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SCATTERING THEORY FOR A PARTICLE COUPLED TO A SCALAR FIELD

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Abstract. We establish soliton-like asymptotics for finite energy solutions to classical particle coupled to a scalar wave field. Any solution that goes to infinity as $t \to \infty$ converges to a sum of traveling wave and of outgoing free wave. The convergence holds in global energy norm. The proof uses a non-autonomous integral inequality method.

1. **Introduction.** Consider a single charge coupled to a scalar wave field and subject to an external potential of compact support in 3-dimensional space. If $q(t) \in \mathbb{R}^3$ denotes the position of the charge at a time t, then the coupled equations read

$$\begin{aligned} &\dot\phi(x,t) = \pi(x,t), && \dot\pi(x,t) = \Delta\phi(x,t) - \rho(x - q(t)), \\ &\dot q(t) = p(t)/(1+p^2(t))^{1/2}, && \dot p(t) = -\nabla V(q(t)) + \int d^3x\, \phi(x,t)\, \nabla\rho(x - q(t)). \end{aligned} \tag{1.1}$$

This is a Hamiltonian system with the Hamiltonian functional

$$\begin{aligned} \mathcal{H}(\phi,\pi,q,p) &= (1+p^2)^{1/2} + V(q) + \frac{1}{2}\int d^3x\Big(|\pi(x)|^2 + |\nabla\phi(x)|^2\Big) \\ &+ \int d^3x\phi(x)\rho(x-q). \end{aligned} \tag{1.2}$$

We have set the mechanical mass of the particle and the speed of wave propagation equal to one. In spirit the interaction term is simply $\phi(q)$. This would result however in an energy which is not bounded from below. Therefore we smoothen

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out the coupling by the function $\rho(x)$. In analogy to the Maxwell-Lorentz equations we call $\rho(x)$ the "charge distribution". We assume the real-valued function $\rho(x)$ to be from the Sobolev space H^1 and of compact support, i.e.

$$\rho, \nabla\rho \in L^2(\mathbb{R}^3)\,, \quad \rho(x) = 0 \quad \text{for} \quad |x| \geq R_\rho\,. \tag{C}$$

The further important assumption is that the norm of ρ in L^2 is sufficiently small,

$$\gamma_\rho \equiv \|\rho\|_{L^2} \ll 1 \tag{1.3}$$

that means *weak* field-particle interaction. We believe this to be an artifact of the mathematical technique employed.

For the potential V we require that it is sufficiently smooth and has a compact support,

$$V \in C^2(\mathbb{R}^3), \quad V(x) = 0 \text{ for } |x| > R_V > 0. \tag{P}$$

Consider the corresponding nonperturbed system with $V \equiv 0$:

$$\begin{aligned} &\dot\phi(x,t) = \pi(x,t), && \dot\pi(x,t) = \Delta\phi(x,t) - \rho(x - q(t)), \\ &\dot q(t) = p(t)/(1+p^2(t))^{1/2}, && \dot p(t) = \int d^3x\,\phi(x,t)\,\nabla\rho(x - q(t)) \end{aligned} \tag{1.4}$$

with the Hamiltonian functional

$$\begin{aligned} \mathcal{H}_0(\phi,\pi,q,p) &= (1+p^2)^{1/2} + \frac{1}{2}\int d^3x\Big(|\pi(x)|^2 + |\nabla\phi(x)|^2\Big) \\ &+ \int d^3x\phi(x)\rho(x-q). \end{aligned} \tag{1.5}$$

The system (1.4) has the set of solutions that correspond to the charge traveling with a uniform velocity, v. Up to translation they are of the form

$$S_v(t) = (\phi_v(x - vt), \pi_v(x - vt), vt, p_v) \tag{1.6}$$

with an arbitrary velocity $v \in V = \{v \in \mathbb{R}^3 : |v| < 1\}$. The components of the traveling solution can be calculated easily in Fourier transform, cf. [10]:

$$\begin{aligned} \phi_v(x) &= -\frac{1}{4\pi}\int \frac{\rho(y)d^3y}{|v(y-x)_\| + \lambda(y-x)_\perp|}\,, \\ \pi_v(x) &= -v\cdot\nabla\phi_v(x),\ \ p_v = v/\lambda. \end{aligned} \tag{1.7}$$

Here we set $\lambda = \sqrt{1-v^2}$ and $x = vx_\| + x_\perp$, where $x_\| \in \mathbb{R}$ and $v \perp x_\perp \in \mathbb{R}^3$ for $x \in \mathbb{R}^3$. In further by "solitons" we mean these traveling solutions to (1.4).

