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On Scattering of Solitons for the Klein–Gordon Equation Coupled to a Particle

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Abstract: We establish the long time soliton asymptotics for the translation invariant nonlinear system consisting of the Klein–Gordon equation coupled to a charged relativistic particle. The coupled system has a six dimensional invariant manifold of the soliton solutions. We show that in the large time approximation any finite energy solution, with the initial state close to the solitary manifold, is a sum of a soliton and a dispersive wave which is a solution of the free Klein–Gordon equation. It is assumed that the charge density satisfies the Wiener condition which is a version of the "Fermi Golden Rule". The proof is based on an extension of the general strategy introduced by Soffer and Weinstein, Buslaev and Perelman, and others: symplectic projection in Hilbert space onto the solitary manifold, modulation equations for the parameters of the projection, and decay of the transversal component.

1. Introduction

Our paper concerns the problem of nonlinear field-particle interaction. A charged particle radiates a field which acts back on the particle. This interaction is responsible for some crucial features of the dynamics: asymptotically uniform motion and stability against small perturbations of the particle, increase of the particle's mass and others (see [1, 11, 26, 37]). The problem has many different appearances: a classical particle coupled to a scalar or Maxwell field, and coupled Maxwell–Schrödinger or Maxwell–Dirac equations, general translation invariant nonlinear hyperbolic PDEs. In all cases the

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goal is to reveal the distinguished role of the soliton solutions, i.e. traveling wave solutions of finite energy. Let us note that the existence of the soliton solutions is proved for nonlinear Klein–Gordon equations with a general nonlinear term [4], and for the coupled Maxwell–Dirac equations [13].

One of the main goals of a mathematical investigation is to study soliton type asymptotics and asymptotic stability of soliton solutions to the equations. First results in this direction have been discovered for the KdV equation and other *completely integrable equations*. For the KdV equation, any solution with sufficiently smooth and rapidly decaying initial data converges to a finite sum of soliton solutions moving to the right, and a dispersive wave moving to the left. A complete survey and proofs can be found in [12].

For nonintegrable equations, the long time convergence of the solution to a soliton part and dispersive wave was obtained first by Soffer and Weinstein in the context of the $U(1)$-invariant Schrödinger equation [31–33]. The extension to translation invariant equations was obtained by Buslaev and Perelman [5, 6] for the 1D Schrödinger equation, and by Miller, Pego and Weinstein for the 1D modified KdV and RLW equations, [27–29]. The techniques introduced by Weinstein [41] play a fundamental role in the proofs of all these results.

In [5, 6] the long time convergence is obtained for the 1D translation invariant and $U(1)$-invariant nonlinear Schrödinger equation. It is shown there that the following asymptotics hold for any finite-energy solution $\psi(x,t)$ with initial data close to a soliton $\psi_{v_0}(x-v_0t-a_0)e^{i\omega_0 t}$:

$$\psi(x,t)=\psi_{v_\pm}(x-v_\pm t-a_\pm)e^{i\omega_\pm t}+W_0(t)\psi_\pm+r_\pm(x,t),\quad t\to\pm\infty. \tag{1.1}$$

Here the first term on the right-hand side is a soliton with parameters $v_\pm, a_\pm, \omega_\pm$ close to v_0, a_0, ω_0, the function $W_0(t)\psi_\pm$ is a dispersive wave which is a solution to the free Schrödinger equation, and the remainder $r_\pm(x,t)$ converges to zero in the global L^2-norm. Recently Cuccagna extended the asymptotics (1.1) to nD Schrödinger equations with $n\ge 3$, [9, 10].

We establish the asymptotics similar to (1.1) for a scalar real-valued Klein–Gordon field $\psi(x)$ in $\mathbb{R}^3$ coupled to a relativistic particle with position q and momentum p governed by

$$\begin{aligned}&\dot\psi(x,t)=\pi(x,t), \qquad \dot\pi(x,t)=\Delta\psi(x,t)-m^2\psi(x,t)-\rho(x-q(t)),\ x\in\mathbb{R}^3,\\ &\dot q(t)=p(t)/\sqrt{1+p^2(t)},\ \dot p(t)=\int\psi(x,t)\,\nabla\rho(x-q(t))dx,\end{aligned} \tag{1.2}$$

where $m>0$ (the case $m=0$ is degenerate and will be considered elsewhere). This is a Hamiltonian system with the Hamiltonian functional

$$\begin{aligned}\mathcal{H}(\psi,\pi,q,p)=&\frac12\int\Big(|\pi(x)|^2+|\nabla\psi(x)|^2+m^2|\psi(x)|^2\Big)dx\\ &+\int\psi(x)\rho(x-q)dx+\sqrt{1+p^2}.\end{aligned} \tag{1.3}$$

The first two equations for the fields are equivalent to the Klein–Gordon equation with the source $\rho(x-q)$. The form of the last two equations in (1.2) is determined by the choice of the relativistic kinetic energy $\sqrt{1+p^2}$ in (1.3). Nevertheless, the system (1.2) is not relativistic invariant.

We have set the maximal speed of the particle equal to one, which is the speed of wave propagation. This is in agreement with the principles of special relativity. Let us also note that the first two equations of (1.2) admit the soliton solutions of finite energy, $\psi_v(x - vt - a), \pi_v(x - vt - a)$, if and only if $|v| < 1$.

The case of a point particle corresponds to $\rho(x) = \delta(x)$ and then the interaction term in the Hamiltonian is simply $\psi(q)$. However, in this case the Hamiltonian is unbounded from below which leads to the ill-posedness of the problem, also known as ultraviolet divergence. Therefore we smooth the coupling by the function $\rho(x)$ following the "extended electron" strategy proposed by M. Abraham [1] for charges coupled to the Maxwell field. In analogy to the Maxwell–Lorentz equations we call ρ the "charge distribution". Let us write the system (1.2) as

$$\dot{Y}(t) = F(Y(t)), \qquad t \in \mathbb{R}, \tag{1.4}$$

where $Y(t) := (\psi(x,t), \pi(x,t), q(t), p(t))$ (below we always deal with column vectors but often write them as row vectors). The system (1.2) is translation-invariant and admits the soliton solutions

$$Y_{a,v}(t) = (\psi_v(x - vt - a), \pi_v(x - vt - a), vt + a, p_v), \quad p_v = v/\sqrt{1 - v^2} \tag{1.5}$$

for all $a, v \in \mathbb{R}^3$ with $|v| < 1$ (see (2.7), (2.10)), where the functions ψ_v, π_v decay exponentially for $m > 0$ (the main difficulty of the case $m = 0$ is provided by very slow decay of the functions). The states $S_{a,v} := Y_{a,v}(0)$ form the solitary manifold

$$\mathcal{S} := \{S_{a,v} : a, v \in \mathbb{R}^3, |v| < 1\}. \tag{1.6}$$

Our main result is the soliton-type asymptotics of type (1.1) for $t \to \pm\infty$,

$$(\psi(x,t), \pi(x,t)) \sim (\psi_{v_\pm}(x - v_\pm t - a_\pm), \pi_{v_\pm}(x - v_\pm t - a_\pm)) + W_0(t)\boldsymbol{\Psi}_\pm \tag{1.7}$$

for solutions to (1.2) with initial data close to the solitary manifold $\mathcal{S}$. Here $W_0(t)$ is the dynamical group of the free Klein–Gordon equation, $\boldsymbol{\Psi}_\pm$ are the corresponding asymptotic scattering states, and the remainder converges to zero in the global energy norm, i.e. in the norm of the Sobolev space $H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. For the particle trajectory we prove that

$$\dot{q}(t) \to v_\pm, \;\; q(t) \sim v_\pm t + a_\pm. \tag{1.8}$$

The results are established under the following conditions on the charge distribution: ρ is a real valued function of the Sobolev class $H^2(\mathbb{R}^3)$, compactly supported, and spherically symmetric, i.e.

$$\rho, \nabla\rho, \nabla\nabla\rho \in L^2(\mathbb{R}^3), \qquad \rho(x) = 0 \;\text{ for }\; |x| \ge R_\rho, \quad \rho(x) = \rho_1(|x|). \tag{1.9}$$

We require that all "modes" of the wave field are coupled to the particle, which is formalized by the Wiener condition

$$\hat{\rho}(k) = (2\pi)^{-3/2} \int e^{ikx} \rho(x) dx \neq 0 \;\text{ for all }\; k \in \mathbb{R}^3. \tag{1.10}$$

It is an analogue of the "Fermi Golden Rule" [7–10, 30, 34, 35]: the coupling term $\rho(x - q)$ is not orthogonal to the eigenfunctions e^{ikx} of the continuous spectrum of the linear part of the equation. As we will see, the Wiener condition (1.10) is very essential for our asymptotic analysis (see Remark 15.5). Generic examples of the coupling function ρ satisfying (1.9) and (1.10) are given in [24].

Remark 1.1. Physically, the Wiener condition means the strong coupling of the particle to the field which leads to *radiation* of the particle. This radiation results in the relaxation of the acceleration $\ddot{q}(t) \to 0$, $t \to \pm\infty$ which provides the asymptotics (1.7) and (1.8). Note that the soliton solutions do not radiate, and the radiation of the particle manifests itself in the decay of the deviation of the solution from the solitary manifold (see (1.18) below).

The problem under investigation was studied earlier in the following two different situations A and B:
A. The asymptotics

$$\dot{q}(t) \to v_\pm, \quad (\psi(x,t), \pi(x,t)) \sim (\psi_{v_\pm}(x - q(t)), \pi_{v_\pm}(x - q(t))) \qquad (1.11)$$

were proved in [25] in the case $m = 0$, under the Wiener condition (1.10), for all finite energy solutions, without the assumption that the initial data are close to $\mathcal{S}$. This means that the solitary manifold is a *global attractor* for the equations (1.2). However, the asymptotics (1.11) were established only in *local energy semi-norms* centered at the particle position $q(t)$. This means that the remainder in (1.11) may contain a dispersive term, similar to the middle term in the right hand side of (1.7), whose energy radiates to infinity as $t \to \pm\infty$ but does not converge to zero. A similar result is established in [16] for coupled Maxwell-Lorentz equations.
B. The asymptotics (1.11), and an analogue of the asymptotics (1.7) in the *global energy norm*, were established in [18] (resp., [15]) also for all finite energy solutions, in the case $m = 0$ (resp., $m > 0$), under the smallness condition on the coupling function, $\|\rho\|_{L^2(\mathbb{R}^3)} \ll 1$. The similar results are established in [17, 37] (resp., [19]) for the coupled Maxwell-Lorentz equations with a moving (resp., rotating) charge. Let us stress that the asymptotics (1.8) for the position was missing in the previous work.

Let us comment on the main difficulties in proving the asymptotic stability of the invariant manifold S and justifying (1.7), (1.8). The method of [16, 25] is based on the Wiener Tauberian Theorem, hence cannot provide a rate of convergence in the velocity asymptotics of (1.8) which is needed to prove (1.7) and the position asymptotics of (1.8). Also the methods of [15, 17–19] are applicable only for a small coupling function $\rho(x)$, and do not provide the position asymptotics in (1.8).

Our approach develops a general strategy introduced in [5, 6, 28, 29] for proving the asymptotic stability of the invariant solitary manifold $\mathcal{S}$. The strategy originates from the techniques in [41] and their developments in [31–33] in the context of the $U(1)$-invariant Schrödinger equation. The approach uses the symplectic geometry methods for the Hamiltonian systems in Hilbert spaces and spectral theory of nonselfadjoint operators.

The invariant manifolds arise automatically for equations with a symmetry Lie group [4, 13, 14]. In particular, our system (1.2) is invariant under translations in $\mathbb{R}^3$. The asymptotic stability of the solitary manifold is studied by a linearization of the dynamics (1.4). The linearization will be made along a special curve on the solitary manifold, $S(t)$, which is the symplectic orthogonal projection of the solution. Then the linearized equation reads

$$\dot{X}(t) = A(t)X(t), \quad t \in \mathbb{R}, \qquad (1.12)$$

where the operator $A(t)$ corresponds to the linearization at the soliton $S(t)$. Furthermore, we consider the "frozen" linearized equation (1.12) with $A(t_1)$ instead of $A(t)$.

The operator $A(t_1)$ has zero eigenvalue, and the frozen linearized equation admits secular solutions linear in t (see (6.24)). The existence of these runaway solutions prohibits the direct application of the Liapunov strategy and is responsible for the instability of the nonlinear dynamics along the manifold $\mathcal{S}$. One crucial observation is that the linearized equation is stable in the *symplectic orthogonal complement* to the tangent space $\mathcal{T}_S$. The complement is invariant under the linearized dynamics since the linearized dynamics is Hamiltonian and leaves the symplectic structure invariant.

Our proofs are based on a suitable extension of the methods in [5, 6, 28, 29]. Let us comment on the main steps.

I. First, we construct the symplectic orthogonal projection $S(t) = \mathbf{\Pi} Y(t)$ of the trajectory $Y(t)$ onto the solitary manifold $\mathcal{S}$. This means that $S(t) \in \mathcal{S}$, and the complement vector $Z(t) := Y(t) - S(t)$ is symplectic orthogonal to the tangent space $\mathcal{T}_{S(t)}$ for every $t \in \mathbb{R}$:

$$Z(t) \nmid \mathcal{T}_{S(t)}, \quad t \in \mathbb{R}. \tag{1.13}$$

So, we get the splitting $Y(t) = S(t) + Z(t)$ and we linearize the dynamics in the *transversal component* $Z(t)$ along the trajectory.

The *soliton component* $S(t) = S_{b(t),v(t)}$ satisfies a *modulation equation*. Namely, in the parametrization $\xi(t) = (c(t), v(t))$ with $c(t) := b(t) - \int_0^t v(s)ds$, we have

$$\dot{\xi}(t) = N_1(\xi(t), Z(t)), \quad |N_1(\xi(t), Z(t))| \le C\|Z(t)\|^2_{-\beta}\,, \tag{1.14}$$

where $\|\cdot\|_{-\beta}$ stands for an appropriate weighted Sobolev norm.

On the other hand, the transversal component satisfies the *transversal equation*

$$\dot{Z}(t) = A(t)Z(t) + N_2(S(t), Z(t)), \tag{1.15}$$

where $A(t) = A_{S(t)}$, and $N_2(S(t), Z(t))$ is a nonlinear part:

$$\|N_2(S(t), Z(t))\|_\beta \le C\|Z(t)\|^2_{-\beta}, \tag{1.16}$$

where $\|\cdot\|_\beta$ is defined similarly to $\|\cdot\|_{-\beta}$. Let us note that the bound (1.16) is not a direct consequence of the linearization, since the function $S(t)$ generally is not a solution of (1.4). The modulation equation and the bound (1.14) play a crucial role in the proof of (1.16).

II. The linearized dynamics (1.12) is nonautonomous. First, let us fix $t = t_1$ in $A(t)$ and consider the corresponding "frozen" linear autonomous equation with $A(t_1)$ instead of $A(t)$. We prove the decay

$$\|X(t)\|_{-\beta} \le \frac{C\|X(0)\|_\beta}{(1+|t|)^{3/2}}, \quad t \in \mathbb{R} \tag{1.17}$$

of the solutions $X(t)$ to the frozen equation for any $X(0) \in \mathcal{Z}_{S_1}$, where $S_1 := S(t_1)$, and $\mathcal{Z}_{S_1}$ is the space of vectors X which are symplectic orthogonal to the tangent space $\mathcal{T}_{S_1}$. Let us stress that the decay holds only for the solutions symplectic orthogonal to the tangent space. Basically, the reason for the decay is the fact that the spectrum of the generator $A(t_1)$ restricted to the space $\mathcal{Z}_{S_1}$ is purely continuous.

III. We combine the decay (1.17) with the bound (1.14) through the nonlinear equation (1.15). This gives the time decay of the transversal component

$$\|Z(t)\|_{-\beta} \le \frac{C(\|Z(0)\|_\beta)}{(1+|t|)^{3/2}}, \quad t \in \mathbb{R}, \tag{1.18}$$

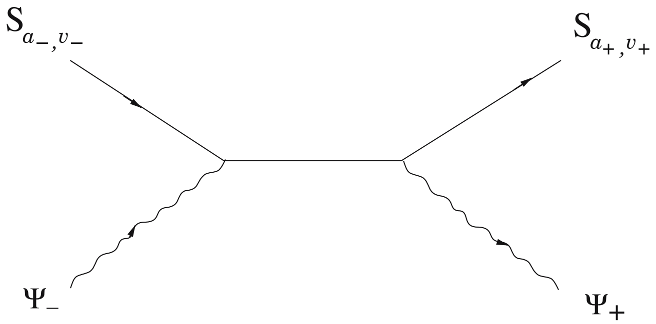

Fig. 1. Wave – particle scattering

if the norm $\|Z(0)\|_\beta$ is sufficiently small. One of the main difficulties in proving the decay (1.18) is the non-autonomous character of the linear part of (1.15). We deduce the decay from Eq. (1.15) written in the "frozen" form

$$\dot{Z}(t) = A(t_1)Z(t) + [A(t) - A(t_1)]Z(t) + N_2(S(t), Z(t)), \quad 0 \le t < t_1, \quad (1.19)$$

with arbitrary large $t_1 > 0$.

IV. The decay (1.18) implies the soliton asymptotics (1.7) and (1.8) by the known techniques of scattering theory.

Remarks 1.2. i) The asymptotic stability of the solitary manifold $\mathcal{S}$ is caused by the *radiation of energy to infinity* which appears as the *local energy decay* for the transversal component, (1.18).

ii) The asymptotics (1.7) can be interpreted as the collision of the incident soliton, with a trajectory $v_-t + a_-$, with an incident wave $W_0(t)\boldsymbol{\Psi}_-$, which results in an outgoing soliton with a new trajectory $v_+t + a_+$, and a new outgoing wave $W_0(t)\boldsymbol{\Psi}_+$. The collision process can be represented by the diagram of Fig. 1. It suggests to introduce the (nonlinear) *scattering operator*

$$\mathbf{S}: \ (v_-, a_-, \boldsymbol{\Psi}_-) \mapsto (v_+, a_+, \boldsymbol{\Psi}_+). \quad (1.20)$$

However, the domain of the operator is an open problem as well as the question on its *asymptotic completeness* (i.e. on its range).

Remarks 1.3. i) The strategy of [5, 6, 28, 29] was further developed in the papers [7–10, 27, 34–36]. Let us stress that these papers contain several assumptions on the discrete and continuous spectrum of the linearized problem. In our case a complete investigation of the spectrum of the linearized problem is given under the Wiener condition and there is no need for any a priori spectral assumptions.

ii) Note that the Wiener condition is indispensable for our proof of the decay (1.17), but only in the proof of Lemma 15.3. Otherwise we use only the fact that the coupling function $\rho(x)$ is not identically zero. The other assumptions on ρ can be weakened: the spherical symmetry is not necessary, and one can assume also that ρ belongs to a weighted Sobolev space rather than having a compact support.

Our paper is organized as follows. In Sect. 2, we formulate the main result. In Sect. 3, we introduce the symplectic projection onto the solitary manifold. The linearized equation is defined in Sect. 4. In Sect. 6, we split the dynamics in two components: along the solitary manifold and in the transversal directions, and we justify the estimate (1.14) concerning the tangential component. The time decay of the transversal component is established in Sects. 7–10 under an assumption on the time decay of the linearized dynamics. In Sect. 11, we prove the main result. Sections 12–18 fill the gap concerning the time decay of the linearized dynamics. In Appendices A and B we collect some routine calculations.

2. Main Results

2.1. Existence of dynamics. To formulate our results precisely, we need some definitions. We introduce a suitable phase space for the Cauchy problem corresponding to (1.2) and (1.3). Let $H^0 = L^2$ be the real Hilbert space $L^2(\mathbb{R}^3)$ with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|_{L^2}$, and let H^1 be the Sobolev space $H^1 = \{\psi \in L^2 : |\nabla\psi| \in L^2\}$ with the norm $\|\psi\|_{H^1} = \|\nabla\psi\|_{L^2} + \|\psi\|_{L^2}$. Let us introduce also the weighted Sobolev spaces H^s_α, $s = 0, 1$, $\alpha \in \mathbb{R}$ with the norms $\|\psi\|_{s,\alpha} := \|(1 + |x|)^\alpha \psi\|_{H^s}$.

