |x|\to\infty. \tag{5.33}$$

Furtheremore, it suffices to suppose the decay (5.33) in an integral sense.

5.3 Extension to Maxwell-Lorentz Eqns

Theorem 5.7 was extended in [47] to the coupled Maxwell-Lorentz Eqns:

$$(5.34)\begin{cases} \operatorname{div} E(x,t)=\rho(x-q(t)), & \operatorname{rot}\, E(x,t) = -\dot{B}(x,t), \\ \operatorname{div} B(x,t) = 0, & \operatorname{rot}\, B(x,t) = \dot{E}(x,t) + \rho(x - q(t))\dot{q}(t), \\ \dot{q}(t) = \dfrac{p(t)}{\sqrt{1+p^2(t)}}, & \dot{p}(t)=\displaystyle\int \Big[E(x,t)+\overline{E}(x)+\dot{q}(t)\wedge(B(x,t)+\overline{B}(x))\Big]\rho(x-q(t))dx. \end{cases}$$

In this case, the relaxation of the acceleration (5.23) solves the problem of *Radiative Damping* in Classical Electrodynamics *for one particle*. The relaxation is claimed in Classical Electrodynamics by Lorentz and others, however it was not proved previously.

Chapter 2

Translation Invariance: Soliton Scattering Asymptotics

1 Translation-Invariant Wave-Particle System

Now we consider the coupled system (5.1) without an external potential, i.e. $V(x) \equiv 0$:

$$(1.1)\qquad \left\{ \begin{array}{rcl} \ddot{\phi}(x,t) & = & \Delta\phi(x,t) - \rho(x-q(t)),\ x \in \mathbb{R}^3 \\ \displaystyle\frac{d}{dt}\frac{\dot{q}(t)}{\sqrt{1-\dot{q}^2}} & = & \displaystyle -\int \nabla\phi(x,t)\rho(x-q(t))\,dx \end{array} \right.$$

In the Hamilton form,

$$(1.2)\qquad \left\{ \begin{array}{ll} \dot{\phi}(x,t) = \pi(x,t), & \dot{\pi}(x,t) = \Delta\phi(x,t) - \rho(x-q(t)), \\ \dot{q}(t) = \displaystyle\frac{p(t)}{\sqrt{1+p^2(t)}}, & \dot{p}(t) = -\nabla V(q(t)) - \displaystyle\int \nabla\phi(x,t)\rho(x-q(t))dx. \end{array} \right.$$

In this case Proposition 5.2 holds as well.

1.1 Soliton Solutions

The system (1.1) is translation invariant. This means that $(\phi(x-a,t), \pi(x-a,t), q(t)+a, p(t)))$ is a solution of the system if $(\phi(x,t), \pi(x,t), q(t), p(t)))$ is one and $a \in \mathbb{R}^3$. Therefore, it is natural to consider the solitons, i.e. traveling wave solutions of the form

$$(1.3)\qquad Y(t) = (\phi_v(x-vt-a), \pi_v(x-vt-a), vt+a, p_v)$$

with fixed $v, a \in \mathbb{R}^3$. Here the shift a obviously is an arbitrary vector of $\mathbb{R}^3$. In contrast, $|v| < 1$ since $\dot{q}(t) = v = p_v/\sqrt{1+p_v^2}$. Obviously, $p_v = v/\sqrt{1-v^2}$.

Lemma 1.1. *i) The soliton solutions exist for arbitrary $v \in \mathbb{R}^3$ with $|v| < 1$ and do not exist for $|v| \geq 1$.*
ii) The functions $\phi_v(x)$, $\phi_v(x)$ are smooth for $|x| > R_\rho$, and (cf. (5.20) of Chapter I)

$$(1.4)\qquad |\partial_x^\alpha \phi_v(x)| + |x||\partial_x^\alpha \pi_v(x)| \leq C_\alpha(v)|x|^{-1-|\alpha|},\quad |x| > R_\rho.$$

Proof First note that $\partial_t\phi(x-vt-a) = -(v\cdot\nabla\phi_v)(x-vt-a)$. Therefore, substituting to the first equation in (1.1), we get

$$(-v\cdot\nabla)^2\phi_v(y) = \Delta\phi_v(y) - \rho_v(y),\ \ y \in \mathbb{R}^3.$$

Let us solve this equation by the Fourier transform:

$$[-k^2+(v\cdot k)^2]\hat{\phi}_v(k)=\hat{\rho}(k),\ \ k\in\mathbb{R}^3.$$

It implies that

$$\hat{\phi}_v(k)=\frac{\hat{\rho}(k)}{-k^2+(v\cdot k)^2}.$$

Hence $|v|<1$: otherwise, $\hat{\phi}_v(k)\notin L^2(\mathbb{R}^3)$. Taking the inverse Fourier transform, we obtain the convolution

$$\phi_v(x)=\rho*E_v(x),\qquad E_v(x):=F^{-1}\frac{1}{-k^2+(v\cdot k)^2} \tag{1.5}$$

For example, for $v=0$ we have

$$E_0(x)=F^{-1}\frac{1}{-k^2}=-\frac{1}{4\pi|x|}.$$

Hence,

$$\phi_0(x)=(\rho*E_0)(x)=-\frac{1}{4\pi}\int\frac{\rho(y)\,dy}{|x-y|}.$$

For general $v\neq 0$ choose the coordinates (k_1,k_2,k_3) in such a way that $v=(\mathrm{v},0,0)$. Then $v\cdot k=\mathrm{v}k_1$ and

$$k^2-(v\cdot k)^2=(1-\mathrm{v}^2)k_1^2+k_2^2+k_3^2. \tag{1.6}$$

Denote by $\gamma=\sqrt{1-\mathrm{v}^2}$ and set $\xi_1:=\gamma k_1$, $\xi_2:=k_2$, $\xi_3:=k_3$. Then the inverse Fourier transform reads

$$\begin{aligned}E_v(x)&=-(2\pi)^{-3}\int\frac{e^{-i(k_1x_1+k_2x_2+k_3x_3)}}{k^2-(v\cdot k)^2}\,dk=-(2\pi)^{-3}\int\frac{e^{-i\frac{\xi_1}{\gamma}x_1+\xi_2x_2+\xi_3x_3}}{|\xi|^2}\frac{d\xi}{\gamma}\\ &=\frac{1}{\gamma}E_0(\frac{x_1}{\gamma},x_2,x_3)=-\frac{1}{\gamma}\frac{1}{4\pi\sqrt{\frac{x_1^2}{\gamma^2}+x_2^2+x_3^2}}.\end{aligned}$$

Hence, for every multiindex $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ the bounds hold

$$|\partial_x^\alpha\mathcal{E}_v(x)|\le C(v)|x|^{-1-|\alpha|},\qquad x\neq 0. \tag{1.7}$$

Now the bounds (1.4) follow from the convolution representation (1.5). ■

2 The Kirchhoff Representations

Consider the Cauchy problem for the wave equation (2.1) (cf. (2.2))

$$\ddot{\varphi}(x,t)=\Delta\varphi(x,t),\ \ \varphi|_{t=0}=\varphi_0(x),\ \dot{\varphi}|_{t=0}=\psi_0(x) \tag{2.1}$$

Denote by $S_t(x)=\{y\in\mathbb{R}^3:\ |y-x|=t\}$

Lemma 2.1. *For $\varphi_0\in C^2(\mathbb{R}^3)$ and $\psi_0\in C^1(\mathbb{R}^3)$ the solution to the Cauchy problem (2.1) can be represented as*

$$\varphi(x,t)=\frac{1}{4\pi t}\int\limits_{S_t(x)}\psi_0(y)\,dS(y)+\sum_{|\alpha|\le 1}t^{|\alpha|-2}\int\limits_{S_t(x)}a_\alpha(x,y)\partial_y^\alpha\varphi_0(y)\,dS(y),$$

where $|a_\alpha(x,y)|\le\mathrm{const}$ for $x,y\in\mathbb{R}^3$.

Proof The solution is given by the Kirchhoff formula (2.8):

$$\varphi(x,t) = \frac{1}{4\pi t}\int\limits_{S_t(x)} \psi_0(y)\,dS(y) + \frac{\partial}{\partial t}\Big(\frac{1}{4\pi t}\int\limits_{S_t(x)} \varphi_0(y)\,dS(y)\Big).$$

Here the last integral can be written as

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\frac{t^2}{4\pi t}\int\limits_{|\omega|=1} \varphi_0(x+t\omega)\,d\omega\Big) &= \frac{1}{4\pi}\int\limits_{|\omega|=1} \varphi_0(x+t\omega)d\omega + \frac{t}{4\pi}\int\limits_{|\omega|=1} \omega\cdot\nabla\varphi_0(x+t\omega)d\omega \\ &= \frac{1}{4\pi t^2}\int\limits_{S_t(x)} \varphi_0(y)dS + \frac{1}{4\pi t}\int\limits_{S_t(x)} \frac{y-x}{t}\cdot\nabla\varphi_0(y)dS. \qquad \blacksquare\end{aligned}$$

Similar arguments imply the following representations for the derivatives,

$$\nabla\varphi(x,t) = \sum_{|\alpha|\le 1} t^{|\alpha|-2}\int\limits_{S_t(x)} b_\alpha(x,y)\partial_y^\alpha\psi_0(y)dS(y) + \sum_{|\alpha|\le 2} t^{|\alpha|-3}\int\limits_{S_t(x)} c_\alpha(x,y)\partial_y^\alpha\varphi_0(y)dS(y), \tag{2.2}$$

$$\dot\varphi(x,t) = \sum_{|\alpha|\le 1} t^{|\alpha|-2}\int\limits_{S_t(x)} d_\alpha(x,y)\partial_y^\alpha\psi_0(y)dS(y) + \sum_{|\alpha|\le 2} t^{|\alpha|-3}\int\limits_{S_t(x)} e_\alpha(x,y)\partial_y^\alpha\varphi_0(y)dS(y), \tag{2.3}$$

where

$$\sup_{\substack{x,y\in\mathbb{R}^3\\ |x-y|=t}} \Big(|b_\alpha(x,y)| + |c_\alpha(x,y)| + |d_\alpha(x,y)| + |e_\alpha(x,y)|\Big) < \infty. \tag{2.4}$$

3 Scattering of Solitons

Introduce the notations

$$F(x,t) = \begin{pmatrix}\phi(x,t)\\ \pi(x,t)\end{pmatrix}, \quad F_v = \begin{pmatrix}\phi_v\\ \pi_v\end{pmatrix}, \quad A = \begin{pmatrix}0 & 1\\ \Delta & 0\end{pmatrix}, \quad \overline{\rho} = \begin{pmatrix}0\\ \rho\end{pmatrix}$$

Then the first equation of (1.1) can be written as

$$\dot F(x,t) = AF(x,t) - \overline{\rho}(x-q(t)). \tag{3.1}$$

In particular, for the soliton (1.3), this gives

$$\frac{d}{dt}F_v(x-vt-a) = AF_v(x-vt-a) - \overline{\rho}(x-vt-a), \tag{3.2}$$

or equivalently,

$$-v\cdot\nabla F_v(y) = AF_v(y) - \overline{\rho}(y), \qquad y\in\mathbb{R}^3. \tag{3.3}$$

Definition 3.1. *i) $\mathcal{F}$ is the Hilbert space of the function $F(x) = (\phi(x),\pi(x))$ endowed with the* **global energy norm**

$$\|F\|_{\mathcal{F}}^2 := \int \Big(|\nabla\phi(x)|^2 + |\pi(x)|^2\Big)\,dx.$$

ii) In the space $\mathcal{F}$ define the **local energy seminorms**

$$\|F(x,t)\|_{\mathcal{F},R}^2 := \int\limits_{|x|<R} \Big(|\nabla\phi(x,t)|^2 + |\pi(x,t)|^2\Big)\,dx, \qquad R>0.$$

Denote by $W_0(t)$ the dynamical group of the wave equation (2.1), i.e. $W_0(t) : (\varphi_0(\cdot), \psi_0(\cdot)) \mapsto (\psi(t,\cdot), \dot\psi(t,\cdot)$. The operators $W_0(t)$ are unitary in the Hilbert space $\mathcal{F}$ by the energy conservation (formula (2.3) of Chapter I).

Let us assume that the coupling function ρ is small:

$$\|\rho\| \le r \ll 1. \tag{3.4}$$

Now we can formulate our main result for the system (1.1).