Let us discuss and summarize now our main results, the precise theorems to be stated in the following section.

Consider the set $\mathcal{S}$ of *scattering* solutions to (1.1) for which $|q(t)| \to \infty$ as $t \to \infty$. Below we discuss the properties of the solutions of the class $\mathcal{S}$. Since only a finite amount of energy can be dissipated to infinity, we have the relaxation of acceleration,

$$\ddot q(t) \to 0,\ \ t \to \pm\infty. \tag{1.8}$$

Moreover, we establish the rate of the convergence $|\ddot q(t)| \sim t^{-1-\sigma}$ with a $\sigma > 0$. This is a crucial point of our asymptotic analysis. It follows that

$$\dot q(t) \to v_\pm,\ \ t \to \pm\infty \tag{1.9}$$

and the fields are asymptotically Coulombic traveling waves in the sense

$$(\phi(x,t),\pi(x,t)) \sim (\phi_{v_\pm}(x-q(t)),\pi_{v_\pm}(x-q(t))), \qquad t\to\pm\infty. \tag{1.10}$$

Since energy is conserved, the convergence here is in the sense of local energy seminorms, cf. Section 2. Further, we establish the corresponding asymptotics in a *global* energy norm,

$$(\phi(x,t),\pi(x,t)) \sim (\phi_{v_\pm}(x-q(t)),\pi_{v_\pm}(x-q(t))+U(t)F_\pm, \quad t\to\pm\infty \tag{1.11}$$

where $U(t)$ is the group of the free wave equation, and $F_\pm$ are scattering states.

At last we suggest simple sufficient conditions for solutions to belong to $\mathcal{S}$.

Note that the results of [9] imply the long-time convergence to the set of solitons (1.6) in the sense of local energy seminorms, as in (1.10). In [9] long-time convergence to the set of solitons (1.6) is established. Here we essentially use the results [9] on integral representation of solutions as well as the existence of dynamics for (1.1) (see also [1]).

Soliton-like asymptotics was proved for some translation invariant completely integrable 1D equations, [13]. Soliton-like asymptotics in *local* energy seminorms was proved for translation invariant system of a scalar field coupled to a particle [10] and for translation invariant 1D kinetic-reaction systems, [6]. Soliton-like asymptotics of type (1.11) in *global* energy norm was proved initially for *small perturbations* of soliton-like solutions to 1D nonlinear Schrödinger translation invariant equations [2, 3].

Soliton-like asymptotics of type (1.11) in energy norm for finite energy solutions of the class $\mathcal{S}$ is proved here for the first time for coupled particle-field equations (1.1).

The physical mechanism is radiation damping [12, 4, 5]: as long as the motion of the particle is accelerated, it loses energy through radiation escaping to infinity.

Note that the orbital stability of the solitons for the system (1.1) was proved in [10]. In [7] a general theory of orbital stability of solitons was developed for general nonlinear relativistic-invariant equations. This orbital stability approach is based on the Liapunov function method and does not take into account the energy radiation to infinity which leads to an asymptotic stability of the solitons. The analysis of the radiation and the convergence to solitons for general nonlinear relativistic-invariant equations is an open problem.

2. **Main results.**

First define a suitable phase space.

Let L^2 be the real Hilbert space $L^2(\mathbb{R}^3)$ with norm $|\cdot|$, and let $\dot H^1$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with norm $\|\phi(x)\| = |\nabla\phi(x)|$. Equivalently, using Sobolev's embedding theorem, $\dot H^1 = \{\phi(x)\in L^6(\mathbb{R}^3) : |\nabla\phi(x)| \in L^2\}$; see [11]. Let $|\phi|_R$ denote the norm in $L^2(B_R)$ for $R>0$, where $B_R=\{x\in\mathbb{R}^3 : |x|\le R\}$. Then the seminorms $\|\phi\|_R = |\nabla\phi|_R$ are continuous on $\dot H^1$.