Definition 2.1. i) *The phase space $\mathcal{E}$ is the real Hilbert space $H^1 \oplus L^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ of states $Y = (\psi, \pi, q, p)$ with the finite norm*

$$\|Y\|_{\mathcal{E}} = \|\psi\|_{H^1} + \|\pi\|_{L^2} + |q| + |p|.$$

ii) *$\mathcal{E}_\alpha$ is the space $H^1_\alpha \oplus H^0_\alpha \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with the norm*

$$\|Y\|_\alpha = \|Y\|_{\mathcal{E}_\alpha} = \|\psi\|_{1,\alpha} + \|\pi\|_{0,\alpha} + |q| + |p|. \tag{2.1}$$

iii) *$\mathcal{F}$ is the space $H^1 \oplus L^2$ of fields $F = (\psi, \pi)$ with the finite norm*

$$\|F\|_{\mathcal{F}} = \|\psi\|_{H^1} + \|\pi\|_{L^2}.$$

Similarly, $\mathcal{F}_\alpha$ is the space $H^1_\alpha \oplus H^0_\alpha$ with the norm

$$\|F\|_\alpha = \|F\|_{\mathcal{F}_\alpha} = \|\psi\|_{1,\alpha} + \|\pi\|_{0,\alpha}. \tag{2.2}$$

Note that we use the same notation for the norms in the space $\mathcal{F}_\alpha$ as in the space $\mathcal{E}_\alpha$ defined in (2.1). We hope it will not create misunderstandings since $\mathcal{F}_\alpha$ is equivalent to the subspace of $\mathcal{E}_\alpha$ which consists of elements of $\mathcal{E}_\alpha$ with zero vector components : $q = p = 0$. It will be always clear from the context if we deal with fields only, and therefore with the space $\mathcal{F}_\alpha$, or with fields-particles, and therefore with elements of the space $\mathcal{E}_\alpha$.

We consider the Cauchy problem for the Hamilton system (1.2) which we write as

$$\dot{Y}(t) = F(Y(t)), \quad t \in \mathbb{R} : \quad Y(0) = Y_0. \tag{2.3}$$

Here $Y(t) = (\psi(t), \pi(t), q(t), p(t))$, $Y_0 = (\psi_0, \pi_0, q_0, p_0)$, and all derivatives are understood in the sense of distributions.

Proposition 2.2 [15]. *Let* (1.9) *hold. Then*

i) *For every $Y_0 \in \mathcal{E}$, the Cauchy problem (2.3) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.*
ii) *For every $t \in \mathbb{R}$, the map $U(t) : Y_0 \mapsto Y(t)$ is continuous on $\mathcal{E}$.*

iii) *The energy is conserved, i.e.*

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y_0), \quad t \in \mathbb{R}, \tag{2.4}$$

and the velocity is bounded,

$$|\dot{q}(t)| \le \overline{v} < 1, \quad t \in \mathbb{R}, \tag{2.5}$$

where $\overline{v} = \overline{v}(Y_0)$.

The proof is based on a priori estimates provided by the fact that the Hamilton functional (1.3) is bounded from below. The latter follows from the bounds

$$-\frac{1}{2m^2}\|\rho\|^2_{L^2} \le \frac{m^2}{2}\|\psi\|^2_{L^2} + \langle\psi, \rho(\cdot - q)\rangle \le \frac{m^2+1}{2}\|\psi\|^2_{L^2} + \frac{1}{2}\|\rho\|^2_{L^2}, \tag{2.6}$$

which imply also that $\mathcal{E}$ is the space of finite energy states.

2.2. Solitary manifold and main result. Let us compute the solitons (1.5). The substitution to (1.2) gives the following stationary equations,

$$\left.\begin{array}{ll} -v\cdot\nabla\psi_v(y) = \pi_v(y), & -v\cdot\nabla\pi_v(y) = \Delta\psi_v(y) - m^2\psi_v(y) - \rho(y) \\ v = \dfrac{p_v}{\sqrt{1+p_v^2}}, & 0 = -\displaystyle\int \nabla\psi_v(y)\rho(y)\,dy \end{array}\right| . \tag{2.7}$$

Then the first two equations imply

$$\Lambda\psi_v(y) := [-\Delta + m^2 + (v\cdot\nabla)^2]\psi_v(y) = -\rho(y), \quad y \in \mathbb{R}^3. \tag{2.8}$$

For $|v| < 1$ the operator Λ is an isomorphism $H^4(\mathbb{R}^3) \to H^2(\mathbb{R}^3)$. Hence (1.9) implies that

$$\psi_v(y) = -\Lambda^{-1}\rho(y) \in H^4(\mathbb{R}^3). \tag{2.9}$$

If v is given and $|v| < 1$, then p_v can be found from the third equation of (2.7). Further, functions ρ and ψ_v are even by (1.9). Thus, $\nabla\psi_v$ is odd and the last equation of (2.7) holds. Hence, the soliton solution (1.5) exists and is defined uniquely for any couple (a, v) with $|v| < 1$.

The function ψ_v can be computed by the Fourier transform. The soliton is given by the formulas

$$\left.\begin{array}{l} \psi_v(x) = -\dfrac{\gamma}{4\pi}\displaystyle\int \frac{e^{-m|\gamma(y-x)_\parallel + (y-x)_\perp|}\rho(y)d^3y}{|\gamma(y-x)_\parallel + (y-x)_\perp|} \\ \pi_v(x) = -v\cdot\nabla\psi_v(x), \quad p_v = \gamma v = \dfrac{v}{\sqrt{1-v^2}} \end{array}\right| . \tag{2.10}$$

Here we set $\gamma = 1/\sqrt{1-v^2}$ and $x = x_\parallel + x_\perp$, where $x_\parallel \| v$ and $x_\perp \perp v$ for $x \in \mathbb{R}^3$.

Let us denote by $V := \{v \in \mathbb{R}^3 : |v| < 1\}$.

Definition 2.3. *A soliton state is* $S(\sigma) := (\psi_v(x-b), \pi_v(x-b), b, p_v)$, *where* $\sigma := (b, v)$ *with* $b \in \mathbb{R}^3$ *and* $v \in V$.

Obviously, the soliton solution admits the representation $S(\sigma(t))$, where

$$\sigma(t) = (b(t), v(t)) = (vt + a, v). \tag{2.11}$$

Definition 2.4. *A solitary manifold is the set* $\mathcal{S} := \{S(\sigma) : \sigma \in \Sigma := \mathbb{R}^3 \times V\}$.

The main result of our paper is the following theorem.

Theorem 2.5. *Let* (1.9) *and* (1.10) *hold. Let* $\beta > 3/2$ *and* $Y(t)$ *be the solution to the Cauchy problem* (2.3) *with the initial state* Y_0 *which is sufficiently close to the solitary manifold:*

$$d_0 := \mathrm{dist}_{\mathcal{E}_\beta}(Y_0, \mathcal{S}) \ll 1. \tag{2.12}$$

Then the asymptotics hold for $t \to \pm\infty$,

$$\dot{q}(t) = v_\pm + \mathcal{O}(|t|^{-2}), \quad q(t) = v_\pm t + a_\pm + \mathcal{O}(|t|^{-3/2}); \tag{2.13}$$

$$\begin{aligned}(\psi(x,t), \pi(x,\bar{t}))\\ = (\psi_{v_\pm}(x - v_\pm t - a_\pm), \pi_{v_\pm}(x - v_\pm t - a_\pm)) + W_0(t)\mathbf{\Psi}_\pm + r_\pm(x,t)\end{aligned} \tag{2.14}$$

with

$$\|r_\pm(t)\|_{\mathcal{F}} = \mathcal{O}(|t|^{-1/2}). \tag{2.15}$$

It suffices to prove the asymptotics (2.14), (2.13) for $t \to +\infty$ since system (1.2) is time reversible.

3. Symplectic Projection

3.1. Symplectic structure and Hamilton form. The system (1.2) reads as the Hamilton system

$$\dot{Y} = J\mathcal{DH}(Y), \quad J := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad Y = (\psi, \pi, q, p) \in \mathcal{E}, \tag{3.1}$$

where $\mathcal{DH}$ is the Fréchet derivative of the Hamilton functional (1.3). Let us identify the tangent space of $\mathcal{E}$, at every point, with the space $\mathcal{E}$. Consider the symplectic form Ω defined on $\mathcal{E}$ by the rule

$$\Omega = \int d\psi(x) \wedge d\pi(x)\, dx + dq \wedge dp.$$

In other words,

$$\Omega(Y_1, Y_2) = \langle Y_1, JY_2 \rangle, \quad Y_1, Y_2 \in \mathcal{E}, \tag{3.2}$$

where

$$\langle Y_1, Y_2 \rangle := \langle \psi_1, \psi_2 \rangle + \langle \pi_1, \pi_2 \rangle + q_1 q_2 + p_1 p_2$$

and $\langle \psi_1, \psi_2 \rangle = \int \psi_1(x)\psi_2(x)dx$ etc. It is clear that the form Ω is non-degenerate, i.e.

$$\Omega(Y_1, Y_2) = 0 \text{ for every } Y_2 \in \mathcal{E} \implies Y_1 = 0.$$

Definition 3.1. i) *The symbol* $Y_1 \nmid Y_2$ *means that* $Y_1 \in \mathcal{E}$, $Y_2 \in \mathcal{E}$, *and* Y_1 *is symplectic orthogonal to* Y_2, *i.e.* $\Omega(Y_1, Y_2) = 0$.

ii) *A projection operator* $\boldsymbol{P} : \mathcal{E} \to \mathcal{E}$ *is said to be symplectic orthogonal if* $Y_1 \nmid Y_2$ *for* $Y_1 \in \mathrm{Ker}\, \boldsymbol{P}$ *and* $Y_2 \in \mathrm{Im}\, \boldsymbol{P}$.

3.2. Symplectic projection onto solitary manifold. Let us consider the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ of the manifold $\mathcal{S}$ at a point $S(\sigma)$. The vectors $\tau_j := \partial_{\sigma_j} S(\sigma)$, where $\partial_{\sigma_j} := \partial_{b_j}$ and $\partial_{\sigma_{j+3}} := \partial_{v_j}$ with $j = 1, 2, 3$, form a basis in $\mathcal{T}_\sigma \mathcal{S}$. In detail,

$$\left.\begin{aligned} \tau_j = \tau_j(v) &:= \ \partial_{b_j} S(\sigma) = (-\partial_j \psi_v(y), -\partial_j \pi_v(y), e_j, 0) \\ \tau_{j+3} = \tau_{j+3}(v) &:= \partial_{v_j} S(\sigma) = (\partial_{v_j} \psi_v(y), \partial_{v_j} \pi_v(y), 0, \partial_{v_j} p_v) \end{aligned}\right| \ j = 1, 2, 3, \tag{3.3}$$

where $y := x - b$ is the "moving frame coordinate", $e_1 = (1, 0, 0)$ etc. Let us stress that the functions τ_j are always regarded as functions of y rather than those of x.

Formulas (2.10) and conditions (1.9) imply that

$$\tau_j(v) \in \mathcal{E}_\alpha, \quad v \in V, \ \ j = 1, \dots, 6, \ \ \forall \alpha \in \mathbb{R}. \tag{3.4}$$

Lemma 3.2. *The matrix with the elements $\Omega(\tau_l(v), \tau_j(v))$ is non-degenerate for any $v \in V$.*

Proof. The elements are computed in Appendix A. As the result, the matrix $\Omega(\tau_l, \tau_j)$ has the form

$$\Omega(v) := (\Omega(\tau_l, \tau_j))_{l,j=1,\dots,6} = \begin{pmatrix} 0 & \Omega^+(v) \\ -\Omega^+(v) & 0 \end{pmatrix}, \tag{3.5}$$

where the 3×3-matrix $\Omega^+(v)$ equals

$$\Omega^+(v) = K + (1 - v^2)^{-1/2} E + (1 - v^2)^{-3/2} v \otimes v. \tag{3.6}$$

Here K is a symmetric 3×3-matrix with the elements

$$\begin{aligned} K_{ij} &= \int dk |\hat{\psi}_v(k)|^2 k_i k_j \frac{k^2 + m^2 + 3(kv)^2}{k^2 + m^2 - (kv)^2} \\ &= \int dk |\hat{\rho}(k)|^2 k_i k_j \frac{k^2 + m^2 + 3(kv)^2}{(k^2 + m^2 - (kv)^2)^3}, \end{aligned} \tag{3.7}$$

where the "hat" stands for the Fourier transform (cf. (1.10)). The matrix K is the integral of the symmetric nonnegative definite matrix $k \otimes k = (k_i k_j)$ with a positive weight. Hence, the matrix K is also nonnegative definite. Since the identity matrix E is positive definite and the matrix $v \otimes v$ is nonnegative definite, the matrix $\Omega^+(v)$ is symmetric and positive definite, hence non-degenerate. Then the matrix $\Omega(\tau_l, \tau_j)$ is also non-degenerate. □

Now we show that in a small neighborhood of the soliton manifold $\mathcal{S}$ a "symplectic orthogonal projection" onto $\mathcal{S}$ is well-defined. Let us introduce the translations $T_a : (\psi(\cdot), \pi(\cdot), q, p) \mapsto (\psi(\cdot - a), \pi(\cdot - a), q + a, p)$, $a \in \mathbb{R}^3$. Note that the manifold $\mathcal{S}$ is invariant with respect to the translations. Let us denote by $v(p) := p/\sqrt{1 + p^2}$ for $p \in \mathbb{R}^3$.

Definition 3.3. i) *For any $\alpha \in \mathbb{R}$ and $\overline{v} < 1$ denote by $\mathcal{E}_\alpha(\overline{v}) = \{Y = (\psi, \pi, q, p) \in \mathcal{E}_\alpha : |v(p)| \le \overline{v}\}$. We set $\mathcal{E}(\overline{v}) := \mathcal{E}_0(\overline{v})$.*
ii) *For any $\tilde{v} < 1$ denote by $\Sigma(\tilde{v}) = \{\sigma = (b, v) : b \in \mathbb{R}^3, |v| \le \tilde{v}\}$.*

Lemma 3.4. *Let* (1.9) *hold, $\alpha \in \mathbb{R}$ and $\overline{v} < 1$. Then*

i) *there exists a neighborhood $\mathcal{O}_\alpha(\mathcal{S})$ of $\mathcal{S}$ in $\mathcal{E}_\alpha$ and a mapping $\mathbf{\Pi} : \mathcal{O}_\alpha(\mathcal{S}) \to \mathcal{S}$ such that $\mathbf{\Pi}$ is uniformly continuous on $\mathcal{O}_\alpha(\mathcal{S}) \cap \mathcal{E}_\alpha(\overline{v})$ in the metric of $\mathcal{E}_\alpha$,*

$$\mathbf{\Pi} Y = Y \text{ for } Y \in \mathcal{S}, \quad \text{and} \quad Y - S \nmid \mathcal{T}_S\mathcal{S}, \text{ where } S = \mathbf{\Pi} Y. \tag{3.8}$$

ii) *$\mathcal{O}_\alpha(\mathcal{S})$ is invariant with respect to the translations T_a, and*

$$\mathbf{\Pi} T_a Y = T_a \mathbf{\Pi} Y, \quad \text{for } Y \in \mathcal{O}_\alpha(\mathcal{S}) \text{ and } a \in \mathbb{R}^3. \tag{3.9}$$

iii) *For any $\overline{v} < 1$ there exists a $\tilde{v} < 1$ s.t. $\mathbf{\Pi} Y = S(\sigma)$ with $\sigma \in \Sigma(\tilde{v})$ for $Y \in \mathcal{O}_\alpha(\mathcal{S}) \cap \mathcal{E}_\alpha(\overline{v})$.*

iv) *For any $\tilde{v} < 1$ there exists an $r_\alpha(\tilde{v}) > 0$ s.t. $S(\sigma) + Z \in \mathcal{O}_\alpha(\mathcal{S})$ if $\sigma \in \Sigma(\tilde{v})$ and $\|Z\|_\alpha < r_\alpha(\tilde{v})$.*

Proof. We have to find $\sigma = \sigma(Y)$ such that $S(\sigma) = \mathbf{\Pi} Y$ and

$$\Omega(Y - S(\sigma), \partial_{\sigma_j} S(\sigma)) = 0, \quad j = 1, \ldots, 6. \tag{3.10}$$

Let us fix an arbitrary $\sigma^0 \in \Sigma$ and note that the system (3.10) involves only 6 smooth scalar functions of Y. Then for Y close to $S(\sigma^0)$, the existence of σ follows by the standard finite dimensional implicit function theorem if we show that the 6×6 Jacobian matrix with elements $M_{lj}(Y) = \partial_{\sigma_l}\Omega(Y - S(\sigma^0), \partial_{\sigma_j} S(\sigma^0))$ is non-degenerate at $Y = S(\sigma^0)$. First note that all the derivatives exist by (3.4). The non-degeneracy holds by Lemma 3.2 and the definition (3.3) since $M_{lj}(S(\sigma^0)) = -\Omega(\partial_{\sigma_l} S(\sigma^0), \partial_{\sigma_j} S(\sigma^0))$. Thus, there exists some neighborhood $\mathcal{O}_\alpha(S(\sigma^0))$ of $S(\sigma^0)$, where $\mathbf{\Pi}$ is well defined and satisfies (3.8), and the same is true in the union $\mathcal{O}'_\alpha(\mathcal{S}) = \cup_{\sigma^0 \in \Sigma} \mathcal{O}_\alpha(S(\sigma^0))$. The identity (3.9) holds for $Y, T_a Y \in \mathcal{O}'_\alpha(\mathcal{S})$, since the form Ω and the manifold $\mathcal{S}$ are invariant with respect to the translations.

It remains to modify $\mathcal{O}'_\alpha(\mathcal{S})$ by the translations: we set $\mathcal{O}_\alpha(\mathcal{S}) = \cup_{b \in \mathbb{R}^3} T_b \mathcal{O}'_\alpha(\mathcal{S})$. Then the second statement obviously holds.

The last two statements and the uniform continuity in the first statement follow by translation invariance and compactness arguments. □

We refer to $\mathbf{\Pi}$ as symplectic orthogonal projection onto $\mathcal{S}$.

Corollary 3.5. *The condition* (2.12) *implies that $Y_0 = S + Z_0$, where $S = S(\sigma_0) = \mathbf{\Pi} Y_0$, and*

$$\|Z_0\|_\beta \ll 1. \tag{3.11}$$

Proof. Lemma 3.4 implies that $\mathbf{\Pi} Y_0 = S$ is well defined for small $d_0 > 0$. Furthermore, the condition (2.12) means that there exists a point $S_1 \in \mathcal{S}$ such that $\|Y_0 - S_1\|_\beta = d_0$. Hence, $Y_0, S_1 \in \mathcal{O}_\beta(\mathcal{S}) \cap \mathcal{E}_\beta(\overline{v})$ with some $\overline{v} < 1$ which does not depend on d_0 for sufficiently small d_0. On the other hand, $\mathbf{\Pi} S_1 = S_1$, hence the uniform continuity of the mapping $\mathbf{\Pi}$ implies that $\|S_1 - S\|_\beta \to 0$ as $d_0 \to 0$. Therefore, finally, $\|Z_0\|_\beta = \|Y_0 - S\|_\beta \le \|Y_0 - S_1\|_\beta + \|S_1 - S\|_\beta \le d_0 + o(1) \ll 1$ for small d_0. □

4. Linearization on the solitary manifold

Let us consider a solution to the system (1.2), and split it as the sum

$$Y(t) = S(\sigma(t)) + Z(t), \tag{4.1}$$

where $\sigma(t) = (b(t), v(t)) \in \Sigma$ is an arbitrary smooth function of $t \in \mathbb{R}$. In detail, denote $Y = (\psi, \pi, q, p)$ and $Z = (\Psi, \Pi, Q, P)$. Then (4.1) means that

$$\left.\begin{array}{l} \psi(x,t) = \psi_{v(t)}(x-b(t)) + \Psi(x-b(t),t),\ q(t) = b(t) + Q(t) \\ \pi(x,t) = \pi_{v(t)}(x-b(t)) + \Pi(x-b(t),t),\ p(t) = p_{v(t)} + P(t) \end{array}\right| . \tag{4.2}$$

Let us substitute (4.2) to (1.2), and linearize the equations in Z. Below we shall choose $S(\sigma(t)) = \mathbf{\Pi} Y(t)$, i.e. $Z(t)$ is symplectic orthogonal to $\mathcal{T}_{S(\sigma(t))}\mathcal{S}$. However, this orthogonality condition is not needed for the formal process of linearization. The orthogonality condition will be important in Sect. 6, where we derive "modulation equations" for the parameters $\sigma(t)$.

