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Effective Dynamics for a Mechanical Particle Coupled to a Wave Field

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Abstract: We consider a particle coupled to a scalar wave field and subject to the slowly varying potential $V(\varepsilon q)$ with small ε. We prove that if the initial state is close, order ε^2, to a soliton (=dressed particle), then the solution stays forever close to the soliton manifold. This estimate implies that over a time span of order ε^{-1} the radiation losses are negligible and that the motion of the particle is governed by the effective Hamiltonian $H_{\text{eff}}(q, P) = E(P) + V(\varepsilon q)$ with energy-momentum relation $E(P)$.

1. Introduction

When a particle interacts with a field its mechanical properties are renormalized, e.g. the particle acquires an effective mass. In the context of charges interacting with the Maxwell field such an effective energy-momentum relation is discussed at length already in the classical work of Abraham [1] and Lorentz [16] with the implicit understanding that this relation determines how the particle responds to external forces. Kramers [14] emphasizes the distinction between bare (appearing in the equation of motion) and physical (observable by outside means) parameters of a charge. His vision has been implemented through the renormalization of quantum electrodynamics. To our knowledge, even on the classical level, it has never been properly settled in which sense and on what scale the dynamics governed by the effective energy-momentum relation is an approximation to the true solution of the coupled equations of motion. To gain some understanding we study here the arguably simplest model, namely a single particle interacting with a scalar wave field.

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Our second source of interest lies in the, by now, long list of examples we have for the emergence of an effective dynamics, to mention only the Boltzmann and Vlasov equation, hydrodynamics [21], homogenization in periodic and random environments [3, 8], interface and vortex dynamics in Ginzburg–Landau theories [11], quantum systems weakly coupled to a heat bath [6], and a quantum particle in the semiclassical limit [10, 18, 22]. Their common thread is a separation of space-time scales together with some sort of local stationarity in such a way that the slowly varying dynamical variables are governed by an effective dynamics. However, the detailed mechanisms differ notably from case to case. Here we add a novel item to the list. It is not covered by the mathematical techniques developed so far.

We consider a scalar wave field $\phi(x)$, in three-dimensional space, coupled to a particle with position q, momentum p, governed by

$$\begin{aligned} &\dot{\phi}(x,t) = \pi(x,t), \qquad &&\dot{\pi}(x,t) = \Delta\phi(x,t) - \rho(x-q(t)), \\ &\dot{q}(t) = p(t)/(1+p^2(t))^{1/2}, \quad &&\dot{p}(t) = \int d^3x\, \phi(x,t)\, \nabla\rho(x-q(t)). \end{aligned} \tag{1.1}$$

This is a Hamiltonian system with the Hamiltonian functional

$$\begin{aligned} \mathcal{H}_0(\phi,\pi,q,p) = (1+p^2)^{1/2} + \frac{1}{2}\int d^3x \Big(|\pi(x)|^2 + |\nabla\phi(x)|^2\Big) \\ + \int d^3x \phi(x)\rho(x-q). \end{aligned} \tag{1.2}$$

We have set the mechanical mass of the particle and the speed of wave propagation equal to one. In spirit the interaction term is simply $\phi(q)$. This would result however in an energy that is not bounded from below. Therefore we smoothen out the coupling by the function $\rho(x)$. In analogy to the Maxwell–Lorentz equations we call $\rho(x)$ the "charge distribution". We assume $\rho(x)$ to belong to the Sobolev space H^1, radial, and compactly supported, i.e.,

$$\rho, \nabla\rho \in L^2(\mathbb{R}^3), \quad \rho(x) = \rho_r(|x|), \quad \rho(x) = 0 \quad \text{for} \quad |x| \geq R_\rho. \tag{C}$$

The system (1.1) has solutions traveling with constant velocity $v, |v| < 1$. They are given by

$$S_v(t) = (\phi_v(x-q-vt), \pi_v(x-q-vt), q+vt, p_v), \quad p_v = v/\sqrt{1-v^2}, \tag{1.3}$$

with

$$\phi_v(x) = -\int \frac{\rho(y)d^3y}{4\pi((1-v^2)(y-x)^2 + (v\cdot(y-x))^2)^{1/2}}, \quad \pi_v(x) = -v\cdot\nabla\phi_v(x). \tag{1.4}$$

To have a short name we call $S_v(t)$ the *soliton* with velocity v centered at $q(t) = q + vt$. We define the normalized energy of a soliton as

$$E_s(v) = \mathcal{H}_0(S_v) - \mathcal{H}_0(S_0), \tag{1.5}$$

$S_v = S_v(0)$, which, using the rotational invariance of ρ, is given by

$$E_s(v) = (1-v^2)^{-1/2} - 1 + 3m_e\left[\frac{2-v^2}{2(1-v^2)} - \frac{1}{2|v|}\log\frac{1+|v|}{1-|v|}\right]. \tag{1.6}$$

Here $3m_e = -\langle\rho, \Delta^{-1}\rho\rangle$, with $\langle\cdot,\cdot\rangle$ the scalar product in $L^2(\mathbb{R}^3)$; we have $m_e < \infty$ by assumption (C). Since the system (1.1) is invariant under spatial translations, the total momentum,

$$\mathcal{P}(\phi, \pi, q, p) = p - \int d^3x\, \pi(x)\, \nabla\phi(x), \tag{1.7}$$

is conserved. Inserting S_v, the total momentum of a soliton is given by

$$P_s(v) = \mathcal{P}(S_v) = v(1-v^2)^{-1/2} + 3m_e v \left[\frac{1}{2v^2(1-v^2)} - \frac{1}{4|v|^3}\log\frac{1+|v|}{1-|v|}\right]. \tag{1.8}$$

The map $v \mapsto P_s(v)$ is invertible from $\mathcal{V} = \{v \in \mathbb{R}^3 : |v| < 1\}$ onto $\mathbb{R}^3$ with the inverse $v_s(P)$; see [12]. Therefore we obtain the *effective energy-momentum relation*

$$E(P) = E_s(v_s(P)). \tag{1.9}$$

Then $E(P)$ is radial. In the nonrelativistic limit (v small) we have

$$E_s(v) \cong \frac{1}{2}(1+m_e)v^2 \quad \text{and} \quad P_s(v) \cong (1+m_e)v \quad \text{for} \quad |v| \ll 1. \tag{1.10}$$

Thus m_e is the additional mass acquired by the particle through the coupling to the field. For large $|P|$ we have the relativistic dependence $E(P) \cong |P|$.

Now let us assume that, at some time t, we have the soliton $S_v(t)$ centered at $q(t)$, $v = \dot{q}(t)$, and that an external force is acting on the particle. This force changes the velocity to $v' \neq v$ and $S_v(t)$ is no longer a solution to the system (1.1). However, if the force is small, so is the difference $v' - v$ and, if the force is slowly varying, the wave field has enough time to reestablish a soliton with new velocity v'. In fact this happens essentially with the speed of wave propagation (one in our case). Geometrically in phase space, we have the 6-dimensional manifold $\mathcal{S}$ of solitons labeled by their center q and velocity v. For zero external force each point in this manifold moves on an orbit $t \mapsto (q + vt, v)$. Under a weak, slowly varying force, the true solution should remain close to the soliton manifold thereby inducing on it an effectively 6-dimensional motion.

With this picture in mind, we add to $\mathcal{H}_0$ in (1.2) the slowly varying potential $V(\varepsilon q)$, $\varepsilon \ll 1$,

$$\mathcal{H}_\varepsilon(\phi, \pi, q, p) = (1+p^2)^{1/2} + V(\varepsilon q) + \frac{1}{2}\int d^3x\Big(|\pi(x)|^2 + |\nabla\phi(x)|^2\Big) + \int d^3x\, \phi(x)\rho(x-q). \tag{1.11}$$

For the potential V we require

$$V \in C^2(\mathbb{R}^3), \quad \inf_{q\in\mathbb{R}^3} V(q) > -\infty, \tag{P}$$

and

$$\sup_{q\in\mathbb{R}^3}\Big(|\nabla V(q)| + |\nabla\nabla V(q)|\Big) < \infty. \tag{U}$$

We remark that, using the conservation of energy, condition (U) can be replaced by

$$V(q) \to \infty \quad \text{as} \quad |q| \to \infty, \tag{U'}$$

i.e., by the assumption that V be confining. In the sequel we study the Hamiltonian dynamics generated by (1.11),

$$\begin{aligned}&\dot\phi(x,t)=\pi(x,t), \qquad \dot\pi(x,t)=\Delta\phi(x,t)-\rho(x-q(t)),\\ &\dot q(t)=p(t)/(1+p^2(t))^{1/2},\ \dot p(t)=-\varepsilon\nabla V(\varepsilon q(t))+\int d^3x\,\phi(x,t)\,\nabla\rho(x-q(t)).\end{aligned} \tag{1.12}$$

The derivatives in (1.12) and below are understood in the sense of distributions. We consider the Cauchy problem for the system (1.12) with initial conditions

$$(\phi(x,0),\pi(x,0),q(0),p(0))=(\phi^0(x),\pi^0(x),q^0,p^0). \tag{1.13}$$

Under our assumptions, the global solution to the Cauchy problem (1.12), (1.13) exists and is unique for initial data with finite energy. The solution depends on ε through the potential and possibly also through the initial conditions. In order as not to overburden our notation, we will mostly suppress this dependence.