Definition 2.1. *i) The phase space $\mathcal{E}$ is the Hilbert space $\dot H^1\oplus L^2\oplus\mathbb{R}^3\oplus\mathbb{R}^3$ of states $Y=(\phi,\pi,q,p)$ with finite norm*

$$\|Y\|_{\mathcal{E}} = \|\phi\| + |\pi| + |q| + |p|\,.$$

ii) $\mathcal{E}_F$ is the space $\mathcal{E}$ endowed with the Fréchet topology defined by the local energy seminorms

$$\|Y\|_R = \|\phi\|_R + |\pi|_R + |q| + |p|, \;\; \forall R>0\,.$$

iii) $\mathcal{F}$ *is the Hilbert space* $\dot{H}^1 \oplus L^2$ *of the fields* $F = (\phi, \pi)$ *with finite norm*

$$\| F\|_{\mathcal{F}} = \|\phi\| + |\pi| .$$

iv) $\mathcal{F}_F$ *is the space* $\mathcal{F}$ *endowed with the Fréchet topology defined by the local energy seminorms*

$$\|F\|_R = \|\phi\|_R + |\pi|_R, \ \ \forall R > 0 .$$

We write the Cauchy problem for the system (1.1) in the form

$$\dot{Y}(t) = \mathbf{F}(Y(t)) , \quad t \in \mathbb{R} , \quad Y(0) = Y^0 , \tag{2.1}$$

where $Y(t) = (\phi(t), \pi(t), q(t), p(t))$ and $Y^0 = (\phi^0, \pi^0, q^0, p^0)$.

Definition 2.2. $\mathcal{E}^\sigma$ *for* $0 \le \sigma \le 1$ *is the set of the states* $(\phi^0(x), \pi^0(x), q, p) \in \mathcal{E}$ *such that*

$$\int\limits_{\{R\le|x|\}} d^3x \, (|\nabla\phi^0(x)|^2 + |\phi^0(x)|^2 + |\pi^0(x)|^2) = \mathcal{O}(R^{-2-2\sigma}) \ \text{as} \ R \to \infty . \tag{2.2}$$

Proposition 2.3. [8, 9, 10] *Let (C), (P) hold and* $Y^0 = (\phi^0(x), \pi^0(x), q^0, p^0) \in \mathcal{E}$. *Then*
(i) *The system (1.1) has a unique solution* $Y(t) = (\phi(x,t), \pi(x,t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{E})$ *with* $Y(0) = Y^0$.
(ii) *The energy is conserved, i.e.*

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y^0) \ \text{ for } t \in \mathbb{R}. \tag{2.3}$$

(iii) *The bound holds*

$$\sup_{t\in\mathbb{R}} |\dot{q}(t)| \le \overline{v} < 1 , \tag{2.4}$$

where $\overline{v}$ *depends on* Y^0 *and* $\delta_\rho := |\langle\rho, \Delta^{-1}\rho\rangle| = \int d^3k \, |\hat{\rho}(k)|^2 |k|^{-2}$.

Denote $F(x,t) = (\phi(x,t), \pi(x,t))$ the field part of a solution to the system (1.1) and $F_v(x) = (\phi_v(x), \pi_v(x))$ the field part of a soliton of the system (1.4). Denote by $U(t)$ the group of the free wave equation on $\mathcal{F}$. The action of this group is isometric on $\mathcal{F}$ according to the corresponding energy conservation law. Put $\gamma_\rho := |\rho|$.

Remark. Note that there exist the functions $\rho(x)$ with bounded values of R_ρ, δ_ρ s.t. $\gamma_\rho \to 0$.

Our main result is the following theorem.

Theorem 2.4. *Let* δ_ρ *be bounded and* γ_ρ *be sufficiently small,* $\gamma_\rho \le \gamma_\rho(\overline{v}, R_\rho)$
Let $Y(t) = (\phi(x,t), \pi(x,t)), q(t), p(t)) \in C(\mathbb{R}, \mathcal{E})$ *be a solution to the system (1.1), let* $Y(0) \in \mathcal{E}^\sigma$ *with some* $\sigma \in (0,1]$ *and let* $Y(t) \in \mathcal{S}$. *Then the relaxation of the acceleration (1.8) holds, and the solution* $Y(t)$ *admits the following long-time asymptotics:*
i) *There exist* $v_\pm = \lim_{t\to\pm\infty} \dot{q}(t) \in \mathcal{V}$ *s.t.*

$$|\dot{q}(t) - v_\pm| \le C \ (1 + |t|)^{-\sigma}, \tag{2.5}$$

$$\|F(x + q(t), t) - F_{v_\pm}(x)\|_R \le C_R(1 + |t|)^{-\sigma}, \quad \forall R > 0. \tag{2.6}$$

ii) *There exist* $F_\pm \in \mathcal{F}$ *such that*

$$\|F(x,t) - F_{v(t)}(x - q(t)) - U(t)F_\pm\|_{\mathcal{F}} \le C(1 + |t|)^{-\sigma}. \tag{2.7}$$

Let us formulate sufficient conditions for a solution $Y(t)$ to belong to $\mathcal{S}$.