Let us proceed to linearization. Setting $y = x - b(t)$ which is the "moving frame coordinate", we obtain from (4.2) and (1.2) that

$$\left.\begin{array}{l} \dot\psi = \dot v \cdot \nabla_v \psi_v(y) - \dot b \cdot \nabla\psi_v(y) + \dot\Psi(y,t) - \dot b \cdot \nabla\Psi(y,t) = \pi_v(y) + \Pi(y,t) \\ \dot\pi = \dot v \cdot \nabla_v \pi_v(y) - \dot b \cdot \nabla\pi_v(y) + \dot\Pi(y,t) - \dot b \cdot \nabla\Pi(y,t) \\ \quad = \Delta\psi_v(y) - m^2\psi_v(y) + \Delta\Psi(y,t) - m^2\Psi(y,t) - \rho(y-Q) \\ \dot q = \dot b + \dot Q = \dfrac{p_v + P}{\sqrt{1+(p_v+P)^2}} \\ \dot p = \dot v \cdot \nabla_v p_v + \dot P = -\langle \nabla(\psi_v(y) + \Psi(y,t)), \rho(y-Q)\rangle. \end{array}\right| \tag{4.3}$$

The equations are linear in Ψ and Π, hence it remains to extract the terms linear in Q and P. First note that $\rho(y-Q) = \rho(y) - Q \cdot \nabla\rho(y) - N_2(Q)$, where $-N_2(Q) = \rho(y-Q) - \rho(y) + Q \cdot \nabla\rho(y)$. The condition (1.9) implies that for $N_2(Q)$ the bound holds,

$$\|N_2(Q)\|_{0,\beta} \le C_\beta(\overline{Q})Q^2, \tag{4.4}$$

uniformly in $|Q| \le \overline{Q}$ for any fixed $\overline{Q}$, where β is the parameter in Theorem 2.5. Second, the Taylor expansion gives

$$\frac{p_v + P}{\sqrt{1+(p_v+P)^2}} = v + \nu(P - v(v\cdot P)) + N_3(v,P),$$

where $\nu := (1+p_v^2)^{-1/2} = \sqrt{1-v^2}$, and

$$|N_3(v,P)| \le C(\tilde v)P^2 \tag{4.5}$$

uniformly with respect to $|v| \le \tilde{v} < 1$. Using Eqs. (2.7), we obtain from (4.3) the following equations for the components of the vector $Z(t)$:

$$\left.\begin{aligned}
\dot{\Psi}(y,t) &= \Pi(y,t) + \dot{b}\cdot\nabla\Psi(y,t) + (\dot{b}-v)\cdot\nabla\psi_v(y) - \dot{v}\cdot\nabla_v\psi_v(y),\\
\dot{\Pi}(y,t) &= \Delta\Psi(y,t) - m^2\Psi(y,t) + \dot{b}\cdot\nabla\Pi(y,t) + Q\cdot\nabla\rho(y)\\
&\quad + (\dot{b}-v)\cdot\nabla\pi_v(y) - \dot{v}\cdot\nabla_v\pi_v(y) + N_2,\\
\dot{Q}(t) &= \nu(E - v\otimes v)P + (v - \dot{b}) + N_3,\\
\dot{P}(t) &= \langle\Psi(y,t),\nabla\rho(y)\rangle + \langle\nabla\psi_v(y), Q\cdot\nabla\rho(y)\rangle - \dot{v}\cdot\nabla_v p_v + N_4(v,Z),
\end{aligned}\right| \tag{4.6}$$

where $N_4(v,Z) = \langle\nabla\psi_v, N_2(Q)\rangle + \langle\nabla\Psi, Q\cdot\nabla\rho\rangle + \langle\nabla\Psi, N_2(Q)\rangle$. Clearly, $N_4(v,Z)$ satisfies the following estimate:

$$|N_4(v,Z)| \le C_\beta(\rho,\tilde{v},\overline{Q})\Big[Q^2 + \|\Psi\|_{1,-\beta}|Q|\Big], \tag{4.7}$$

uniformly in $|v| \le \tilde{v}$ and $|Q| \le \overline{Q}$ for any fixed $\tilde{v} < 1$. We can write Eqs. (4.6) as

$$\dot{Z}(t) = A(t)Z(t) + T(t) + N(t), \quad t \in \mathbb{R}. \tag{4.8}$$

Here the operator $A(t) = A_{v,w}$ depends on two parameters, $v = v(t)$, and $w = \dot{b}(t)$ and can be written in the form

$$A_{v,w}\begin{pmatrix}\Psi\\ \Pi\\ Q\\ P\end{pmatrix} := \begin{pmatrix} w\cdot\nabla & 1 & 0 & 0\\ \Delta - m^2 & w\cdot\nabla & \nabla\rho\cdot & 0\\ 0 & 0 & 0 & B_v\\ \langle\cdot,\nabla\rho\rangle & 0 & \langle\nabla\psi_v,\cdot\nabla\rho\rangle & 0\end{pmatrix}\begin{pmatrix}\Psi\\ \Pi\\ Q\\ P\end{pmatrix}, \tag{4.9}$$

where $B_v = \nu(E - v\otimes v)$. Furthermore, $T(t) = T_{v,w}$ and $N(t) = N(\sigma, Z)$ are given by

$$T_{v,w} = \begin{pmatrix}(w-v)\cdot\nabla\psi_v - \dot{v}\cdot\nabla_v\psi_v\\ (w-v)\cdot\nabla\pi_v - \dot{v}\cdot\nabla_v\pi_v\\ v - w\\ -\dot{v}\cdot\nabla_v p_v\end{pmatrix}, \quad N(\sigma,Z) = \begin{pmatrix}0\\ N_2(Z)\\ N_3(v,Z)\\ N_4(v,Z)\end{pmatrix}, \tag{4.10}$$

where $v = v(t)$, $w = w(t)$, $\sigma = \sigma(t) = (b(t), v(t))$, and $Z = Z(t)$. Estimates (4.4), (4.5) and (4.7) imply that

$$\|N(\sigma,Z)\|_\beta \le C(\tilde{v},\overline{Q})\|Z\|^2_{-\beta}, \tag{4.11}$$

uniformly in $\sigma \in \Sigma(\tilde{v})$ and $\|Z\|_{-\beta} \le r_{-\beta}(\tilde{v})$ for any fixed $\tilde{v} < 1$.

Remarks 4.1. i) The term $A(t)Z(t)$ in the right-hand side of Eq. (4.8) is linear in $Z(t)$, and $N(t)$ is a *high order term* in $Z(t)$. On the other hand, $T(t)$ is a zero order term which does not vanish at $Z(t) = 0$ since $S(\sigma(t))$ generally is not a soliton solution if (2.11) fails to hold (though $S(\sigma(t))$ belongs to the solitary manifold).

ii) Formulas (3.3) and (4.10) imply:

$$T(t) = -\sum_{l=1}^{3}[(w-v)_l\tau_l + \dot{v}_l\tau_{l+3}], \tag{4.12}$$

and hence $T(t) \in \mathcal{T}_{S(\sigma(t))}\mathcal{S}$, $t \in \mathbb{R}$. This fact suggests an unstable character of the nonlinear dynamics *along the solitary manifold*.

5. Linearized equation

Here we collect some Hamiltonian and spectral properties of the operator (4.9). First, let us consider the linear equation

$$\dot{X}(t) = A_{v,w}X(t), \quad t \in \mathbb{R} \tag{5.1}$$

with arbitrary fixed $v \in V = \{v \in \mathbb{R}^3 : |v| < 1\}$ and $w \in \mathbb{R}^3$. Let us define the space $\mathcal{E}^+ := H^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$.

Lemma 5.1. i) *For any $v \in V$ and $w \in \mathbb{R}^3$, Eq.* (5.1) *can be represented as the Hamiltonian system* (cf. (3.1)),

$$\dot{X}(t) = JD\mathcal{H}_{v,w}(X(t)), \quad t \in \mathbb{R}, \tag{5.2}$$

where $D\mathcal{H}_{v,w}$ is the Fréchet derivative of the Hamiltonian functional,

$$\begin{aligned}\mathcal{H}_{v,w}(X) = \frac{1}{2}\int \Big[|\Pi|^2 + |\nabla\Psi|^2 + m^2|\Psi|^2\Big]dy + \int \Pi w \cdot \nabla\Psi dy \\ + \int \rho(y)Q\cdot\nabla\Psi dy + \frac{1}{2}P\cdot B_vP - \frac{1}{2}\langle Q\cdot\nabla\psi_v(y), Q\cdot\nabla\rho(y)\rangle, \\ X = (\Psi, \Pi, Q, P) \in \mathcal{E}.\end{aligned} \tag{5.3}$$

ii) *The energy conservation law holds for the solutions $X(t) \in C^1(\mathbb{R}, \mathcal{E}^+)$,*

$$\mathcal{H}_{v,w}(X(t)) = \text{const}, \quad t \in \mathbb{R}. \tag{5.4}$$

iii) *The skew-symmetry relation holds,*

$$\Omega(A_{v,w}X_1, X_2) = -\Omega(X_1, A_{v,w}X_2), \quad X_1, X_2 \in \mathcal{E}. \tag{5.5}$$

Proof. i) Equation (5.1) reads as follows:

$$\frac{d}{dt}\begin{pmatrix}\Psi \\ \Pi \\ Q \\ P\end{pmatrix} = \begin{pmatrix}\Pi + w\cdot\nabla\Psi \\ \Delta\Psi - m^2\Psi + w\cdot\nabla\Pi + Q\cdot\nabla\rho \\ B_vP \\ -\langle\nabla\Psi, \rho\rangle + \langle\nabla\psi_v, Q\cdot\nabla\rho\rangle\end{pmatrix}. \tag{5.6}$$

The first three equations correspond to the Hamilton form since

$$\Pi + w\cdot\nabla\Psi = D_\Pi\mathcal{H}_{v,w}, \quad \Delta\Psi - m^2\Psi + w\cdot\nabla\Pi + Q\cdot\nabla\rho = -D_\Psi\mathcal{H}_{v,w}, \quad B_vP = \nabla_P\mathcal{H}_{v,w}.$$

Let us check that the last equation has also the Hamilton form, i.e. $-\langle\nabla\Psi, \rho\rangle + \langle\nabla\psi_v, Q\cdot\nabla\rho\rangle = -\nabla_Q\mathcal{H}_{v,w}$. First we note that $-\langle\partial_j\Psi, \rho\rangle = -\partial_{Q_j}\int \rho Q\cdot\nabla\Psi dx$. It remains to show that

$$\langle\partial_j\psi_v, Q\cdot\nabla\rho\rangle = \partial_{Q_j}\frac{1}{2}\langle Q\cdot\nabla\psi_v, Q\cdot\nabla\rho\rangle. \tag{5.7}$$

Indeed,

$$\begin{aligned}\partial_{Q_j}\frac{1}{2}\langle Q\cdot\nabla\psi_v, Q\cdot\nabla\rho\rangle &= \frac{1}{2}\langle\partial_j\psi_v, Q\cdot\nabla\rho\rangle + \frac{1}{2}\langle Q\cdot\nabla\psi_v, \partial_j\rho\rangle \\ &= \frac{1}{2}\langle\partial_j\psi_v, Q\cdot\nabla\rho\rangle + \frac{1}{2}\langle\partial_j\psi_v, Q\cdot\nabla\rho\rangle,\end{aligned} \tag{5.8}$$

where we have integrated twice by parts. Then (5.7) follows.

ii) The energy conservation law follows by (5.2) and the chain rule for the Fréchet derivatives:

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{H}_{v,w}(X(t)) = \langle D\mathcal{H}_{v,w}(X(t)), \dot{X}(t)\rangle \\
&= \langle D\mathcal{H}_{v,w}(X(t)), JD\mathcal{H}_{v,w}(X(t))\rangle = 0, \quad t \in \mathbb{R},
\end{aligned}
\tag{5.9}
$$

since the operator J is skew-symmetric by (3.1), and $D\mathcal{H}_{v,w}(X(t)) \in \mathcal{E}$ for $X(t) \in \mathcal{E}^+$.

iii) The skew-symmetry holds since $A_{v,w}X = JD\mathcal{H}_{v,w}(X)$, and the linear operator $X \mapsto D\mathcal{H}_{v,w}(X)$ is symmetric as the Fréchet derivative of a quadratic form. □

Remark 5.2. One can obtain (5.3) by expanding $\mathcal{H}(S_{b,v} + X)$ to a power series in X up to second order terms. As a result, $\mathcal{H}_{v,w}(X)$ is the quadratic part of the Taylor series complemented by the second integral on the right-hand side of (5.3) arising from the left-hand side of (3.1).

Lemma 5.3. *The operator $A_{v,w}$ acts on the tangent vectors $\tau_j(v)$ to the solitary manifold as follows:*

$$
\begin{aligned}
A_{v,w}[\tau_j(v)] &= (w - v) \cdot \nabla\tau_j(v), \\
A_{v,w}[\tau_{j+3}(v)] &= (w - v) \cdot \nabla\tau_{j+3}(v) + \tau_j(v), \quad j = 1, 2, 3.
\end{aligned}
\tag{5.10}
$$

Proof. In detail, we have to show that

$$
A_{v,w}\begin{pmatrix} -\partial_j\psi_v \\ -\partial_j\pi_v \\ e_j \\ 0 \end{pmatrix} = \begin{pmatrix} (v - w) \cdot \nabla\partial_j\psi_v \\ (v - w) \cdot \nabla\partial_j\pi_v \\ 0 \\ 0 \end{pmatrix},
$$

$$
A_{v,w}\begin{pmatrix} \partial_{v_j}\psi_v \\ \partial_{v_j}\pi_v \\ 0 \\ \partial_{v_j}p_v \end{pmatrix} = \begin{pmatrix} (w - v) \cdot \nabla\partial_{v_j}\psi_v \\ (w - v) \cdot \nabla\partial_{v_j}\pi_v \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\partial_j\psi_v \\ -\partial_j\pi_v \\ e_j \\ 0 \end{pmatrix}.
\tag{5.11}
$$

Indeed, differentiate Eqs. (2.7) in b_j and v_j, and obtain that the derivatives of the soliton state in parameters satisfy the following equations:

$$
\left.
\begin{aligned}
&-v \cdot \nabla\partial_j\psi_v = \partial_j\pi_v, \qquad -v \cdot \nabla\partial_j\pi_v = \Delta\partial_j\psi_v - m^2\partial_j\psi_v - \partial_j\rho, \\
&-\partial_j\psi_v - v \cdot \nabla\partial_{v_j}\psi_v = \partial_{v_j}\pi_v, \qquad -\partial_j\pi_v - v\cdot\nabla\partial_{v_j}\pi_v = \Delta\partial_{v_j}\psi_v - m^2\partial_{v_j}\psi_v, \\
&\partial_{v_j}p_v = e_j(1 - v^2)^{-1/2} + v\frac{v_j}{(1 - v^2)^{3/2}}, \qquad 0 = -\langle\nabla\partial_{v_j}\psi_v, \rho\rangle,
\end{aligned}
\right|
\tag{5.12}
$$

for $j = 1, 2, 3$. Then (5.11) follows from (5.12) by definition of A in (4.9). □

We shall apply Lemma 5.1 mainly to the operator $A_{v,v}$ corresponding to $w = v$. In that case the linearized equation has the following additional specific features.

Lemma 5.4. *Let us assume that $w = v \in V$. Then*

i) *The tangent vectors* $\tau_j(v)$ *with* $j = 1, 2, 3$ *are eigenvectors, and* $\tau_{j+3}(v)$ *are root vectors of the operator* $A_{v,v}$*, that correspond to the zero eigenvalue, i.e.*

$$A_{v,v}[\tau_j(v)] = 0, \quad A_{v,v}[\tau_{j+3}(v)] = \tau_j(v), \quad j = 1, 2, 3. \tag{5.13}$$

ii) *The Hamiltonian function* (5.3) *is nonnegative definite since*

$$\mathcal{H}_{v,v}(X) = \frac{1}{2}\int \Big[|\Pi + v\cdot\nabla\Psi|^2 + |\Lambda^{1/2}\Psi - \Lambda^{-1/2}Q\cdot\nabla\rho|^2\Big]dx + \frac{1}{2}P\cdot B_v P \geq 0. \tag{5.14}$$

Here Λ *is the operator* (2.8) *which is symmetric and nonnegative definite in* $L^2(\mathbb{R}^3)$ *for* $|v| < 1$*, and* $\Lambda^{1/2}$ *is the nonnegative definite square root defined in the Fourier representation.*

Proof. The first statement follows from (5.10) with $w = v$. In order to prove ii) we rewrite the integral in (5.14) as follows:

$$\frac{1}{2}\langle\Pi + v\cdot\nabla\Psi, \Pi + v\cdot\nabla\Psi\rangle + \frac{1}{2}\langle\Lambda^{1/2}\Psi - \Lambda^{-1/2}Q\cdot\nabla\rho, \Lambda^{1/2}\Psi - \Lambda^{-1/2}Q\cdot\nabla\rho\rangle$$

$$= \frac{1}{2}\langle\Pi, \Pi\rangle + \langle\Pi, v\cdot\nabla\Psi\rangle + \frac{1}{2}\langle v\cdot\nabla\Psi, v\cdot\nabla\Psi\rangle$$

$$+\frac{1}{2}\langle\Lambda\Psi, \Psi\rangle - \langle\Psi, Q\cdot\nabla\rho\rangle + \frac{1}{2}\langle\Lambda^{-1}Q\cdot\nabla\rho, Q\cdot\nabla\rho\rangle, \tag{5.15}$$

since the operator $\Lambda^{1/2}$ is symmetric in $L^2(\mathbb{R}^3)$. Now all the terms of the expression (5.15) can be identified with the corresponding terms in (5.3) since

$$\frac{1}{2}\langle\Lambda\Psi, \Psi\rangle = \frac{1}{2}\langle[-\Delta + m^2 + (v\cdot\nabla)^2]\Psi, \Psi\rangle, \quad \Lambda^{-1}\rho = -\psi_v \tag{5.16}$$

by (2.8) and (2.9). □

Remark 5.5. In Sect. 14 we will apply Lemma 5.4 ii) together with energy conservation (5.4) to prove the analyticity of the resolvent $(A_{v,v} - \lambda)^{-1}$ for $\mathrm{Re}\,\lambda > 0$.

Remark 5.6. For a soliton solution of the system (1.2) we have $\dot b = v$, $\dot v = 0$, and hence $T(t) \equiv 0$. Thus, Eq. (5.1) is the linearization of system (1.2) on a soliton solution. In fact, we linearize (1.2) on a trajectory $S(\sigma(t))$, where $\sigma(t)$ is nonlinear with respect to t, rather than on a soliton solution. We shall show below that $T(t)$ is quadratic in $Z(t)$ if we choose $S(\sigma(t))$ to be the symplectic orthogonal projection of $Y(t)$. In this case, (5.1) is a linearization of (1.2) again.

6. Symplectic decomposition of the dynamics

Here we decompose the dynamics in two components: along the manifold $\mathcal{S}$ and in transversal directions. Equation (4.8) is obtained without any assumption on $\sigma(t)$ in (4.1). We are going to specify $S(\sigma(t)) := \mathbf{\Pi}Y(t)$. However, in this case we must know that

$$Y(t) \in \mathcal{O}_\alpha(\mathcal{S}), \quad t \in \mathbb{R}, \tag{6.1}$$

with some $\mathcal{O}_\alpha(\mathcal{S})$ defined in Lemma 3.4. It is true for $t = 0$ by our main assumption (2.12) with sufficiently small $d_0 > 0$. Then $S(\sigma(0)) = \mathbf{\Pi} Y(0)$ and $Z(0) = Y(0) - S(\sigma(0))$ are well defined. We shall prove below that (6.1) holds with $\alpha = -\beta$ if d_0 is sufficiently small. First, the a priori estimate (2.5) together with Lemma 3.4 iii) imply that $\mathbf{\Pi} Y(t) = S(\sigma(t))$ with $\sigma(t) = (b(t), v(t))$, and

$$|v(t)| \le \tilde{v} < 1, \quad t \in \mathbb{R} \tag{6.2}$$

if $Y(t) \in \mathcal{O}_{-\beta}(\mathcal{S})$. Denote by $r_{-\beta}(\tilde{v})$ the positive number in Lemma 3.4 iv) which corresponds to $\alpha = -\beta$. Then $S(\sigma) + Z \in \mathcal{O}_{-\beta}(\mathcal{S})$ if $\sigma = (b, v)$ with $|v| < \tilde{v}$ and $\|Z\|_{-\beta} < r_{-\beta}(\tilde{v})$. Note that (2.5) implies $\|Z(0)\|_{-\beta} < r_{-\beta}(\tilde{v})$ if d_0 is sufficiently small. Therefore, $S(\sigma(t)) = \mathbf{\Pi} Y(t)$ and $Z(t) = Y(t) - S(\sigma(t))$ are well defined for small times $t \ge 0$, such that $\|Z(t)\|_{-\beta} < r_{-\beta}(\tilde{v})$. This argument can be formalized by the following standard definition.