We assume the initial state to be close to a soliton. Since the force is slowly varying, near the particle such a wave field should persist. Indeed, we prove that

$$\|(\phi(q(t)+x,t),\pi(q(t)+x,t))-(\phi_{v(t)}(x),\pi_{v(t)}(x))\|_R\le C_R\varepsilon, \quad \forall R>0, \tag{1.14}$$

uniformly in $t\in\mathbb{R}$ (with the norm $\|\cdot\|_R$ being defined by the field energy in a ball of radius R), provided a smallness condition on ρ is satisfied. Presumably, this condition is an artifact of our method.

In (1.12) the external force is $\mathcal{O}(\varepsilon)$. So is the self-force, since according to (1.14) the field ϕ deviates from the soliton only by $\mathcal{O}(\varepsilon)$. Then $\ddot q$ is of order ε, whereas $\dot q$ is of order 1. The effective energy-momentum relation should be visible on a time scale $\mathcal{O}(1)$. Therefore we define the comparison dynamics through the effective Hamiltonian

$$H_{\text{eff}}(Q,P)=E(P)+V(\varepsilon Q)$$

with the corresponding equations of motion,

$$\dot Q(t)=\nabla E(P(t)), \qquad \dot P(t)=-\varepsilon\nabla V(\varepsilon Q(t)), \tag{1.15}$$

suppressing again the ε-dependence of $(Q(t),P(t))$. Since the energy-momentum relation $E(P)$ depends on the charge distribution only through m_e, the effective dynamics is a structure independent property of the coupled system particle+field in the sense of the Kramers [14].

The particle loses energy through radiation, which is proportional to $\ddot q^2$ and thus $\mathcal{O}(\varepsilon^2)$. Therefore the comparison dynamics should be a valid approximation over a time scale ε^{-1}, i.e., over *any* time interval of duration $\varepsilon^{-1}\tau$. At time t_0 the comparison dynamics is adjusted to the true solution through the initial conditions

$$Q(t_0)=q(t_0), \qquad P(t_0)=P_s(\dot q(t_0)). \tag{1.16}$$

Let $(Q(t),P(t))$ be the solution to (1.15) with these initial values. We then establish that, for $|t-t_0|=\mathcal{O}(\varepsilon^{-1})$,

$$|q(t)-Q(t)|=\mathcal{O}(1), \quad |\dot q(t)-\dot Q(t)|=\mathcal{O}(\varepsilon), \quad |\ddot q(t)-\ddot Q(t)|=\mathcal{O}(\varepsilon^2) \tag{1.17}$$

uniformly in t_0. This is our main result.

In the proof, we stick for a while to the traditional route. One solves the inhomogeneous wave equation and inserts the solution into the self-force. Thereby the force on the particle depends on its past history, but not on the field. If one expands this force at $q(t)$ up to second order, one recovers the term missing in the full energy-momentum relation. To justify such a procedure mathematically we have to know *a priori* that

$$|\ddot{q}(t)| \sim \varepsilon \qquad \text{and} \qquad |\dddot{q}(t)| \sim \varepsilon^2 \tag{1.18}$$

uniformly in t, which requires an estimate of the field difference (1.14) and a similar one to handle $\dddot{q}\,(t)$. Our experience from the past is confirmed, namely a direct analysis of the exact delay equation for $q(t)$ is hopeless. To make progress one has to switch back and forth between particle and field.

2. Main Results

To formulate our results precisely, we need some definitions. We introduce the phase space suitable for the Cauchy problem corresponding to (1.12) and (1.13).

Let L^2 be the real Hilbert space $L^2(\mathbb{R}^3)$ with norm $\|\cdot\|$, and let $\dot{H}^1$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with norm $\|\phi(x)\| = \|\nabla\phi(x)\|$. Equivalently, using Sobolev's embedding theorem, $\dot{H}^1 = \{\phi(x) \in L^6(\mathbb{R}^3) : |\nabla\phi(x)| \in L^2\}$; see [15]. Let $\|\phi\|_R$ denote the norm in $L^2(B_R)$ for $R > 0$, where $B_R = \{x \in \mathbb{R}^3 : |x| \le R\}$. Then the seminorms $\|\phi\|_R = \|\nabla\phi\|_R$ are continuous on $\dot{H}^1$.

Definition 2.1. i) *The phase space $\mathcal{E}$ is the Hilbert space $\dot{H}^1 \oplus L^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ of states $Y = (\phi, \pi, q, p)$ with finite norm*

$$\|\, Y\|_{\mathcal{E}} = \|\phi\| + \|\pi\| + |q| + |p|.$$

ii) *$\mathcal{E}_F$ is the space $\mathcal{E}$ endowed with the Fréchet topology defined by the local energy seminorms*

$$\|Y\|_R = \|\phi\|_R + \|\pi\|_R + |q| + |p|, \;\; \forall R > 0.$$

iii) *$\mathcal{F}$ is the Hilbert space $\dot{H}^1 \oplus L^2$ of the fields $\Phi = (\phi, \pi)$ with finite norm*

$$\|\,\Phi\|_{\mathcal{F}} = \|\phi\| + \|\pi\|.$$

iv) *$\mathcal{F}_F$ is the space $\mathcal{F}$ endowed with the Fréchet topology defined by the local energy seminorms*

$$\|\Phi\|_R = \|\phi\|_R + \|\pi\|_R, \;\; \forall R > 0.$$

A point in phase space is referred to as state. We write the Cauchy problem (1.12), (1.13) in the form

$$\dot{Y}(t) = \mathbf{F}(Y(t)), \quad t \in \mathbb{R}, \quad Y(0) = Y^0, \tag{2.1}$$

where $Y(t) = (\phi(t), \pi(t), q(t), p(t))$ and $Y^0 = (\phi^0, \pi^0, q^0, p^0)$. As already mentioned, we mostly suppress the ε-dependence of the solutions, of the vector field $\mathbf{F}$, and of the initial conditions.

The following lemma is proved analogously to the corresponding result in [13].

Lemma 2.2. *Let (C), (P), and (U), resp. (U'), hold. Then for every $Y^0 \in \mathcal{E}$, $|\varepsilon| \le 1$, the Cauchy problem (2.1) has a unique solution $Y \in C(\mathbb{R}, \mathcal{E})$ with speed bounded as*

$$\sup_{t\in\mathbb{R}} |\dot q(t)| \le \overline{v} < 1. \tag{2.2}$$

The bound $\overline{v} = \overline{v}(Y^0)$ is uniform in $|\varepsilon| \le 1$ and for initial values Y^0 in bounded subsets of $\mathcal{E}$.

If the effective dynamics is approximately valid, then the field should be close to the soliton centered at $q(t)$ with velocity $v(t) = \dot q(t)$. We therefore consider the difference

$$Z(x,t) = \Phi(x,t) - \Phi_*(x,t), \tag{2.3}$$

where

$$\Phi(x,t) = (\phi(x,t), \pi(x,t)), \quad \Phi_*(x,t) = \Phi_{v(t)}(x - q(t))$$

and $\Phi_v(x) = (\phi_v(x), \pi_v(x))$ is the field part of the soliton. Defining $\overline{\rho}(x) = (0, \rho(x))$ and $A(\phi,\pi) = (\pi, \Delta\phi)$, it follows that Φ and Z satisfy the equations of motion

$$\dot\Phi(x,t) = A\Phi(x,t) - \overline{\rho}(x - q(t)), \tag{2.4}$$

$$\dot Z(x,t) = AZ(x,t) - B(x,t), B(x,t) = \dot p(t)\cdot\nabla_p \Phi_{v(t)}(x - q(t)). \tag{2.5}$$

Here, according to the chain rule,

$$\nabla_p \Phi_v = \nabla_v \Phi_v \, dv(p), \tag{2.6}$$

where $dv(p)$ is the differential of the map $p \mapsto v(p) = p/\sqrt{1+p^2}$. In Cartesian coordinates, $dv(p)$ is just the Jacobi matrix $\partial v_i/\partial p_j$.