Theorem 3.2. *Let the condition (5.4) of Chapter I, and (3.4) hold, and $\sigma \in (0,1]$. Then for every initial state $(\phi_0, \pi_0, q_0, p_0) \in \mathcal{E}^\sigma$, the corresponding solution satisfies the following asymptotics:*
i) the relaxation of the acceleration holds,

$$|\ddot q(t)| \le C(1+|t|)^{-\sigma-1}. \tag{3.5}$$

ii) The velocity has the limits $v_\pm \in \mathbb{R}^3$ as $t \to \pm\infty$, and

$$|\dot q(t) - v_\pm| \le C(1+|t|)^{-\sigma}. \tag{3.6}$$

iii) The fields converge in the **local energy seminorms** *centered at the particle,*

$$\|F(q(t)+x,t) - F_{v_\pm}(x)\|_{\mathcal{F},R} \le C_R(1+|t|)^{-1-\sigma}. \tag{3.7}$$

iv) The **scattering asymptotics** *hold (cf. (0.33))*

$$F(x,t) = F_{v_\pm}(x - q(t)) + W_0(t)\Psi_\pm + r_\pm(x,t), \tag{3.8}$$

with some scattering states $\Psi_\pm \in \mathcal{E}$, and the remainder is small in the **global energy norm***:*

$$\|r_\pm(\cdot,t)\|_{\mathcal{F}} \to 0, \quad t \to \pm\infty. \tag{3.9}$$

Remarks 3.3. *i) The asymptotics $q(t) \sim v_\pm t + a_\pm$ are not proved yet.*
ii) We suppose that the smallness condition (3.4) can be replaced by the **Wiener condition** *(5.18) used in [46, 47, 48]. We have proved in [46] that (cf. (3.5)-(3.7))*

$$\ddot q(t) \to 0, \quad \dot q(t) \to v_\pm, \quad \|F(q(t)+x,t) - F_{v_\pm}(x)\|_{\mathcal{F},R} \to 0, \qquad t \to \pm\infty, \tag{3.10}$$

under the Wiener condition. However, the proving of the asymptotics (3.5)-(3.9), under the Wiener condition, is still an open problem.
iii) The condition on initial data can be modified by (5.33) of Chapter I.

On Inelastic Scattering of Solitons

The asymptotics (3.8) means an *inelastic scattering* of the solitons by the nonlinear interaction: the *incident scattering data* (F_{v_-}, Ψ_-) at $t = -\infty$ transfer to the *outgoing scattering data* (F_{v_+}, Ψ_+) at $t = +\infty$, see Fig. 2.1. This suggests to introduce the *nonlinear scattering operator*

$$\mathbf{S} : (v_-, b_-, \Psi_-) \mapsto (v_+, b_+, \Psi_+), \tag{3.11}$$

if we would know the asymptotics $q(t) \sim v_\pm t + b_\pm$ for $t \to \pm\infty$. However the justification of this asymptotics and definition is an open question. The *elastic scattering* corresponds to the case $v_+ = v_-$.

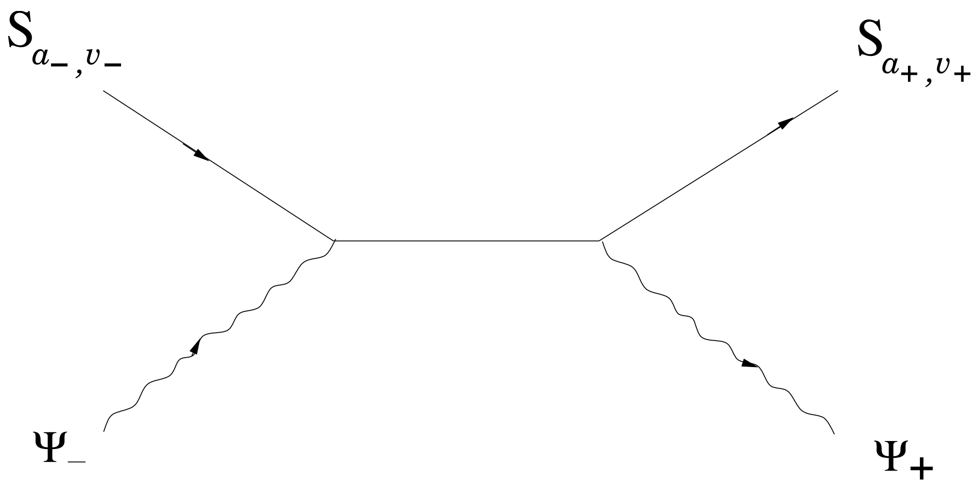

Figure 2.1: Wave-particle scattering.

4 Integral Inequality Method

We prove Theorem 3.2.
Step i) Define $v(t) := \dot{q}(t)$ and

$$Z(x,t) := F(x,t) - F_{v(t)}(x - q(t)).$$

Let us differentiate:

$$\begin{aligned}\partial_t\Big(F_{v(t)}(x-q(t))\Big) &= -v(t)\nabla_x\Big(F_{v(t)}(x-q(t))\Big) \\ &\quad +\dot{v}(t)\cdot\nabla_v\Big(F_{v(t)}(x-q(t))\Big).\end{aligned}$$

Then (3.3) implies that

$$\begin{aligned}\partial_t\Big(F_{v(t)}(x-q(t))\Big) &= AF_{v(t)}(x-q(t)) - \overline{\rho}(x-q(t)) \\ &\quad +\dot{v}(t)\cdot\nabla_v\Big(F_{v(t)}(x-q(t))\Big).\end{aligned}$$

Subtracting this formula from (3.1), we get that

$$\dot{Z}(x,t) = AZ(x,t) - \dot{v}(t)\cdot\nabla_v\Big(F_{v(t)}(x-q(t))\Big). \tag{4.1}$$

Step ii) Let us derive from (4.1) an integral inequality for $\|Z(\cdot,t)\|_R$. First, write the Duhamel integral representation:

$$Z(t) = W_0(t)Z(0) - \int_0^t W_0(t-s)\Big[\dot{v}(s)\cdot\nabla_v\Big(F_{v(s)}(\cdot - q(s))\Big)\Big]\,ds. \tag{4.2}$$

Denote by Q_t the shift $f(x) \mapsto f(q(t)+x)$. Then (4.2) implies that

$$\|Q_tZ(t)\|_{\mathcal{F},R} \le \|Q_tW_0(t)Z(0)(t)\|_{\mathcal{F},R} + \int_0^t |\dot{v}(s)|\,\Big\|Q_tW_0(t-s)\Big[\nabla_v\Big(F_{v(s)}(\cdot - q(s))\Big)\Big]\Big\|_{\mathcal{F},R}\,ds. \tag{4.3}$$

We will estimate the RHS by the following three lemmas.

Lemma 4.1. *For $R > R_\rho$ the bounds hold,*

$$|\dot v(t)| \le C\|Q_t Z(t)\|_{\mathcal{F},R}\,\|\rho\|. \tag{4.4}$$

Proof First, (1.2) implies that

$$v(t) = \dot q(t) = \frac{p(t)}{\sqrt{1+p^2(t)}}. \tag{4.5}$$

Therefore,

$$|\dot v(t)| \le C|\dot p(t)|. \tag{4.6}$$

Second, for the solitons, $v(t) \equiv v$, and second equation of (1.1) implies that

$$0 = \frac{d}{dt}\frac{v}{\sqrt{1-v^2}} = \int \nabla\phi_v(x-q)\rho(x-q)dx, \ \ \forall v, q \in \mathbb{R}^3.$$

Therefore,

$$\dot p(t) = -\int \nabla\phi(x,t)\rho(x-q(t))\,dx \ = \ -\int \nabla\Big(\phi(x,t) - \phi_{v(t)}(x-q(t))\Big)\rho(x-q(t))dx \\ -\int \nabla Z_1(x,t)\rho(x-q(t))dx \tag{4.7}$$

since $\phi(x,t) - \phi_{v(t)}(x-q(t)) = Z_1(x,t)$. Hence, finally,

$$|\dot p(t)| \le \int\limits_{|x-q(t)|\le R_\rho} |\nabla Z_1(x,t)\rho(x-q)|dx \le \|Q_t Z\|_{\mathcal{F},R}\,\|\rho\| \qquad \blacksquare$$

Lemma 4.2. *Let the initial state satisfy (5.20). Then the bounds hold*

$$\|[Q_t W_0(t)Z(0)](t)\|_{\mathcal{F},R} \le C_0(1+|t|)^{-3/2-\sigma}. \tag{4.8}$$

Proof Denoting $[W_0(t)Z(0)](x) := (\varphi(x,t), \dot\varphi(x,t))$, we get

$$\|Q_t W_0(t)Z(0)\|_{\mathcal{F},R} \ \sim \ \|\nabla\varphi(q(t)+x,t)\|_{\mathcal{F},R} + \|\dot\varphi(q(t)+x,t)\|_{\mathcal{F},R} \\ \le \ C(R)\Big(\max_{|x|\le R}|\nabla\varphi(q(t)+x,t)| + \max_{|x|\le R}|\dot\varphi(q(t)+x,t)|\Big). \tag{4.9}$$

The Kirchhoff representation (2.2) implies that

$$\nabla\varphi(q(t)+x,t) \ = \sum_{|\alpha|\le 1} t^{|\alpha|-2} \int\limits_{S_t(q(t)+x)} b_\alpha(x,y)\partial_y^\alpha\Big(\pi_0(y) - \pi_{v(0)}(y)\Big)dS(y) \\ + \sum_{|\alpha|\le 2} t^{|\alpha|-3} \int\limits_{S_t(q(t)+x)} c_\alpha(x,y)\partial_y^\alpha\Big(\phi_0(y) - \phi_{v(0)}(y)\Big)dS(y) \tag{4.10}$$

At last, the a priori bounds (5.8) of Chapter I imply that $|q(t)| \le \overline{v}t + |q(0)|$, where $\overline{v} < 1$. Hence,

$$|y| \ge |y-q(t)| - |q(t)| \ge t - \overline{v}t - |q_0| = (1-\overline{v})t - c \tag{4.11}$$

for $y \in S_t(q(t)+x)$. Now (4.10) implies the time-decay

$$|\nabla\varphi(q(t)+x,t)| \le Ct^{-3/2-\sigma} \tag{4.12}$$

by the space-decay (5.20) of Chapter I and the bounds (1.4). Similar time-decay follows for $\dot\varphi(q(t)+x,t)$ from the Kirchhoff representation (2.3). Therefore, (4.9) implies (4.8). $\blacksquare$

Lemma 4.3. *For $|v| \leq \overline{v}$ the uniform bounds hold,*

$$\|[Q_t W_0(t-s)]\nabla_v F_v(\cdot - q(s))\|_{\mathcal{F},R} \leq \frac{C(\overline{v})\|\rho\|}{1+|t-s|^2}. \tag{4.13}$$

Proof The bounds follow similarly to (4.8) since the soliton fields F_v, and all their derivatives in v, satisfy the uniform estimates (1.4) corresponding to (5.20) of Chapter I with $\sigma = 1/2$.
Step iii) Denote by $n(t) := \|Q_t Z(t)\|_{\mathcal{F},R}$. Then (4.3) and Lemmas 4.1-4.3 imply the desired integral inequality

$$n(t) \leq C_0(1+|t|)^{-3/2-\sigma} + \int_0^t \frac{C_1\|\rho\|^2 n(s)ds}{1+|t-s|^2} \tag{4.14}$$

Multiply both sides by $(1+|t|)^{1+\sigma}$ and denote by $N(t) := (1+|t|)^{1+\sigma} n(t)$. Then we get

$$N(t) \leq C_0(1+|t|)^{-1/2} + \int_0^t \frac{C_1\|\rho\|^2(1+|t|)^{1+\sigma}N(s)ds}{(1+|t-s|^2)(1+|s|)^{1+\sigma}} \tag{4.15}$$

Let us denote

$$I_\sigma(t) := \int_0^t \frac{(1+|t|)^{1+\sigma}ds}{(1+|t-s|^2)(1+|s|)^{1+\sigma}}. \tag{4.16}$$

Then (4.15) implies that

$$N(t) \leq C_0 + C_1 r^2 \max_{s\in[0,t]} N(s) I_\sigma(t) \tag{4.17}$$

since $\|\rho\| \leq r$ by (3.4).