Definition 6.1. *Let t_* be the "exit time",*

$$t_* = \sup\{t > 0 : \|Z(s)\|_{-\beta} < r_{-\beta}(\tilde{v}), \quad 0 \le s \le t\}. \tag{6.3}$$

One of our main goals is to prove that $t_* = \infty$ if d_0 is sufficiently small. This would follow if we shall show that

$$\|Z(t)\|_{-\beta} < r_{-\beta}(\tilde{v})/2, \quad 0 \le t < t_*. \tag{6.4}$$

Note that

$$|Q(t)| \le \overline{Q} := r_{-\beta}(\tilde{v}), \quad 0 \le t < t_*. \tag{6.5}$$

Now by (4.11), the term $N(t)$ in (4.8) satisfies the following estimate:

$$\|N(t)\|_\beta \le C_\beta(\tilde{v}) \|Z(t)\|^2_{-\beta}, \quad 0 \le t < t_*. \tag{6.6}$$

6.1. Longitudinal Dynamics: Modulation Equations. From now on we fix the decomposition $Y(t) = S(\sigma(t)) + Z(t)$ for $0 < t < t_*$ by setting $S(\sigma(t)) = \mathbf{\Pi} Y(t)$ which is equivalent to the symplectic orthogonality condition of type (3.8),

$$Z(t) \nmid \mathcal{T}_{S(\sigma(t))}\mathcal{S}, \quad 0 \le t < t_*. \tag{6.7}$$

This enables us to drastically simplify the asymptotic analysis of the dynamical equation (4.8) for the transversal component $Z(t)$. As the first step, we derive the longitudinal dynamics, i.e. find the "modulation equations" for the parameters $\sigma(t)$. Let us derive a system of ordinary differential equations for the vector $\sigma(t)$. For this purpose, let us write (6.7) in the form

$$\Omega(Z(t), \tau_j(t)) = 0, \quad j = 1, \dots, 6, \quad 0 \le t < t_*, \tag{6.8}$$

where the vectors $\tau_j(t) = \tau_j(\sigma(t))$ span the tangent space $\mathcal{T}_{S(\sigma(t))}\mathcal{S}$. Note that $\sigma(t) = (b(t), v(t))$, where

$$|v(t)| \le \tilde{v} < 1, \quad 0 \le t < t_*, \tag{6.9}$$

by Lemma 3.4 iii). It would be convenient for us to use some other parameters (c, v) instead of $\sigma = (b, v)$, where $c(t) = b(t) - \int_0^t v(\tau)d\tau$ and

$$\dot c(t) = \dot b(t) - v(t) = w(t) - v(t), \quad 0 \le t < t_*. \tag{6.10}$$

We do not need an explicit form of the equations for (c, v) but rather the following statement:

Lemma 6.2. *Let $Y(t)$ be a solution to the Cauchy problem (2.3), and (4.1), (6.8) hold. Then $(c(t), v(t))$ satisfies the equation*

$$\begin{pmatrix} \dot c(t) \\ \dot v(t) \end{pmatrix} = \mathcal{N}(\sigma(t), Z(t)), \quad 0 \le t < t_*, \tag{6.11}$$

where

$$\mathcal{N}(\sigma, Z) = \mathcal{O}(\|Z\|^2_{-\beta}) \tag{6.12}$$

uniformly in $\sigma \in \Sigma(\tilde v)$.

Proof. We differentiate the orthogonality conditions (6.8) in t, and obtain

$$0 = \Omega(\dot Z, \tau_j) + \Omega(Z, \dot\tau_j) = \Omega(AZ + T + N, \tau_j) + \Omega(Z, \dot\tau_j), \quad 0 \le t < t_*. \tag{6.13}$$

First, let us compute the principal (i.e. non-vanishing at $Z = 0$) term $\Omega(T, \tau_j)$. For $j = 1, 2, 3$ one has by (4.12), (3.5),

$$\Omega(T, \tau_j) = -\sum_l (\dot c_l \Omega(\tau_l, \tau_j) + \dot v_l \Omega(\tau_{l+3}, \tau_j)) = \sum_l \Omega(\tau_j, \tau_{l+3})\dot v_l = \sum_l \Omega^+_{jl}\dot v_l,$$

where the matrix Ω^+ is defined by (3.6). Similarly,

$$\begin{aligned}\Omega(T, \tau_{j+3}) &= -\sum_l (\dot c_l \Omega(\tau_l, \tau_{j+3}) + \dot v_l \Omega(\tau_{l+3}, \tau_{j+3})) \\ &= \sum_l \Omega(\tau_{j+3}, \tau_l)\dot c_l = -\sum_l \Omega^+_{jl}\dot c_l.\end{aligned}$$

As the result, we have by (3.5),

$$\Omega(T, \tau) = \begin{pmatrix} 0 & \Omega^+(v) \\ -\Omega^+(v) & 0 \end{pmatrix} \begin{pmatrix} \dot c \\ \dot v \end{pmatrix} = \Omega(v) \begin{pmatrix} \dot c \\ \dot v \end{pmatrix} \tag{6.14}$$

in the vector form.

Second, let us compute $\Omega(AZ, \tau_j)$. The skew-symmetry (5.5) implies that $\Omega(AZ, \tau_j) = -\Omega(Z, A\tau_j)$. Then for $j = 1, 2, 3$, we have by (5.10),

$$\Omega(AZ, \tau_j) = -\Omega(Z, \dot c \cdot \nabla\tau_j), \tag{6.15}$$

and similarly,

$$\begin{aligned}\Omega(AZ, \tau_{j+3}) &= -\Omega(Z, \dot c \cdot \nabla\tau_{j+3} + \tau_j) = -\Omega(Z, \dot c \cdot \nabla\tau_{j+3}) - \Omega(Z, \tau_j) \\ &= -\Omega(Z, \dot c \cdot \nabla\tau_{j+3}),\end{aligned} \tag{6.16}$$

since $\Omega(Z, \tau_j) = 0$.

Finally, let us compute the last term $\Omega(Z, \dot{\tau}_j)$. For $j = 1, \dots, 6$ one has $\dot{\tau}_j = \dot{b} \cdot \nabla_b \tau_j + \dot{v} \cdot \nabla_v \tau_j = \dot{v} \cdot \nabla_v \tau_j$ since the vectors τ_j do not depend on b according to (3.3). Hence,

$$\Omega(Z, \dot{\tau}_j) = \Omega(Z, \dot{v} \cdot \nabla_v \tau_j). \tag{6.17}$$

As the result, by (6.14)–(6.17), Eq. (6.13) becomes

$$0 = \Omega(v) \begin{pmatrix} \dot{c} \\ \dot{v} \end{pmatrix} + \mathcal{M}_0(\sigma, Z) \begin{pmatrix} \dot{c} \\ \dot{v} \end{pmatrix} + \mathcal{N}_0(\sigma, Z), \tag{6.18}$$

where the matrix $\mathcal{M}_0(\sigma, Z) = \mathcal{O}(\|Z\|_{-\beta})$, and $\mathcal{N}_0(\sigma, Z) = \mathcal{O}(\|Z\|_{-\beta}^2)$ uniformly in $\sigma \in \Sigma(\tilde{v})$ and $\|Z\|_{-\beta} < r_{-\beta}(\tilde{v})$. Then, since $\Omega(v)$ is invertible by Lemma 3.2, and $\|Z\|_{-\beta}$ is small, we can resolve (6.18) with respect to the derivatives and obtain Eq. (6.11) with $\mathcal{N} = \mathcal{O}(\|Z\|_{-\beta}^2)$ uniformly in $\sigma \in \Sigma(\tilde{v})$. □

Remark 6.3. Equations (6.11), (6.12) imply that the soliton parameters $c(t)$ and $v(t)$ are *adiabatic invariants* (see [3]).

6.2. Decay for the transversal dynamics. In Sect. 11 we shall show that our main Theorem 2.5 can be derived from the following time decay of the transversal component $Z(t)$:

Proposition 6.4. *Let all conditions of Theorem 2.5 hold. Then $t_* = \infty$, and*

$$\|Z(t)\|_{-\beta} \le \frac{C(\rho, \overline{v}, d_0)}{(1 + |t|)^{3/2}}, \quad t \ge 0. \tag{6.19}$$

We shall derive (6.19) in Sects. 7–11 from our Eq. (4.8) for the transversal component $Z(t)$. This equation can be specified by using Lemma 6.2. Indeed, the lemma implies that

$$\|T(t)\|_{\beta} \le C(\tilde{v}) \|Z(t)\|_{-\beta}^2, \quad 0 \le t < t_*, \tag{6.20}$$

by (4.10) since $w - v = \dot{c}$. Thus, Eq. (4.8) becomes

$$\dot{Z}(t) = A(t) Z(t) + \tilde{N}(t), \quad 0 \le t < t_*, \tag{6.21}$$

where $A(t) = A_{v(t), w(t)}$, and $\tilde{N}(t) := T(t) + N(t)$ satisfies the estimate

$$\|\tilde{N}(t)\|_{\beta} \le C \|Z(t)\|_{-\beta}^2, \quad 0 \le t < t_*. \tag{6.22}$$

In the remaining part of our paper we mainly analyze the **basic equation** (6.21) to establish the decay (6.19). We are going to derive the decay using the bound (6.22) and the orthogonality condition (6.7).

Let us comment on two main difficulties in proving (6.19). The difficulties are common for the problems studied in [5]. First, the linear part of the equation is non-autonomous, hence we cannot apply directly known methods of scattering theory. Similarly to the approach of [5], we reduce the problem to the analysis of the *frozen* linear equation,

$$\dot{X}(t) = A_1 X(t), \quad t \in \mathbb{R}, \tag{6.23}$$

where A_1 is the operator A_{v_1,v_1} defined by (4.9) with $v_1 = v(t_1)$ for a fixed $t_1 \in [0, t_*)$. Then we estimate the error by the method of majorants.

Second, even for the frozen equation (6.23), the decay of type (6.19) for all solutions does not hold without the orthogonality condition of type (6.7). Namely, by (5.13) Eq. (6.23) admits the *secular solutions*

$$X(t) = \sum_1^3 C_j \tau_j(v_1) + \sum_1^3 D_j[\tau_j(v_1)t + \tau_{j+3}(v_1)]. \tag{6.24}$$

The solutions lie in the tangent space $\mathcal{T}_{S(\sigma_1)}\mathcal{S}$ with $\sigma_1 = (b_1, v_1)$ (for an arbitrary $b_1 \in \mathbb{R}$) that suggests an unstable character of the nonlinear dynamics *along the solitary manifold* (cf. Remark 4.1 ii)). Thus, the orthogonality condition (6.7) eliminates the secular solutions. We shall apply the corresponding projection to kill the unstable "longitudinal terms" in the basic equation (6.21).

Definition 6.5. i) *For $v \in V$, denote by $\mathbf{\Pi}_v$ the symplectic orthogonal projection of $\mathcal{E}$ onto the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$, and write $\boldsymbol{P}_v = \mathbf{I} - \mathbf{\Pi}_v$.*

ii) *Denote by $\mathcal{Z}_v = \boldsymbol{P}_v\mathcal{E}$ the space symplectic orthogonal to $\mathcal{T}_{S(\sigma)}\mathcal{S}$ with $\sigma = (b, v)$ (for an arbitrary $b \in \mathbb{R}$).*

Note that by the linearity,

$$\mathbf{\Pi}_v Z = \sum \mathbf{\Pi}_{jl}(v)\tau_j(v)\Omega(\tau_l(v), Z), \quad Z \in \mathcal{E}, \tag{6.25}$$

with some smooth coefficients $\mathbf{\Pi}_{jl}(v)$. Hence, the projector $\mathbf{\Pi}_v$ does not depend on b (in the variable $y = x - b$), and this explains the choice of the subindex in $\mathbf{\Pi}_v$ and $\boldsymbol{P}_v$.

We have now the symplectic orthogonal decomposition

$$\mathcal{E} = \mathcal{T}_{S(\sigma)}\mathcal{S} + \mathcal{Z}_v, \quad \sigma = (b, v), \tag{6.26}$$

and the symplectic orthogonality (6.7) can be represented in the following equivalent forms,

$$\mathbf{\Pi}_{v(t)}Z(t) = 0, \;\; \boldsymbol{P}_{v(t)}Z(t) = Z(t), \quad 0 \le t < t_*. \tag{6.27}$$

Remark 6.6. The tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ is invariant under the operator $A_{v,v}$ by Lemma 5.4 i), hence the space $\mathcal{Z}_v$ is also invariant by (5.5), namely: $A_{v,v}Z \in \mathcal{Z}_v$ on a dense domain of $Z \in \mathcal{Z}_v$.

In Sects. 12–18 below we will prove the following proposition which is one of the main ingredients to proving (6.19). Let us consider the Cauchy problem for Eq. (6.23) with $A = A_{v,v}$ for a fixed $v \in V$. Recall that the parameter $\beta > 3/2$ is also fixed.

Proposition 6.7. *Let* (1.9) *and* (1.10) *hold, $|v| \le \tilde{v} < 1$, and $X_0 \in \mathcal{E}$. Then*

i) *Equation* (6.23)*, with $A_1 = A = A_{v,v}$, admits a unique solution $e^{At}X_0 := X(t) \in C(\mathbb{R}, \mathcal{E})$ with the initial condition $X(0) = X_0$.*

ii) *For $X_0 \in \mathcal{Z}_v \cap \mathcal{E}_\beta$, the solution $X(t)$ has the following decay:*

$$\|e^{At}X_0\|_{-\beta} \le \frac{C(\beta, \tilde{v})}{(1 + |t|)^{3/2}}\|X_0\|_\beta, \quad t \in \mathbb{R}. \tag{6.28}$$

Remark 6.8. The decay is provided by two fundamental facts which we will establish below:

i) the null root space of the generator A coincides with the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$, where $\sigma = (b, v)$ (for an arbitrary $b \in \mathbb{R}$), and

ii) the spectrum of A in the space $\mathcal{Z}_v$ is purely continuous.

7. Frozen Form of Transversal Dynamics

Now let us fix an arbitrary $t_1 \in [0, t_*)$, and rewrite Eq. (6.21) in a "frozen form"

$$\dot{Z}(t) = A_1 Z(t) + (A(t) - A_1) Z(t) + \tilde{N}(t), \quad 0 \le t < t_*, \tag{7.1}$$

where $A_1 = A_{v(t_1),v(t_1)}$ and

$$A(t) - A_1 = \begin{pmatrix} [w(t)-v(t_1)]\cdot\nabla & 0 & 0 & 0 \\ 0 & [w(t)-v(t_1)]\cdot\nabla & 0 & 0 \\ 0 & 0 & 0 & B_{v(t)} - B_{v(t_1)} \\ 0 & 0 & \langle\nabla(\psi_{v(t)} - \psi_{v(t_1)}), \nabla\rho\rangle & 0 \end{pmatrix}.$$

The next trick is important since it enables us to kill the "bad terms" $[w(t)-v(t_1)]\cdot\nabla$ in the operator $A(t) - A_1$.

Definition 7.1. *Let us change the variables* $(y, t) \mapsto (y_1, t) = (y + d_1(t), t)$*, where*

$$d_1(t) := \int_{t_1}^{t} (w(s) - v(t_1)) ds, \quad 0 \le t \le t_1. \tag{7.2}$$

Next, let us write

$$Z_1(t) = (\Psi(y_1 - d_1(t), t), \Pi(y_1 - d_1(t), t), Q(t), P(t)). \tag{7.3}$$

Then we obtain the final form of the "frozen equation" for the transversal dynamics

$$\dot{Z}_1(t) = A_1 Z_1(t) + B_1(t) Z_1(t) + N_1(t), \quad 0 \le t \le t_1, \tag{7.4}$$

where $N_1(t) = \tilde{N}(t)$ expressed in terms of $y = y_1 - d_1(t)$, and

$$B_1(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{v(t)} - B_{v(t_1)} \\ 0 & 0 & \langle\nabla(\psi_{v(t)} - \psi_{v(t_1)}), \nabla\rho\rangle & 0 \end{pmatrix}.$$

At the end of this section, we will derive appropriate bounds for the "remainder terms" $B_1(t)Z_1(t)$ and $N_1(t)$ in (7.4). First, note that we have by Lemma 6.2,

$$|B_{v(t)} - B_{v(t_1)}| \le |\int_{t_1}^{t} \dot{v}(s) \cdot \nabla_v B_{v(s)} ds| \le C \int_{t}^{t_1} \|Z(s)\|_{-\beta}^2 ds. \tag{7.5}$$

Similarly,

$$|\langle\nabla(\psi_{v(t)} - \psi_{v(t_1)}), \nabla\rho\rangle| \le C \int_{t}^{t_1} \|Z(s)\|_{-\beta}^2 ds. \tag{7.6}$$

Let us recall the following well-known inequality: for any $\alpha \in \mathbb{R}$,

$$(1 + |y + x|)^\alpha \le (1 + |y|)^\alpha (1 + |x|)^{|\alpha|}, \quad x, y \in \mathbb{R}^3. \tag{7.7}$$

Lemma 7.2. *For* $(\Psi, \Pi, Q, P) \in \mathcal{E}_\alpha$ *with any* $\alpha \in \mathbb{R}$ *the following estimate holds:*

$$\|(\Psi(y_1 - d_1), \Pi(y_1 - d_1), Q, P)\|_\alpha \le \|(\Psi, \Pi, Q, P)\|_\alpha (1 + |d_1|)^{|\alpha|}, \quad d_1 \in \mathbb{R}^3. \tag{7.8}$$

Proof. Let us check the estimate only for one component, say, for Π. One has by (7.7),

$$\begin{aligned}\|\Pi(y_1 - d_1, t)\|^2_{0,\alpha} &= \int |\Pi(y_1 - d_1, t)|^2 (1 + |y_1|)^{2\alpha} dy_1 \\ &= \int |\Pi(y, t)|^2 (1 + |y + d_1|)^{2\alpha} dy \\ &\le \int |\Pi(y, t)|^2 (1 + |y|)^{2\alpha} (1 + |d_1|)^{2|\alpha|} dy \le (1 + |d_1|)^{2|\alpha|} \|\Pi\|^2_{0,\alpha} ,\end{aligned}$$

and the lemma is proved. □

Corollary 7.3. *The following bound holds:*

$$\|N_1(t)\|_\beta \le (1 + |d_1(t)|)^\beta \|Z(t)\|^2_{-\beta}, \quad 0 \le t \le t_1. \tag{7.9}$$

Indeed, applying the previous lemma, we obtain from (6.22) that

$$\|N_1(t)\|_\beta \le (1 + |d_1(t)|)^\beta \|\tilde{N}(t, Z(t))\|_\beta \le (1 + |d_1(t)|)^\beta \|Z(t)\|^2_{-\beta}.$$

Corollary 7.4. *The following bound holds:*

$$\|B_1(t) Z_1(t)\|_\beta \le C \|Z(t)\|_{-\beta} \int_t^{t_1} \|Z(\tau)\|^2_{-\beta} d\tau, \quad 0 \le t \le t_1. \tag{7.10}$$

For the proof we apply Lemma 7.2 to (7.5) and (7.6) and use the fact that $B_1(t) Z_1(t)$ depends only on the finite-dimensional components of $Z_1(t)$.

8. Integral Inequality

Equation (7.4) can be represented in the integral form:

$$Z_1(t) = e^{A_1 t} Z_1(0) + \int_0^t e^{A_1(t-s)} [B_1 Z_1(s) + N_1(s)] ds, \quad 0 \le t \le t_1. \tag{8.1}$$

We apply the symplectic orthogonal projection $\boldsymbol{P}_1 := \boldsymbol{P}_{v(t_1)}$ to both sides, and get

$$\boldsymbol{P}_1 Z_1(t) = e^{A_1 t} \boldsymbol{P}_1 Z_1(0) + \int_0^t e^{A_1(t-s)} \boldsymbol{P}_1 [B_1 Z_1(s) + N_1(s)] ds.$$

We have used here that $\boldsymbol{P}_1$ commutes with the group $e^{A_1 t}$ since the space $\mathcal{Z}_1 := \boldsymbol{P}_1 \mathcal{E}$ is invariant with respect to $e^{A_1 t}$ by Remark 6.6. Applying (6.28) we obtain

$$\begin{aligned}\|\boldsymbol{P}_1 Z_1(t)\|_{-\beta} \le &\frac{C}{(1+t)^{3/2}} \|\boldsymbol{P}_1 Z_1(0)\|_\beta \\ &+ C \int_0^t \frac{1}{(1 + |t - s|)^{3/2}} \|\boldsymbol{P}_1 [B_1 Z_1(s) + N_1(s)]\|_\beta ds.\end{aligned} \tag{8.2}$$

The operator $\boldsymbol{P}_1 = \mathbf{I} - \boldsymbol{\Pi}_1$ is continuous in $\mathcal{E}_\beta$ by (6.25). Hence, from (8.2) and (7.9), (7.10), we obtain

$$\|\boldsymbol{P}_1 Z_1(t)\|_{-\beta} \le \frac{C(\overline{d}_1(0))}{(1+t)^{3/2}}\|Z(0)\|_\beta + C(\overline{d}_1(t))\int_0^t \frac{1}{(1+|t-s|)^{3/2}} \times\left[\|Z(s)\|_{-\beta}\int_s^{t_1}\|Z(\tau)\|_{-\beta}^2 d\tau + \|Z(s)\|_{-\beta}^2\right]ds, \quad 0\le t\le t_1, \tag{8.3}$$

where $\overline{d}_1(t) := \sup_{0\le s\le t}|d_1(s)|$.