Put $h(t) = \frac{1}{2}\int d^3x(|\nabla\phi(x,t)|^2 + |\pi(x,t)|^2)$ — the energy of the field part of a solution. Recall that *supp* $V \subset \{x : |x| \le R_V\}$. Put $G = \sup_{x\in\mathbb{R}^3} |\nabla V(x)|$, $v(t) := \dot{q}(t)$.

Theorem 2.5. *Consider solutions $Y(t)$ to the system (1.1) with initial data $Y(0) \in \mathcal{E}^\sigma$ with some $\sigma \in (0,1]$, let $\dot{q}(0) = v(0)$. Let $R_V, G, |q(0)|, h(0)$ and δ_ρ be bounded. Then for $|v(0)|$ close enough to 1 and sufficiently small γ_ρ the solution $Y(t)$ to the system (1.1) belongs to $\mathcal{S}$, that is*

$$\lim_{t\to\pm\infty} |q(t)| = \infty.$$

3. **Integral inequality argument.** Consider a solution $Y(t) \in \mathcal{S}$ to the system (1.1). If t is sufficiently large, then $Y(t)$ obeys the nonperturbed equations (1.4). Since the system (1.1) is invariant with respect to time translations, we may assume that $Y(t)$ obeys the equations (1.4) for $t \ge 0$.

If the soliton-like asymptotics is approximately valid, then the field should be close to the soliton centered at $q(t)$ with velocity $v(t) = \dot{q}(t)$. We therefore consider the difference

$$Z(x,t) = F(x,t) - F_{v(t)}(x - q(t)), \tag{3.1}$$

where $v(t) \equiv \dot{q}(t)$. Defining $\overline{\rho}(x) = (0, \rho(x))$ and $A(\phi,\pi) = (\pi, \Delta\phi)$ we obtain that F obeys the equations of motion

$$\dot{F}(x,t) = AF(x,t) - \overline{\rho}(x - q(t)). \tag{3.2}$$

On the other hand, for the soliton field F_v with a fixed v, the equation holds

$$-v \cdot \nabla F_v(x - q(t)) = AF_v(x - q(t)) - \overline{\rho}(x - q(t)).$$

Then for Z we have the equation

$$\dot{Z}(x,t) = AZ(x,t) - \dot{p}(t) \cdot \nabla_p F_{v(t)}(x - q(t)). \tag{3.3}$$

Here, according to the chain rule,

$$\nabla_p F_v = \nabla_v F_v \, dv(p), \tag{3.4}$$

where $dv(p)$ is the differential of the map $p \mapsto v(p) = p/\sqrt{1+p^2}$. In the Cartesian coordinate system $dv(p)$ is represented by the Jacobi matrix $\partial v_i/\partial p_j$.

Lemma 3.1. *Under the assumptions of Theorem 2.4 the following bound holds for any $R > 0$:*

$$\|Z(\cdot + q(t), t)\|_R \le C_R(1 + |t|)^{-1-\sigma}, \tag{3.5}$$

where C_R depends also on initial data, $\overline{v}$, and R_ρ.

Proof: First, we prove the estimate with $R = R_\rho$. Definition (3.1) imply $Z(\cdot, t) \in \mathcal{F}$. Solving equations (3.3) we get the mild solution representation:

$$Z(t) = U(t)Z(0) - \int_0^t U(t-s)[\dot{p}(s) \cdot \nabla_p F_{v(s)}(\cdot - q(s))]\, ds \tag{3.6}$$

with $U(t)$ the group generated by the free wave equation in $\dot{H}^1 \oplus L^2$.

Denote by $Z_1(x,t) = \phi(x,t) - \phi_{v(t)}(x - q(t))$ the first component of $Z(x,t)$ and observe that $\langle \phi_v(x), \nabla\rho(x)\rangle = 0$ for $|v| < 1$ because the soliton (1.6) is a solution to (1.4). Then (1.1) (coinciding to (1.4) for $t \ge 0$) implies

$$\dot{p}(t) = \langle Z_1(x + q(t), t), \nabla\rho(x)\rangle . \tag{3.7}$$

Thus we obtain,

$$|\dot{p}(t)| \le C\|Z(\cdot + q(t), t)\|_{R_\rho} |\rho|. \tag{3.8}$$

Let us denote $\overline{\pi}_v = \nabla_p \pi_v$, $\overline{\phi}_v = \nabla_p \phi_v$, $S_{t-s}(x) = \{y: \ |y-x| = t-s\}$, and

$$(\overline{\phi}(\cdot, t, s), \overline{\pi}(\cdot, t, s)) = U(t-s)[\nabla_p F_{v(s)}(\cdot - q(s))]. \tag{3.9}$$