Theorem 2.3. *Let the conditions of Lemma 2.2 hold and let $\|\rho\|$ be sufficiently small, $\|\rho\| \le \delta(\overline{v}, R_\rho)$. Then for every $R > 0$ there exists C_R such that*

$$\sup_{t\in\mathbb{R}} \|Z(\cdot + q(t), t)\|_R \le C_R(\|Z(0)\|_{\mathcal{F}} + \varepsilon). \tag{2.7}$$

For the unperturbed, $\varepsilon = 0$, system our theorem states that the distance between the true solution and the soliton manifold

$$\mathcal{S} = \{(\phi_v(x-q), \pi_v(x-q), q, p_v) : \; q \in \mathbb{R}^3, \; v \in \mathcal{V}\} \tag{2.8}$$

remains bounded in time. This property is called orbital stability, which has been established for the system (1.1) in [12] and for related equations in [7, 2] using the Liapunov method in combination with energy and momentum conservation. For $\varepsilon > 0$ such an argument breaks down, since the Hamiltonian vector field is no longer parallel to $\mathcal{S}$. To have a stability result as (2.7) we therefore need to exploit that through radiation damping the solution is "pushed" towards $\mathcal{S}$. In other words, through the free wave equation a small deviation from the soliton is transported to infinity, which also shows that we are not allowed to replace the local energy norm in (2.7) by the global one. An adequate mathematical argument is provided by the nonautonomous integral equation method [4, 5, 19, 20], which has been used to prove the convergence to the soliton manifold in the context of the nonlinear Schrödinger equation.

If we assume that initially $\|Z(0)\|_{\mathcal{F}} \le C\varepsilon$, then according to (2.7) the solution remains $\mathcal{O}(\varepsilon)$ close to $\mathcal{S}$ for all times. Thus it remains to characterize the motion along $\mathcal{S}$ as given by the particle trajectory $q(t)$. To obtain its approximate equation of motion we have to estimate the self-force. By Theorem 2.3 it is of $\mathcal{O}(\varepsilon)$. To control the error, $\mathcal{O}(\varepsilon^2)$, the solution has to be slowly varying in time with outgoing fields, which we formalize through the notion of an *adiabatic family* of solutions $Y_\varepsilon(t) = (\phi_\varepsilon(t), \pi_\varepsilon(t), q_\varepsilon(t), p_\varepsilon(t))$.

We denote by $U(t)$ the dynamical group on $\mathcal{F}$ generated by the free wave equation and set

$$\Phi^0 = (\phi_\varepsilon(0), \pi_\varepsilon(0)), \qquad (\phi^0_\varepsilon(\cdot,t), \pi^0_\varepsilon(\cdot,t)) = U(t)\Phi^0. \tag{2.9}$$

Definition 2.4. *A family of solutions $Y_\varepsilon(t) \in C(\mathbb{R}, \mathcal{E})$, $0 < \varepsilon \leq 1$, to the system (1.12) is called adiabatic, if there exist constants $a, T_0 > 0$, and $\overline{v} < 1$, such that the following bounds hold:*

$$\sup_{t\in\mathbb{R}} |\dot{q}_\varepsilon(t)| \leq \overline{v}, \tag{2.10}$$

$$\sup_{t\in\mathbb{R}} |\ddot{q}_\varepsilon(t)| \leq a\varepsilon, \tag{2.11}$$

$$\sup_{t\in\mathbb{R}} |\dddot{q}_\varepsilon(t)| \leq a\varepsilon^2, \tag{2.12}$$

$$| < \phi^0_\varepsilon(x,t), \nabla\rho(x-q) > | \leq a\varepsilon^2 \quad \textit{for} \quad |q| < |t| - T_0. \tag{2.13}$$

This definition is time-invariant, i.e., a family of solutions $Y_\varepsilon(t+\theta)$ is adiabatic for any $\theta \in \mathbb{R}$ if it is for some θ.

Our main result is the following

Theorem 2.5. *Let the assumptions of Theorem 2.3 hold and let $Y_\varepsilon(t) \in C(\mathbb{R}, \mathcal{E})$ be an adiabatic family of solutions to (1.12). Let $(Q(t), P(t))$ be the comparison dynamics (1.15) with initial values (1.16). Then for any $\tau > 0$ there exists $C = C(\tau)$ such that for $|t - t_0| \leq \varepsilon^{-1}\tau$,*

$$|q(t) - Q(t)| \leq C, \quad |\dot{q}(t) - \dot{Q}(t)| \leq C\varepsilon, \quad |\ddot{q}(t) - \ddot{Q}(t)| \leq C\varepsilon^2. \tag{2.14}$$

The constant $C(\tau)$ can be chosen independently of t_0.

Of course, we still need a criterion for initial states, that ensures the corresponding family of solution trajectories is adiabatic. The following theorem provides sufficient conditions, which in particular show that any initial soliton $(\phi_v(x-q^0), \pi_v(x-q^0), q^0, p_v)$ defines an adiabatic family of solutions and that the set of adiabatic families of solutions is nonempty and open in an appropriate topology.

We set $(\varphi^0(x), \psi^0(x)) = Z^0(x) = Z(x,0)$ with corresponding Fourier transforms $(\hat{\varphi}^0(k), \hat{\psi}^0(k))$, and we let

$$\dot{p}(0) = -\varepsilon\nabla V(\varepsilon q(0)) + \int d^3x\, \phi(x,0)\, \nabla\rho(x - q(0)).$$

Theorem 2.6. *Let there exist $a^0 > 0$ such that for the initial states $Y^0_\varepsilon = Y^0 = (\phi^0, \pi^0, q^0, p^0) \in \mathcal{E}$, $0 < \varepsilon \leq 1$, the following bounds hold:*

$$\|Y^0(x)\|_{\mathcal{E}} \leq a^0, \tag{2.15}$$

$$\|Z^0(x)\|_{\mathcal{F}} \leq a^0\varepsilon, \tag{2.16}$$

$$\|\nabla Z^0(x)\|_{\mathcal{F}} + |\dot{p}(0)| \leq a^0\varepsilon^2, \tag{2.17}$$

$$\int d^3k \left(|k||\hat{\varphi}^0(k)| + |\hat{\psi}^0(k)|\right) |\hat{\rho}(k)| \leq a^0\varepsilon^2, \tag{2.18}$$

$$\int d^3k\, |k| \Big(|k||\nabla[\hat{\varphi}^0(k)\hat{\rho}(k)]| + |\nabla[\hat{\psi}^0(k)\hat{\rho}(k)]|\Big) \leq a^0\varepsilon, \tag{2.19}$$

and let $\|\rho\|$ be sufficiently small, $\|\rho\| \le \delta(a^0, R_\rho)$. Then the family of solutions $Y_\varepsilon(t) \in C(\mathbb{R}, \mathcal{E})$ to the Cauchy problem (2.1) is adiabatic.

Thus Theorem 2.6, in essence, requires that the deviation from the soliton has sufficient smoothness and decay.

Our paper is organized as follows. Theorem 2.3 is proved in Sect. 3, and Theorem 2.6 is established in Sect. 4. In Sect. 5 we compute the self-force, and in Sect. 6 we complete the proof of Theorem 2.5. Section 7 concerns the translation invariant system (1.1). In Appendix A we collect Fourier space computations. Finally, in Appendix B, we list some remarks on the Hamiltonian structure.

3. Stability of the Soliton Manifold

We prove Theorem 2.3 and establish first the required bound for $R = R_\rho$ from (C).