Definition 8.1. *Let t_*' be the exit time*

$$t_*' = \sup\{t\in[0,t_*) : \overline{d}_1(s)\le 1,\ 0\le s\le t\}. \tag{8.4}$$

Now (8.3) implies that for $t_1 < t_*'$,

$$\|\boldsymbol{P}_1 Z_1(t)\|_{-\beta} \le \frac{C}{(1+t)^{3/2}}\|Z(0)\|_\beta + C_1\int_0^t \frac{1}{(1+|t-s|)^{3/2}} \times\left[\|Z(s)\|_{-\beta}\int_s^{t_1}\|Z(\tau)\|_{-\beta}^2 d\tau + \|Z(s)\|_{-\beta}^2\right]ds, \quad 0\le t\le t_1. \tag{8.5}$$

9. Symplectic Orthogonality

Finally, we are going to change $\boldsymbol{P}_1 Z_1(t)$ by $Z(t)$ in the left-hand side of (8.5). We shall prove that this change is possible indeed by using again the smallness condition (2.12). For the justification we reduce the exit time further. First, introduce the "majorant"

$$m(t) := \sup_{s\in[0,t]}(1+s)^{3/2}\|Z(s)\|_{-\beta}, \quad t\in[0,t_*). \tag{9.1}$$

Denote by ε a fixed positive number (which will be specified below).

Definition 9.1. *Let t_*'' be the exit time*

$$t_*'' = \sup\{t\in[0,t_*') : m(s)\le\varepsilon,\ 0\le s\le t\}. \tag{9.2}$$

The following important bound (9.3) enables us to change the norm of $\boldsymbol{P}_1 Z_1(t)$ on the left-hand side of (8.5) by the norm of $Z(t)$.

Lemma 9.2. *For sufficiently small $\varepsilon > 0$, we have*

$$\|Z(t)\|_{-\beta} \le C\|\boldsymbol{P}_1 Z_1(t)\|_{-\beta}, \quad 0\le t\le t_1, \tag{9.3}$$

for any $t_1 < t_*''$, where C depends only on ρ and $\overline{v}$.

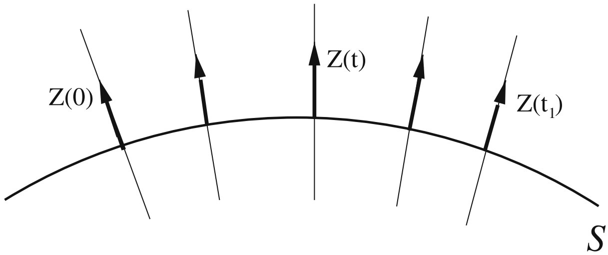

Fig. 2. Symplectic orthogonality

Proof. The proof is based on the symplectic orthogonality (6.27), i.e.

$$\mathbf{\Pi}_{v(t)} Z(t) = 0, \quad t \in [0, t_1], \tag{9.4}$$

and on the fact that all the spaces $\mathcal{Z}(t) := \boldsymbol{P}_{v(t)}\mathcal{E}$ are almost parallel for all t (see Fig. 2). Namely, we first note that $\|Z(t)\|_{-\beta} \le C\|Z_1(t)\|_{-\beta}$ by Lemma 7.2, since $|d_1(t)| \le 1$ for $t \le t_1 < t_*'' < t_*'$. Therefore, it suffices to prove that

$$\|Z_1(t)\|_{-\beta} \le 2\|\boldsymbol{P}_1 Z_1(t)\|_{-\beta}, \quad 0 \le t \le t_1. \tag{9.5}$$

This estimate will follow from

$$\|\mathbf{\Pi}_{v(t_1)} Z_1(t)\|_{-\beta} \le \frac{1}{2}\|Z_1(t)\|_{-\beta}, \quad 0 \le t \le t_1\,, \tag{9.6}$$

since $\boldsymbol{P}_1 Z_1(t) = Z_1(t) - \mathbf{\Pi}_{v(t_1)} Z_1(t)$. To prove (9.6), we write (9.4) as

$$\mathbf{\Pi}_{v(t),1} Z_1(t) = 0, \quad t \in [0, t_1], \tag{9.7}$$

where $\mathbf{\Pi}_{v(t),1} Z_1(t)$ is $\mathbf{\Pi}_{v(t)} Z(t)$ expressed in terms of the variable $y_1 = y + d_1(t)$. Hence, (9.6) follows from (9.7) if the difference $\mathbf{\Pi}_{v(t_1)} - \mathbf{\Pi}_{v(t),1}$ is small uniformly in t, i.e.

$$\|\mathbf{\Pi}_{v(t_1)} - \mathbf{\Pi}_{v(t),1}\| < 1/2, \quad 0 \le t \le t_1. \tag{9.8}$$

It remains to justify (9.8) for any sufficiently small $\varepsilon > 0$. We will need the formula (6.25) and the following relation which follows from (6.25):

$$\mathbf{\Pi}_{v(t),1} Z_1(t) = \sum \mathbf{\Pi}_{jl}(v(t))\tau_{j,1}(v(t))\Omega(\tau_{l,1}(v(t)), Z_1(t)), \tag{9.9}$$

where $\tau_{j,1}(v(t))$ are the vectors $\tau_j(v(t))$ expressed via the variables y_1. In detail (cf. (3.3)),

$$\left.\begin{array}{l} \tau_{j,1}(v) := (-\partial_j \psi_v(y_1 - d_1(t)), -\partial_j \pi_v(y_1 - d_1(t)), e_j, 0), \\ \tau_{j+3,1}(v) := (\partial_{v_j} \psi_v(y_1 - d_1(t)), \partial_{v_j} \pi_v(y_1 - d_1(t)), 0, \partial_{v_j} p_v), \end{array}\right| \quad j = 1, 2, 3, \tag{9.10}$$

where $v = v(t)$. Since $|d_1(t)| \le 1$, and the functions $\nabla \tau_j$ are smooth and rapidly decaying at infinity, Lemma 7.2 implies that

$$\|\tau_{j,1}(v(t)) - \tau_j(v(t))\|_\beta \le C|d_1(t)|^\beta, \quad 0 \le t \le t_1 \tag{9.11}$$

for all $j = 1, 2, \ldots, 6$. Furthermore,

$$\tau_j(v(t)) - \tau_j(v(t_1)) = \int_t^{t_1} \dot{v}(s) \cdot \nabla_v \tau_j(v(s))ds,$$

and therefore

$$\|\tau_j(v(t)) - \tau_j(v(t_1))\|_\beta \leq C \int_t^{t_1} |\dot{v}(s)|ds, \quad 0 \leq t \leq t_1. \tag{9.12}$$

Similarly,

$$\begin{aligned} |\mathbf{\Pi}_{jl}(v(t)) - \mathbf{\Pi}_{jl}(v(t_1))| &= |\int_t^{t_1} \dot{v}(s) \cdot \nabla_v \mathbf{\Pi}_{jl}(v(s))ds| \\ &\leq C \int_t^{t_1} |\dot{v}(s)|ds, \quad 0 \leq t \leq t_1, \end{aligned} \tag{9.13}$$

since $|\nabla_v \mathbf{\Pi}_{jl}(v(s))|$ is uniformly bounded by (6.9). Hence, the bounds (9.8) will follow from (6.25), (9.9) and (9.11)–(9.13) if we shall prove that $|d_1(t)|$ and the integral on the right-hand side of (9.12) can be made as small as desired by choosing a sufficiently small $\varepsilon > 0$.

To estimate $d_1(t)$, note that

$$w(s) - v(t_1) = w(s) - v(s) + v(s) - v(t_1) = \dot{c}(s) + \int_s^{t_1} \dot{v}(\tau)d\tau \tag{9.14}$$

by (6.10). Hence, the definitions (7.2), (9.1), and Lemma 6.2 imply that

$$\begin{aligned} |d_1(t)| = |\int_{t_1}^t (w(s) - v(t_1))ds| &\leq \int_t^{t_1} \left(|\dot{c}(s)| + \int_s^{t_1} |\dot{v}(\tau)|d\tau\right) ds \\ &\leq Cm^2(t_1) \int_t^{t_1} \left(\frac{1}{(1+s)^3} + \int_s^{t_1} \frac{d\tau}{(1+\tau)^3}\right) ds \leq Cm^2(t_1) \leq C\varepsilon^2, \quad 0 \leq t \leq t_1, \end{aligned} \tag{9.15}$$

since $t_1 < t_*''$. Similarly,

$$\int_t^{t_1} |\dot{v}(s)|ds \leq Cm^2(t_1) \int_t^{t_1} \frac{ds}{(1+s)^3} \leq C\varepsilon^2, \quad 0 \leq t \leq t_1. \tag{9.16}$$

The proof is completed. □

10. Decay of Transversal Component

Here we prove Proposition 6.4.
Step i) We fix $\varepsilon > 0$ and $t_*'' = t_*''(\varepsilon)$ for which Lemma 9.2 holds. Then a bound of type (8.5) holds with $\|\boldsymbol{P}_1 Z_1(t)\|_{-\beta}$ replaced by $\|Z(t)\|_{-\beta}$ on the left-hand side:

$$\begin{aligned} \|Z(t)\|_{-\beta} &\leq \frac{C}{(1+t)^{3/2}} \|Z(0)\|_\beta \\ &+ C \int_0^t \frac{1}{(1+|t-s|)^{3/2}} \left[\|Z(s)\|_{-\beta} \int_s^{t_1} \|Z(\tau)\|_{-\beta}^2 d\tau + \|Z(s)\|_{-\beta}^2\right] ds, \quad 0 \leq t \leq t_1 \end{aligned} \tag{10.1}$$

for $t_1 < t'_*$. This implies an integral inequality for the majorant

$$m(t) := \sup_{s\in[0,t]} (1+s)^{3/2}\|Z(s)\|_{-\beta}\ .$$

Namely, multiplying (10.1) by $(1+t)^{3/2}$ and taking the supremum in $t \in [0, t_1]$, we get

$$m(t_1) \le C\|Z(0)\|_\beta + C \sup_{t\in[0,t_1]} \int_0^t \frac{(1+t)^{3/2}}{(1+|t-s|)^{3/2}} \times \left[\frac{m(s)}{(1+s)^{3/2}} \int_s^{t_1} \frac{m^2(\tau)d\tau}{(1+\tau)^3} + \frac{m^2(s)}{(1+s)^3} \right] ds$$

for $t_1 < t''_*$. Taking into account that $m(t)$ is a monotone increasing function, we get

$$m(t_1) \le C\|Z(0)\|_\beta + C[m^3(t_1) + m^2(t_1)]I(t_1), \quad t_1 < t''_*, \tag{10.2}$$

where

$$I(t_1) = \sup_{t\in[0,t_1]} \int_0^t \frac{(1+t)^{3/2}}{(1+|t-s|)^{3/2}} \left[\frac{1}{(1+s)^{3/2}} \int_s^{t_1} \frac{d\tau}{(1+\tau)^3} + \frac{1}{(1+s)^3} \right] ds \le \overline{I} < \infty, \quad t_1 \ge 0.$$

Therefore, (10.2) becomes

$$m(t_1) \le C\|Z(0)\|_\beta + C\overline{I}[m^3(t_1) + m^2(t_1)], \quad t_1 < t''_*. \tag{10.3}$$

This inequality implies that $m(t_1)$ is bounded for $t_1 < t''_*$, and moreover,

$$m(t_1) \le C_1\|Z(0)\|_\beta, \quad t_1 < t''_*, \tag{10.4}$$

since $m(0) = \|Z(0)\|_\beta$ is sufficiently small by (3.11).
Step ii) The constant C_1 in the estimate (10.4) does not depend on t_*, t'_* and t''_* by Lemma 9.2. We choose a small d_0 in (2.12) such that $\|Z(0)\|_\beta < \varepsilon/(2C_1)$. This is possible by (3.11). Then the estimate (10.4) implies that $t''_* = t'_*$, and therefore (10.4) holds for all $t_1 < t'_*$. Then the bound (9.15) holds for all $t < t'_*$. Choose a small ε such that the right-hand side in (9.15) does not exceed one. Then $t'_* = t_*$. Therefore, (10.4) holds for any $t_1 < t''_* = t_*$, hence (6.4) also holds if $\|Z(0)\|_\beta$ is sufficiently small. Finally, this implies that $t_* = \infty$. Hence we also have $t''_* = t'_* = \infty$, and (10.4) holds for any $t_1 > 0$ if d_0 is sufficiently small. □

11. Soliton Asymptotics

Here we prove our main Theorem 2.5 under the assumption that the decay (6.19) holds. Let us first prove the asymptotics (2.13) for the vector components, and then the asymptotics (2.14) for the fields.

Asymptotics for the vector components. It follows from (4.3) that $\dot q = \dot b + \dot Q$, and from (6.21), (6.22), and (4.9) that $\dot Q = B_{v(t)}P + \mathcal{O}(\|Z\|^2_{-\beta})$. Thus,

$$\dot q = \dot b + \dot Q = v(t) + \dot c(t) + B_{v(t)}P(t) + \mathcal{O}(\|Z\|^2_{-\beta}). \tag{11.1}$$

Equation (6.11) together with estimates (6.12) and (6.19) imply that

$$|\dot c(t)| + |\dot v(t)| \le \frac{C_1(\rho, \overline{v}, d_0)}{(1+t)^3}, \quad t \ge 0. \tag{11.2}$$

Therefore, $c(t) = c_+ + \mathcal{O}(t^{-2})$ and $v(t) = v_+ + \mathcal{O}(t^{-2})$, $t \to \infty$. Since $|P| \le \|Z\|_{-\beta}$, the estimate (6.19) together with relations (11.2) and (11.1) imply that

$$\dot q(t) = v_+ + \mathcal{O}(t^{-3/2}). \tag{11.3}$$

Similarly,

$$b(t) = c(t) + \int_0^t v(s)ds = v_+ t + a_+ + \mathcal{O}(t^{-1}), \tag{11.4}$$

and hence the second part of (2.13) follows:

$$q(t) = b(t) + Q(t) = v_+ t + a_+ + \mathcal{O}(t^{-1}), \tag{11.5}$$

since $Q(t) = \mathcal{O}(t^{-3/2})$ by (6.19).

Asymptotics for the fields. We apply the approach developed in [18, 23]. For the field part of the solution, $F(t) = (\psi(x,t), \pi(x,t))$, in the original variable x, let us define the accompanying soliton field as $F_{\mathrm{v(t)}}(t) = (\psi_{\mathrm{v(t)}}(x - q(t)), \pi_{\mathrm{v(t)}}(x - q(t)))$, where we now set $\mathrm{v}(t) = \dot q(t)$, cf. (11.1). Then for the difference $Z(t) = F(t) - F_{\mathrm{v(t)}}(t)$ we obtain easily, from the first two equations of the system (1.2), the inhomogeneous Klein–Gordon equation [23, (2.5)],

$$\dot Z(t) = A_0 Z(t) - \dot{\mathrm{v}} \cdot \nabla_{\mathrm{v}} F_{\mathrm{v}(t)}(t), \quad A_0(\psi, \pi) = (\pi, (\Delta - m^2)\psi).$$

Then

$$Z(t) = W_0(t)Z(0) - \int_0^t W_0(t-s)[\dot{\mathrm{v}}(s) \cdot \nabla_{\mathrm{v}} F_{\mathrm{v}(s)}(s)]ds, \tag{11.6}$$

where $W_0(t)$ is the dynamical group of free Klein–Gordon equation. To obtain the asymptotics (2.14) it suffices to prove that $Z(t) = W_0(t)\boldsymbol{\Psi}_+ + r_+(t)$ for some $\boldsymbol{\Psi}_+ \in \mathcal{F}$ and that $\|r_+(t)\|_{\mathcal{F}} = \mathcal{O}(t^{-1/2})$. This is equivalent to the asymptotics

$$W_0(-t)Z(t) = \boldsymbol{\Psi}_+ + r'_+(t), \quad \|r'_+(t)\|_{\mathcal{F}} = \mathcal{O}(t^{-1/2}), \tag{11.7}$$

since $W_0(t)$ is a unitary group on the Sobolev space $\mathcal{F}$ by the energy conservation for the free Klein–Gordon equation. Finally, the asymptotics (11.7) hold since (11.6) implies that

$$W_0(-t)Z(t) = Z(0) - \int_0^t W_0(-s)R(s)ds, \quad R(s) = \dot{\mathrm{v}}(s) \cdot \nabla_{\mathrm{v}} F_{\mathrm{v}(s)}(s), \tag{11.8}$$

where the integral on the right-hand side of (11.8) converges in the Hilbert space $\mathcal{F}$ with the rate $\mathcal{O}(t^{-1/2})$. The latter holds since $\|W_0(-s)R(s)\|_{\mathcal{F}} = \mathcal{O}(s^{-3/2})$ by the unitarity of $W_0(-s)$ and the decay rate $\|R(s)\|_{\mathcal{F}} = \mathcal{O}(s^{-3/2})$. Let us prove this rate of decay. It suffices to prove that $|\dot{\mathrm{v}}(s)| = \mathcal{O}(s^{-3/2})$, or equivalently $|\dot{p}(s)| = \mathcal{O}(s^{-3/2})$. Substitute (4.2) to the last equation of (1.2) and obtain

$$\begin{aligned}\dot{p}(t) &= \int \left[\psi_{v(t)}(x - b(t)) + \Psi(x - b(t), t)\right] \nabla\rho(x - b(t) - Q(t))dx \\ &= \int \psi_{v(t)}(y)\nabla\rho(y)dy + \int \psi_{v(t)}(y)\left[\nabla\rho(y - Q(t)) - \nabla\rho(y)\right]dy \\ &\quad + \int \Psi(y, t)\nabla\rho(y - Q(t))dy. \qquad (11.9)\end{aligned}$$

The first integral on the right-hand side is zero by the stationary equations (2.7). The second integral is $\mathcal{O}(t^{-3/2})$, which follows from the conditions (1.9) on ρ and the asymptotics $Q(t) = \mathcal{O}(t^{-3/2})$. Finally, the third integral is $\mathcal{O}(t^{-3/2})$ by estimate (6.19). This completes the proof. □

12. Decay for the Linearized Dynamics

In the remaining section, we prove Proposition 6.7 to complete the proof of the main result (Theorem 2.5). Here we discuss the general strategy of proving the proposition. We apply the Fourier–Laplace transform

$$\tilde{X}(\lambda) = \int_0^\infty e^{-\lambda t} X(t)dt, \quad \mathrm{Re}\,\lambda > 0 \qquad (12.1)$$

to (6.23). According to Proposition 6.7, we can expect that the solution $X(t)$ is bounded in the norm $\|\cdot\|_{-\beta}$. Then the integral (12.1) converges and is analytic for $\mathrm{Re}\,\lambda > 0$, and

$$\|\tilde{X}(\lambda)\|_{-\beta} \le \frac{C}{\mathrm{Re}\,\lambda}, \quad \mathrm{Re}\,\lambda > 0. \qquad (12.2)$$

Let us derive an equation for $\tilde{X}(\lambda)$ which is equivalent to the Cauchy problem for (6.23) with the initial condition $X(0) = X_0 \in \mathcal{E}_{-\beta}$. We shall write A and v instead of A_1 and v_1 in all the remaining part of the paper. Applying the Fourier–Laplace transform to (6.23), we get that

$$\lambda\tilde{X}(\lambda) = A\tilde{X}(\lambda) + X_0, \quad \mathrm{Re}\,\lambda > 0. \qquad (12.3)$$

Let us stress that (12.3) is equivalent to the Cauchy problem for the functions $X(t) \in C_b([0, \infty); \mathcal{E}_{-\beta})$. Hence the solution $X(t)$ is given by

$$\tilde{X}(\lambda) = -(A - \lambda)^{-1}X_0, \quad \mathrm{Re}\,\lambda > 0 \qquad (12.4)$$

if the resolvent $R(\lambda) = (A - \lambda)^{-1}$ exists for $\mathrm{Re}\,\lambda > 0$.

