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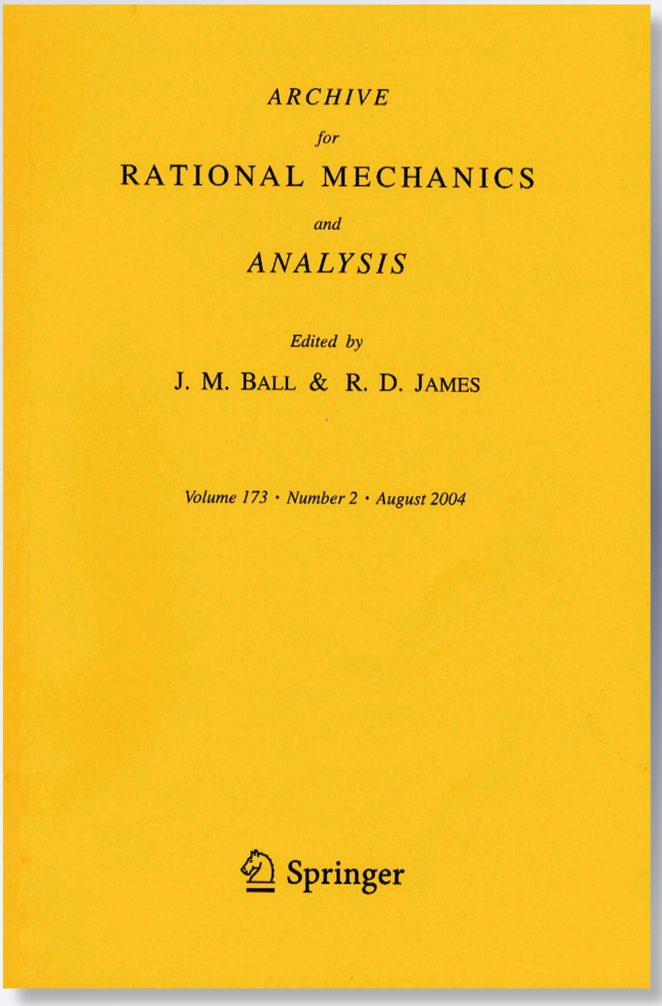

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On Asymptotic Stability of Kink for Relativistic Ginzburg–Landau Equations

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Abstract

We prove the asymptotic stability of kink for the nonlinear relativistic wave equations of the Ginzburg–Landau type in one space dimension: for any odd initial condition in a small neighborhood of the kink, the solution, asymptotically in time, is the sum of the kink and dispersive part described by the free Klein–Gordon equation. The remainder converges to zero in a global norm.

1. Introduction

We prove the asymptotic stability of kinks for relativistic nonlinear wave equations with two-well potentials of Ginzburg–Landau type. We consider the equation

$$\ddot{\psi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \quad x \in \mathbb{R} \tag{1.1}$$

where $\psi(x,t)$ is real, and $F(\psi) = -U'(\psi)$. We assume $U(\psi)$ similar to the Ginzburg–Landau potential $U_0(\psi) = (\psi^2-1)^2/4$ which corresponds to the cubic equation with $F(\psi) = \psi - \psi^3$.

Condition U1. *$U(\psi)$ is a real smooth even function which satisfies the following conditions*

$$U(\psi) > 0 \quad \text{for } \psi \neq \pm a, \tag{1.2}$$

$$U(\psi) = \frac{m^2}{2}(\psi \mp a)^2 + \mathcal{O}(|\psi \mp a|^{14}), \quad x \to \pm a \tag{1.3}$$

with some $a, m > 0$. In a vector form, Equation (1.1) reads

$$\begin{cases} \dot{\psi}(x,t) = \pi(x,t), \\ \dot{\pi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \quad x \in \mathbb{R}. \end{cases} \tag{1.4}$$

Formally, this is a Hamiltonian system with the Hamilton functional

$$\mathcal{H}(\psi,\pi)=\int\left[\frac{|\pi(x)|^2}{2}+\frac{|\psi'(x)|^2}{2}+U(\psi(x))\right]\mathrm{d}x. \tag{1.5}$$