Lemma 3.1. *Under the assumptions of Theorem 2.3, the bound (2.7) holds for $R = R_\rho$,*

$$\|Z(\cdot + q(t), t)\|_{R_\rho} \le C(\|Z(0)\|_{\mathcal{F}} + \varepsilon). \tag{3.1}$$

Proof. Solving Eq. (2.5) by Fourier transform we get the mild solution representation

$$Z(t) = U(t)Z(0) - \int_0^t U(t-s)[\dot{p}(s) \cdot \nabla_p \Phi_{v(s)}(\cdot - q(s))]\, ds, \tag{3.2}$$

with $U(t)$ being the group generated by the free wave equation in $\dot{H}^1 \oplus L^2$. By conservation of energy for the wave equation

$$\|[U(t)Z(0)](\cdot + q(t))\|_{R_\rho} \le \|[U(t)Z(0)](\cdot + q(t))\|_{\mathcal{F}} = \|Z(0)\|_{\mathcal{F}}. \tag{3.3}$$

We denote by $\varphi(x,t) = \phi(x,t) - \phi_{v(t)}(x - q(t))$ the first component of $Z(x,t)$ and observe that $\langle \phi_v(x), \nabla\rho(x)\rangle = 0$ for $|v| < 1$ because the soliton (1.3) is a solution to (1.1). Then (1.12) implies

$$\dot{p}(t) = -\varepsilon \nabla V(\varepsilon q(t)) + \langle \varphi(x + q(t), t), \nabla\rho(x)\rangle. \tag{3.4}$$

Thus with assumption (U) we obtain,

$$|\dot{p}(t)| \le C\Big(\varepsilon + \|Z(\cdot + q(t), t)\|_{R_\rho} \|\rho\|\Big). \tag{3.5}$$

We further introduce $\overline{\pi}_v = \nabla_p \pi_v$, $\overline{\phi}_v = \nabla_p \phi_v$, $S_{t-s}(x) = \{y :\ |y - x| = t - s\}$, and

$$(\overline{\phi}(\cdot, t, s), \overline{\pi}(\cdot, t, s)) = U(t-s)[\nabla_p \Phi_{v(s)}(\cdot - q(s))]. \tag{3.6}$$

Then Kirchhoff's formula for $U(t-s)$ implies the representation

$$\begin{aligned} \nabla\overline{\phi}(x,t,s) = &\sum_{|\alpha|\le 1} (t-s)^{|\alpha|-2} \int_{S_{t-s}(x)} d^2y\, a_\alpha(x-y) \partial_y^\alpha \overline{\pi}_{v(s)}(y - q(s)) \\ &+ \sum_{|\alpha|\le 2} (t-s)^{|\alpha|-3} \int_{S_{t-s}(x)} d^2y\, b_\alpha(x-y) \partial_y^\alpha \overline{\phi}_{v(s)}(y - q(s)), \end{aligned} \tag{3.7}$$

and a similar representation for $\overline{\pi}(x,t,s)$. The coefficients $a_\alpha(\cdot)$, $b_\alpha(\cdot)$ are bounded and sums are taken over multiindices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with integers $\alpha_j \geq 0$. Therefore $\nabla\overline{\phi}(x+q(t),t,s)$ and $\overline{\pi}(x+q(t),t,s)$ can be represented as integrals of type (3.7) over the shifted sphere $S_{t-s}(x+q(t))$. If $|x| \leq R_\rho$, we have on this sphere

$$\begin{aligned} |y-q(s)| &= |(y-x-q(t)) + (x+q(t)-q(s))| \\ &\geq (t-s) - |x| - \overline{v}(t-s) \geq (1-\overline{v})(t-s) - R_\rho \end{aligned} \tag{3.8}$$

by the bound (2.2) on $\dot{q}(t)$. On the other hand, the integral representation (1.4) yields by Cauchy–Schwarz

$$\begin{aligned} \sup_{|v|\leq\overline{v}} \sup_{|x|\geq 2R_\rho} \Big[&|x||\overline{\phi}_v(x)| + |x|^2(|\nabla\overline{\phi}_v(x)| + |\overline{\pi}_v(x)|) + \\ &|x|^3(|\nabla\nabla\overline{\phi}_v(x)| + |\nabla\overline{\pi}_v(x)|)\Big] \leq C(\overline{v}, R_\rho)\|\rho\| < \infty. \end{aligned} \tag{3.9}$$

Inserting (3.9) and (3.8) in Kirchhoff's formula for $\nabla\overline{\phi}(x+q(t),t,s)$, we obtain the pointwise bound

$$\begin{aligned} |\nabla\overline{\phi}(x+q(t),t,s)| &\leq \sum_{|\alpha|\leq 1} (t-s)^{|\alpha|-2} \frac{C_1(\overline{v},R_\rho)\|\rho\|(t-s)^2}{(1+|t-s|)^{|\alpha|+2}} \\ &\quad + \sum_{|\alpha|\leq 2} (t-s)^{|\alpha|-3} \frac{C_1(\overline{v},R_\rho)\|\rho\|(t-s)^2}{(1+|t-s|)^{|\alpha|+1}} \\ &\leq \frac{C_2(\overline{v},R_\rho)\|\rho\|}{1+(t-s)^2} \end{aligned} \tag{3.10}$$

for $|x| \leq R_\rho$ and provided $t-s \geq 3R_\rho/(1-\overline{v})$. Therefore (3.10) implies for large $t-s$, together with a similar bound for $\overline{\pi}(x+q(t),t,s)$, the integral estimate

$$\|(\overline{\phi}(x+q(t),t,s), \overline{\pi}(x+q(t),t,s)\|_{R_\rho} \leq \frac{C_3(\overline{v},R_\rho)\|\rho\|}{1+(t-s)^2}. \tag{3.11}$$

On the other hand, for bounded $t-s$ this integral estimate follows directly from (3.6) by energy conservation for the map $U(t-s)$, since $\|\nabla_p\Phi_v\|_{\mathcal{F}} \leq C(\overline{v},R_\rho)\|\rho\|$ by (C). Finally, (3.5) and (3.11) imply

$$\begin{aligned} &\|\dot{p}(s)\cdot(\overline{\phi}(x+q(t),t,s), \overline{\pi}(x+q(t),t,s)\|_{R_\rho} \\ &\leq C_4(\overline{v},R_\rho)\|\rho\| \frac{\varepsilon + \|Z(\cdot+q(s),s)\|_{R_\rho}\|\rho\|}{1+(t-s)^2}, \end{aligned} \tag{3.12}$$

and combining (3.2) and (3.3) we arrive at

$$\begin{aligned} &\|Z(\cdot+q(t),t)\|_{R_\rho} \\ &\leq \|Z(0)\|_{\mathcal{F}} + C_4(\overline{v},R_\rho)\|\rho\| \int_0^t \frac{\varepsilon + \|Z(\cdot+q(s),s)\|_{R_\rho}\|\rho\|}{1+(t-s)^2}\,ds, \quad t\geq 0. \end{aligned} \tag{3.13}$$

Thus, denoting $M(t) = \max_{0\leq s\leq t}\|Z(q(s)+x,s)\|_{R_\rho}$, we have

$$M(t) \leq \|Z(0)\|_{\mathcal{F}} + C_5(\overline{v},R_\rho)\|\rho\|(\varepsilon + \|\rho\|M(t)).$$

We choose now $\|\rho\|$ so small that $C_5(\overline{v},R_\rho)\|\rho\|^2 < 1$. Then (3.1) follows for $t \geq 0$. $\square$

We claim that the bound (3.1) implies (2.7) for any $R > 0$. Indeed, (3.11)-(3.13) hold with the norm $\|\cdot\|_R$ instead of $\|\cdot\|_{R_\rho}$ on the *left* hand sides and with $C_i(\overline{v}, \rho, R)$ instead of $C_i(\overline{v}, \rho)$ on the right hand sides. Then (3.13) with this generalization and (3.1) imply (2.7).

4. Adiabatic Solutions

We prove Theorem 2.6. The bound (2.10) is the assertion of Lemma 2.2. Concerning (2.13), we have

$$U(t)\Phi^0 = U(t)\Phi_{v(0)}(\cdot - q^0) + U(t)Z^0. \tag{4.1}$$

Moreover, $U(t)\Phi_{v(0)}(x - q^0) = 0$ for $|x - q^0| < |t| - R_\rho$ by Kirchhoff's formula, since we have the representation

$$\Phi_{v(0)}(x) = -\int_{-\infty}^{0} [U(-s)\overline{\rho}(\cdot - q^0 - v(0)s)](x)\, ds. \tag{4.2}$$

Therefore with the choice $T_0 = 2R_\rho + |q^0|$ (2.13) holds for the first component of $[U(t)\Phi_{v(0)}](x)$. With the choice $T_0 = 0$, (2.18) implies (2.13) for the first component of $U(t)Z^0$, as can be seen in Fourier space representation.

Thus it remains to prove (2.11) and (2.12).