Let us comment on our following strategy in proving the decay (6.19). We shall first construct the resolvent $R(\lambda)$ for $\mathrm{Re}\,\lambda > 0$ and prove that this resolvent is a continuous operator on $\mathcal{E}_{-\beta}$. Then $\tilde{X}(\lambda) \in \mathcal{E}_{-\beta}$ and is an analytic function for $\mathrm{Re}\,\lambda > 0$. After this we must justify that there exists a (unique) function $X(t) \in C([0, \infty); \mathcal{E}_{-\beta})$ satisfying (12.1).

The analyticity of $\tilde{X}(\lambda)$ and the Paley-Wiener arguments (see [22]) should provide the existence of a $\mathcal{E}_{-\beta}$ – valued distribution $X(t)$, $t \in \mathbb{R}$, with a support in $[0, \infty)$. Formally,

$$X(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega t} \tilde{X}(i\omega + 0) d\omega, \quad t \in \mathbb{R}. \tag{12.5}$$

However, to establish the continuity of $X(t)$ for $t \geq 0$, we need additional bound for $\tilde{X}(i\omega+0)$ for large values of $|\omega|$. Finally, for the time decay of $X(t)$, we need additional information on the smoothness and decay of $\tilde{X}(i\omega + 0)$. More precisely, we must prove that the function $\tilde{X}(i\omega + 0)$ has the following properties:

i) it is smooth outside $\omega = 0$ and $\omega = \pm\mu$, where $\mu = \mu(v) > 0$,
ii) it decays in a certain sense as $|\omega| \to \infty$,
iii) it admits the Puiseux expansion at $\omega = \pm\mu$,
iv) it is analytic at $\omega = 0$ if $X_0 \in \mathcal{Z}_v := \boldsymbol{P}_v\mathcal{E}$ and $X_0 \in \mathcal{E}_\beta$.

Then the decay (6.19) would follow from the Fourier–Laplace representation (12.5).

We shall check in detail properties of the type i)–iv) only for the last two components $\tilde{Q}(\lambda)$ and $\tilde{P}(\lambda)$ of the vector $\tilde{X}(\lambda) = (\tilde{\Psi}(\lambda), \tilde{\Pi}(\lambda), \tilde{Q}(\lambda), \tilde{P}(\lambda))$. The properties provide the decay (6.19) for the vector components $Q(t)$ and $P(t)$ of the solution $X(t)$.

However, we will not prove the properties of the type i)–iv) for the field components $\Psi(x, \lambda)$ and $\Pi(x, \lambda)$. The decay (6.19) for the field components is deduced in Sect. 18 directly from the time-dependent field equations of the system (6.23), using the decay of the component $Q(t)$ and a version of the strong Huygens principle for the Klein–Gordon equation.

13. Constructing the Resolvent

To justify the representation (12.4), we construct the resolvent as a bounded operator in $\mathcal{E}_{-\beta}$ for $\mathrm{Re}\,\lambda > 0$. We shall write $(\Psi(y), \Pi(y), Q, P)$ instead of $(\tilde{\Psi}(y, \lambda), \tilde{\Pi}(y, \lambda), \tilde{Q}(\lambda), \tilde{P}(\lambda))$ to simplify the notations. Then (12.3) reads

$$(A-\lambda)\begin{pmatrix}\Psi\\ \Pi\\ Q\\ P\end{pmatrix} = -\begin{pmatrix}\Psi_0\\ \Pi_0\\ Q_0\\ P_0\end{pmatrix}, \text{ where } A\begin{pmatrix}\Psi\\ \Pi\\ Q\\ P\end{pmatrix} = \begin{pmatrix}\Pi + v\cdot\nabla\Psi\\ \Delta\Psi - m^2\Psi + v\cdot\nabla\Pi + Q\cdot\nabla\rho\\ B_v P\\ -\langle\nabla\Psi, \rho\rangle + \langle\nabla\psi_v, Q\cdot\nabla\rho\rangle\end{pmatrix}.$$

This gives the system of equations

$$\left.\begin{aligned} \Pi(y) + v\cdot\nabla\Psi(y) - \lambda\Psi(y) &= -\Psi_0(y)\\ \Delta\Psi(y) - m^2\Psi(y) + v\cdot\nabla\Pi(y) + Q\cdot\nabla\rho(y) - \lambda\Pi(y) &= -\Pi_0(y)\\ B_v P - \lambda Q &= -Q_0\\ -\langle\nabla\Psi(y), \rho(y)\rangle + \langle\nabla\psi_v(y), Q\cdot\nabla\rho(y)\rangle - \lambda P &= -P_0 \end{aligned}\right| \quad y \in \mathbb{R}^3. \tag{13.1}$$

Step i) Let us study the first two equations. In the Fourier space they become

$$\left.\begin{aligned} \hat{\Pi}(k) - ivk\hat{\Psi}(k) - \lambda\hat{\Psi}(k) &= -\hat{\Psi}_0(k)\\ (-k^2 - m^2)\hat{\Psi}(k) - (ivk + \lambda)\hat{\Pi}(k) &= -\hat{\Pi}_0(k) + iQk\hat{\rho}(k) \end{aligned}\right| \quad k \in \mathbb{R}^3. \tag{13.2}$$

Let us invert the matrix of the system and obtain

$$\begin{pmatrix} -(ivk+\lambda) & 1 \\ -(k^2+m^2) & -(ivk+\lambda) \end{pmatrix}^{-1} = [(ivk+\lambda)^2+k^2+m^2]^{-1} \begin{pmatrix} -(ivk+\lambda) & -1 \\ k^2+m^2 & -(ivk+\lambda) \end{pmatrix}.$$

Taking the inverse Fourier transform, we obtain the corresponding fundamental solution

$$G_\lambda(y) = \begin{pmatrix} v\cdot\nabla-\lambda & -1 \\ -\Delta+m^2 & v\cdot\nabla-\lambda \end{pmatrix} g_\lambda(y), \tag{13.3}$$

where $g_\lambda(y)$ is the unique tempered fundamental solution of the determinant

$$D = D(\lambda) = -\Delta + m^2 + (-v\cdot\nabla+\lambda)^2. \tag{13.4}$$

From now on we use the system of coordinates in x-space in which $v = (|v|, 0, 0)$, hence $vk = |v|k_1$, and

$$g_\lambda(y) = F^{-1}_{k\to y}\frac{1}{k^2+m^2+(ivk+\lambda)^2} = F^{-1}_{k\to y}\frac{1}{k^2+m^2+(i|v|k_1+\lambda)^2}, \quad y\in\mathbb{R}^3. \tag{13.5}$$

Note that the denominator does not vanish for $\operatorname{Re}\lambda > 0$. This implies

Lemma 13.1. *The operator G_λ with the integral kernel $G_\lambda(y-y')$ is continuous as an operator from $H^1(\mathbb{R}^3)\oplus L^2(R^3)$ to $H^2(\mathbb{R}^3)\oplus H^1(R^3)$ for* $\operatorname{Re}\lambda > 0$.

Thus, formulas (13.2) and (13.3) imply the convolution representation

$$\left.\begin{aligned} \Psi &= -(v\cdot\nabla-\lambda)g_\lambda * \Psi_0 + g_\lambda * \Pi_0 + (g_\lambda * \nabla\rho)\cdot Q \\ \Pi &= -(-\Delta+m^2)g_\lambda * \Psi_0 - (v\cdot\nabla-\lambda)g_\lambda * \Pi_0 - (v\cdot\nabla-\lambda)(g_\lambda * \nabla\rho)\cdot Q \end{aligned}\right|. \tag{13.6}$$

Step ii) Let us compute $g_\lambda(y)$ explicitly. First consider the case $v=0$. The fundamental solution of the operator $-\Delta+m^2+\lambda^2$ is

$$g_\lambda(y) = \frac{e^{-\kappa|y|}}{4\pi|y|}, \tag{13.7}$$

where

$$\kappa^2 = m^2+\lambda^2, \quad \operatorname{Re}\kappa > 0 \text{ for } \operatorname{Re}\lambda > 0. \tag{13.8}$$

Thus, in the case $v=0$ we have

$$G_\lambda(y-y') = \begin{pmatrix} -\lambda & -1 \\ -\Delta+m^2 & -\lambda \end{pmatrix} \frac{e^{-\sqrt{\lambda^2+m^2}|y-y'|}}{4\pi|y-y'|}.$$

For general $v = (|v|, 0, 0)$ with $|v| < 1$ the denominator in (13.5), which is the Fourier symbol of D, reads

$$\hat{D}(k) = k^2 + m^2 + (i|v|k_1 + \lambda)^2$$
$$= (1 - v^2)k_1^2 + k_2^2 + k_3^2 + 2i|v|k_1\lambda + \lambda^2 + m^2$$
$$= (1 - v^2)(k_1 + \frac{i|v|\lambda}{1 - v^2})^2 + k_2^2 + k_3^2 + \kappa^2, \tag{13.9}$$

where

$$\kappa^2 = \frac{v^2\lambda^2}{1 - v^2} + \lambda^2 + m^2 = \frac{\lambda^2}{1 - v^2} + m^2. \tag{13.10}$$

Therefore, setting $\gamma := 1/\sqrt{1 - v^2}$, we have

$$\kappa = \gamma\sqrt{\lambda^2 + \mu^2}, \quad \mu := m/\gamma. \tag{13.11}$$

Return to x-space:

$$D = -\frac{1}{\gamma^2}(\nabla_1 + \gamma\kappa_1)^2 - \nabla_2^2 - \nabla_3^2 + \kappa^2, \quad \kappa_1 := \gamma|v|\lambda. \tag{13.12}$$

Define $\tilde{y}_1 := \gamma y_1$ and $\tilde{\nabla}_1 := \partial/\partial\tilde{y}_1$. Then

$$D = -(\tilde{\nabla}_1 + \kappa_1)^2 - \nabla_2^2 - \nabla_3^2 + \kappa^2. \tag{13.13}$$

Thus, its fundamental solution is

$$g_\lambda(y) = \frac{e^{-\kappa|\tilde{y}| - \kappa_1\tilde{y}_1}}{4\pi|\tilde{y}|}, \quad \tilde{y} := (\gamma y_1, y_2, y_3), \tag{13.14}$$

where we choose $\operatorname{Re}\kappa > 0$ for $\operatorname{Re}\lambda > 0$. Let us note that

$$0 < \operatorname{Re}\kappa_1 < \operatorname{Re}\kappa, \quad \operatorname{Re}\lambda > 0. \tag{13.15}$$

This inequality follows from the fact that the fundamental solution decays exponentially by the Paley–Wiener arguments since the quadratic form (13.9) does not vanish in a complex neighborhood of the real space $\mathbb{R}^3$ for $\operatorname{Re}\lambda > 0$. Let us state the result which we obtained above.

Lemma 13.2. i) *The operator* $D = D(\lambda)$ *is invertible in* $L^2(\mathbb{R}^3)$ *for* $\operatorname{Re}\lambda > 0$ *and its fundamental solution* (13.14) *decays exponentially.*

ii) *Formulas* (13.14) *and* (13.11) *imply that, for every fixed* y*, the Green function* $g_\lambda(y)$ *admits an analytic continuation* (*in the variable* λ) *to the Riemann surface of the algebraic function* $\sqrt{\lambda^2 + \mu^2}$ *with the branching points* $\pm i\mu$.

Step iii) Let us now proceed with the last two equations (13.1),

$$-\lambda Q + B_v P = -Q_0, \quad \langle \nabla\psi_v, Q\cdot\nabla\rho\rangle - \langle\nabla\Psi,\rho\rangle - \lambda P = -P_0. \tag{13.16}$$

Let us eliminate the field Ψ by the first equation (13.6). Namely, rewrite the equation in the form $\Psi(x) = \Psi_1(Q) + \Psi_2(\Psi_0, \Pi_0)$, where

$$\Psi_1(Q) = Q\cdot(g_\lambda * \nabla\rho), \quad \Psi_2(\Psi_0,\Pi_0) = -(v\cdot\nabla - \lambda)g_\lambda * \Psi_0 + g_\lambda * \Pi_0. \tag{13.17}$$

Then we have

$$\langle\nabla\Psi,\rho\rangle = \langle\nabla\Psi_1(Q),\rho\rangle + \langle\nabla\Psi_2(\Psi_0,\Pi_0),\rho\rangle,$$

and the last equation in (13.16) becomes

$$\langle\nabla\psi_v, Q\cdot\nabla\rho\rangle - \langle\nabla\Psi_1(Q),\rho\rangle - \lambda P = -P_0 + \langle\nabla\Psi_2(\Psi_0,\Pi_0),\rho\rangle =: -P_0'.$$

Let us first compute the term $\langle\nabla\psi_v, Q\cdot\nabla\rho\rangle = \sum_j\langle\nabla\psi_v, Q_j\partial_j\rho\rangle = \sum_j\langle\nabla\psi_v, \partial_j\rho\rangle Q_j$. Applying the Fourier transform $F_{y\to k}$, the Parseval identity, and (A.5) we see that

$$\begin{aligned}\langle\partial_i\psi_v,\partial_j\rho\rangle &= \langle -ik_i\hat\psi_v(k), -ik_j\hat\rho(k)\rangle = \langle k_i\hat\psi_v(k), k_j\hat\rho(k)\rangle\\ &= -\langle\frac{k_i\hat\rho(k)}{k^2+m^2-(|v|k_1)^2}, k_j\hat\rho(k)\rangle = -\int\frac{k_ik_j|\hat\rho(k)|^2dk}{k^2+m^2-(|v|k_1)^2} =: -K_{ij}.\end{aligned} \tag{13.18}$$

As the result, $\langle\nabla\psi_v, Q\cdot\nabla\rho\rangle = -KQ$, where K is the 3×3 matrix with the matrix elements K_{ij}. The matrix K is diagonal and positive definite since $\hat\rho(k)$ is spherically symmetric and not identically zero by (1.10).

Let us now compute the term $-\langle\nabla\Psi_1,\rho\rangle = \langle\Psi_1,\nabla\rho\rangle$. We have

$$\langle\Psi_1,\partial_i\rho\rangle = \langle\sum_j(g_\lambda*\partial_j\rho)Q_j,\partial_i\rho\rangle = \sum_j\langle g_\lambda*\partial_j\rho,\partial_i\rho\rangle Q_j = \sum_j H_{ij}(\lambda)Q_j,$$

since $\Psi_1 = Q\cdot(g_\lambda*\nabla\rho)$, and by the Parseval identity again, we have

$$\begin{aligned}H_{ij}(\lambda) :&= \langle g_\lambda*\partial_j\rho,\partial_i\rho\rangle = \langle i\hat g_\lambda(k)k_j\hat\rho(k), ik_i\hat\rho(k)\rangle\\ &= \langle\frac{ik_j\hat\rho(k)}{k^2+m^2+(i|v|k_1+\lambda)^2}, ik_i\hat\rho(k)\rangle = \int\frac{k_ik_j|\hat\rho(k)|^2dk}{k^2+m^2+(i|v|k_1+\lambda)^2}.\end{aligned} \tag{13.19}$$

The matrix H is well defined for $\operatorname{Re}\lambda > 0$ since the denominator does not vanish (or $g_\lambda(x)$ exponentially decays). The matrix H is diagonal similarly to K. Indeed, if $i\neq j$, then at least one of these indexes is not equal to one, and the integrand in (13.19) is odd with respect to the corresponding variable.

As the result, $-\langle\nabla\Psi_1,\rho\rangle = HQ$, where H is the diagonal matrix with matrix elements H_{jj}, $1\le j\le 3$. Finally, Eqs. (13.16) become

$$M(\lambda)\begin{pmatrix}Q\\P\end{pmatrix} = \begin{pmatrix}Q_0\\P_0'\end{pmatrix}, \quad \text{where } M(\lambda) = \begin{pmatrix}\lambda E & -B_v\\ K-H(\lambda) & \lambda E\end{pmatrix}, \tag{13.20}$$

where the matrices K and $H(\lambda)$ are diagonal.
Step iv) Assume for a moment that the matrix $M(\lambda)$ is invertible for $\operatorname{Re}\lambda > 0$ (later we shall prove that this the case indeed). Then

$$\begin{pmatrix} Q \\ P \end{pmatrix} = M^{-1}(\lambda) \begin{pmatrix} Q_0 \\ P_0' \end{pmatrix}, \quad \operatorname{Re}\lambda > 0. \tag{13.21}$$

Finally, formulas (13.21) and (13.6) give the expression of the resolvent $R(\lambda) = (A - \lambda)^{-1}$, $\operatorname{Re}\lambda > 0$.

Lemma 13.3. *The matrix function* $M(\lambda)$ $(M^{-1}(\lambda))$ *admits an analytic (meromorphic) continuation from the domain* $\operatorname{Re}\lambda > 0$ *to the Riemann surface of the function* $\sqrt{\lambda^2 + \mu^2}$.

Proof. The analytic continuation of $M(\lambda)$ exists by Lemma 13.2 ii) and the convolution expressions in (13.19) since the function $\rho(x)$ is compactly supported by (1.9). The inverse matrix is then meromorphic since it exists for large $\operatorname{Re}\lambda$: this follows from (13.20) since $H(\lambda) \to 0$ as $\operatorname{Re}\lambda \to \infty$ by (13.19). □

14. Analyticity in the Half-Plane

Here we prove the following proposition.

Proposition 14.1. *The operator-valued function* $R(\lambda) : \mathcal{E} \to \mathcal{E}$ *is analytic for* $\operatorname{Re}\lambda > 0$.

Proof. It suffices to prove that the operator $A - \lambda : \mathcal{E} \to \mathcal{E}$ has bounded inverse operator for $\operatorname{Re}\lambda > 0$. Recall that $A = A_{v,v}$ where $|v| < 1$.
Step i) Let us prove that $\operatorname{Ker}(A - \lambda) = 0$ for $\operatorname{Re}\lambda > 0$. Indeed, assume that the vector $X_\lambda = (\Psi_\lambda, \Pi_\lambda, Q_\lambda, P_\lambda) \in \mathcal{E}$ satisfies the equation $(A - \lambda)X_\lambda = 0$, that is X_λ is a solution to (13.1) with $\Psi_0 = \Pi_0 = 0$ and $Q_0 = P_0 = 0$. We must prove that $X_\lambda = 0$.

Let us first show that $P_\lambda = 0$. Indeed, the trajectory $X := X_\lambda e^{\lambda t} \in C(\mathbb{R}, \mathcal{E})$ is the solution to the equation $\dot{X} = AX$ of type (5.1) with $w = v$. Then $\mathcal{H}_{v,v}(X(t))$ grows exponentially by (5.14), since the matrix B_v is positive. This growth contradicts the conservation of $\mathcal{H}_{v,v}$, which follows from Lemma 5.1 ii) because $X(t) \in C^1(\mathbb{R}, \mathcal{E}^+)$. The latter inclusion follows from Lemma 13.1 since $(\Psi_\lambda, \Pi_\lambda)$ satisfies Eqs. (13.6) with $\Psi_0 = \Pi_0 = 0$ and $Q = Q_\lambda$.

We now have $\lambda Q_\lambda = B_v P_\lambda = 0$ by the third equation of (13.1), and hence $Q_\lambda = 0$ because $\lambda \neq 0$. Finally, $\Psi_\lambda = 0$, $\Pi_\lambda = 0$ by Eqs. (13.6) with $Q = Q_\lambda = 0$.
Step ii) One has

$$(A - \lambda)\begin{pmatrix} \Psi \\ \Pi \\ Q \\ P \end{pmatrix} = \begin{pmatrix} v\cdot\nabla - \lambda & 1 & 0 & 0 \\ \Delta - m^2 & v\cdot\nabla - \lambda & \cdot\nabla\rho & 0 \\ 0 & 0 & -\lambda & B_v \\ \langle\cdot, \nabla\rho\rangle & 0 & \langle\nabla\psi_v, \cdot\nabla\rho\rangle & -\lambda \end{pmatrix}\begin{pmatrix} \Psi \\ \Pi \\ Q \\ P \end{pmatrix}.$$

Thus, $A - \lambda = A_0 + T$, where

$$A_0 = \begin{pmatrix} v\cdot\nabla - \lambda & 1 & 0 & 0 \\ \Delta - m^2 & v\cdot\nabla - \lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot\nabla\rho & 0 \\ 0 & 0 & 0 & B_v \\ \langle\cdot, \nabla\rho\rangle & 0 & \langle\nabla\psi_v, \cdot\nabla\rho\rangle & 0 \end{pmatrix}.$$

The operator T is finite-dimensional, and the operator A_0^{-1} is bounded on $\mathcal{E}$ by Lemma 13.1. Finally, $A - \lambda = A_0(I + A_0^{-1}T)$, where $A_0^{-1}T$ is a compact operator. Since we know that $\mathrm{Ker}\,(I + A_0^{-1}T) = 0$, the operator $(I + A_0^{-1}T)$ is invertible by the Fredholm theory. □

Corollary 14.2. *The matrix* $M(\lambda)$ *of* (13.20) *is invertible for* $\mathrm{Re}\,\lambda > 0$.