Then Kirchhoff's formula for $U(t-s)$ implies the representation

$$\begin{aligned} \nabla\overline{\phi}(x,t,s) = & \sum_{|\alpha|\le 1} (t-s)^{|\alpha|-2} \int_{S_{t-s}(x)} d^2y \, a_\alpha(x-y) \partial_y^\alpha \overline{\pi}_{v(s)}(y - q(s)) \\ & + \sum_{|\alpha|\le 2} (t-s)^{|\alpha|-3} \int_{S_{t-s}(x)} d^2y \, b_\alpha(x-y) \partial_y^\alpha \overline{\phi}_{v(s)}(y - q(s)) \end{aligned} \tag{3.10}$$

and a similar representation for $\overline{\pi}(x,t,s)$. The coefficients $a_\alpha(\cdot)$, $b_\alpha(\cdot)$ are bounded and the sums run over the multiindices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with integers $\alpha_j \ge 0$. Therefore $\nabla\overline{\phi}(x + q(t), t, s)$ and $\overline{\pi}(x + q(t), t, s)$ can be represented as integrals of type (3.10) over the shifted sphere $S_{t-s}(x + q(t))$ and with $x + q(t)$ substituted to $a_\alpha(x-y)$ and $b_\alpha(x-y)$ instead of x. If $|x| \le R_\rho$, we have on this sphere

$$\begin{aligned} |y - q(s)| &= |(y - x - q(t)) + (x + q(t) - q(s))| \\ &\ge (t-s) - |x| - \overline{v}(t-s) \ge (1-\overline{v})(t-s) - R_\rho \end{aligned} \tag{3.11}$$

by the bound (2.4) on $\dot{q}(t)$. On the other hand, the integral representation (1.7) yields by Cauchy-Schwarz

$$\sup_{|v|\le\overline{v}} \sup_{|x|\ge 2R_\rho} \Big[|x||\overline{\phi}_v(x)| + |x|^2(|\nabla\overline{\phi}_v(x)| + |\overline{\pi}_v(x)|) + |x|^3(|\nabla\nabla\overline{\phi}_v(x)| + |\nabla\overline{\pi}_v(x)|)\Big] \le C(\overline{v}, R_\rho)|\rho| < \infty\,. \tag{3.12}$$

Inserting (3.12) and (3.11) in Kirchhoff's formula for $\nabla\overline{\phi}(x + q(t), t, s)$, we obtain the pointwise bound

$$\begin{aligned} |\nabla\overline{\phi}(x + q(t), t, s)| \ \le\ & \sum_{|\alpha|\le 1} (t-s)^{|\alpha|-2} \frac{C_1(\overline{v}, R_\rho)|\rho|(t-s)^2}{(1 + |t-s|)^{|\alpha|+2}} \\ & + \sum_{|\alpha|\le 2} (t-s)^{|\alpha|-3} \frac{C_1(\overline{v}, R_\rho)|\rho|(t-s)^2}{(1 + |t-s|)^{|\alpha|+1}} \le \frac{C_2(\overline{v}, R_\rho)|\rho|}{1 + (t-s)^2} \end{aligned} \tag{3.13}$$

for $|x| \le R_\rho$ and provided $t - s \ge 3R_\rho/(1 - \overline{v})$. Therefore (3.13) implies for large $t - s$, together with similar bound for $\overline{\pi}(x + q(t), t, s)$, the integral estimate

$$\|(\overline{\phi}(x + q(t), t, s), \overline{\pi}(x + q(t), t, s)\|_{R_\rho} \le \frac{C_3(\overline{v}, R_\rho)|\rho|}{1 + (t-s)^2}\,. \tag{3.14}$$

On the other hand, for bounded $t - s$ this integral estimate follows from (3.9) by energy conservation for the map $U(t-s)$ since $\|\nabla_p F_v\|_{\mathcal{F}} \le C(\overline{v}, R_\rho)|\rho|$ by (C). Finally, (3.8) and (3.14) imply

$$\begin{aligned} & \|\dot{p}(s) \cdot (\overline{\phi}(x + q(t), t, s), \overline{\pi}(x + q(t), t, s)\|_{R_\rho} = \\ & \le C_4(\overline{v}, R_\rho)|\rho| \, \frac{\|Z(\cdot + q(s), s)\|_{R_\rho} \, |\rho|}{1 + (t-s)^2}. \end{aligned} \tag{3.15}$$