Proposition 4.1. *For small $\|\rho\|$, the following bounds hold:*

$$\sup_{t\in\mathbb{R}} |\dot{v}(t)| \le C(a^0, R_\rho)\, \varepsilon, \tag{4.3}$$

$$\sup_{t\in\mathbb{R}} |\ddot{v}(t)| \le C(a^0, R_\rho)\, \varepsilon^2. \tag{4.4}$$

Proof. The estimate (4.3) follows from (3.5), (3.1), and (2.16). To obtain (4.4), we differentiate (3.4) using (C),

$$\ddot{p}(t) = -\varepsilon^2 v(t) \cdot \nabla\, \nabla V(\varepsilon q(t)) + M(t), \tag{4.5}$$

where $M(t) = \langle L(t)\varphi(x + q(t), t), \nabla\rho(x)\rangle$ and $L(t) = \partial_t + v(t)\cdot\nabla$. Then (U) implies

$$|\ddot{p}(t)| \le C(\varepsilon^2 + |M(t)|). \tag{4.6}$$

Therefore (4.4) will be a consequence of

Lemma 4.2. *We have*

$$\sup_{t\in\mathbb{R}} |M(t)| \le C(a^0, R_\rho)\varepsilon^2 \tag{4.7}$$

for small $\|\rho\|$.

Proof. We extend the method of the previous section. Denoting $\Xi(x,t) = L(t)Z(x,t)$, we have $M(t) = \langle \Xi(x,t), \nabla\rho_*(x-q(t))\rangle$, where $\rho_*(x) = (\rho(x),0)$. To obtain an equation for $\Xi(t)$ we apply the differential operator $L(t)$ to (2.5) in the sense of distributions to find

$$\dot\Xi(x,t) = A\Xi(x,t) - L(t)B(x,t) + \dot v(t)\cdot\nabla Z(x,t). \tag{4.8}$$

Here $\dot v(t)\cdot\nabla Z(\cdot,t)\in C(\mathbb{R},\mathcal{F})$ due to (3.2), (2.17), and (C). Also $L(t)B(\cdot,t)\in C(\mathbb{R},\mathcal{F})$ because

$$L(t)B(x,t) = \ddot p(t)\cdot\nabla_p\Phi_{v(t)}(x-q(t)) + (\dot p(t)\cdot\nabla_p)^2\Phi_{v(t)}(x-q(t)). \tag{4.9}$$

Moreover, assumptions (2.17) and (C) imply $\Xi(\cdot,0)\in\mathcal{F}$, since

$$\Xi(x,t) = A\Phi(x,t) - \overline{\rho}(x-q(t)) + v(t)\cdot\nabla\Phi(x,t) - \dot p(t)\cdot\nabla_p\Phi_{v(t)}(x-q(t)) \tag{4.10}$$

by definition of Z in (2.3) and by (2.5). Therefore, using the Fourier transform to solve the linear nonhomogeneous equation (4.8), we get the following integral representation, similar to (3.2),

$$\Xi(x,t) = U(t)\Xi(\cdot,0) - \int_0^t U(t-s)L(s)B(s)ds + \int_0^t \dot v(s)\cdot U(t-s)\nabla Z(s)\,ds, \tag{4.11}$$

where both integrals converge in $\mathcal{F}$. Hence (C) implies

$$\begin{aligned} M(t) = {} & \langle U(t)\Xi(\cdot,0), \nabla\rho_*(\cdot-q(t))\rangle \\ & - \int_0^t \langle U(t-s)L(s)B(s), \nabla\rho_*(\cdot-q(t))\rangle ds \\ & + \int_0^t \dot v(s)\cdot\langle U(t-s)\nabla Z(s), \nabla\rho_*(\cdot-q(t))\rangle\, ds. \end{aligned} \tag{4.12}$$

We analyze the three summands separately.

(i) For the first summand we prove the bound

$$\sup_{t\ge 0} |\langle U(t)\Xi(\cdot,0), \nabla\rho_*(\cdot-q(t))\rangle| \le C_1(a^0)\|\rho\|\,\varepsilon^2. \tag{4.13}$$

Equation (4.10) implies $\|\Xi(\cdot,0)\|_{\mathcal{F}} \le C(a^0)\varepsilon^2$ by assumptions (2.17) and (C). Energy conservation then yields the uniform bound (4.13).

(ii) For the second summand in (4.12) we will obtain

$$\begin{aligned} & \left|\int_0^t \langle U(t-s)L(s)B(s), \nabla\rho_*(\cdot-q(t))\rangle\, ds\right| \\ & \le C_2(a^0,R_\rho)\,\|\rho\|^2 \int_0^t \frac{\varepsilon^2+|M(s)|}{1+(t-s)^2}\,ds, \quad t\ge 0. \end{aligned} \tag{4.14}$$

Equations (4.9), (4.6), and (4.3) result in $L(t)B(x,t) = e(x,t)+m(x,t)$, where again by (C),

$$\sup_{t\ge 0}\|e(x,t)\|_{\mathcal{F}} \le C(a^0,R_\rho)\|\rho\|\,\varepsilon^2, \quad \|m(x,t)\|_{\mathcal{F}} \le C(a^0,R_\rho)\|\rho\|\,|M(t)|\,.$$

Therefore (4.14) follows by repeating the arguments from (3.6)–(3.12).

(iii) For the third summand in (4.12) we will prove

$$\sup_{t\geq 0}\left|\int_0^t \dot v(s)\cdot\langle U(t-s)\nabla Z(s),\nabla\rho_*(\cdot-q(t))\rangle\,ds\right|\leq C_3(a^0,\rho)\,\varepsilon^2. \quad (4.15)$$

Taking the gradient of (3.2) yields

$$U(t-s)\nabla Z(s)=U(t)\nabla Z(0)-\int_0^s \dot p(\tau)\cdot U(t-\tau)\nabla[\nabla_p\Phi_{v(\tau)}(\cdot-q(\tau))]\,d\tau. \quad (4.16)$$

For the first term, by partial integration in polar coordinates of the Fourier representation, (2.19) and (2.18) imply that

$$|\langle U(t)\nabla Z(0),\nabla\rho_*(\cdot-q(t))\rangle|\leq C(a^0)t^{-1}\,\varepsilon.$$

The integral is oscillatory due to the bound (2.2). The justification for this partial integration comes from an appropriate averaging process. To bound the second term we note, similarly to (3.11),

$$\|U(t-\tau)\nabla[\nabla_p\Phi_{v(\tau)}(\cdot-q(\tau))]\|_{R_\rho}\leq\frac{C(a^0,R_\rho)\|\rho\|}{1+(t-\tau)^3}, \quad (4.17)$$

since the bounds of type (3.9) hold for $\nabla\nabla_p\Phi_v(x)$ with an additional power of $|x|$ on the left hand side. Then (4.16)-(4.17) and (4.3) imply (4.15).

Finally we substitute (4.13), (4.14), and (4.15) into (4.12) to obtain the integral inequality

$$|M(t)|\leq C(a^0,\rho)\varepsilon^2+C(a^0,R_\rho)\,\|\rho\|^2\int_0^t\frac{\varepsilon^2+|M(s)|}{1+(t-s)^2}\,ds,\quad t\geq 0.$$

Therefore (4.7) for $t\geq 0$ follows, provided that $\|\rho\|\leq\delta(a^0,R_\rho)$. □

5. Inertial Representation of the Self-Force

We study the self-action term

$$F_s(t)=\int d^3x\,\phi(x,t)\,\nabla\rho(x-q(t)).$$

Denote $T_1=2R_\rho(1-\overline{v})^{-1}$, where $\overline{v}<1$ is the bound from (2.10), and $\overline{T}=\max(T_0,T_1)$ with T_0 from (2.13). We also introduce the field part of the total momentum,

$$P_{\rm f}(v)=P_s(v)-p_v, \quad (5.1)$$

cf. (1.8), (1.3). The corresponding “effective mass”, $m_{\rm f}(v)$, is given by the differential

$$dP_{\rm f}(v)=:m_{\rm f}(v).$$

Lemma 5.1. *Let the assumptions of Theorem 2.5 hold. Then*

$$F_s(t)=-m_{\rm f}(\dot q(t))\ddot q(t)+f_s(t),\quad |f_s(t)|\leq C\varepsilon^2,\quad \textit{for}\quad |t|\geq\overline{T}. \quad (5.2)$$

Proof. We note that by (1.12) and (2.9), $\phi(x,t) = \phi^0(x,t) + \phi^r(x,t)$, where $\phi^0(x,t)$ is a solution to the free wave equation defined in (2.9), while ϕ^r is the retarded potential

$$\phi^r(x,t) = -\frac{1}{4\pi}\int_0^t \frac{ds}{t-s}\int_{|x-y|=t-s} d^2y\, \rho(y-q(s)). \tag{5.3}$$