Exercise 4.4. *Check that for $0 < \sigma \leq 1$, we have*

$$\sup_{t\in\mathbb{R}} I_\sigma(t) \leq B_\sigma < \infty. \tag{4.18}$$

Now (4.17) implies that

$$M(t) \leq C_0 + C_1 r^2 B_\sigma M(t), \tag{4.19}$$

where $M(t) := \max_{s\in[0,t]} N(s)$. Therefore,

$$M(t)(1 - C_1 r^2 B_\sigma) \leq C_0 \tag{4.20}$$

Hence we get for $r \ll 1$ that

$$\sup_{t\geq 0} M(t) < \infty \tag{4.21}$$

which implies Theorem 3.2 *iii)*:

$$\|Z(q(t)+x,t)\|_{\mathcal{F},R} \leq C(1+|t|)^{-1-\sigma}.$$

Now Lemma I implies Thm 3.2 *i)*. Hence also Thm 3.2 *ii)* is proved.
Step iv) To prove Thm 3.2 *iv)*, it suffices to check the bounds

$$\|Z(x,t) - W_0(t)F_\pm\|_{\mathcal{F}} \leq C(1+|t|)^{-\sigma} \qquad t \in \mathbb{R}$$

with appropriate scattering states $F_\pm$. This is equivalent to bounds

$$\|W_0(-t)Z(x,t) - F_\pm\|_{\mathcal{F}} \leq C(1+|t|)^{-\sigma}$$

since the group $W_0(t)$ is an isometry in the global energy norm $\|\cdot\|_{\mathcal{F}}$ of the Hilbert space $\mathcal{F}$. Let ua apply $W_0(-t)$ to the Duhamel integral equation (4.2):

$$W_0(-t)Z(t) = Z(0) - \int_0^t W_0(-s)[\dot{v}(s)\cdot\nabla_v F_{v(s)}(\cdot - q(s))]ds.$$

The a priori bounds $|v(s)| \le \overline{v} < 1$ and (1.4) provide that

$$\sup_{s\in\mathbb{R}} \|\nabla_v F_{v(s)}(\cdot - q(s))\|_{\mathcal{F}} < \infty. \tag{4.22}$$

Therefore, Thm 3.2 *i)* implies the convergence of the integral in the norm $\|\cdot\|_{\mathcal{F}}$ at the stated rate. ■

Chapter 3

Rotation Invariance: Attraction to Nonlinear Eigenfunctions

1 Rotation-Invariant Equations

Consider nonlinear Klein-Gordon equations of the type

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - m^2\psi(x,t) + f(x,\psi(x,t)), \qquad x \in \mathbb{R}^d. \tag{1.1}$$

Assume that $\psi(x,t) \in \mathbb{C}^n$ and

$$f(x,\psi) = -\nabla_\psi V(x,\psi), \qquad \psi \in \mathbb{C}^n, \tag{1.2}$$

where the gradient is understood in the real sense, i.e. with respect to $(\operatorname{Re}\psi, \operatorname{Im}\psi) \in \mathbb{R}^{2n}$. Then the equation is formally a Hamiltonian system with the Hamilton functional

$$\mathcal{H}(\psi,\dot{\psi}) = \int \Big(\frac{|\dot{\psi}(x)|^2}{2} + \frac{|\nabla\psi(x)|^2}{2} + V(x,\psi(x))\Big)dx. \tag{1.3}$$

Further, assume that the potential is radial, i.e.

$$V(x,\psi) = \mathcal{V}(x,|\psi|), \qquad \psi \in \mathbb{C}^n. \tag{1.4}$$

Then the equation is G-invariant with respect to the unitary group $U(n)$ since

$$f(x,g\psi) = gf(x,\psi), \qquad g \in U(N), \quad x \in \mathbb{R}^d, \quad \psi \in \mathbb{C}^n. \tag{1.5}$$

This identity holds since the differentiation of (1.4) gives that

$$f(x,\psi) = -\nabla_\psi \mathcal{V}(x,|\psi|) = -\mathcal{V}'(x,|\psi|)\frac{\psi}{|\psi|} = a(x,|\psi|)\psi, \tag{1.6}$$

where $\mathcal{V}'(x,r) := \partial_r\mathcal{V}(x,r)$ and $a(x,r) := \mathcal{V}'(x,r)/r$.

Definition 1.1. Solitary Waves *are the solutions of type $e^{i\Omega t}\psi(x)$, with an Hermitian $n\times n$-matrix Ω.*

Example 1.2. For $n = 1$, the group $U(1) = \{e^{i\theta} : \theta \in [0,2\pi]\}$. Hence (1.5) becomes the $U(1)$-symmetry:

$$f(x,e^{i\theta}\psi) = e^{i\theta}f(x,\psi), \quad \theta \in [0,2\pi], \ x \in \mathbb{R}^d, \ \psi \in \mathbb{C}. \tag{1.7}$$

Then the solitary waves are the solutions of type $e^{i\omega t}\psi(x)$, with $\omega \in \mathbb{R}$. Substitution to (1.1) gives the *nonlinear eigenvalue problem*

$$-\omega^2\psi(x) = \Delta\psi(x) - m^2\psi(x) + f(x,\psi(x)), \qquad x \in \mathbb{R}^d. \tag{1.8}$$

Existence of Solitary Waves

For $n = 1$ the existence of the solitary waves for a general class of equations (1.1) with $f(x,\psi) \equiv f(\psi)$ has been proved by H.Beresticky & P.Lions, [3]: the nonzero solitary waves $\psi(x)e^{i\omega t}$ exist for an interval of $\omega \in (-m, m)$ for a wide class of the potential $\mathcal{V}(|\psi|)$.
For general systems (1.1) with $n > 1$ the existence of the solitary waves is not proved. However, the existence is proved for the coupled Maxwell-Dirac equations (0.21) by M.Esteban, V.Georgiev and E.Séré, [14]. The system is $U(1)$-invariant with respect to the representation $g_* : (A(x), \psi(x)) \mapsto (A(x), e^{i\theta}\psi(x))$ of the rotations $g = e^{i\theta}$. In this case solitary waves are the solutions $(\phi(\mathbf{x}), e^{i\omega t}\psi(\mathbf{x}))$, where $t = x_0/c$. The nonzero solitary waves exist for $\omega\hbar \in (0, mc^2)$.

Problem of Attraction to Solitary Waves

The results on the existence of the solitary waves suggest the following problem:

To prove that each finite energy solution of the $U(1)$-invariant equations of type (1.1) converges to a solitary wave in the long-time limit:

(1.9) $$\psi(x,t) \sim \psi_\pm(x)e^{i\omega_\pm t}, \quad t \to \pm\infty.$$

The problem is inspired by Bohr's basic postulate on transitions to Quantum Stationary States. Namely, the asymptotics (1.9) correspond to the Bohr "quantum jump"

(1.10) $$\begin{array}{ccc} |\omega_-\rangle & -\,-\,-\,-\,-\,-\,-\,-\,- > & |\omega_+\rangle \\ t=-\infty & & t=+\infty \end{array}$$

Below we give a partial result on the convergence for a special type of the function $f(x,\psi) = \delta(x)F(\psi)$. More precisely, we will establish the attraction to the set (0.28) which is the set of all solitary waves (1.9).

2 Klein-Gordon Equation Coupled to a Nonlinear Oscillator

Consider the 1D equation (1.1) with the particular form of the nonlinear interaction $f(x,\psi) = \delta(x)f(\psi)$:

(2.1) $$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \delta(x)F(\psi(0,t)),\ x \in \mathbb{R}$$

We assume that $m > 0$ and

(2.2) $$F(\psi) = -\nabla\mathcal{V}(|\psi|).$$

Then (2.1) formally is a Hamiltonian system with the Hamilton functional

(2.3) $$H(\psi,\dot{\psi}) = \frac{1}{2}\int \left[|\dot{\psi}(x)|^2 + |\nabla\psi(x)|^2 + m^2|\psi(x)|^2\right] dx + \mathcal{V}(|\psi(0)|),$$

and it is $U(1)$-invariant. We assume that the equation is **strictly nonlinear** in the following sense:

(2.4) $$\mathcal{V}(|\psi|) = \sum_{k\le N} \mathcal{V}_k|\psi|^{2k}, \qquad \mathcal{V}_N > 0,\ N \ge 2.$$

Then $F(\psi)$ is the polynomial in ψ and $\overline{\psi}$ of a degree **at least three**.

2.1 Existence of Dynamics

Introduce the phase space for the equation (2.1) (cf. Definition 3.1). Now $\|\cdot\|$ resp. $\|\cdot\|_R$ stands for the norm in $L^2 := L^2(\mathbb{R}, \mathbb{C})$ resp. $L^2_R := L^2(-R, R; \mathbb{C})$.

Definition 2.1. *i) $\mathcal{E}$ is the Hilbert space of complex-valued functions $\{Y(x) = (\psi(x), \pi(x)) : \psi, \psi' \in L^2, \pi \in L^2\}$ with the* **global energy norm**

$$\|Y\|_{\mathcal{E}} = \|\psi'\| + \|\psi\| + \|\pi\|, \tag{2.5}$$

where ψ' means the derivative in the sense of distributions.
ii) $\mathcal{E}_F$ is the linear space $\mathcal{E}$ endowed with the topology defined by the **local energy seminorms**

$$\|Y\|_{\mathcal{E},R} = \|\psi'\|_R + \|\psi\|_R + \|\pi\|_R, \quad R > 0. \tag{2.6}$$

Consider the Cauchy problem for the equation (2.1) with the initial conditions

$$\psi|_{t=0} = \psi_0(x);\ \dot{\psi}|_{t=0} = \pi_0(x). \tag{2.7}$$

Denote by $Y(t) := (\psi(x,t), \dot{\psi}(x,t))$ and $Y_0 := (\phi_0, \pi_0)$. Then the Cauchy problem (2.1), (2.7) reads

$$\dot{Y}(t) = \mathbf{F}(Y(t)), \quad t \in \mathbb{R};\ \ Y(0) = Y_0 \tag{2.8}$$

Proposition 2.2. *([42]) Let the condition (2.4) be fulfilled. Then*
i) For every $Y_0 \in \mathcal{E}$ the Cauchy problem (2.8) admits a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.
ii) The map $W(t) : Y_0 \mapsto Y(t)$ is continuous in $\mathcal{E}$ and $\mathcal{E}_F$.
iii) The energy is conserved,

$$\mathcal{H}(Y(t)) = \text{const}, \qquad t \in \mathbb{R}. \tag{2.9}$$

iv) The a priori bound holds,

$$\sup_{t\in\mathbb{R}} \|Y(t)\|_{\mathcal{E}} < \infty. \tag{2.10}$$

2.2 Solitary Waves

Definition 2.3. *i) Solitary waves for the equation (2.1) are the solutions $\psi_\omega(x)e^{i\omega t}$, with $\psi_\omega, \psi'_\omega \in L^2$.*
ii) S denotes the set of all the functions ψ_ω, and $\mathcal{S}$ denotes the corresponding set $\{\psi_\omega, i\omega\psi_\omega\} \subset \mathcal{E}$.

Proposition 2.4. *S is the set of all functions $\psi_\omega(x)$ of the form $Ce^{-\kappa|x|}$, where $\omega \in [-m, m]$, $\kappa := \sqrt{m^2 - \omega^2} \geq 0$ and C is a solution to the equation*

$$F(C) = 2\kappa C. \tag{2.11}$$

In addition, $C = 0$ when $\kappa = 0$.

Proof We have $\kappa^2 = m^2 - \omega^2$. Substituting $\psi_\omega(x)e^{i\omega t}$ to (2.1), we get the corresponding **stationary nonlinear eigenvalue problem**

$$-\omega^2\psi_\omega(x) = \psi''_\omega(x) - m^2\psi_\omega(x) + \delta(x)F(\psi_\omega(0)), \qquad x \in \mathbb{R}. \tag{2.12}$$

In particular, it implies that $\psi''_\omega(x) = \kappa^2\psi_\omega(x)$, $x \neq 0$, hence $\psi_\omega(x) = C_\pm e^{-\kappa|x|}$, $\pm x > 0$. Since $\psi'_\omega(x) \in L^2$, the function $\psi_\omega(x)$ is continuous, hence $C_- = C_+ = C$ and $\psi_\omega(x) = Ce^{-\kappa|x|}$, $x \in \mathbb{R}$. Furthermore, $\psi_\omega(x) \in L^2$, therefore κ is real and nonnegative:

$$\psi_\omega(x) = Ce^{-\kappa|x|}, \quad \kappa = \sqrt{m^2 - \omega^2} \geq 0, \quad \omega \in [-m,\ m]. \tag{2.13}$$

At last, we get an algebraic equation for the constant C equating the coefficients of $\delta(x)$ in both sides of (2.12):

$$0 = \psi'_\omega(0+) - \psi'_\omega(0-) + F(\psi_\omega(0)). \tag{2.14}$$

This implies $0 = -2\kappa C + F(C)$, or equivalently, (2.11). ■

Our main result is the following theorem.

Theorem 2.5. *([42]) Let (2.4) be fulfilled. Then for any solution $Y(t) = (\psi(\cdot,t), \dot\psi(\cdot,t)) \in C(\mathbb{R}, \mathcal{E})$ to (2.1), the attraction holds,*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}, \quad t \to \pm\infty. \tag{2.15}$$

Remark 2.6. *i) The theorem gives a partial solution to the problem (1.9). However, the asymptotics (1.9) are not proved yet even for the particular form nonlinear term.*
ii) An equation (2.1) with a polynomial potential (2.4) is "well posed" only if the leading coefficient $\mathcal{V}_N > 0$. The condition (2.4) means that $N > 2$ which corresponds to an "open dense set" among all the polynomial functions $\mathcal{V}(|\psi|)$. In this sense, the attraction (2.15) holds for "almost all" reasonable equations (2.1) that agrees with the conjecture **C** *from Introduction.*

3 Dispersive and Bound Components

3.1 The First Splitting

Split the solution in two components, $\psi(x,t) = \psi_0(x,t) + \psi_1(x,t)$. Namely, set $F_0(t) := F(\psi(0,t))$ and define the components as the solutions to the problems

$$\begin{cases} \ddot\psi_0(x,t) = \psi''_0(x,t) - m^2\psi_0(x,t), \\ \psi_0|_{t=0} = \psi_0, \quad \dot\psi_0|_{t=0} = \pi_0, \end{cases} \tag{3.1}$$

$$\begin{cases} \ddot\psi_1(x,t) = \psi''_1(x,t) - m^2\psi_1(x,t) + \delta(x)F_0(t), \\ \psi_1|_{t=0} = 0, \quad \dot\psi_1|_{t=0} = 0. \end{cases} \tag{3.2}$$

Lemma 3.1. *$\psi_0(x,t)$ is a* **dispersive component** *of the solution, i.e. for every $R > 0$ we have*

$$\int\limits_{|x|<R} \Big(|\dot\psi_0(x,t)|^2 + m^2|\psi'_0(x,t)|^2 + |\psi_0(x,t)|^2\Big)dx \to 0, \qquad t \to \pm\infty. \tag{3.3}$$

Proof Use the well known fundamental solution of the Klein-Gordon equation [41].