Now, let us bound the first term on right hand side of (3.6), more precisely, we should estimate $\|[U(t)Z(0)](\cdot + q(t), t)\|_{R_\rho}$. Since $Y(0) \in \mathcal{E}^\sigma$ by assumption,

$F(x,0) = (\phi^0(x), \pi^0(x))$ satisfies the bounds (2.2) with a $\sigma \in (0,1]$. Applying the well-known energy inequality for the free wave equation, we get

$$\|[U(t)F(0)](\cdot + q(t), t)\|_{R_\rho} \leq \|F(\cdot + q(t), 0)\|_{t+R_\rho}.$$

Further, from the strong Huygen's principle it follows that for $t > R_\rho$ the solution $[U(t)F(0)](\cdot + q(t), t)$ does not change inside the ball B_{R_ρ} if one replaces $F(x,0)$ by zero outside a small neighborhood of the spherical layer $L := \{t - R_\rho \leq |x - q(t)| \leq t + R_\rho\}$. Hence,

$$\|[U(t)F(0)](\cdot + q(t), t)\|^2_{R_\rho} \leq C \int\limits_L d^3x (|\nabla \phi^0(x)|^2 + |\phi^0(x)|^2 + |\pi^0(x)|^2).$$

Note that for x belonging to this layer of integration we have the following lower bound due to (2.4):

$$|x| = |x - q(t) + q(t)| \geq |x - q(t)| - |q(t)| \geq t - R_\rho - \overline{v}t - |q^0| \geq (1 - \overline{v})t - |q^0| - R_\rho.$$

Then the condition (2.2) for $F(0)$ implies, for sufficiently large t,

$$\|[U(t)F(0)](\cdot + q(t), t)\|_{R_\rho} \leq \frac{C(F(0), q^0, \bar{v}, R_\rho)}{(1 + |t|)^{1+\sigma}}. \tag{3.16}$$

On the other hand, from (1.7) it follows by direct computation, cf. [9], that $U(t)F_{v(0)}$ satisfies the same bounds (3.16) with a $\sigma = 1$. Hence, $U(t)Z = U(t)F - U(t)F_{v(0)}$ satisfy the bound

$$\|[U(t)Z(0)](\cdot + q(t), t)\|_{R_\rho} \leq \frac{C}{(1 + |t|)^{1+\sigma}} \tag{3.17}$$

with the same σ as in (3.16), where C depends on initial data, $\bar{v}$, and R_ρ. For bounded t this estimate follows from the energy conservation for the free wave equation. Combining (3.6) to (3.15) and (3.17) we arrive at

$$\begin{aligned} \|Z(\cdot + q(t), t)\|_{R_\rho} \quad &\leq \quad \frac{C(Z(0), q^0, \bar{v}, R_\rho)}{(1 + |t|)^{1+\sigma}} + \\ &\gamma_\rho^2 C_4(\overline{v}, R_\rho) \int_0^t \frac{\|Z(\cdot + q(s), s)\|_{R_\rho}}{1 + (t - s)^2}\, ds, \quad t \geq 0. \end{aligned} \tag{3.18}$$

Therefore, setting $M(t) = \max_{0 \leq s \leq t} (1 + |s|)^{1+\sigma} \|Z(\cdot + q(s), s)\|_{R_\rho}$, we have

$$M(t) \leq C_0(Z(0), q^0, \overline{v}, R_\rho) + \gamma_\rho^2 C(\overline{v}, R_\rho) I_\sigma M(t)$$

where

$$I_\sigma = \sup_{t \geq 0} (1 + |t|)^{1+\sigma} \int_0^t \frac{(1 + |s|)^{-1-\sigma}}{(1 + |t - s|^2)}\, ds < \infty \text{ for } \sigma \in (0,1]\,.$$

It remains to choose $\gamma_\rho^2 C(\overline{v}, R_\rho) I_\sigma < 1$, then (3.5) with $R = R_\rho$ follows.

At last, we claim that the bound (3.5) with $R = R_\rho$ implies (3.5) for any $R > 0$. Indeed, (3.14)-(3.18) hold with the norm $\| \cdot \|_R$ instead of $\| \cdot \|_{R_\rho}$ on the *left* hand sides and with $C_i(\overline{v}, \rho, R)$ instead of $C_i(\overline{v}, \rho)$ on the *right* hand sides. Then (3.18) with this generalization and (3.5) with $R = R_\rho$ imply (3.5) for any $R > 0$.