We decompose accordingly $F_s(t) = F^0(t) + F^r(t)$, with

$$F^0(t) = \langle \phi^0(\cdot,t), \nabla\rho(\cdot - q(t))\rangle, \quad F^r(t) = \langle \phi^r(\cdot,t), \nabla\rho(\cdot - q(t))\rangle. \tag{5.4}$$

From (2.10) we conclude that $|q(t) - q^0| \le \overline{v}t$, and therefore

$$F^0(t) = \mathcal{O}(\varepsilon^2) \quad \text{for} \quad t \ge T_0 \tag{5.5}$$

by (2.13), since the solution is adiabatic. Hence

$$F_s(t) = F^r(t) + \mathcal{O}(\varepsilon^2) \quad \text{for} \quad t \ge T_0. \tag{5.6}$$

Equations (5.3) and (5.4) imply

$$F^r(t) = -\frac{1}{4\pi}\int_0^t \frac{ds}{t-s}\int d^3x \int_{|x-y|=t-s} d^2y\, \rho(y-q(s))\,\nabla\rho(x-q(t)). \tag{5.7}$$

Now observe that for all $t, T \ge T_1$ the $\displaystyle\int_0^t ds(\dots)$-integral in (5.7) may be changed to a $\displaystyle\int_{t-T}^t ds(\dots)$-integral, since

$$\rho(y-q(s))\,\nabla\rho(x-q(t)) = 0 \quad \text{if} \quad |x-y| = t-s \ge T_1. \tag{5.8}$$

Indeed, $\rho(y-q(s))\,\nabla\rho(x-q(t)) \neq 0$ implies $|y-q(s)| < R_\rho$ and $|x-q(t)| < R_\rho$. Therefore $|x-y| < 2R_\rho + \overline{v}(t-s)$, since $|q(t)-q(s)| \le \overline{v}(t-s)$ by (2.2). Substituting $|x-y|$ by $t-s$ we obtain $t-s < 2R_\rho/(1-\overline{v}) = T_1$.

Next we fix $t, T \ge T_1$ and substitute in (5.7) the Taylor expansion

$$q(s) = q(t) - \dot{q}(t)(t-s) + \frac{1}{2}\ddot{q}(t)(t-s)^2 + \mathcal{O}(\varepsilon^2)$$

according to (2.11)–(2.12). Then

$$F^r(t) = -\frac{1}{4\pi}\int_{t-T}^{t} \frac{ds}{t-s}\int d^3x \int_{|x-y|=t-s} d^2y\, \rho\Big(y - q(t) + \dot{q}(t)(t-s) - \frac{1}{2}\ddot{q}(t)(t-s)^2 + \mathcal{O}(\varepsilon^2)\Big)\nabla\rho(x-q(t)).$$

Combining with (5.6) we finally obtain

$$F_s(t) = -\frac{1}{4\pi}\int_{t-T}^{t} \frac{ds}{t-s}\int d^3x \int_{|x-y|=t-s} d^2y\, \Big[\rho(y-q(t)+\dot{q}(t)(t-s)) - \frac{1}{2}(t-s)^2\,\ddot{q}(t)\cdot\nabla\rho(y-q(t)+\dot{q}(t)(t-s))\Big]\nabla\rho(x-q(t)) + f_s(t) \tag{5.9}$$

with $f_s(t)$ satisfying (5.2). The integral does not depend on T provided $T, t > T_1$, which reflects the strong Huyghen's principle. We will show in Appendix A by taking the limit $T \to \infty$ that the integral in (5.9) in fact equals $-m_{\mathrm{f}}(\dot q)\ddot q$. Then (5.2) follows for $t \geq \overline{T}$. □

6. The Adiabatic Limit

We complete the proof of Theorem 2.5. We first ensure the existence of the effective dynamics.

Lemma 6.1. *Define $E(P)$ through (1.9), and let the potential V satisfy (U). Then for every initial state $(Q(0), P(0)) \in \mathbb{R}^3 \times \mathbb{R}^3$ the Hamiltonian system*

$$\dot Q(t) = \nabla E(P(t)), \quad \dot P(t) = -\varepsilon \nabla V(\varepsilon Q(t)) \tag{6.1}$$

has a unique solution $(Q(t), P(t)) \in C(\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3)$. Moreover, $|\ddot Q(t)|$ and $|\dddot Q(t)|$ are bounded uniformly in t.

Proof. Both $\nabla\nabla E(P)$ and $\nabla\nabla V(Q)$ are bounded and $H_{\mathrm{eff}}(P, Q)$ is bounded from below. □

Let $m(v) = dP_s(v)$. From Lemma 5.1, together with definitions (1.8), (5.1) and the equations of motion (1.12), we conclude that

$$m(\dot q(t))\ddot q(t) = -\varepsilon \nabla V(\varepsilon q(t)) + f_s(t). \tag{6.2}$$

We want to rewrite (6.2) in a Hamiltonian form. For this purpose we introduce $\Pi(t) = P_s(\dot q(t))$, which yields $m(\dot q(t))\ddot q(t) = \dot\Pi(t)$. To obtain $\dot q$ as a function of Π we have to invert the map $v \mapsto P_s(v)$.

Lemma 6.2. *The inverse function to $P_s(v)$ is given by*

$$v_s(P) = \nabla E(P). \tag{6.3}$$

Proof. Using the chain rule, Eq. (9.1) states

$$v = \nabla E_s(v)\,(dP_s(v))^{-1} = \nabla E(P_s(v)). \qquad \square$$

With these definitions, (6.2) becomes

$$\dot q(t) = \nabla E(\Pi(t)), \quad \dot\Pi(t) = -\varepsilon \nabla V(\varepsilon q(t)) + f_s(t). \tag{6.4}$$

Let $q^\varepsilon(t) = \varepsilon q(\varepsilon^{-1}t), Q^\varepsilon(t) = \varepsilon Q(\varepsilon^{-1}t)$ and $\Pi^\varepsilon(t) = \Pi(\varepsilon^{-1}t), P^\varepsilon(t) = P(\varepsilon^{-1}t)$. Then (6.4) and (6.1) read

$$\begin{aligned} \dot q^\varepsilon(t) &= \nabla E(\Pi^\varepsilon(t)), \quad \dot\Pi^\varepsilon(t) = -\nabla V(\varepsilon q^\varepsilon(t)) + \varepsilon^{-1} f_s(\varepsilon t), \\ \dot Q^\varepsilon(t) &= \nabla E(P^\varepsilon(t)), \quad \dot P^\varepsilon(t) = -\nabla V(\varepsilon Q^\varepsilon(t)). \end{aligned}$$

Since $\nabla\nabla E$ and $\nabla\nabla V$ are bounded, and $|f_s(\varepsilon t)| \leq C\varepsilon^2$ for $|t| \geq \varepsilon\overline{T}$, from a Gronwall argument for $r(t) = |q^\varepsilon(t) - Q^\varepsilon(t)| + |\Pi^\varepsilon(t) - P^\varepsilon(t)|$, we conclude that

$$r(t) \leq C(r_0 + \varepsilon)e^{C|t-t_0|}. \tag{6.5}$$

Here $r_0 := r(t_0) = 0$ due to (1.16), if $|t_0| > \overline{T}$, otherwise $r_0 := r(\pm\varepsilon\overline{T}) = \mathcal{O}(\varepsilon)$, since $q^\varepsilon(t), Q^\varepsilon(t), \Pi^\varepsilon(t), P^\varepsilon(t)$ change by $\mathcal{O}(\varepsilon)$ over the time interval $|t| \leq \varepsilon\overline{T}$. Therefore, (6.5) implies the first two bounds of (2.14). The third bound follows from the second order equation (6.2) for $\ddot q$ and a similar equation for $\ddot Q$.

7. The Translation Invariant Case

For $V = 0$ the velocity $\dot{q}(t)$ of the particle should, after a transient period, stabilize at some definite v dressed by the corresponding soliton field. Such a result was established in [12], where we only had to assume the Wiener condition $\hat{\rho}(k) \neq 0$. The technique developed here avoids this condition at the prize of $\|\rho\| \ll 1$ and obtains even a bound on the rate of convergence. We denote $Z(0) = (\varphi^0(x), \psi^0(x))$.