3.2 A Priori Bounds

Take into account the energy conservation:

$$\mathcal{H} := \frac{1}{2}\int [|\dot\psi|^2 + |\psi'|^2 + m^2|\psi|^2]dx + V(\psi(0,t)) = \text{const}$$

It implies a priori bounds

$$\sup_{t\in\mathbb{R}} |\psi(0,t)| < \infty, \ \sup_{t\in\mathbb{R}} \|\psi(\cdot,t)\|_1 < \infty. \tag{3.4}$$

Similarly, $\sup\limits_{t\in\mathbb{R}} \|\psi_0(\cdot,t)\|_1 < \infty$, and hence,

$$\sup_{t\in\mathbb{R}} \|\psi_1(\cdot,t)\|_1 < \infty. \tag{3.5}$$

3.3 Absolute Continuous Spectrum

Further we consider the long-time asymptotics only for $t \to \infty$ and introduce the Fourier-Laplace transform of the **bound component**

$$\tilde{\psi}_1(x,\omega) := \mathcal{F}^+_{t\to\omega}\psi_1 = \int_0^\infty e^{i\omega t}\psi_1(x,t)dt.$$

By (3.5), the integral converges in $H_1(\mathbb{R})$ for $\omega \in \mathbb{C}^+ := \{z \in \mathbb{C}: \ \mathrm{Im}\, z > 0\}$ and satisfies the bounds

$$\sup_{\mathrm{Im}\,\omega > \varepsilon} \|\tilde{\psi}_1(\cdot,t)\|_1 < \frac{1}{\varepsilon}, \qquad \varepsilon > 0. \tag{3.6}$$

The equation (3.2) implies the **stationary Helmholtz equation**

$$-\omega^2\tilde{\psi}_1(x,\omega) = \tilde{\psi}_1''(x,\omega) - m^2\tilde{\psi}_1(x,\omega) + \delta(x)\tilde{F}_0(\omega), \qquad \omega \in \mathbb{C}^+. \tag{3.7}$$

Therefore,

$$\tilde{\psi}_1(x,\omega) = z(\omega)e^{ik(\omega)|x|}, \ \ \omega \in \mathbb{C}^+, \tag{3.8}$$

where $k(\omega) := \sqrt{\omega^2 - m^2}$. By (3.6), we have to choose the branch with $\mathrm{Im}\, k(\omega) > 0$ for $\omega \in \mathbb{C}^+$ (this corresponds to *Limiting Absorption Principle*).

Proposition 3.2. *The distribution $z(\omega + i0)$, $\omega \in \mathbb{R}$, is absolutely continuous for $|\omega| > m$, and*

$$\int_{|\omega|>m} |z(\omega)|^2(|k(\omega)|^2 + |k(\omega)|)d\omega < \infty. \tag{3.9}$$

The proposition is a nonlinear version of Kato's theorem on the absence of the embedded discrete spectrum. The proof is based on (3.5) and the Paley-Wiener technique.

3.4 The Second Splitting

Now the Fourier representation reads,

$$\begin{aligned}\psi_1(x,t) &= \frac{1}{2\pi}\int_{|\omega|>m} e^{-i\omega t}z(\omega)e^{ik(\omega)|x|}d\omega + \frac{1}{2\pi}\langle z_0(\omega)e^{-\kappa(\omega)|x|}, e^{-i\omega t}\rangle \\ &=: d(x,t) + b(x,t),\end{aligned} \tag{3.10}$$

where z_0 is a distribution with $\mathrm{supp}\ z_0 \subset [-m,m]$, and

$$\kappa(\omega) := -ik(\omega) = \sqrt{m^2 - \omega^2} \geq 0, \qquad \omega \in [-m,m].$$

The function d is also the **dispersive component**:

Lemma 3.3. *For every $R > 0$,*

$$\int_{|x|<R} \Big(|\dot{d}(x,t)|^2 + m^2|d'(x,t)|^2 + |d(x,t)|^2\Big)dx \to 0, \qquad t \to \infty.$$

Proof This follows from (3.9) by the same arguments as Lemma 3.1. ■

Remark 3.4. The function $\kappa(\omega)$ is not smooth at the end points $\pm m$ of the supp $z(\omega)$. Hence, the scalar product in the definition of $b(x,t)$ in (3.10) requires a justification with an appropriate theory of *Quasimeasures and Multiplicators* (TQM) from [42]. Namely, the distribution $z_0(\omega)$ is a *Quasimeasure* while $e^{-\kappa(\omega)|x|}$ is the corresponding *Multiplicator*.

4 Spectral Analysis of Bound Component

Here we prove the main Theorem 2.5.

4.1 Compactness and Omega-Limiting Trajectories

By definition (3.10)

$$b(x,t) = \frac{1}{2\pi}\langle z_0(\omega)e^{-\kappa(\omega)|x|}, e^{-i\omega t}\rangle, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{4.1}$$

Proposition 4.1. *3.3 i) For each sequence $s_k \to \infty$ there exists a subsequence $s_{k'} \to \infty$ such that*

$$\beta(x,t) = \lim_{s_{k'}\to\infty} b(x, s_{k'} + t), \qquad (x,t) \in \mathbb{R}^2 \tag{4.2}$$

uniformly in $|x| + |t| \leq R$ for any $R > 0$. Similarly, for every $m, n = 0, 1, ...,$ we have

$$\partial_t^m \partial_x^n b(x, s_{k'} + t) \to \partial_t^m \partial_x^n \beta(x,t), \qquad x \neq 0, t \in \mathbb{R} \tag{4.3}$$

uniformly in $|x| + |t| \leq R$ for any $R > 0$.
ii) The representation holds

$$\beta(x,t) = \frac{1}{2\pi}\langle \zeta(\omega)e^{-\kappa(\omega)|x|}, e^{-i\omega t}\rangle, \qquad (x,t) \in \mathbb{R}^2, \tag{4.4}$$

where

$$\text{supp}\ \zeta \subset [-m, m]. \tag{4.5}$$

Proof The proposition follows by Arcela's Theorem and the TQM from [42]:

$$\zeta(\omega) = \lim_{k'\to\infty} z_0(\omega)e^{-i\omega s_{k'}},$$

where the convergence holds in the space of Quasimeasures. ■

4.2 Spectral Inclusion

First, (4.4) implies (by the TQM) the *Spectral Identity*

$$\text{Spec}\,\beta(x,\cdot) = \text{supp}\ \zeta, \qquad x \in \mathbb{R}, \tag{4.6}$$

where $\text{Spec}\,\beta(x,\cdot) := \text{supp}\ \tilde{\beta}(x,\cdot)$. Next, the limiting trajectory $\beta(x,t)$ is a solution to the equation (2.1) (while $b(x,t)$ is NOT !):

$$\ddot{\beta}(x,t) = \beta''(x,t) - m^2\beta(x,t) + \delta(x)F(\beta(0,t)), \qquad (x,t) \in \mathbb{R}^2. \tag{4.7}$$

This follows from (2.1) and (4.2), (4.3) since both dispersive components decay to zero. The equation implies the crucial Spectral Inclusion. Namely, (4.7) implies that

$$\text{Spec}\,F(\beta(0,\cdot)) \ \subset \text{supp}\ \zeta \tag{4.8}$$

since the spectra of all linear terms in (4.7) lie in supp ζ by (4.6). Finally, denote $\gamma(t) := \beta(0,\cdot)$. Then $\zeta = \tilde{\gamma}$ by (4.4), hence $\text{Spec}\,\gamma = \text{supp}\ \zeta$ and (4.8) leads to the Spectral Inclusion:

$$\text{Spec}\,F(\gamma(\cdot)) \subset \text{Spec}\,\gamma. \tag{4.9}$$

The Inclusion plays the role of an "equation" for the determination of the spectrum of the solution β.

4.3 Titchmarsh Theorem

Let us use (4.9) to reduce the spectrum to one point.

Theorem 4.2. *Let us assume that*
a) The potential $\mathcal{V}(|\psi|)$ *satisfies the condition (2.4),*
b) $\gamma(t)$ *is a complex-valued bounded continuous function in* $\mathbb{R}$*,*
c) $\mathrm{Spec}\,\gamma\subset[-m,m]$ *and (4.9) holds.*

Then
i) $|\gamma(t)|=\mathrm{const},\ \ t\in\mathbb{R}.$
ii) $\tilde\gamma(\omega)=C\delta(\omega-\omega_+)$*, where* $\omega_+\in[-m,m]$.

Proof The condition (2.4) implies that

$$F(\psi)=-\nabla_\psi\mathcal{V}(|\psi|)=a(|\psi|)\psi, \tag{4.10}$$

where a is the polynomial of **at least second degree**. Denote $\Phi(t):=F(\gamma(t))$ and $A(t):=a(|\gamma(t)|)$. Then (4.10) implies that $A(t)\gamma(t)\equiv\Phi(t)$, hence

$$\tilde A*\tilde\gamma=\tilde\Phi. \tag{4.11}$$

Now the Spectral Inclusion (4.9) reads

$$\mathrm{supp}\,[\tilde A*\tilde\gamma]\subset\mathrm{supp}\,\tilde\gamma. \tag{4.12}$$

Lemma 4.3. supp $\tilde A=\{0\}$.

Proof supp $\tilde A\subset[s_-,s_+]$, supp $\tilde\gamma\subset[\sigma_-,\sigma_+]$, and $s_\pm\in$ supp $\tilde A$, $\sigma_\pm\in$ supp $\tilde\gamma$. Therefore,

$$s_-+\sigma_-,\ s_++\sigma_+\in\mathrm{supp}\,\tilde A*\tilde\gamma$$

by the **Titchmarsh Convolution Theorem** [70] (see [28, Thm 4.3.3] for highly simplified proof and multidimensional generalization). Now (4.12) implies that

$$s_-+\sigma_-,\ s_++\sigma_+\in[\sigma_-,\sigma_+],$$

hence $s_\pm=0$ since $s_-\le s_+$ and $\sigma_-\le\sigma_+$. ■

Proof of Theorem 4.2 i) Lemma 4.3 implies that $A(t)$ is a polynomial, hence $A(t)=\mathrm{const}$ since $A(t):=a(|\gamma(t)|)$ is a bounded function. Hence, $|\gamma(t)|=\mathrm{const}$ since $|\gamma(t)|$ is a continuous function while the function a is a polynomial of the degree at least two. ■

Proof of Theorem 4.2 ii) Since $\gamma(t)\overline{\gamma(t)}=\mathrm{const}$, we have

$$\mathrm{supp}\,\tilde\gamma*\tilde{\overline\gamma}\subset\{0\}.$$

On the other hand,

$$\mathrm{supp}\,\tilde{\overline\gamma}=-\mathrm{supp}\,\tilde\gamma.$$

Therefore,

$$\mathrm{supp}\,\tilde\gamma=\{\omega_+\}$$

by the Titchmarsh Convolution Thm, etc. ■

Remarks i) We have used essentially that the spectrum of γ is bounded.
ii) We used the same for $F(\gamma(\cdot))$. For this purpose we have assumed the polynomial character of the nonlinear term.

Finally, Theorem 4.2 imply (2.15) since each omega-limiting point of the trajectory $Y(t)$ lies in $\mathcal{S}$. Theorem 2.5 is proved. ■

5 On Linear and Nonlinear Radiation

In our proof of Theorem 2.5 we have used the following two distinct mechanisms of energy radiation.

5.1 Linear Dispersive Radiation

The decay of the dispersive component, like $t^{-1/2}$, relies on the dispersion of the waves for the linear Klein-Gordon equation. The decay follows by the stationary phase method applied to the Fourier integral representation of the solution. It means the spreading of the wave packets with the group velocities $\pm\nabla\omega(k)$, where the dispersion relation $\omega(k) = \sqrt{k^2 + m^2} \geq m$ corresponds to the momentum $k \in \mathbb{R}$.

5.2 Nonlinear Multiplicative Radiation

The spectrum of any omega-limiting trajectory reduces to a unique frequency $\omega \in [-m, m]$. The reduction is provided by the following purely nonlinear mechanism.