Proof of Theorem 2.4: i) (3.5) with $R = R_\rho$ and (3.8) imply

$$|\dot{p}(t)| \leq C(1 + |t|)^{-1-\sigma} \Longleftrightarrow |\ddot{q}(t)| \leq C_1(1 + |t|)^{-1-\sigma}. \tag{3.19}$$

Then there exist the limits (1.9), and (2.5) follows. Therefore, (3.5) implies (2.6).

ii) We have to prove that $\|Z(x,t)-U(t)F_\pm\|_{\mathcal{F}} \le C(1+|t|)^{-\sigma}$. This is equivalent to $\|U(-t)Z(x,t) - F_\pm\|_{\mathcal{F}} \le C(1+|t|)^{-\sigma}$ since the group $U(t)$ is isometric in $\mathcal{F}$. Apply $U(-t)$ to integral equation (3.6) and get

$$U(-t)Z(t) = Z(0) - \int_0^t U(-s)[\dot p(s)\cdot\nabla_p F_{v(s)}(\cdot - q(s))]\,ds.$$

The condition (2.4) provides that the norm of $F_{v(s)}(\cdot - q(s))$ in $\mathcal{F}$ is bounded uniformly with respect to s. Then (3.19) implies the convergence of the integral in $\mathcal{F}_s$ at the stated rate. Theorem 2.4 is proved.

4. **Constructing scattering solutions.** In this section we prove Theorem 2.5. Since the system (1.1) is time-invertible, we consider only the case $t\to+\infty$. Let $|q(0)|\le R$. Without loss of generality we suppose that $R > R_V$. Consider the particle with initial data $q(0), v(0)$. Introduce $e = v(0)/|v(0)|$. The orthogonal projections of the vectors $v(t), p(t), q(t)$ onto e read $v_e(t)e$, $p_e(t)e$, $q_e(t)e$ respectively with $v_e(t) := v(t)\cdot e$, $p_e(t) := p(t)\cdot e$, $q_e(t) := q(t)\cdot e$, here dot means the scalar product in $\mathbb{R}^3$. Note that the vectors $v(t)$ and $p(t)$, $v_e(t)$ and $p_e(t)$ are of the same directions and $v_e(0) = |v(0)|$, $p_e(0) = |p(0)|$. Introduce the layer in $\mathbb{R}^3$, $L(e,R_V) := \{x : |x\cdot e| \le R_V\}$, then $supp\,V \subset\ L(e,R_V)$.

The statement of the Theorem follows from the three Propositions below. Since the system (1.1) is invariant with respect to time translations, we start from $t = 0$ in each Proposition.

Proposition 4.1. *Let $|q(0)| > R_V$, $|v(0)|$ be close enough to 1, let e be directed toward $L(e,R_V)$. Then the particle enters $L(e,R_V)$ at a certain moment τ with $|v_e(\tau)|$ close to 1.*

Proposition 4.2. *Let $|q(0)| \le R_V$, let $|v(0)|$ be close to 1. Then the particle leaves $L(e,R_V)$ at a certain moment τ such that $|v_e(\tau)| > 0$ and $v_e(\tau)e$ is directed outside $L(e,R_V)$.*

Proposition 4.3. *Let $|q(0)| \ge R_V$, $|v(0)| > 0$ and e is directed outside $L(e,R_V)$. Then the particle never enters $L(e,R_V)$ and $|q_e(t)| \to \infty$ as $t\to+\infty$.*

Proof of Proposition 4.1: For $v_e(t)$ we have the estimate

$$v_e(t) \ge v_e(0) - \int_0^t |\dot v(s)|ds = |v(0)| - \int_0^t |\dot v(s)|ds.$$

Since outside $L(e,R_v)$ the free equations (1.4) are satisfied, the following estimate (see (3.5) and (3.8)) is valid:

$$|\dot v(t)| \le \frac{C\gamma_\rho}{(1+|t|)^{\sigma+1}} \tag{4.1}$$

with a finite C determined by initial data, $\overline{v}$, and R_ρ. Thus,

$$v_e(t) \ge |v(0)| - \int_0^\infty \frac{C\gamma_\rho dt}{(1+|t|)^{\sigma+1}} = |v(0)| - \frac{C\gamma_\rho}{\sigma},$$

and we obtain the required result for sufficiently small γ_ρ.

Proof of Proposition 4.2: First we check that the growth of the field energy is not very fast. Recall that $h(t) = \frac{1}{2}\int d^3x(|\nabla\phi(x,t)|^2 + |\pi(x,t)|^2)$.