Proposition 7.1. *Let $\|\rho\|$ be sufficiently small, $\|\rho\| \le \delta(\overline{v}, R_\rho)$, and assume for some $\sigma \in (0, 1]$,*

$$|\varphi^0(x)| + |x|(|\nabla\varphi^0(x)| + |\psi^0(x)|) + |x|^2(|\nabla\nabla\varphi^0(x)| + |\nabla\psi^0(x)|) = \mathcal{O}(|x|^{-\sigma}) \quad \textit{as} \quad |x| \to \infty. \tag{7.1}$$

Then the solution to (1.1) satisfies

$$\|Z(\cdot + q(t), t)\|_R \le C_R(1 + |t|)^{-1-\sigma}, \quad \forall R > 0. \tag{7.2}$$

Corollary 7.2. *Under the same assumptions the acceleration is bounded as*

$$|\ddot{q}(t)| \le C(1 + |t|)^{-1-\sigma}. \tag{7.3}$$

Therefore, the limits $\lim_{t\to\pm\infty} \dot{q}(t) = v_\pm \in \mathcal{V}$ *exist, and*

$$|\dot{q}(t) - v_\pm| \le C(1 + |t|)^{-\sigma}. \tag{7.4}$$

Proof. Equations (7.1) and (3.2)–(3.11) with $\varepsilon = 0$ imply, similarly to (3.13),

$$\|Z(\cdot + q(t), t)\|_{R_\rho} \le C(1 + |t|)^{-1-\sigma} + C(\overline{v}, \rho)\|\rho\|^2 \int_0^t \frac{\|Z(q(s) + x, s)\|_{R_\rho}}{(1 + |t - s|)^2}\, ds$$

for $t \ge 0$. Therefore, setting $M(t) = \max_{0\le s\le t}(1 + |t|)^{1+\sigma}\|Z(q(s) + x, s)\|_{R_\rho}$, we find

$$M(t) \le C + C(\overline{v}, \rho)\|\rho\|^2 I_\sigma M(t),$$

where

$$I_\sigma = \sup_{t\ge 0}(1 + |t|)^{1+\sigma} \int_0^t \frac{(1 + |s|)^{-1-\sigma}}{(1 + |t - s|)^2}\, ds < \infty \quad \text{for} \quad \sigma \in (0, 1].$$

It remains to choose $C(\overline{v}, \rho)\|\rho\|^2 I_\sigma < 1$, then (7.2) with $R = R_\rho$ follows for $t \ge 0$. The corollary is a consequence of (3.4) with $\varepsilon = 0$. □

Remark. Soliton-like asymptotics are established in [17] for some translation invariant 1D completely integrable equations, in [4, 5] for small perturbations of soliton solutions to 1D translation invariant nonlinear Schrödinger equations, and in [19, 20] for $U(1)$-invariant 2D and 3D nonlinear Schrödinger equations with a potential term decaying like a power decay at infinity; [9] studies soliton-like asymptotics for 1D translation invariant nonlinear reaction systems.

8. Appendix A. Fourier Integrals

As usual, we denote by $\hat{f}(k) = (2\pi)^{-3/2} \int d^3x\, e^{ikx} f(x)$ the Fourier transform of $f(x)$.

Solitons: The soliton (1.4) has the Fourier transform

$$\begin{cases} \hat{\phi}_v(k) = -\dfrac{\hat{\rho}(k)}{k^2 - (k\cdot v)^2}, \\ \hat{\pi}_v(k) = -\dfrac{ik\cdot v\ \hat{\rho}(k)}{k^2-(k\cdot v)^2}. \end{cases} \tag{8.1}$$

Energy-momentum relation: Inserting (8.1) in (1.2) and (1.7), the energy and the total momentum of a soliton with velocity v are, respectively,

$$\mathcal{H}_0(S_v) = (1-v^2)^{-1/2} + \frac{1}{2}\int d^3k\,|\hat{\rho}(k)|^2 \frac{3(k\cdot v)^2 - k^2}{[k^2-(k\cdot v)^2]^2}, \tag{8.2}$$

$$P_s(v) = v(1-v^2)^{-1/2} + \int d^3k\,|\hat{\rho}(k)|^2 \frac{k\cdot v}{[k^2-(k\cdot v)^2]^2} k. \tag{8.3}$$

After some calculations, this yields (1.6) and (1.8).

Field mass: Equation (8.3) implies that the effective mass due to the coupling to the field is given by

$$m_{\rm f}(v) = dP_{\rm f}(v) = \int d^3k\,|\hat{\rho}(k)|^2\, \frac{k^2+3(k\cdot v)^2}{[k^2-(k\cdot v)^2]^3}\, k\otimes k, \quad |v|<1. \tag{8.4}$$

Self-force: We compute the integral (5.9) by switching to Fourier space. The wave propagator in Fourier space is multiplication by $|k|^{-1}\sin|k|t$. Hence

$$\begin{aligned} F_s(t) = &\int d^3k\,|\hat{\rho}(k)|^2 ik \int_{t-T}^t ds\, e^{-ik\cdot\dot{q}(t)(t-s)}\Big[1 - \frac{1}{2}\ddot{q}(t)\cdot(-ik)(t-s)^2\Big] \\ &\times |k|^{-1}\sin|k|(t-s) + f_s(t). \end{aligned} \tag{8.5}$$

We evaluate this integral by taking the limit as $T\to\infty$, recalling that the integral does not depend on T provided $T \ge T^1$. We set $F_s(t) = I_1(T) + I_2(T) + f_s(t)$.

In (8.5) we integrate over s. Setting $v = \dot{q}(t)$ and $k_\pm = -k\cdot v \pm |k|$, the first integral reads

$$\begin{aligned} I_1(T) &= \int d^3k\,|\hat{\rho}(k)|^2 ik \int_{t-T}^t ds\, e^{-ik\cdot\dot{q}(t)(t-s)}\, \frac{\sin|k|(t-s)}{|k|} \\ &= i\int d^3k\,|\hat{\rho}(k)|^2 \left[\frac{k}{k^2-(k\cdot v)^2} - \frac{k}{2|k|}\left(\frac{e^{ik_+T}}{k_+} - \frac{e^{ik_-T}}{k_-}\right)\right] \\ &= -i\int d^3k\,|\hat{\rho}(k)|^2 \frac{k}{2|k|}\left(\frac{e^{ik_+T}}{k_+} - \frac{e^{ik_-T}}{k_-}\right) =: I_1^+(T) + I_1^-(T). \end{aligned}$$

Introducing polar coordinates $\nu = |k|$, $\theta = k/|k|$, we have

$$\begin{aligned} I_1^+(T) &= -i \int d^3k\, |\hat\rho(k)|^2\, \frac{k}{2|k|}\, \frac{e^{ik_+T}}{k_+} \\ &= -\frac{i}{2} \int_{|\theta|=1} d^2\theta\, \frac{\theta}{\theta\cdot v + 1} \int_0^\infty d\nu\, \nu\, |\hat\rho(\nu\theta)|^2\, e^{i(\theta\cdot v+1)\,\nu T}. \end{aligned} \tag{8.6}$$

The integral converges absolutely, since $\hat\rho(k)$ is smooth with all derivatives in $L^2(\mathbb{R}^3)$ by assumption (C). Therefore, integrating by parts twice in the ν-integral yields $|I_1^+(T)| \le CT^{-2}$ because $|v| = |\dot q(t)| < 1$.

The same argument applies to $I_1^-(T)$ and it follows that

$$|I_1(T)| \le CT^{-2} \to 0 \quad \text{as} \quad T \to \infty. \tag{8.7}$$

The second integral reads

$$\begin{aligned} I_2(T) &= -\frac{1}{2} \int d^3k\, |\hat\rho(k)|^2 k\ \ddot q(t)\cdot k \int_{t-T}^t ds\, e^{-ik\cdot\dot q(t)(t-s)}(t-s)^2\, \frac{\sin|k|(t-s)}{|k|} \\ &= -\int d^3k\, |\hat\rho(k)|^2\, k\ \ddot q(t)\cdot k \left[\frac{k^2+3(k\cdot v)^2}{(k^2-(k\cdot v)^2)^3} + \frac{1}{2|k|}\left(\frac{e^{ik_+T}}{k_+^3} - \frac{e^{ik_-T}}{k_-^3}\right) \right. \\ &\qquad \left. -\frac{iT}{2|k|}\left(\frac{e^{ik_+T}}{k_+^2} - \frac{e^{ik_-T}}{k_-^2}\right) - \frac{T^2}{4|k|}\left(\frac{e^{ik_+T}}{k_+} - \frac{e^{ik_-T}}{k_-}\right)\right]. \end{aligned}$$

The integrals containing T are again oscillatory and vanish as $T \to \infty$. Therefore, comparing with (8.4), we conclude

$$I_2(T) \to -m_{\mathrm{f}}(\dot q(t))\ddot q(t) \quad \text{as} \quad T \to \infty. \tag{8.8}$$

Hence (5.2) follows from (8.7) and (8.8).