15. Regularity on the Imaginary Axis

Next step should be an investigation of the limit values of the resolvent $R(\lambda)$ at the imaginary axis $\lambda = i\omega$, $\omega \in \mathbb{R}$, that is necessary for proving the decay (6.19) of the solution $X(t) = (\Psi(t), \Pi(t), Q(t), P(t))$.

Let us first describe the continuous spectrum of the operator $A = A_{v,v}$ on the imaginary axis. By definition, the continuous spectrum corresponds to $\omega \in \mathbb{R}$ such that the resolvent $R(i\omega + 0)$ is not a bounded operator on $\mathcal{E}$. By the formulas (13.6), this is the case if the Green function $g_\lambda(y - y')$ fails to have exponential decay. This is equivalent to the condition that $\mathrm{Re}\,\kappa = 0$, where κ is given by (13.11): $\kappa = \gamma\sqrt{\mu^2 - \omega^2}$. Thus, $i\omega$ belongs to the continuous spectrum if (cf. (13.11))

$$|\omega| \geq \mu = m\sqrt{1 - v^2}.$$

By Lemma 13.3, the limit matrix

$$M(i\omega) := M(i\omega + 0) = \begin{pmatrix} i\omega E & -B_v \\ K - H(i\omega + 0) & i\omega E \end{pmatrix}, \quad \omega \in \mathbb{R}, \tag{15.1}$$

exists, and its entries are continuous functions of $\omega \in \mathbb{R}$, smooth for $|\omega| < \mu$ and $|\omega| > \mu$. Recall that the point $\lambda = 0$ belongs to the discrete spectrum of the operator A by Lemma 5.4 i), and hence $M(i\omega + 0)$ is (probably) not invertible either at $\omega = 0$.

Proposition 15.1. *Let* (1.9) *and* (1.10) *hold, and* $|v| < 1$. *Then the limit matrix* $M(i\omega{+}0)$ *is invertible for* $\omega \neq 0$, $\omega \in \mathbb{R}$.

Proof. Let us consider the three possible cases $0 < |\omega| < \mu$, $|\omega| = \mu$, and $|\omega| > \mu$ separately. Let us recall that the matrices K and H are diagonal with the entries

$$K_{jj} = \int \frac{k_j^2 |\hat{\rho}(k)|^2 dk}{k^2 + m^2 - (|v|k_1)^2}, \tag{15.2}$$

$$H_{jj}(\lambda) = \int \frac{k_j^2 |\hat{\rho}(k)|^2 dk}{k^2 + m^2 + (i|v|k_1 + \lambda)^2}, \quad \mathrm{Re}\,\lambda > 0, \tag{15.3}$$

and $H_{22} = H_{33}$. Since $v = (|v|, 0, 0)$, the matrix B_v is also diagonal:

$$B_v := \nu(E - v \otimes v) = \begin{pmatrix} \nu^3 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \tag{15.4}$$

since $\nu^2 := 1 - v^2$. Let us denote $F(\omega) := -K + H(i\omega + 0)$ which is also diagonal, and let $F_\| := F_{11}(\omega)$, and $F_\perp := F_{22}(\omega) = F_{33}(\omega)$. Then by (15.1)

$$\det M(i\omega) = \det \begin{pmatrix} i\omega E & -B_v \\ -F(\omega) & i\omega E \end{pmatrix} = -(\omega^2 + \nu^3 F_\|)(\omega^2 + \nu F_\perp)^2, \quad \omega \in \mathbb{R}. \tag{15.5}$$

The formula for the determinant is obvious since both matrices $F(\omega)$ and B_v are diagonal, hence the matrix $M(i\omega)$ is equivalent to three independent matrices 2×2. Namely, let us transpose the columns and rows of the matrix $M(i\omega)$ in the order (142536). Then we get the matrix with three 2×2 blocks on the main diagonal. Therefore, the determinant of $M(i\omega)$ is simply a product of the determinants of the three matrices.

I. First, let us consider the case $0 < |\omega| < \mu$. Then the invertibility of $M(i\omega)$ follows from (15.5) by the following lemma. □

Lemma 15.2. *For* $0 < |\omega| < \mu$*, the matrix* $F(\omega)$ *is positive definite, i.e.* $F_{jj}(\omega) > 0$*,* $j = 1, 2, 3$.

Proof. First, let us check that the denominator in (15.3) is positive for $\lambda = i\omega$ with $|\omega| < \mu$. Indeed, it equals $m^2 + k^2 - (\omega + |v|k_1)^2$ and we have to prove that $m^2 + k^2 > \omega^2 + 2\omega|v|k_1 + v^2k_1^2$. By the condition $|\omega| < \mu = m\sqrt{1 - v^2}$, it suffices to prove that $m^2+k^2 \geq m^2(1-v^2)+2\omega|v|k_1+v^2k_1^2$. This is equivalent to $k_2^2+k_3^2+m^2v^2+k_1^2(1-v^2) \geq 2m|v|k_1\sqrt{1 - v^2}$, which is evidently true. Thus,

$$F_{jj}(\omega) = \int k_j^2|\hat{\rho}(k)|^2 dk \left(\frac{1}{m^2 + k^2 - (|v|k_1 + \omega)^2} - \frac{1}{m^2 + k^2 - (|v|k_1)^2} \right),$$
$$j = 1, 2, 3.$$

Let us prove that $F_{jj}(\omega) > 0$. Indeed, since $\hat{\rho}(k) = \hat{\rho}(-k)$, we obtain that

$$F_{jj}(\omega) = \int dk_2 dk_3 \int_0^{+\infty} k_j^2|\hat{\rho}(k)|^2 \left(\frac{1}{m^2 + k^2 - (|v|k_1 + \omega)^2} + \frac{1}{m^2 + k^2 - (|v|k_1 - \omega)^2} - \frac{2}{m^2 + k^2 - (|v|k_1)^2} \right) dk_1. \tag{15.6}$$

Now it suffices to prove that the expression in brackets is positive (or positive infinite) under the conditions

$$|v| < 1, \quad 0 < |\omega| \leq \mu = m\sqrt{1 - v^2}. \tag{15.7}$$

This is proved in Appendix B. □

II. $\omega = \pm\mu$. For example consider the case $\omega = \mu$. Then formula (15.3) reads (see (13.9)):

$$H_{jj}(i\mu) = \int \frac{k_j^2|\hat{\rho}(k)|^2 dk}{k_2^2 + k_3^2 + (\nu k_1 - m|v|)^2}.$$

Now the integrand has a unique singular point. The singularity is integrable, and therefore the terms $F_{jj}(\mu)$ are finite. Furthermore, the terms are positive by the integral representation (15.6) again. Hence, the matrix $M(i\mu)$ is invertible.

III. $|\omega| > \mu$. Here we apply another argument: the invertibility of $M(i\omega)$ follows from (15.5) by the methods used in [40, Chapter VII, formula (58)].

Lemma 15.3. *If* (1.10) *holds and if* $|\omega| > \mu$, *then the imaginary part of the matrix* $\frac{\omega}{|\omega|}F(\omega)$ *is negative definite, i.e.* $\frac{\omega}{|\omega|}\text{Im}\, F_{jj}(\omega) < 0$, $j = 1, 2, 3$.

Proof. Since $F(\omega) = -K + H(i\omega + 0)$, where the matrix K is real, it suffices to study the matrix $H(i\omega + 0)$. For $\varepsilon > 0$, we have

$$H_{jj}(i\omega + \varepsilon) = \int \frac{k_j^2|\hat\rho(k)|^2 dk}{k_1^2 + k_2^2 + k_3^2 - (|v|k_1 + \omega - i\varepsilon)^2 + m^2}, \quad j = 1, 2, 3. \quad (15.8)$$

Consider the denominator

$$\hat D_\varepsilon(k) = k^2 + m^2 - (|v|k_1 + \omega - i\varepsilon)^2.$$

It was shown above that $\hat D_0(k) \neq 0$ if $|\omega| < \mu$, and $\hat D_0(k)$ vanishes at one point if $|\omega| = \mu$. On the other hand, for $|\omega| > \mu$ the denominator $\hat D_0(k)$ vanishes on the ellipsoid

$$T_\omega = \left\{ k : (\nu k_1 - \frac{|v|\omega}{\nu})^2 + k_2^2 + k_3^2 = R^2 := \frac{\omega^2 - \mu^2}{\nu^2} \right\},$$

where $\nu = \sqrt{1 - v^2}$. We shall show below that the Plemelj formula for C^1-functions implies that

$$\text{Im}\, H_{jj}(i\omega + 0) = -\frac{\omega}{|\omega|}\pi \int_{T_\omega} \frac{k_j^2|\hat\rho(k)|^2}{|\nabla \hat D_0(k)|} dS, \quad (15.9)$$

where dS is the element of the surface area. This immediately implies the statement of the lemma since the integrand in (15.9) is positive by the Wiener condition (1.10).

Let us justify (15.9) for $\omega > \mu > 0$ (the case $\omega < -\mu < 0$ can be treated similarly). Let $\zeta \in C_0^\infty(\mathbb{R}^3)$ be a nonnegative cut off function equal to one when $|\hat D_0(k)| < \delta$ and vanishing when $|\hat D_0(k)| > 2\delta$. We fix a small δ and split the integral (15.8) in two parts: with the factor ζ in the integrand and with the factor $1 - \zeta$. The limit of the second term as $\varepsilon \to 0$ is real. Hence, we have to calculate the imaginary part only for

$$H_{jj}^{(\delta)}(i\omega + 0) = \lim_{\varepsilon \to 0} \int \zeta(k) \frac{k_j^2|\hat\rho(k)|^2 dk}{\hat D_\varepsilon(k)}. \quad (15.10)$$

Denote $a(k) = \sqrt{k^2 + m^2}$ and $b(k) = |v|k_1 + \omega$. Then

$$\frac{1}{\hat D_\varepsilon(k)} = \frac{1}{a^2 - (b - i\varepsilon)^2} = \frac{1}{2a(a - b + i\varepsilon)} + \frac{1}{2a(a + b - i\varepsilon)}. \quad (15.11)$$

Note that $\hat D_0(k) \neq 0$ if $b(k) = 0$. Thus, $b(k) \neq 0$ on T_ω, and therefore $b(k) \neq 0$ on the support of ζ if $\delta \ll 1$. Since $b(k) > 0$ when $v = 0$ ($\nu = 1$), we get that $b(k) > 0$ on the support of ζ for all v with $|v| < 1$.

We split the integral in (15.10) in two terms according to (15.11). Then the second term is real for $\varepsilon = 0$. Now it remains to calculate the imaginary part of $h(i\omega + 0)$, where

$$h(i\omega + \varepsilon) := \int \zeta(k) \frac{k_j^2|\hat\rho(k)|^2 dk}{2a(a - b + i\varepsilon)}. \quad (15.12)$$

One can rewrite (15.12) as the iterated integral: over the surfaces $T_\omega^\alpha = \{k \in \mathbb{R}^3 : a(k) - b(k) = \alpha, \ |\hat{D}_0(k)| < \delta\}$ and over α. Then we get

$$h(i\omega+\varepsilon) = \int \frac{u(\alpha)}{\alpha + i\varepsilon} d\alpha, \ \ u(\alpha) = \int_{T_\omega^\alpha} \zeta(k) \frac{k_j^2 |\hat{\rho}(k)|^2 dS}{2a|\nabla(a-b)|},$$

and therefore

$$\operatorname{Im} h(i\omega+0) = -\pi u(0) = -\pi \int_{T_\omega} \zeta(k) \frac{k_j^2 |\hat{\rho}(k)|^2 dS}{2a|\nabla(a-b)|}.$$

This implies (15.9), since $\hat{D}_0(k) = a^2 - b^2$ and $|\nabla \hat{D}_0(k)| = 2a|\nabla(a-b)|$ on T_ω. □

This completes the proofs of Lemma 15.3 and Proposition 15.1.

Corollary 15.4. *Proposition* 15.1 *implies that the matrix $M^{-1}(i\omega)$ is smooth with respect to $\omega \in \mathbb{R}$ outside the three points $\omega = 0, \pm\mu$.*

Remark 15.5. The proof of Lemma 15.3 is the unique point in the paper where the Wiener condition is indispensable. In Lemma 15.2 we use only that the coupling function $\rho(x)$ is not identically zero.

16. Singular Spectral Points

Recall that the formula (13.21) expresses the Fourier–Laplace transforms $\tilde{Q}(\lambda), \tilde{P}(\lambda)$. Hence, the components are given by the Fourier integral

$$\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \frac{1}{2\pi} \int e^{i\omega t} M^{-1}(i\omega+0) \begin{pmatrix} Q_0 \\ P_0' \end{pmatrix} d\omega \tag{16.1}$$

which converges in the sense of distributions. It remains to prove the continuity and decay of the vector components. Corollary 15.4 by itself is insufficient to prove the convergence and decay of the integral. Namely, we need additional information about the regularity of the matrix $M^{-1}(i\omega)$ at the singular points $\omega = 0, \pm\mu$ and about some bounds at $|\omega| \to \infty$. We shall study the points separately.
I. Consider first the points $\pm\mu$.

Lemma 16.1. *The matrix $M^{-1}(i\omega)$ admits the following Puiseux expansion in a neighborhood of $\pm\mu$: there exists an $\varepsilon_\pm > 0$ s.t.*

$$M^{-1}(i\omega) = \sum_{k=0}^{\infty} R_k^{\pm} (\omega \mp \mu)^{k/2}, \ \ |\omega \mp \mu| < \varepsilon_\pm, \quad \omega \in \mathbb{R}. \tag{16.2}$$

Proof. It suffices to prove a similar expansion for $M(i\omega)$. Then (16.2) holds for $M^{-1}(i\omega)$ as well, since the matrices $M(\pm i\mu)$ are invertible. The asymptotics for $M(i\omega)$ holds by the convolution representation in (13.19):

$$H_{ij}(\lambda) = \langle g_\lambda * \partial_j \rho, \partial_i \rho \rangle, \tag{16.3}$$

since g_λ admits the corresponding Puiseux expansions by formula (13.14). □

II. Second, we study the asymptotic behavior of $M^{-1}(\lambda)$ at infinity. Let us recall that $M^{-1}(\lambda)$ was originally defined for $\mathrm{Re}\,\lambda > 0$, but it admits a meromorphic continuation to the Riemann surface of the function $\sqrt{\lambda^2+\mu^2}$ (see Lemma 13.3).

The following proposition is a very particular case of a general fundamental theorem about the bound for the truncated resolvent on the continuous spectrum. The bound plays a crucial role in the study of the long-time asymptotics of general linear hyperbolic PDEs, [40].

Proposition 16.2. *We can find a matrix R_0 and a matrix-function $R_1(\omega)$ such that*

$$M^{-1}(i\omega) = \frac{R_0}{\omega} + R_1(\omega), \ \ |\omega| \ge \mu + 1, \quad \omega \in \mathbb{R},$$

where

$$|\partial_\omega^k R_1(\omega)| \le \frac{C_k}{|\omega|^2}, \quad |\omega| \ge \mu + 1, \quad \omega \in \mathbb{R} \tag{16.4}$$

for every $k = 0, 1, 2, ...,$.

Proof. By the structure (15.1) of the matrix $M(i\omega)$ it suffices to prove the following estimate for the elements of the matrix $H(i\omega) := H(i\omega + 0)$:

$$|\partial_\omega^k H_{jj}(i\omega)| \le \frac{C_k}{|\omega|}, \quad \omega \in \mathbb{R}, \ \ |\omega| \ge \mu + 1, \ \ j = 1, 2, 3. \tag{16.5}$$

Let us rewrite (16.3) as

$$H_{ij}(\lambda) = \langle D^{-1}(\lambda)\partial_j\rho, \partial_i\rho\rangle, \quad \mathrm{Re}\,\lambda > 0, \tag{16.6}$$

where $D(\lambda)$ is the operator (13.4), and $D^{-1}(\lambda)$ is a bounded operator on $L^2(\mathbb{R}^3)$. Let us denote by B_R the ball $\{x \in \mathbb{R}^3 : |x| < R\}$. Estimate (16.5) immediately follows from a more general bound

$$\|\partial_\omega^k D^{-1}(i\omega + 0) f\|_{L^2(B_R)} \le \frac{C_k(R)}{|\omega|}\|f\|_{L^2(B_R)}, \quad \omega \in \mathbb{R}, \ \ |\omega| \ge \mu + 1 \tag{16.7}$$

which holds for every $R > 0$ and all functions $f(y) \in L^2_R := \{f(y) \in L^2(\mathbb{R}^3) : \mathrm{supp}\, f \subset B_R\}$. Namely, by (1.9) the asymptotics (16.5) follows from the bound (16.7) applied to the function $f(y) = \partial_j\rho(y) \in L^2_R$ with $R \ge R_\rho$. The bound (16.7) follows from a general estimate [38, Thm 3] (see also [2, the bound (A.2')] ,[21, Thm 8.1], [39, Thm 3]).

III. Finally, consider the point $\omega = 0$ which is the most singular. This is an isolated pole of a finite degree by Lemma 13.3, and hence the Laurent expansion holds,

$$M^{-1}(i\omega) = \sum_{k=0}^{n} L_k\omega^{-k-1} + h(\omega), \quad |\omega| < \varepsilon_0, \tag{16.8}$$

where L_k are 6×6 complex matrices, $\varepsilon_0 > 0$, and $h(\omega)$ is an analytic matrix-valued function for complex ω with $|\omega| < \varepsilon_0$.

17. Time Decay of the Vector Components

Here we prove the decay (6.19) for the components $Q(t)$ and $P(t)$.

Lemma 17.1. *Let $X_0 \in \mathcal{Z}_{v,\beta}$. Then $Q(t)$, $P(t)$ are continuous and the following bound holds:*

$$|Q(t)| + |P(t)| \le \frac{C(\rho, \overline{v}, d_0)}{(1+|t|)^{3/2}}, \quad t \ge 0. \tag{17.1}$$

Proof. Expansions (16.2), (16.4), and (16.8) imply the convergence of the Fourier integral (16.1) in the sense of distributions to a continuous function of $t \ge 0$. Let us prove the decay (17.1). We know that the linearized dynamics admits the secular solutions without decay, see (6.24). The formulas (3.3) give the corresponding components $Q_S(t)$ and $P_S(t)$ of the secular solutions,

$$\begin{pmatrix} Q_S(t) \\ P_S(t) \end{pmatrix} = \sum_1^3 C_j \begin{pmatrix} e_j \\ 0 \end{pmatrix} + \sum_1^3 D_j \left[\begin{pmatrix} e_j \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ \partial_{v_j} p_v \end{pmatrix} \right]. \tag{17.2}$$

We claim that the symplectic orthogonality condition leads to (17.1). Let us split the Fourier integral (16.1) into three terms by using the partition of unity $\zeta_1(\omega)+\zeta_2(\omega)+\zeta_3(\omega) = 1$, $\omega \in \mathbb{R}$:

$$\begin{aligned} \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} &= \frac{1}{2\pi} \int e^{i\omega t} (\zeta_1(\omega) + \zeta_2(\omega) + \zeta_3(\omega)) M^{-1}(i\omega + 0) \begin{pmatrix} Q_0 \\ P_0' \end{pmatrix} d\omega \\ &= I_1(t) + I_2(t) + I_3(t), \end{aligned} \tag{17.3}$$

where the functions $\zeta_k(\omega) \in C^\infty(\mathbb{R})$ are supported by

$$\left.\begin{array}{l} \operatorname{supp} \zeta_1 \subset \{\omega \in \mathbb{R} : \varepsilon_0/2 < |\omega| < \mu + 2\} \\ \operatorname{supp} \zeta_2 \subset \{\omega \in \mathbb{R} : |\omega| > \mu + 1\} \\ \operatorname{supp} \zeta_3 \subset \{\omega \in \mathbb{R} : |\omega| < \varepsilon_0\} \end{array}\right| . \tag{17.4}$$

Then

i) The function $I_1(t) \in C^\infty(\mathbb{R})$ decays like $(1+|t|)^{-3/2}$ by the Puiseux expansion (16.2).
ii) The function $I_2(t) \in C[0,\infty)$ decays faster than any power of t due to Proposition 16.2.
iii) Finally, the function $I_3(t)$ generally does not decay if $n \ge 0$ in the Laurent expansion (16.8).