Namely, an omega-limiting trajectory does not radiate the energy to infinity since the total amount of energy is finite. In other words, any omega-limiting trajectory is *radiationless*. On the other hand, the radiation of the trajectory does not vanish if its spectrum contains at least two distinct points. This follows from the polynomial character of the nonlinear interaction. Let us explain this on the simplest case when the spectrum contains just two distinct points ω_1 and ω_2. Namely, the nonlinear polynomial $F(\psi)$ contains then all frequencies $\omega_i + n\Delta\omega$ where $i = 1, 2$, $\Delta\omega = \omega_2 - \omega_1$ and $n = 0, \pm 1, ...$ Obviously, $|\omega_i + n\Delta\omega| > m$ for large $|n|$, hence $\omega_i + n\Delta\omega$ belongs to the continuous spectrum that provides the energy radiation that contradicts the radiationless character of the omega-limiting trajectory. Formal argument: $|\omega_i + n\Delta\omega| > m$ contradicts the spectal inclusions (4.5) and (4.8).

Remark 5.1. *This mechanism of energy radiation has been discovered numerically by F.Collino and T.Fouquet (Project ONDES, INRIA, Rocquencourt) in 1999 (see Section 4.1.5). We have formalized this mechanism, in the proof of Theorem 2.5 above, by using the Titchmarsh theorem.*

Thus, the inclusion (4.9) means the absence of energy flow from low to higher modes. This characterization of the limiting "radiationless" trajectories serves as a criterion for the determination of the global attractor.

Chapter 4

Lorentz Invariance: Numerical Observations

We describe our numerical observations of the soliton-type asymptotics for the Lorentz invariant wave and Klein-Gordon Eqns, and adiabatic effective dynamics of the solitons in slowly varying potentials.

1 Relativistic Ginzburg-Landau Equation

We consider real solutions to 1D relativistic nonlinear wave equation of type

$$\ddot{\psi}(x,t) = \psi''(x,t) + f(\psi(x,t)), \quad x \in \mathbb{R}, \tag{1.1}$$

with the nonlinear term $f \in C^1(\mathbb{R})$. The equation is translation invariant and Lorentz invariant. Formally it is a Hamiltonian system with the Hamilton functional

$$\mathcal{H}(\psi,\pi) = \int [\frac{|\pi(x)|^2}{2} + \frac{|\psi'(x)|^2}{2} + U(\psi(x))]dx, \tag{1.2}$$

where the potential $U(\psi) = -\int_0^\psi f(\varphi)d\varphi + \text{const}$. We assume $U(\psi)$ be "two-well" potential similar to the Ginzburg-Landau one $U(\psi) \sim (1-\psi^2)^2$. We assume, more generally, that

$$U \in C^2(\mathbb{R}), \qquad U(a_-) = U(a_+) = 0, \qquad \text{and } \; U(\psi) > 0 \; \text{ for } \; \psi \neq a_\pm, \tag{1.3}$$

with some $a_- < a_+$. Then $f(a_\pm) = 0$ and the constant functions $s_\pm(x) := a_\pm$ are stationary finite energy solutions to (1.1).

1.1 Kink Solutions

We assume the points $a_\pm$ be nondegenerate local minima of the potential $U(\psi)$,

$$m_\pm^2 = U''(a_\pm) > 0. \tag{1.4}$$

Then there exists a "kink", i.e. a nonconstant finite energy stationary solution $s(x)$ to (1.1),

$$0 = s''(x) + f(s(x)), \quad x \in \mathbb{R}; \quad s(x) \not\equiv \text{const}; \quad H(s,0) < \infty. \tag{1.5}$$

The moving kikns, with velocities $v \in \mathbb{R}$, exist for $|v| < 1$ and are obtained by the Lorentz transformation

$$(x,t) \mapsto \gamma_v(x - vt, t - vx), \tag{1.6}$$

where $\gamma_v = 1/\sqrt{1-v^2}$ corresponds to the Lorentz contraction. Namely, for any shift $q \in \mathbb{R}$ the traveling kink

$$\psi(x,t) = s(\gamma_v(x - vt - q)) \tag{1.7}$$

also is a finite energy solution to (1.1).

We suppose that for the kink solutions, the asymptotics (0.33) has to be modified as follows:

$$Y(x,t) \approx \sum_{k=1}^{N_\pm} \zeta\Big(\frac{x - v_\pm^k t}{l(t)}\Big) Y_{v_\pm^k}(x - v_\pm^k t) + \eta_\pm(x,t) W_0(t)\Psi_\pm, \qquad t \to \pm\infty, \tag{1.8}$$

where $l(t) := \log(|t| + 2)$, the function $\zeta \in C_0^\infty(\mathbb{R})$, $\zeta(x) = 1$ for $|x| \le 1$, and

$$\sum_{k=1}^{N_\pm} \zeta\Big(\frac{x - v_\pm^k t}{l(t)}\Big) + \eta_\pm(x,t) \equiv 1.$$

The asymptotics (1.8) mean that
i) The kinks contribute to the union of intervals

$$I_\pm(t) := \cup_{k=1}^{N_\pm} \Big[v_\pm^k t - l(t), v_\pm^k t + l(t) \Big],$$

of total lenghth $\sim \log t$.
ii) Outside the intervals, the solution is close to the dispersive wave $W_0(t)\Psi_\pm$ as $t \to \pm\infty$.

Note that the energy of the dispersive wave in the set $I_\pm(t)$ decays to zero. This follows by the stationary phase method (see (1.25) below).

1.2 Existence of Dynamics

Let us rewrite (1.1) as first order system

$$\dot\psi(x,t) = \pi(x,t), \quad \dot\pi(x,t) = \psi''(x,t) + f(\psi(x,t)). \tag{1.9}$$

We consider the Cauchy problem for the system (1.9) with the initial conditions

$$\psi|_{t=0} = \psi_0(x), \quad \dot\psi|_{t=0} = \pi_0(x), \quad x \in \mathbb{R}. \tag{1.10}$$

Let us define the phase space $\mathcal{E}$ of finite energy states for the wave equation (1.1). For any $p \in [1, \infty]$ let us denote by L^p the space $L^p(\mathbb{R})$ endowed with the norm $\|\cdot\|_p$. For any $R > 0$ denote by $\|\cdot\|_{p,R}$ the norm in the space $L^p(-R,R)$.

Definition 1.1. *i) E is the Hilbert space of $(\psi,\pi) \in L^2 \oplus L^2$ with finite 'energy norm'*

$$\|(\psi,\pi)\|_E = \|\psi'\|_2 + \|\psi\|_\infty + \|\pi\|_2. \tag{1.11}$$

ii) E_F is the space E endowed with the (Fréchet) topology defined by the 'local energy seminorms'

$$\|(\psi,\pi)\|_{E,R} = \|\psi'\|_{2,R} + \|\psi\|_{\infty,R} + \|\pi\|_{2,R}, \qquad R > 0. \tag{1.12}$$

iii) The phase space $\mathcal{E}$ is the set of $(\psi,\pi) \in E$ with the finite energy $\mathcal{H}(\psi,\pi) < \infty$, endowed with the topology of E. The space $\mathcal{E}_F$ is the set $\mathcal{E}$ endowed with the topology of E_F.

Remark 1.2. *The space $\mathcal{E}_F$ is metrizable. For example, the convergence in $\mathcal{E}_F$ is equivalent to the convergence w.r.t. the metric*

$$\rho(Y_1,Y_2) = \sum_{R=1}^\infty 2^{-R} \frac{\|Y_1 - Y_2\|_{\mathcal{E},R}}{1 + \|Y_1 - Y_2\|_{\mathcal{E},R}}.$$

It is easy to check that $(s_\pm(x),0),(s(\pm x),0)\in\mathcal{E}$. Let $S_v=(\psi_v,\pi_v)$ denote the initial state of the soliton (1.7) with $|v|<1$ and $q=0$,

$$S_v(x)=(s(\gamma_v x),-\gamma_v v s(\gamma_v x)). \tag{1.13}$$

The conditions (1.3), (1.4) imply that $S_v(x-q)\in\mathcal{E}$ for any $|v|<1$ and $q\in\mathbb{R}$. The following lemma is obvious.

Lemma 1.3. *Let the conditions (1.3), (1.4) hold. Then Hamilton functional $\mathcal{H}$ is continuous on the phase space $\mathcal{E}$.*

The existence and uniqueness of the solutions to the Cauchy problem (1.9), (1.10) is well known and can be proved by the methods [54, 59, 68].

Proposition 1.4. *Let the conditions (1.3), (1.4) hold. Then*
i) for every initial datum $(\psi_0(x),\pi_0(x))\in\mathcal{E}$ there exists the unique solution $(\psi(x,t),\dot\psi(x,t))\in C(\mathbb{R},\mathcal{E})$ to the problem (1.9), (1.10).
ii) The energy is conserved,

$$\mathcal{H}(\psi(\cdot,t),\pi(\cdot,t))=\text{const},\quad t\in\mathbb{R}. \tag{1.14}$$

iii) The trajectory is bounded,

$$\sup_{t\in\mathbb{R}}\|(\psi(\cdot,t),\pi(\cdot,t)\|_E<\infty. \tag{1.15}$$

1.3 Numerical Observations

Let us describe the results of our numerical experiments and give an identification of the details in terms of equation (1.1).

We have observed the asymptotics of type (0.33) for finite energy solutions of the equations (1.1) with the polynomial potential of the Ginzburg-Landau type

$$U(\psi)=\frac{(|\psi|^2-1)^2}{4}. \tag{1.16}$$

Then (1.1) reads

$$\ddot\psi(x,t)=\psi''(x,t)-|\psi(x,t)|^2\psi(x,t)+\psi(x,t), \tag{1.17}$$

For the potential (1.16) the conditions (1.3), (1.4) hold with $a_\pm=\pm1$, and the (standing) kink solutions are

$$s(x)=\pm\tanh\tilde{x},\qquad \tilde{x}:=x/\sqrt{2}, \tag{1.18}$$

up to translation. We have chosen different initial functions ψ_0,π_0 with the following properties:

$$|\psi_0(x)|,|\pi_0(x)|\sim1,\qquad \operatorname{supp}\psi_0',\operatorname{supp}\pi_0\subset[-20,20],\qquad |\psi(x)|\equiv1\ \text{ for }\ |x|\ge20. \tag{1.19}$$

We use the numerical second order scheme with $\Delta t\sim\Delta x\sim0.01,0.001$. In all cases (more than 100 initial functions), we have observed the asymptotics of type (0.33) for $t\ge100$, with the number of the solitons $N_+=0,1,...,5$.

Example 1.5. Figure 4.1 represents a solution of the equation (1.1) with the potential (1.16).
• **Space and Time:** The space variable x (horizontal axis) and the time axis t (vertical axis). The space is two times contracted at time $t=20$ and $t=60$.
• **Colors:** The distribution of the colors corresponds to the range of the solution as follows,

ψ	$(-\infty,-1.01)$	$[-1.01,-0.99)$	$[-0.99,-0.8)$	$[-0.8,0.8]$	$(0.8,0.99]$	$(0.99,1.01]$	$(1.01,\infty)$
Color	*White*	*Blue*	*Grey*	*Yellow*	*Grey*	*Red*	*White*

• **Transient Phase:** for $t \in [0, 20]$ we observe a chaotic behavior *without radiation.*
• **Asymptotic Phase:** for $t > 20$ we observe the asymptotics of type (0.33) with $N_+ = 3$.
• **Kinks** The three *Yellow color oscillating strips* represent the trajectories of the kinks. The Yellow strip around a trajectory correspond to the 'support' of a kink (with the values $\psi \in [-0.8, 0.8]$).
• **Dispersive Wave** The *hyperbolic and rectilinear* Blue-White and Red-White trajectories outside the Yellow strips represent the dispersive wave (the values $\psi \approx \pm 1$) The rectilinear trajectories mean the decay of the dispersive wave to the *wave packets*, propagating uniformly, with distinct group velocities.

1.4 Kink Oscillations and Linearized Equation

• **Lorentz-Einstein dilation** The boundaries of the yellow strips oscillate with different periods:
1) For the left kink, with the velocity $v_l \approx -0.24$, the period $T_l \approx 5.3$: about 45 of the periods between t=60 and t=300.
2) For the central kink, with the velocity $v_c \approx 0.02$, the period $T_c \approx 5.1$: about 47 of the periods between t=60 and t=300.
3) For the right kink, with the velocity $v_r \approx 0.88$, the period $T_r \approx 8.8$: about 15 of the periods between t=60 and t=190.
The ratio $T_c/T_l \approx 5.1/5.3 = 0.96$ corresponds to the Lorentz-Einstein dilation $\sqrt{1-v_l^2}/\sqrt{1-v_c^2} \approx \sqrt{1-v_l} \approx 0.97$. The error $0.96 - 0.97$ is about 1%.
The ratio $T_c/T_r \approx 5.1/8.8 = 0.58$ approximately corresponds to the Lorentz-Einstein dilation $\sqrt{1-v_r^2}/\sqrt{1-v_c^2} \approx \sqrt{1-v_r^2} \approx 0.48$. The error $0.58 - 0.48$ is about 17% and probably is due to the nonlinear interaction.