9. Appendix B. The Hamiltonian Structure

Energy-momentum relation: In Sect. 6 we used the identity

$$v\, dP_s(v) = \nabla E_s(v), \quad |v| < 1. \tag{9.1}$$

While obtained from the explicit expressions (8.2), (8.3), resp. (1.6), (1.8), this identity should be understood as a direct consequence of the conservation of total momentum, i.e., of the translation invariance of (1.1).

Our argument uses the canonical transformation [12]

$$\begin{aligned} \mathcal{T} : (\phi, \pi, q, p) &\mapsto (\Phi(x), \Pi(x), Q, P) \\ &= (\phi(q+x), \pi(q+x), q, p - <\pi(x), \nabla\phi(x)>). \end{aligned}$$

In new variables the Hamiltonian (1.2) reads

$$\begin{aligned}\mathcal{H}_P(\Phi,\Pi) &= \mathcal{H}_0\Big(\Phi(x-Q),\Pi(x-Q),Q,P+<\Pi(x),\nabla\Phi(x)>\Big)\\ &= \int d^3x\,\Big(\frac{1}{2}|\Pi(x)|^2+\frac{1}{2}|\nabla\Phi(x)|^2+\Phi(x)\rho(x)\Big)\\ &\quad+\Big(1+\Big(P+<\Pi(x),\nabla\Phi(x)>\Big)^2\Big)^{1/2}.\end{aligned}$$

$\mathcal{H}_P$ is bounded from below and has its unique minimum at the point (ϕ_v,π_v), the soliton at velocity $v=v_s(P)$, with minimal value $\mathcal{H}_P(\phi_v,\pi_v)=E_s(v)+\mathcal{H}_0(S_0)$; see [12]. Differentiating in v we obtain

$$\begin{aligned}\nabla E_s(v) &= \langle\frac{\delta\mathcal{H}_P}{\delta\Phi}(\phi_v,\pi_v),\nabla_v\phi_v\rangle+\langle\frac{\delta\mathcal{H}_P}{\delta\Pi}(\phi_v,\pi_v),\nabla_v\pi_v\rangle\\ &\quad+\nabla_P\mathcal{H}_P(\phi_v,\pi_v)\,dP_s(v)\\ &= v\,dP_s(v),\end{aligned}$$

since (ϕ_v,π_v) is a critical point of $\mathcal{H}_P$ and the first two terms vanish, while $v=\dot{Q}=\nabla_P\mathcal{H}_P(\phi_v,\pi_v)$ because $\mathcal{T}$ is a canonical transformation.

Correspondence of the Hamiltonian structures: Definitions (1.5), (1.8), and (1.9) imply that the Hamiltonian functional $\mathcal{H}_\varepsilon$ of (1.11) restricted to the soliton $S_v=(\phi_v(x-q),\pi_v(x-q),q,p_v)$ becomes

$$\mathcal{H}_\varepsilon(S_v)=E(P)+V(\varepsilon q)+\mathcal{H}_0(S_0)=H_{\rm eff}(q,P)+\mathcal{H}_0(S_0) \tag{9.2}$$

with $P=P_s(v)$. Thus the effective Hamiltonian can be understood as the restriction of $\mathcal{H}_\varepsilon$ to the soliton manifold. We need in addition the appropriate choice of the canonical variables to write the Hamilton's equations in standard form (1.15). For general reasons one expects the conserved quantities to play a distinguished role. In our case this suggests P and q as canonical variables. The next lemma gives an inherent geometrical meaning to this choice, which might be valuable in a more general context.

Lemma 9.1. *The canonical structure $P\,dq$ on the soliton manifold $\mathcal{S}$ is the restriction of the full canonical form $p\,dq+<\phi,d\pi>$, i.e.,*

$$P\,dq=(p\,dq+\langle\phi,d\pi\rangle)\Big|_{\mathcal{S}}.$$

Proof. We have $p\,dq+\langle\phi,d\pi\rangle=P\,dQ+\langle\Phi,d\Pi\rangle$, since $\mathcal{T}$ is a canonical transformation, and

$$\langle\Phi,d\Pi\rangle\Big|_{\mathcal{S}}=\langle\phi_v,d\pi_v\rangle=\langle\phi_v,\nabla_v\pi_v\rangle\,dv=0$$

by antisymmetry in Fourier space and since $|\hat\rho(-k)|=|\hat\rho(k)|$. □

References

1. Abraham, M.: *Theorie der Elektrizität, Band 2: Elektromagnetische Theorie der Strahlung*. Leipzig: Teubner, 1905
2. Bambusi, D., Galgani, L.: Some rigorous results on the Pauli-Fierz model of classical electrodynamics. Ann. Inst. H. Poincaré, Phys. Theor. **58**, 155–171 (1993)
3. Bensoussan, A., Lions, J.L., Papanicolaou, G.: *Asymptotic Analysis for Periodic Structures*. Studies in Mathematics and its Applications, Vol. **5**, Amsterdam: North-Holland, 1978
4. Buslaev, V.S., Perelman, G.S.: On nonlinear scattering of states which are close to a soliton. In: *Méthodes Semi-Classiques, Vol.2 Colloque International (Nantes, juin 1991)*, Asterisque **208**, 1992, pp. 49–63
5. Buslaev, V.S., Perelman, G.S.: On the stability of solitary waves for nonlinear Schrödinger equations. Trans. Amer. Math. Soc. **164**, 75–98 (1995)
6. Davies, E.B.: *Quantum Theory of Open Systems*. London: Academic Press, 1976
7. Grillakis, M., Shatah, J., Strauss, W.A.: Stability theory of solitary waves in the presence of symmetry I and II. J. Func. Anal. **74**, 160–197 (1987); **94**, 308–348 (1990)
8. De Masi, A., Ferrari, P.A., Goldstein, S., Wick, W.D.: An invariance principle for reversible Markov processes. Application to random environments. J. Stat. Phys. **55**, 787–855 (1989)
9. Fleckinger, J., Komech, A.: On soliton-like asymptotics for 1D nonlinear reaction systems. Russian J. Math. Phys. **5**, 295–307 (1997)
10. Hagedorn, G.A.: A time dependent Born–Oppenheimer approximation. Commun. Math. Phys. **77**, 1–19 (1980)
11. Jerrard, R.L., Soner, H.M.: Dynamics of Ginzburg–Landau Vortices. Preprint, 1995
12. Komech, A., Spohn, H.: Soliton-like asymptotics for a classical particle interacting with a scalar wave field. Nonlinear Anal. **33**, 13–24 (1998)
13. Komech, A., Spohn, H., Kunze, M.: Long-time asymptotics for a classical particle interacting with a scalar wave field. Comm. Partial Differential Equations **22**, 307–335 (1997)
14. Kramers, H.A.: Non-relativistic Quantum-Electrodynamics and correspondence Principle. In: *Solvay Conference 1948, Rapport et Discussions, Bruxelles, 1950* pp. 241–265; in: Kramers, H.A.: *Collected Scientific Papers*. Amsterdam: North-Holland, 1956, pp. 845–869
15. Lions, J.L.; *Problèmes aux Limites dans les Equations aux Dérivées Partielles*. Montréal: Presses de l'Univ. Montréal, 1962
16. Lorentz, H.A.: *Theory of Electrons*, 2nd edition 1915. Reprinted by New York: Dover, 1952
17. Novikov, S.P., Manakov, S.V., Pitaevskii, L.P., Zakharov, V.E.: *Theory of Solitons: The Inverse Scattering Method*. Consultants Bureau, 1984
18. Robert, D.: *Autour de l'Approximation Semi-Classique*, Progress in Mathematics, Vol. **68** Basel: Birkhäuser, 1987
19. Soffer, A., Weinstein, M.I.: Multichannel nonlinear scattering for nonintegrable equations. Commun. Math. Phys. **133**, 119–146 (1990)
20. Soffer, A., Weinstein, M.I.: Multichannel nonlinear scattering for nonintegrable equations II. The case of anisotropic potentials and data. J. Differ. Eqs. **98**, 376–390 (1992)
21. Spohn, H.: *Large Scale Dynamics of Interacting Particles*. Berlin: Springer, 1991
22. Spohn, H.: Long time asymptotics for quantum particles in a periodic potential. Phys. Rev. Lett. **77**, 1198–1201 (1996)

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