Namely, the contribution of the analytic function $h(\omega)$ is a smooth function of $t \in \mathbb{R}$, and decays faster than any power of t. On the other hand, the contribution of the Laurent series,

$$\begin{pmatrix} Q_L(t) \\ P_L(t) \end{pmatrix} := \frac{1}{2\pi} \int e^{i\omega t} \zeta_3(\omega) \sum_{k=0}^n L_k (\omega - i0)^{-k-1} \begin{pmatrix} Q_0 \\ P_0' \end{pmatrix} d\omega, \quad t \in \mathbb{R}, \tag{17.5}$$

is a polynomial function of $t \in \mathbb{R}$ of a degree $\le n$, modulo smooth functions of $t \in \mathbb{R}$ decaying faster than any power of t. This follows by the Cauchy theorem applied to the integral (17.5) if we change the integral over $\omega \in [-\varepsilon_0/2, \varepsilon_0/2]$, where $\zeta_3(\omega) \equiv 1$, by the integral over the semicircle $e^{i\theta}\varepsilon_0/2$, $\theta \in [\pi, 0]$. Let us note that the formula (17.2) gives an example of polynomial functions arising from (17.5).

We must show that the symplectic orthogonality condition eliminates the polynomial functions. Our main difficulty is that we know nothing about the order n of the pole and about the Laurent coefficients L_k of the matrix $M^{-1}(i\omega)$ at $\omega = 0$. Our crucial observation has the following form:

a) The components (17.2) of the secular solutions form a linear space $\mathcal{L}_S$ of dimension $\dim \mathcal{L}_S = 6$.
b) The polynomial functions in (17.5) belong to a linear space $\mathcal{L}_L$ of dimension $\dim \mathcal{L}_L \le 6$ since $(Q_0, P_0') \in \mathbb{R}^6$.
c) $\mathcal{L}_S \subset \mathcal{L}_L$ since any function (17.2) admits a representation of the form (17.5). The validity of this representation follows from the fact that the secular solutions (6.24) can be reproduced by our calculations with the Laplace transform.

Therefore, we can conclude that

$$\mathcal{L}_L = \mathcal{L}_S. \tag{17.6}$$

Let us show that the secular solutions are forbidden since $X_0 \in \mathcal{Z}_{v,\beta}$, and hence the polynomial terms in (17.5) vanish, which implies the decay (17.1).

First, the constructed vector components $Q(t)$ and $P(t)$ are continuous functions of $t \ge 0$. Hence, the corresponding field components $\Psi(t)$ and $\Pi(t)$ can be constructred by solving the first two equations of (6.23), where A_1 is given by (4.9) with $w = v = v(t_1)$ (see (18.1) below). Proposition 18.1 i) in the next section implies that $X(t) \in C(\mathbb{R}, \mathcal{E})$.

Second, the condition $X_0 \in \mathcal{Z}_{v,\beta}$ Implies that the entire trajectory $X(t)$ lies in $\mathcal{Z}_{v,\beta}$. This follows from the invariance of the space $\mathcal{Z}_{v,\beta}$ under the generator $A_{v,v}$ (cf. Remark 6.6). In other words, $X(t) = \boldsymbol{P}_v X(t)$.

On the other hand, identity (17.6) implies that $X(t)$ can be corrected by a secular solution $X_S(t)$ s.t. the corresponding components $Q_\Delta(t)$ and $P_\Delta(t)$ of the difference $\Delta(t) := X(t) - X_S(t)$ decay at the rate $(1 + |t|)^{-3/2}$. Note that $\boldsymbol{P}_v\Delta(t) = \boldsymbol{P}_v X(t) = X(t)$ since $\boldsymbol{P}_v X_S(t) = 0$.

Further, the difference $\Delta(t) \in C(\mathbb{R}, \mathcal{E})$ is a solution to the linearized equation (6.23). Hence, the corresponding norms of the field components of $\Delta(t)$ also decay like $(1 + |t|)^{-3/2}$ that follows from Proposition 18.1 ii). Therefore, $\|\Delta(t)\|_{-\beta} \le C(1 + |t|)^{-3/2}$, hence the components $Q(t)$ and $P(t)$, of $X(t) = \boldsymbol{P}_v\Delta(t)$ also decay like $(1 + |t|)^{-3/2}$. □

18. Time Decay of Fields

In Sects. 12–17 we denote by $X(t)$ the solution to the linearized equation (6.23) with a fixed initial condition X_0. Here we consider an arbitrary solution $X(t) = (\Psi(\cdot, t), \Pi(\cdot, t), Q(t), P(t))$ of the linearized equation. We shall prove a proposition which can be applied to the solution $X(t)$ from previous sections as well as to the solution $\Delta(t)$ above. Let us study the field part of the solution, $F(t) = (\Psi(\cdot, t), \Pi(\cdot, t))$, solving the first two equations from the system (6.23). These two equations have the form

$$\dot{F}(t) = \begin{pmatrix} v \cdot \nabla & 1 \\ \Delta - m^2 & v \cdot \nabla \end{pmatrix} F(t) + \begin{pmatrix} 0 \\ Q(t) \cdot \nabla\rho \end{pmatrix}. \tag{18.1}$$

We shall assume that the vector components decay,

$$|Q(t)| \le \frac{C(\rho, \overline{v}, d_0)}{(1+|t|)^{3/2}}, \quad t \ge 0. \tag{18.2}$$

Proposition 6.7 is reduced now to the following assertion.

Proposition 18.1. i) *Let* $Q(t) \in C([0,\infty); \mathbb{R}^3)$*, and* $F_0 \in \mathcal{F}$*. Then Eq.* (18.1) *admits a unique solution* $F(t) \in C([0,\infty); \mathcal{F})$ *with the initial condition* $F(0) = F_0$*.*

ii) *If* $F_0 \in \mathcal{F}_\beta$ *and if the decay* (18.2) *holds, then the corresponding fields also decay uniformly with respect to* v*:*

$$\|F(t)\|_{-\beta} \le \frac{C(\rho, \overline{v}, \tilde{v}, d_0, \|F_0\|_\beta)}{(1+|t|)^{3/2}}, \quad t \ge 0, \tag{18.3}$$

for $|v| \le \tilde{v}$ *with any* $\tilde{v} \in (0,1)$*.*

Proof. Step i) The statement i) follows from the Duhamel representation

$$F(t) = W(t)F_0 + \left[\int_0^t W(t-s)\begin{pmatrix} 0 \\ Q(s)\cdot\nabla\rho \end{pmatrix} ds\right], \qquad t \ge 0, \tag{18.4}$$

where $W(t)$ is the dynamical group of the modified Klein–Gordon equation

$$\dot{F}(t) = \begin{pmatrix} v\cdot\nabla & 1 \\ \Delta - m^2 & v\cdot\nabla \end{pmatrix} F(t). \tag{18.5}$$

The group $W(t)$ can be expressed through the group $W_0(t)$ of the standard Klein–Gordon equation

$$\dot{\Phi}(t) = \begin{pmatrix} 0 & 1 \\ \Delta - m^2 & 0 \end{pmatrix} \Phi(t). \tag{18.6}$$

Namely, the problem (18.6) corresponds to (18.5), when $v = 0$, and it is easy to see that

$$[W(t)F(0)](x) = [W_0(t)F(0)](x + vt), \quad x \in \mathbb{R}^3, \ t \in \mathbb{R}. \tag{18.7}$$

Denote by $W(x-y,t)$ and $W_0(x-y,t)$ the (distribution) integral matrix kernels of the operators $W(t)$ and $W_0(t)$ respectively. Then (18.7) implies that

$$W(x-y,t) = W_0(x-y+vt,t), \quad x, y \in \mathbb{R}^3, \ t \in \mathbb{R}. \tag{18.8}$$

The identity (18.7) implies also the energy conservation law for the group $W(t)$. Namely, for $(\Psi(\cdot,t), \Pi(\cdot,t)) = W(t)F(0)$ we have

$$\int [|\Pi(x,t) - v\cdot\nabla\Psi(x,t)|^2 + |\nabla\Psi(x,t)|^2 + m^2|\Psi(x,t)|^2]dx = \text{const}, \qquad t \in \mathbb{R}.$$

In particular, this gives that

$$\|W(t)F_0\|_{\mathcal{F}} \le C(\overline{v})\|F_0\|_{\mathcal{F}}, \ \ t \in \mathbb{R}. \tag{18.9}$$

This estimate and (18.4) imply the statement i).

Step ii) The statement ii) follows from the Duhamel representation (18.4) and the next lemma.

Lemma 18.2. *For any $\beta > 3/2$, $\overline{v} < 1$ and $F_0 \in \mathcal{F}_\beta$, the following decay holds:*

$$\|W(t)F_0\|_{-\beta} \le \frac{C(\beta,\overline{v})}{(1+t)^{3/2}}\|F_0\|_\beta, \qquad t \ge 0, \tag{18.10}$$

for the dynamical group $W(t)$ corresponding to the modified Klein–Gordon equation (18.5) *with $|v| < \overline{v}$.*

Proof. The lemma can be proved by general methods of Jensen and Kato [21] relying on the fundamental Agmon estimate [2, the bound (A.2')]. We give an independent short proof for the convenience of the reader.

Step i) The matrix kernel $W_0(x-y,t)$ of the group $W_0(t)$ can be written explicitly since the solution to (18.6) has the form (see [22])

$$\Psi(\cdot,t) = \left[\frac{\partial}{\partial t}R(t) * \Psi_0 + R(t) * \Pi_0\right], \quad \Pi(\cdot,t) = \dot{\Psi}(\cdot,t). \tag{18.11}$$

Here $R(t) = R(\cdot,t) = R_0(\cdot,t) + R_m(\cdot,t)$, and

$$R_0(x,t) = \frac{\delta(t-|x|)}{4\pi t}, \quad R_m(x,t) = -\frac{m}{4\pi}\frac{J_1^+(m\sqrt{t^2-|x|^2})}{\sqrt{t^2-|x|^2}},$$

where

$$J_1^+(m\sqrt{s}) := \begin{cases} J_1(m\sqrt{s}), & s \ge 0 \\ 0 & s < 0, \end{cases}$$

and J_1 is the Bessel function of order 1. From here and well known asymptotics of the Bessel function it follows that

$$\begin{aligned} W_0(z,t) &= 0, \qquad |z| > t, \\ |\partial_z^\alpha W_0(z,t)| &\le C(\delta)(1+t)^{-3/2}, \quad |z| \le (1-\delta)t, \end{aligned}$$

for $t \ge 1$, $|\alpha| \le 1$ and any $\delta > 0$. From the last two relations and (18.8) it follows that, for any $\overline{v} < 1$ and $\varepsilon = \frac{1-\overline{v}}{2}$, the following estimates hold for the matrix kernel $W(z,t)$ of the group $W(t)$:

$$W(z,t) = 0, \qquad |z| > (1+\overline{v})t, \tag{18.12}$$

$$|\partial_z^\alpha W(z,t)| \le C(\overline{v})(1+t)^{-3/2}, \qquad |z| \le \varepsilon t, \qquad |\alpha| < 1. \tag{18.13}$$

Step ii) Let us fix an arbitrary $t \ge 1$, and split the initial function F_0 in two terms, $F_0 = F'_{0,t} + F''_{0,t}$ such that

$$\|F'_{0,t}\|_\beta + \|F''_{0,t}\|_\beta \le C\|F_0\|_\beta, \qquad t \ge 1, \tag{18.14}$$

and

$$F'_{0,t}(x) = 0, \qquad |x| > \frac{\varepsilon t}{2}, \tag{18.15}$$

$$F''_{0,t}(x) = 0, \qquad |x| < \frac{\varepsilon t}{4}, \tag{18.16}$$

where $\varepsilon > 0$ is defined in (18.13). The estimate for $W(t)F''_{0,t}$ follows by (18.9), (18.16) and (18.14):

$$\|W(t)F''_{0,t}\|_{-\beta} \le \|W(t)F''_{0,t}\|_{\mathcal{F}} \le C\|F''_{0,t}\|_{\mathcal{F}} \le C_1(\varepsilon)\|F''_{0,t}\|_{\beta}(1+t)^{-\beta} \le C_2(\varepsilon)\|F_0\|_{\beta}(1+t)^{-\beta}, \qquad t \ge 1. \tag{18.17}$$

Step iii) It remains to estimate $W(t)F'_{0,t}$. We split the operator $W(t)$, for $t > 1$, in two terms:

$$W(t) = (1-\zeta)W(t) + \zeta W(t),$$

where ζ is the operator of multiplication by the function $\zeta(|x|/t)$ such that $\zeta = \zeta(s) \in C_0^\infty(\mathbb{R})$, $\zeta(s) = 1$ for $|s| < \varepsilon/4$, $\zeta(s) = 0$ for $|s| > \varepsilon/2$. Since

$$|\partial_x^\alpha \zeta(|x|/t)| \le C, \qquad |\alpha| \le 1, \qquad t \ge 1,$$

and $1 - \zeta(|x|/t) = 0$ for $|x| < \varepsilon t/4$, we have, for $t \ge 1$,

$$||(1-\zeta)W(t)F'_{0,t}||_{-\beta} \le C_3(\varepsilon)(1+t)^{-\beta}||(1-\zeta)W(t)F'_{0,t}||_{\mathcal{F}} \le C_4(\varepsilon)(1+t)^{-\beta}||W(t)F'_{0,t}||_{\mathcal{F}}.$$

From here, (18.9) and (18.14) it follows that

$$||(1-\zeta)W(t)F'_{0,t}||_{-\beta} \le C_5(\varepsilon)(1+t)^{-\beta}||F'_{0,t}||_{\mathcal{F}} \le C_6(\varepsilon)(1+t)^{-\beta}||F_0||_{\mathcal{F}}, \qquad t \ge 1. \tag{18.18}$$

Step iv) Thus, in order to complete the proof of Lemma 18.2, it remains to receive a similar estimate for $\zeta W(t)F'_{0,t}$. Let $\chi_{\varepsilon t/2}$ be the characteristic function of the ball $|x| \le \varepsilon t/2$. We will use the same notation for the operator of multiplication by this characteristic function. From (18.15) it follows that

$$\zeta W(t)F'_{0,t} = \zeta W(t)\chi_{\varepsilon t/2}F'_{0,t}.$$

The matrix kernel $W'(x, y, t)$ of the operator $\zeta W(t)\chi_{\varepsilon t/2}$ is equal to

$$W'(x, y, t) = \zeta(|x|/t)W(x-y, t)\chi_{\varepsilon t/2}(y).$$

Since $\zeta(|x|/t) = 0$ for $|x| > \varepsilon t/2$ and $\chi_{\varepsilon t/2}(y) = 0$ for $|y| > \varepsilon t/2$, the estimate (18.13) implies that

$$|\partial_x^\alpha W'(x, y, t)| \le C(\overline{v})(1+t)^{-3/2}, \qquad |\alpha| < 1, \qquad t \ge 1. \tag{18.19}$$

The norm of the operator $\zeta W(t)\chi_{\varepsilon t/2} : \mathcal{F}_\beta \to \mathcal{F}_{-\beta}$ is equivalent to the norm of the operator

$$(1+|x|)^{-\beta}\zeta W(t)\chi_{\varepsilon t/2}(1+|y|)^{-\beta} : \mathcal{F} \to \mathcal{F}.$$

The norm of the later operator does not exceed the sum in α, $|\alpha| \le 1$ of the norms of operators

$$\partial_x^\alpha[(1+|x|)^{-\beta}\zeta W(t)\chi_{\varepsilon t/2}(1+|y|)^{-\beta}] : L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}) \to L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3). \tag{18.20}$$

From (18.19) it follows that operators (18.20) are Hilbert-Schmidt operators since $\beta > 3/2$, and their Hilbert-Schmidt norms do not exceed $C(1+t)^{-3/2}$. Hence

$$||\zeta W(t) F'_{0,t}||_{-\beta} \le C(\overline{v})(1+t)^{-3/2}||F'_{0,t}||_{\beta} \le C_7(\overline{v})(1+t)^{-3/2}||F_{0,t}||_{\beta}, \quad t \ge 1. \tag{18.21}$$

The last estimate above is due to (18.14). Finally, the estimates (18.21), (18.18) and (18.17) imply (18.10). □

A. Appendix: Computing Symplectic Form

Here we justify the formulas (3.5)–(3.7) for the matrix Ω. For $j, l = 1, 2, 3$ it follows from (3.3) and (3.2) that

$$\Omega(\tau_j, \tau_l) = \langle \partial_j \psi_v, \partial_l \pi_v \rangle - \langle \partial_j \pi_v, \partial_l \psi_v \rangle, \tag{A.1}$$

$$\Omega(\tau_{j+3}, \tau_{l+3}) = \langle \partial_{v_j} \psi_v, \partial_{v_l} \pi_v \rangle - \langle \partial_{v_j} \pi_v, \partial_{v_l} \psi_v \rangle, \tag{A.2}$$

and

$$\Omega(\tau_j, \tau_{l+3}) = -\langle \partial_j \psi_v, \partial_{v_l} \pi_v \rangle + \langle \partial_j \pi_v, \partial_{v_l} \psi_v \rangle + e_j \cdot \partial_{v_l} p_v. \tag{A.3}$$

Let us transfer to the Fourier representation. Set

$$\hat{\psi}(k) := (2\pi)^{-3/2} \int e^{ikx} \psi(x) dx. \tag{A.4}$$

It is easy to compute that

$$\hat{\psi}_v(k) = -\frac{\hat{\rho}(k)}{k^2 + m^2 - (kv)^2}, \quad \hat{\pi}_v(k) = i(kv)\hat{\psi}_v(k). \tag{A.5}$$

Further, differentiating, we obtain

$$\partial_{v_j} \hat{\psi}_v = \frac{2(kv)k_j}{k^2 + m^2 - (kv)^2} \hat{\psi}_v, \quad \partial_{v_j} \hat{\pi}_v = ik_j \frac{k^2 + m^2 + (kv)^2}{k^2 + m^2 - (kv)^2} \hat{\psi}_v, \quad j = 1, 2, 3, \tag{A.6}$$

and

$$\partial_{v_j} p_v := \frac{e_j}{\sqrt{1 - v^2}} + \frac{v_j}{(1 - v^2)^{3/2}} v, \quad j = 1, 2, 3.$$

Then for $j, l = 1, 2, 3$ we see from (A.1) by the Parseval identity that

$$\Omega(\tau_j, \tau_l) = -2i \int k_j k_l (kv) |\hat{\psi}_v|^2 \, dk = 0, \tag{A.7}$$

since the integrand is odd in k. Similarly, by (A.2),

$$\Omega(\tau_{j+3}, \tau_{l+3}) = -4i \int \frac{k_j k_l (kv)(k^2 + m^2 + (kv)^2)|\hat{\psi}_v|^2}{(k^2 + m^2 - (kv)^2)^2} = 0. \tag{A.8}$$

Finally, by (A.3),

$$\Omega(\tau_j, \tau_{l+3}) = \int dk|\hat{\psi}_v|^2 k_j k_l \left[\frac{k^2+m^2+(kv)^2}{k^2+m^2-(kv)^2} + \frac{2(kv)^2}{k^2+m^2-(kv)^2}\right] + e_j \cdot \partial_{v_l} p_v$$
$$= \int dk|\hat{\psi}_v|^2 k_j k_l \frac{k^2+m^2+3(kv)^2}{k^2+m^2-(kv)^2} + e_j \cdot \left(\frac{e_l}{\sqrt{1-v^2}} + \frac{v_l v}{(1-v^2)^{3/2}}\right). \tag{A.9}$$

This completes the proof of (3.5)–(3.7).

B. Appendix: Positivity of the Matrix *F*

Here we justify the inequality used above in the proof of Lemma 15.2:

$$\frac{1}{m^2+k^2-(|v|k_1+\omega)^2} + \frac{1}{m^2+k^2-(|v|k_1-\omega)^2} - \frac{2}{m^2+k^2-(|v|k_1)^2} > 0$$

under the conditions (15.7):

$$|v| < 1, \quad 0 < |\omega| \le \mu = m\sqrt{1-v^2}. \tag{B.1}$$

Let us denote $M^2 := m^2 + k^2$, $r_\pm := |v|k_1 \pm \omega$, and $r := |v|k_1$. Then the inequality reads, after cancellation by $2M$,

$$\frac{1}{M-r_+} + \frac{1}{M-r_-} - \frac{2}{M-r} + \frac{1}{M+r_+} + \frac{1}{M+r_-} - \frac{2}{M+r} > 0. \tag{B.2}$$

The sum of the first three terms in (B.2) can be written as

$$\frac{1}{N-\omega} + \frac{1}{N+\omega} - \frac{2}{N} = \frac{2\omega^2}{(N+\omega)(N-\omega)N}, \tag{B.3}$$

where $N := M - r$. It is easy to check that $N \pm \omega \ge 0$ and $N > 0$ under conditions (B.1). Hence, the sum (B.3) is positive (or positive infinite). Similarly, the sum of the last three terms in (B.2) also is positive.

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