We suppose that the oscillations are provided by the eigenvalue $\omega_1 = \sqrt{3/2} \approx 1.224$ of the linearized equation, and

$$T_c \approx \frac{2\pi}{\omega_1} = 5.133.... \tag{1.20}$$

The error $5.133 - 5.1$ is about 0.6%.
• **Spectrum of Linearized Equation** Let us derive the equation for small perturbation of the kink with the zero velocity. Namely, substitute $\psi(x,t) = s(x) + \varphi(x,t)$ into (1.17). Neglecting the terms $\mathcal{O}(|\varphi|^2)$, we formally obtain the linearized equation

$$\begin{aligned} \ddot{\varphi}(x,t) = -H\varphi(x,t) : &= \varphi''(x,t) - 3s^2(x)\varphi(x,t) + \varphi(x,t) \\ &= \varphi''(x,t) - 2\varphi(x,t) - V(x)\varphi(x,t), \end{aligned} \tag{1.21}$$

where the potential $V(x) := 3s^2(x) - 3 \le 0$ and the 'Schrödinger operator' $H := -\dfrac{d^2}{dx^2} + 2 + V(x)$.

The continuous spectrum of H is evidently the interval $[2, \infty)$. The discrete spectrum contains at least two points: $\lambda = 0$ and $\lambda = 3/2$. Namely,

I. The zero point corresponds to the eigenfunction $\psi_0(x) = s'(x)$ which is the groundstate since it is positive. Namely, differentiating the stationary equation (1.5), we obtain $0 = Hs'(x)$.

II. The point $\lambda = 3/2$ is the next eigenvalue since it corresponds to the eigenfunction $\psi_1(x) = \dfrac{\sinh(\tilde{x})}{\cosh^2(\tilde{x})}$ where $\tilde{x} := x/\sqrt{2}$ (it is easy to check by direct calculation), and the eigenfunction has one zero point.

The point $\lambda = 3/2$ provides the oscillatory solutions $\mathrm{Re}\,\psi_1(x)e^{\pm i\omega_1 t}$ where $\omega_1 := \sqrt{3/2}$. We suppose that the oscillatory solutions are responsible for the oscillations of the kinks. For

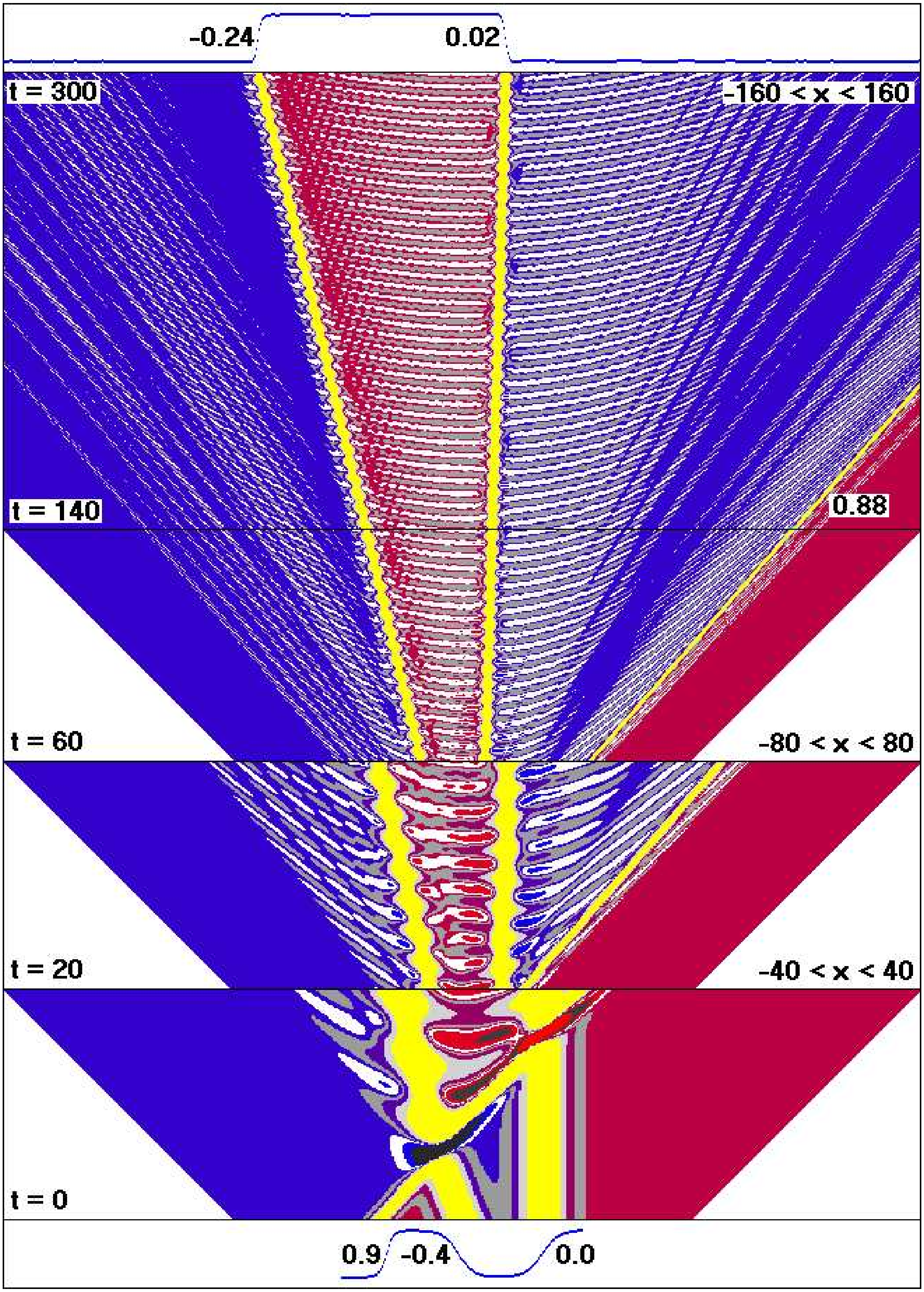

Figure 4.1: Formation of kinks

example, for the central kink, the deviation of the periods of the oscillations from $2\pi/\omega_1$ is about 0.6%.

1.5 Dispersive Wave

• **Hyperbolic Structure** The dispersive wave is detected by the hyperbolic level lines (see fig. 4.1) typical for the solutions of the Klein-Gordon equation. Namely, outside the yellow strips surrounding the kinks, the observed solution is close to the stationary solutions $s_\pm(x) \equiv \pm 1$: the difference is ~ 0.01. Hence, the solution approximately satisfies the linearized equations, corresponding to $s_\pm(x)$, which is the Klein-Gordon equation (1.21) with the potential $V(x) := 3s_\pm^2(x) - 3 \equiv 0$:

$$\ddot\varphi(x,t) = \varphi''(x,t) - 2\varphi(x,t). \tag{1.22}$$

Each finite energy solution is a convolution with the fundamental solution which is the Bessel function of the Lorentz interval $t^2 - x^2$, [41]. Hence, the level lines of any finite energy solution are asymptotically described by the equation $t^2 - x^2 = \text{const}$.

Remark 1.6. *The identification of the dispersive wave with the Bessel functions has been checked numerically with a high precision by Yulian Radvogin. Some of the numerical results are published in [4].*

• **Dispersion Relation** The dispersive wave, outside the kinks, decays like $t^{-1/2}$. This follows by method of stationary phase [15]. Therefore, the density of energy decays like t^{-1}. Hence, the total energy between the moving kinks is about constant since the distance is of order t. We have checked numerically that the energy of the dispersive wave between the kinks does not decay to zero and converges to a nonzero limit.

This corresponds to the dispersion relation $\omega(k) = \pm\sqrt{k^2+m^2}$ of the Klein-Gordon equation. Namely, all group velocities $v \in (-1,1)$ are allowed since $v = \nabla\omega(k)$, [43]. For example, the energy of the dispersive wave between the left and central kinks, is transported by the harmonics with the wave numbers satisfying the inequalities

$$-0.24 \approx v_l < \nabla\omega(k) < v_c \approx 0.22. \tag{1.23}$$

For the linear Klein-Gordon equation (1.22), the energy in the sector (1.23) converges to a limit which generally is not zero. Namely, the total energy for the linear equation reads (cf. (1.2))

$$\frac{1}{2}\int (|\dot\phi(x,t)|^2 + |\phi'(x,t)|^2 + 2|\phi(x,t)|^2)dx = \frac{1}{2}\int (|\dot{\hat\phi}(k,t)|^2 + |k\hat\phi(k,t)|^2 + 2|\hat\phi(k,t)|^2)dk \tag{1.24}$$

by the Parseval identity, where $\hat\phi(k,t)$ stands for the Fourier transform

$$\hat\phi(k,t) := \frac{1}{\sqrt{2\pi}}\int e^{ikx}\phi(x)dx.$$

The energy in any region $v_l t + o(t) < x < v_c t + o(t)$ converges to the corresponding limit:

$$\lim_{t\to\pm\infty}\frac{1}{2}\int\limits_{v_l t+o(t)<x<v_c t+o(t)} (|\dot\phi(x,t)|^2 + |\phi'(x,t)|^2 + 2|\phi(x,t)|^2)dx$$

$$= \frac{1}{2}\int\limits_{v_l<\nabla\omega(k)<v_c} (|\dot{\hat\phi}(k,0)|^2 + |k\hat\phi(k,0)|^2 + 2|\hat\phi(k,0)|^2)dk \tag{1.25}$$

if the initial functions are sufficiently smooth. This can be proved by the methods [43, Section "Schrödinger Equation and Geometric Optics"].

• **Discrete Spectrum of Group Velocities** The dispersive wave decays to the discrete wave packets (see fig. 4.1). This demonstrates the discreteness of the corresponding group velocities. We suppose that the discreteness is due to the polynomial character of the nonlinear term. Namely, let us consider the solution around a moving kink. After a Lorentz transformation, we can assume that the velocity of the kink is zero. Then the oscillations of the kink (probably) correspond to the solutions $\mathrm{Re}\,\psi_1(x)e^{\pm i\omega_1 t}$ of the linearized equation. The polynomial nonlinear term produces all the frequencies $n\omega_1$ with $n = \pm 3, \pm 5, \ldots$. Therefore, it is natural to think that the observed discrete spectrum of the group velocities is described by $v_n = \pm\nabla\omega(k_n)$ where the wave numbers k_n satisfy the dispersion relation $n\omega_1 = \pm\sqrt{k_n^2 + m^2}$. However, a satisfactory numerical identification of the wave packets is not done yet.

1.6 Linear and Nonlinear Radiative Mechanism

Our experiments suggest that the attraction to the kinks is due to the radiation induced by the oscillations. This mechanism can be explained by the equation (1.17).

Namely, let us represent the equation as the linearized Klein-Gordon equation (1.21) excited by the source which includes the nonlinear term:

$$\ddot{\psi}(x,t) + H\psi(x,t) = -|\psi(x,t)|^2\psi(x,t) + 3\psi(x,t) + V(x)\psi(x,t), \quad (1.26)$$

It is well-known from the scattering theory [43] that the long-time asymptotics and the radiation depend on the time-spectrum of the source:

• For ω in the continuous spectrum of the linear Klein-Gordon equation, $\omega \in \mathbb{R} \setminus (-\sqrt{2}, \sqrt{2})$, the harmonics $e^{i\omega t}$ in the source generates the radiation to infinity that means the *long range scattering*.

• Otherwise, for $\omega \in (-\sqrt{2}, \sqrt{2})$, the harmonics generates the forced oscillation without radiation to infinity that means the *short range scattering*.

The numerical experiments [11] illustrate this theory for small perturbation of the standing kink (1.18). Namely

I. For small times, the time-spectrum of the solution $\psi(0,t)$ contains mainly two points $\pm\omega_1 \in (-\sqrt{2}, \sqrt{2})$, and the radiation is not observed.

II. For large times, the time-spectrum of the solution $\psi(0,t)$ contains new points $\pm 3\omega_1, \pm 5\omega_1, \ldots \in \mathbb{R} \setminus (-\sqrt{2}, \sqrt{2})$, and the radiation of corresponding frequencies is observed. The frequencies are generated from ω_1 by the cubic polynomial in RHS of (1.26).

Our experiment also demonstrates the difference in the radiation of the dispersive wave for $t < 20$ and $t > 20$ (see fig. 4.1).

Summarizing: The nonlinear term translates the harmonics from the spectral gap $(-\sqrt{2}, \sqrt{2})$ into the continuous spectrum $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$. Then the linear Klein-Gordon dynamics disperses the energy at infinity. This *radiative mechanism* plays the role of a dissipation and is responsible for the convergence to the global attractor in reversible Hamilton equations.

In [42] the convergence to the minimal global point attractor is proved for $U(1)$-invariant 1D nonlinear Klein-Gordon equation with the nonlinear term $\delta(x)F(\psi)$ concentrated at one point $x = 0$. The function $F(\psi) = g(|\psi|^2)\psi$ where g is a polynomial of order ≥ 1. The proof is based on the detailed analysis of the radiative mechanism by the Titchmarsh Convolution theorem [28, Thm 4.3.3].