Lemma 4.4.

$$h(t) \le (\sqrt{h(0)} + \sqrt{2}\gamma_\rho t)^2. \tag{4.2}$$

Proof: Multiply the equation $\ddot{\phi} = \Delta\phi - \rho$ by $\dot{\phi}$ and integrate over $\mathbb{R}^3$. We obtain $\dot{h}(t) = -\int d^3x\rho\dot{\phi}$ and hence $\dot{h}(t) \le \sqrt{2}\gamma_\rho\sqrt{h}$. Integrating this differential inequality in t we come to $\sqrt{h(t)} \le \sqrt{h(0)} + \sqrt{2}\gamma_\rho t$ which proves (4.2).

Let us now prove the Proposition. Recall that $v = p/\sqrt{1+p^2}$ and hence, $p = v/\sqrt{1-v^2}$. Thus, $|v|$ is close to 1 if and only if $|p|$ is large. From the equation

$$\dot{p}(t) = -\nabla V(q(t)) + \int d^3x\,\phi(x,t)\,\nabla\rho(x-q(t))$$

we obtain, due to (4.2), $|\dot{p}| \le G + \|\phi\|\gamma_\rho \le G + (2h(t))^{1/2}\gamma_\rho \le G + ((2h(0))^{1/2} + 2\gamma_\rho t)\gamma_\rho = G_1 + 2\gamma_\rho^2 t$ with $G_1 := G + (2h(0))^{1/2}\gamma_\rho$. The conditions of the theorem imply that G_1 is bounded. We obtain the following lower and upper bounds,

$$p_e(t) \ge p_e(0) - \int_0^t |\dot{p}(s)|ds \ge |p(0)| - G_1 t - \gamma_\rho^2 t^2 = P - f(t),$$

$$|p(t)| \le |p(0)| + \int_0^t |\dot{p}(s)|ds \le |p(0)| + G_1 t + \gamma_\rho^2 t^2 = P + f(t),$$

where $P := |p(0)|$, $f(t) := G_1 t + \gamma_\rho^2 t^2$. These estimates imply for $v_e(t)$

$$\begin{aligned} v_e(t) &= \frac{p_e(t)}{|p(t)|}\left(1 + \frac{1}{|p(t)|^2}\right)^{-1/2} \\ &\ge \frac{P - f(t)}{P + f(t)}\left(1 - \frac{1}{(P-f(t))^2}\right) = \frac{1 - a^2 - 2af(t) + a^2 f^2(t)}{1 - a^2 f^2(t)} \\ &\ge (1 - a^2 - 2af(t) + a^2 f^2(t))(1 + a^2 f^2(t)) = 1 - a^2 + g(t), \end{aligned} \tag{4.3}$$

where $a := P^{-1}$, $g(t) := -2af(t) + (2a^2 - a^4)f^2(t) - 2a^3 f^3(t) + a^4 f^4(t)$.

The corresponding estimate for $q_e(t)$ is

$$q_e(t) \ge q_e(0) + (1-a^2)t + \int_0^t g(s)ds. \tag{4.4}$$

Take sufficiently large P, that is small a, then from the estimates (4.4), (4.3) the statement of the proposition follows.

Proof of Proposition 4.3: We claim that there exist such small $\gamma_\rho > 0$, $\underline{v} > 0$ that $\forall\ t > 0\ v_e(t) \ge \underline{v}$.

Indeed, put $T = \sup\{t > 0 :\ v_e(t) > \underline{v}\}$. If $\underline{v} < v_e(0)/2$, then, by continuity, $T > 0$.

Further, it is possible to choose such small $\gamma_\rho > 0$, $\underline{v} > 0$ that $T = +\infty$. Note that for $t \in [0, T]$ the free equations (1.4) are satisfied, hence the estimate (4.1) is valid. Take

$$0 < \underline{v} < v_e(0) - \int_0^\infty \frac{C\gamma_\rho}{(1+|t|)^{\sigma+1}} = v_e(0) - \frac{C\gamma_\rho}{\sigma},$$

the choice is possible for sufficiently small γ_ρ. If $T < +\infty$, then $v_e(T) > \underline{v}$, hence, by continuity, $v_e(T + \varepsilon) > \underline{v}$ for some $\varepsilon > 0$. This contradicts to the definition of T. Thus, $T = +\infty$. Hence, for $t > 0$ one obtains $q_e(t) \geq q_e(0) + \underline{v}t$.

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