2 Relativistic Klein-Gordon Equations

2.1 Soliton Solutions

Further, we have observed the asymptotics of type (0.35) for complex solutions to relativistic 1D nonlinear Klein-Gordon equation

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - \psi(x,t) + f(\psi(x,t)). \qquad x \in \mathbb{R}. \tag{2.1}$$

We assume that $f(\psi) = -\nabla U(|\psi|)$ with a polynomial potential

$$U(|\psi|) = a|\psi|^{2m} - b|\psi|^{2n}, \tag{2.2}$$

where $a, b > 0$ and $m > n = 2, 3, \ldots$. Then

$$f(\psi) = 2am|\psi|^{2m-2}\psi - 2bn|\psi|^{2n-2}\psi. \tag{2.3}$$

The equation (2.1) is the Hamilton dynamical system with the Hamilton functional

$$\mathcal{H}(\psi,\pi) = \int [\frac{|\pi(x)|^2}{2} + \frac{|\psi'(x)|^2}{2} + \frac{|\psi(x)|^2}{2} + U(|\psi(x)|)]dx, \tag{2.4}$$

Further, Eqn (2.1) is translation invariant and $U(1)$-invariant, hence generally admits the solitary wave solutions $e^{i\omega t}\psi_{v,\omega}(x - vt)$. First consider the standing solitary waves, i.e. with $v = 0$. Substitution into (2.1) gives

$$-\omega^2\psi_{0,\omega}(x) = \psi''_{0,\omega}(x) - \psi_{0,\omega}(x) + f(\psi_{0,\omega}(x)), \quad x \in \mathbb{R}, \tag{2.5}$$

since $f(e^{i\theta}\psi) \equiv e^{i\theta}f(\psi)$ by (2.3). The equation (1.1) can be solved explicitly, and the finite energy solitary waves generally exist for a range of $\omega \in I \subset [-1, 1]$ and decay exponentially at infinity. For the concreteness we denote by $\phi_{0,\omega}(x)$ an even solution to (2.5).

For $|v| < 1$ the solitary waves are obtained by the Lorentz transformation (1.6): the moving solitary wave is equal to

$$\phi_{v,\omega}(x,t) := e^{i\omega\gamma_v(t-vx)}\phi_{0,\omega}(\gamma_v(x - vt)). \tag{2.6}$$

The *total energy* of the soliton coincides with the *kinetic energy* of classical relativistic particle:

$$\mathcal{H}(\phi_{v,\omega}(\cdot,t), \dot{\phi}_{v,\omega}(\cdot,t)) = \frac{E_0(\omega)}{\sqrt{1-v^2}}, \tag{2.7}$$

where generically $E_0(\omega) > 0$. This is proved in [13] for general relativistic equations in all dimensions.

Remark 2.1. *The solution (2.6) can be rewritten in the form* $e^{i\tilde{\omega}t}\psi(x - vt)$ *with* $\tilde{\omega} := \gamma_v\omega$ *(cf. (0.29)).*

2.2 Numerical Observations

We have tried the following values of a, b, m, n:

N	a	m	b	n
1	1	3	0.61	2
2	10	4	2.1	2
3	10	6	8.75	5

We have chosen different "smooth" initial functions ψ_0, π_0 with the support in a bounded interval $[-20, 20]$. We use the numerical second order scheme with $\Delta x, \Delta t \sim 0.01, 0.001$. In all cases we have observed the asymptotics of type (0.35) with the number of solitons $N_+ = 0, 1, 3$ for $t \geq 100$.

3 Adiabatic Effective Dynamics of Solitons

3.1 Effective Hamiltonian

We also consider the equation (2.1) with an external slowly varying potential,

$$\ddot{\psi}(x,t) = \psi''(x,t) - \psi(x,t) + f(\psi(x,t)) - V(x)\psi(x,t), \quad x \in \mathbb{R}, \tag{3.1}$$

The equation is the Hamilton dynamical system with the Hamilton functional

$$\mathcal{H}_V(\psi,\pi) = \int [\frac{|\pi(x)|^2}{2} + \frac{|\psi'(x)|^2}{2} + \frac{|\psi(x)|^2}{2} + U(|\psi(x)|) + V(x)\frac{|\psi(x)|^2}{2}]dx, \tag{3.2}$$

Note that the function (2.6) is not a solution to (3.1) if $V(x) \not\equiv 0$. However, we have observed numerically the solutions close to the solitary manifold for all times, i.e.

$$\psi(x,t) \approx \phi_{v(t),\omega(t)}(x - q(t), 0), \tag{3.3}$$

where $\phi_{v,\omega}$ stands for the function (2.6). The numerical experiments suggest an effective adiabatic dynamics for the parameters q, v, ω of the soliton if the potential $V(x)$ is slowly varying, i.e.

$$\varepsilon := \max |V'(x)| \ll 1. \tag{3.4}$$

Namely, let us choose the initial point from the soliton manifold, i.e.

$$\psi(x,0) = \phi_{v(0),\omega(0)}(x - q(0), 0), \qquad \dot{\psi}(x,0) = \dot{\phi}_{v(0),\omega(0)}(x - q(0), 0), \tag{3.5}$$

with some initial parameters $q(0), v(0), \omega(0)$. If $V(x) \equiv \text{const}$, the solution remains the soliton with the parameters $q(t) = q(0) + v(0)t, v(t) = v(0), \omega(t) = \omega(0)$. Further, let us assume (3.4) hold, i.e. $V(x) \approx \text{const}$. Then it is natural to expect an adiabatic effective dynamics of the parameters if the initial point is sufficiently close to the solitary manifold.

Let us determine the corresponding effective Hamilton functional. Namely, substitute the function (3.3) to the Hamiltonian (3.2), and denote the relativistic momentum $p := v/\sqrt{1-v^2}$. Then (2.7) implies that

$$\mathcal{H}_V(\psi(\cdot,t), \dot{\psi}(\cdot,t)) \approx \frac{E_0(\omega(t))}{\sqrt{1-v^2(t)}} + V(q(t))I(p(t),\omega(t)), \quad I(p,\omega) := \frac{1}{2}\int \phi^2_{v,\omega}(x,0)dx \tag{3.6}$$

since the soliton $\phi_{v(t),\omega(t)}(x - q(t), 0)$ is concentrated near the point $q(t)$. Then we define the effective Hamiltonian as follows:

$$\mathcal{H}_{\text{eff}}(Q,P,\Omega) := E_0(\Omega)\sqrt{1+P^2} + V(Q)I(P,\Omega). \tag{3.7}$$

It is natural to expect that the functions $q(t), \omega(t)$ are close to the corresponding components of a trajectory of the Hamiltonian system

$$\begin{cases} \dot{Q} = \nabla_P \mathcal{H}_{\text{eff}}(Q,P,\Omega), & \dot{P} = -\nabla_Q \mathcal{H}_{\text{eff}}(Q,P,\Omega) \\ \dot{\Theta} = \nabla_\Omega \mathcal{H}_{\text{eff}}(Q,P,\Omega), & \dot{\Omega} = -\nabla_\Theta \mathcal{H}_{\text{eff}}(Q,P,\Omega) = 0 \end{cases} \tag{3.8}$$

since the effective Hamiltonian does not depend on the phase variable Θ. Therefore, $\Omega = \text{const}$, and $Q(t)$ is a solution of two first equations with the intitial conditions $Q(0) = q(0)$, $P(0) = p(0)$ and the fixed $\Omega = \omega(0)$. Finally, we expect that $q(t)$ is close to $Q(t)$ in the adiabatic limit, i.e.

$$|q(t) - Q(t)| \le 1, \qquad |t| \le C\varepsilon^{-1}. \tag{3.9}$$

Our numerical observations confirm qualitatively the adiabatic effective dynamics. The mathematical proof is still open problem. Similar asymptotics are proved for the solitons of the nonlinear Schrödinger and Hartree Eqns, [17, 18], and for the particle coupled to scalar or Maxwell field [44, 50].

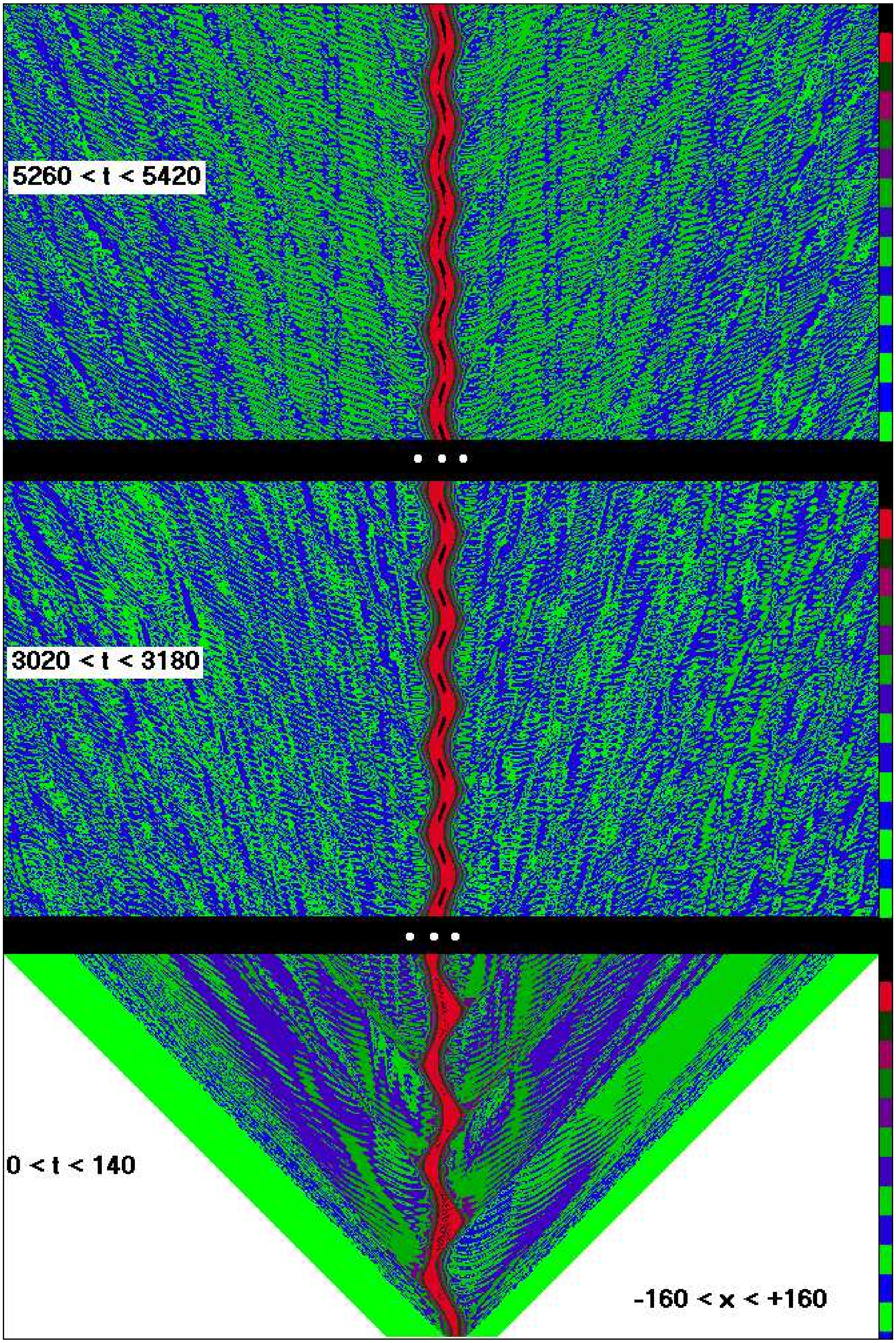

Figure 4.2: Adiabatic effective dynamics of a soliton

3.2 Numerical Observation

Figure 4.2 represents a solution of the equation (3.1) with the potential (2.2) where $a = 10$, $m = 6$ and $b = 8.75$, $n = 5$. We choose $V(x) = -0.2\cos(0.31x)$ and the following initial conditions:

$$\psi(x,0) = \phi_{v(0),\omega(0)}(x - q(0), 0), \qquad \dot{\psi}(x,0) = 0, \tag{3.10}$$

where $v(0) = 0$, $\omega(0) = 0.6$ and $q(0) = 5.0$. Note that the initial state does not belong to the solitary manifold (3.5) since $\omega(0) \neq 0$. The effective width (half-amplitude) of the soliton is in the interval $[4.4, 5.6]$. The width is sufficiently small w.r.t. the period $2\pi/0.31 \sim 20$ of the potential: it is confirmed by the numerical observation. Namely,

- Blue-green sites represent the amplitudes $|\psi(x,t)| < 0.01$; red sites represent the amplitudes $|\psi(x,t)| \in [0.4, 0.8]$.
- The trajectory of the soliton in the figure 4.2 ('red snake') is similar to the oscillation of the classical particle.
- For $0 < t < 140$ the solution is not very close to the solitary manifold and we observe an intensive radiation.
- For $3020 < t < 3180$ the solution approaches to the solitary manifold and the radiation is less intensive. The amplitude of the soliton oscillations is almost constant for a large time that corresponds to the effective Hamilton dynamics (3.8).
- On the other hand, for $5260 < t < 5420$ the amplitude of the soliton oscillations is two-times smaller. Hence, the amplitude decays on a larger time scale that contradicts the effective Hamilton dynamics (3.8). Therefore, the Hamilton dynamics may be efficient only in an adiabatic limit like $t \sim \varepsilon^{-1}$.
- The deviation from the Hamilton dynamics is caused by the radiation which plays the role of a dissipation.
- We observe the radiation with the discrete spectrum of the group velocities like fig 4.1. The magnitude of the solution at the soliton is of order ~ 1 while the radiation field is less 0.01, so its density of the energy is less 0.0001.

Bibliography

[1] A.V.Babin, M.I.Vishik, Attractors of Evolutionary Equations, North-Holland, Amsterdam, 1992.

[2] A.Bensoussan, C.Iliine, A.Komech, Breathers for a relativistic nonlinear wave equation, *Arch. Rat. Mech. Anal.* **165** (2002), 317-345.

[3] H.Berestycki, P.L.Lions, *Arch. Rat. Mech. and Anal.* **82** (1983), no.4, 313-375.

[4] N.Yu.Bessonova, Yu.B.Radvogin, On the frontal asymptotics for the nonlinear Klein-Gordon equation, Prepint No. 127, Keldysh Institute of Applied Mathematics RAS, Moscow, 1995 [Russian].

[5] H.Bethe, Intermediate Quantum Mechanics, Benjamin, NY, 1964.

[6] H.Bethe, E.Salpeter, Quantum Mechanics of One- and Two-Electron Atoms, Berlin, Springer, 1957.

[7] V.S.Buslaev, G.S.Perelman, Scattering for the nonlinear Schrdinger equation: States close to a soliton, *St. Petersbg. Math. J.* **4** (1993), no.6, 1111-1142.

[8] V.S.Buslaev, G.S.Perelman, On the stability of solitary waves for nonlinear Schrödinger equations, *Amer. Math. Soc. Trans.* (2) **164** (1995), 75-98.

[9] V.S.Buslaev, C.Sulem, On asymptotic stability of solitary waves for nonlinear Schrödinger equations, *Ann. Inst. Henri Poincare, Anal. Non Lineaire* **20** (2003), no.3, 419-475.

[10] G.M.Chadam, Asymptotics for $\Box u = m^2 u + G(x,t,u,u_x,u_t)$, *Ann. Scuola Norm. Sup. Pisa Fis. Mat.* **26** (1972), 33-95.

[11] F.Collino, T.Fouquet, private communication.

[12] S.Cuccagna, Stabilization of solutions to nonlinear Schrödinger equations, *Comm. Pure Appl. Math.* **54** (2001), no.9, 1110-1145.

[13] T.Dudnikova, A.Komech, H.Spohn, Energy-momentum relation for solitary waves of relativistic wave equation, *Russian Journal Math. Phys.* **9** (2002), no.2, 153-160.

[14] M.Esteban, V.Georgiev, E.Sere, Stationary solutions of the Maxwell-Dirac and the Klein-Gordon-Dirac equations, *Calc. Var. Partial Differ. Equ.* **4** (1996), no.3, 265-281.

[15] M. Fedoriuk, Asymptotic Methods for Partial Differential Equations, Encyclopaedia of Mathematical Sciences Vol. 34, Springer, Berlin, 1999.

[16] C.Foias, G.R.Sell, R.Temam, Inertial manifolds for nonlinear evolutionary equations, *J. Differ. Equations* **73** (1988), no.2, 309-353.

[17] J.Fröhlich, S.Gustafson, B.L.G.Jonsson, I.M.Sigal, Solitary wave dynamics in an external potential, arXiv:math-ph/0309053v1.

[18] J.Frölich, T.P.Tsai, H.T.Yau, On the point-particle (Newtonian) limit of the non-linear Hartree equation, *Comm. Math. Physics* **225** (2002), no.2, 223-274.

[19] M.Gell-Mann, *Phys. Rev.* **125** (1962), 1067.

[20] J.Ginibre, G.Velo, Time decay of finite energy solutions of the nonlinear Klein-Gordon and Schrödinger equations, *Ann. Inst. Henri Poincaré* **43** (1985), 399-442.

[21] R.T.Glassey, W.A.Strauss, Decay of classical Yang-Mills fields, *Comm. Math. Phys.* **65** (1979), 1-13.

[22] R.T.Glassey, W.A.Strauss, Decay of Yang-Mills field coupled to a scalar field, *Comm. Math. Phys.* **67** (1979), 51-67.

[23] M.Grillakis, J.Shatah, W.A.Strauss, Stability theory of solitary waves in the presence of symmetry, I; II. *J. Func. Anal.* **74** (1987), no.1, 160-197; **94** (1990), no.2, 308-348.

[24] Y.Guo, K.Nakamitsu, W.Strauss, Global finite-energy solutions of the Maxwell-Schrdinger system, *Commun. Math. Phys.* **170**(1995), no.1, 181-196.

[25] J.Hale, Asymptotic Behavior of Dissipative Systems, AMS, Providence, 1988.

[26] W.Heisenberg, Der derzeitige Stand der nichtlinearen Spinortheorie der Elementarteilchen, *Acta Phys. Austriaca* **14** (1961), 328-339.

[27] D.Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, 840, Springer, Berlin, 1981.

[28] L.Hörmander, The Analysis of Linear Partial Differential Operators I, Distribution Theory and Fourier Analysis, Springer, Berlin, 1983.

[29] L.Hörmander, On the fully nonlinear Cauchy problem with small data, pp. 51-81. "Microlocal Analysis and Nonlinear Waves", *in* IMA Vol. Math. Appl., Vol.30, Springer, Berlin, 1991.

[30] V.Imaikin, A.Komech, H.Spohn, Soliton-like asymptotics and scattering for a particle coupled to Maxwell field, *Russian Journal of Mathematical Physics* **9** (2002), no.4, 428-436.

[31] V.Imaikin, A.Komech, H.Spohn, Rotating charge coupled to the Maxwell field: scattering theory and adiabatic limit, *Monatsh. Math.* **142** (2004), no.1-2, 143-156.

[32] V.Imaikin, A.Komech, P.Markowich, Scattering of solitons of the Klein-Gordon equation coupled to a classical particle, *Journal of Mathematical Physics* **44** (2003), no.3, 1202-1217.

[33] V.Imaikin, A.Komech, H.Spohn, Scattering theory for a particle coupled to a scalar field, *Journal of Discrete and Continuous Dynamical Systems* **10** (2003), no.1&2, 387-396.

[34] S.Klainerman, Long-time behavior of solutions to nonlinear evolution equations, *Arch. Rat. Mech. Anal.* **78** (1982), 73-98.

[35] On the stabilization of interaction of a string with a nonlinear oscillator, *Moscow Univ. Math. Bull.* **46** (1991), no. 6, 34-39.

[36] A.I.Komech, On stabilization of string-nonlinear oscillator interaction, *J. Math. Anal. Appl.* **196** (1995), 384-409.

[37] A.I.Komech, On the stabilization of string-oscillators interaction, *Russian Journal of Math.Phys.* **3** (1995), 227-248.

[38] A.I.Komech, On transitions to stationary states in one-dimensional nonlinear wave equations, *Arch. Rat. Mech. Anal.* **149** (1999), no.3, 213-228.

[39] A.Komech, Attractors of non-linear Hamiltonian one-dimensional wave equations, *Russ. Math. Surv.* **55** (2000), no.1, 43-92.

[40] A.Komech, On transitions to stationary states in Hamiltonian nonlinear wave equations, *Phys. Letters A* **241** (1998), 311-322.

[41] A.Komech, Linear Partial Differential Equations with Constant Coefficients, p.127-260 in: Yu.V.Egorov, A.Komech, M.A.Shubin, Elements of the Modern Theory of Partial Differential Equations, Springer, Berlin, 1999.

[42] A.Komech, On attractor of a singular nonlinear $U(1)$-invariant Klein-Gordon equation, p. 599-611 in: Proceedings of the $3^r d$ ISAAC Congress, Freie Universität Berlin, Berlin, 2003.

[43] A.Komech, Lectures on Classical Fields and Quantum Mechanics. Part I, Rapport de recherche de l'INRIA, RR-4736, INRIA, Rocquencourt, 2003.

[44] A.Komech, M.Kunze, H.Spohn, Effective Dynamics for a mechanical particle coupled to a wave field, *Comm. Math. Phys.* **203** (1999), 1-19.

[45] A.Komech, N.Mauser, A.Vinnichenko, On Attraction to Solitons in Relativistic Nonlinear Wave Equations, submitted to *Russian J. Math. Phys.*, 2004.

[46] A.Komech, H.Spohn, Soliton-like asymptotics for a classical particle interacting with a scalar wave field, *Nonlinear Analysis* **33** (1998), no.1, 13-24.

[47] A.Komech, H.Spohn, Long-time asymptotics for the coupled Maxwell-Lorentz equations, *Comm. Partial Diff. Equs.* **25** (2000), no.3/4, 558-585.

[48] A.Komech, H.Spohn, M.Kunze, Long-time asymptotics for a classical particle interacting with a scalar wave field, *Comm. Partial Diff. Equs.* **22** (1997), no.1/2, 307-335.

[49] A.Komech, B.Vainberg, On asymptotic stability of stationary solutions to nonlinear wave and Klein-Gordon equations, *Arch. Rat. Mech. Anal.* **134** (1996), 227-248.

[50] M.Kunze, H.Spohn, Adiabatic limit for the Maxwell-Lorentz equations, *Ann. Henri Poincaré* **1** (2000), no.4, 625-653.

[51] H.Lamb, On a peculiarity of the wave-system due to the free vibrations of a nucleus in an extended medium, *Proc. London Math. Soc.* **32** (1900), 208-211.

[52] P.D.Lax, R.S.Phillips, "Scattering Theory", Academic Press, New York, 1967.

[53] J.L.Lions, "Problèmes aux Limites dans les Équations aux Dérivées Partielles", Presses de l'Univ. de Montréal, Montréal, 1962.

[54] J.L.Lions, "Quelques Mèthodes de Rèsolution des Problémes aux Limites non Linéaires", Paris, Dunod, 1969.

[55] C.S.Morawetz, The decay of solutions to exterior initial-boundary value problem for the wave equation, *Comm. Pure Appl. Math.* **14** (1961), 561-568.

[56] C.S.Morawetz, W.A.Strauss, Decay and scattering of solutions of a nonlinear relativistic wave equation, *Comm. Pure Appl. Math.* **25** (1972), 1-31.

[57] S.P.Novikov, S.V.Manakov, L.P.Pitaevskii, V.E.Zakharov, Theory of Solitons: The Inverse Scattering Method, Consultants Bureau [Plenum], New York, 1984.

[58] C.-A.Pillet, C.E.Wayne, Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations, *J. Differ. Equations* **141** (1997), No.2, 310-326.

[59] M.Reed, "Abstract Non-Linear Wave Equations", Lecture Notes in Mathematics 507, Springer, Berlin, 1976.

[60] M.Reed, B.Simon, *Methods of Modern Mathematical Physics III: Scattering Theory*, Academic Press, New York, 1979.

[61] W.Rudin, "Functional Analysis", McGraw Hill, New York, 1977.

[62] G.Scharf, "From Electrostatics to Optics", Springer, Berlin, 1994.

[63] L.Schiff, Quantum Mechanics, McGraw-Hill, NY, 1955.

[64] I.Segal, Dispersion for nonlinear relativistic wave equations, *Ann. Sci. Ecole Norm. Sup.* **1** (1968), 459-497.

[65] A.Soffer, M.I.Weinstein, Multichannel nonlinear scattering for nonintegrable equations, *Comm. Math. Phys.* **133** (1990), 119-146.

[66] A.Soffer, M.I.Weinstein, Multichannel nonlinear scattering for nonintegrable equations. II. The case of anisotropic potentials and data. *J. Differential Equations* **98** (1992) no. 2 376.

[67] W.A.Strauss, Decay and asymptotics for $\Box u = F(u)$, *J. Funct. Anal.* **2** (1968), 409-457.

[68] W.A.Strauss, Nonlinear Invariant Wave Equations, Lecture Notes in Phys., Vol.73, Springer, Berlin, 1978.

[69] R.Temam, "Infinite-Dimensional Dynamical Systems in Mechanics and Physics", Applied Mathematical Sciences, Springer, Berlin, 1988.

[70] E.C.Titchmarsh, The zeros of certain integral functions, *Proc. of the London Math. Soc.* **25** (1926), 283-302.

[71] B.R.Vainberg, Asymptotic Methods in Equations of Mathematical Physics, Gordon and Breach, New York, 1989.