The corresponding stationary equation reads

$$s''-U'(s)=0. \tag{1.6}$$

There is an odd finite energy solution $s(x)$ (a "kink") to (1.6) such that

$$s(0)=0,\quad s(x)\to\pm a \quad \text{as } x\to\pm\infty. \tag{1.7}$$

The condition **U1** implies that $(s(x)\mp a)''\sim m^2(s(x)\mp a)$ for $x\to\pm\infty$, hence

$$s(x)\mp a\sim C\mathrm{e}^{-m|x|},\quad x\to\pm\infty. \tag{1.8}$$

The generator of linearized equations near the kink reads (see Section 2)

$$A=\begin{pmatrix}0 & 1\\ -H & 0\end{pmatrix}$$

where H is the Schrödinger operator

$$\begin{aligned}H&=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-F'(s)=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+m^2+V(x),\quad V(x)=-F'(s(x))-m^2\\ &=U''(s(x))-m^2.\end{aligned} \tag{1.9}$$

By (1.8), we have

$$V(x)\sim C(s(x)\mp a)^{12}\sim C\mathrm{e}^{-12m|x|},\quad x\to\pm\infty. \tag{1.10}$$

The continuous spectrum of H coincides with $[m^2,\infty)$. Not to overburden the exposition, we consider only odd solutions $\psi(-x,t)=-\psi(x,t)$. We assume the following spectral condition:

Condition U2. *The discrete spectrum of H, restricted to the subspace of odd functions, consists of only one simple eigenvalue $\lambda_1<m^2$ with $4\lambda_1>m^2$, and the edge point $\lambda=m^2$ is neither eigenvalue nor resonance for H.*

We assume also a non-degeneracy condition, the "Fermi Golden Rule" introduced by SIGAL [24]. The condition provides a strong coupling of the nonlinear term with the eigenfunctions of the continuous spectrum and the energy radiation.

Condition U3. *The non-degeneracy condition holds (see condition* (1.0.11) *in* [3]*)*

$$\int_0^\infty \varphi_{4\lambda_1}(x)F''(s(x))\varphi_{\lambda_1}^2(x)\,\mathrm{d}x\neq 0, \tag{1.11}$$

where $\varphi_{\lambda_1}(x)$ and $\varphi_{4\lambda_1}(x)$ are the odd eigenfunctions of a discrete and continuous spectrum corresponding to λ_1 and $4\lambda_1$ respectively.

The Ginzburg–Landau potential $U_0(\psi) = (\psi^2 - 1)^2/4$ satisfies **U1–U3** except (1.3). In Appendix C we construct small perturbations of $U_0(\psi)$ which satisfy **U1–U3** including (1.3).

Our main result is the following asymptotics

$$(\psi(x,t), \dot{\psi}(x,t)) \sim (s(x), 0) + W_0(t)\Phi_\pm, \quad t \to \pm\infty \tag{1.12}$$

for solutions to (1.4) with odd initial data close to the kink $S(x) = (s(x), 0)$. Here $W_0(t)$ is the dynamical group of the free Klein–Gordon equation, $\Phi_\pm$ are the corresponding asymptotic states, and the remainder converges to zero $\sim t^{-1/3}$ in $H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$.

Remark 1.1. We consider the solutions close to the kink, $\psi(x,t) = s(x) + \phi(x,t)$, with small perturbations $\phi(x,t)$. For such solutions, (1.3) and (1.8) mean that Equation (1.1) is almost linear for large $|x|$. This fact is helpful for application of dispersive properties of the corresponding linearized equation.

Let us comment on previous results in this field.

- *The Schrödinger equation* The asymptotics of type (1.12) were established for the first time by Soffer and Weinstein [25,26] (see also [20]) for nonlinear $U(1)$-invariant Schrödinger equation with a potential for small initial states, if the nonlinear coupling constant is sufficiently small.

The results have been extended by Buslaev and Perelman [1] to the translation invariant one-dimensional nonlinear $U(1)$-invariant Schrödinger equation. The initial states are sufficiently close to the solitary waves with the unique eigenvalue $\lambda = 0$ in the discrete spectrum of the corresponding linearized dynamics. The novel techniques [1] are based on the “separation of variables” along the solitary manifold and in transversal directions. The symplectic projection allows exclusion from the transversal dynamics of the unstable directions corresponding to the zero discrete spectrum of the linearized dynamics. The extensions to higher dimensions were obtained in [4,13,23,30].

Similar techniques were developed by Miller, Pego and Weinstein for the one-dimensional modified KdV and RLW equations, [18,19]. These techniques were motivated by the investigation of soliton asymptotics for integrable equations (a survey can be found in [8,9]), and by the methods introduced in [25,26,32].

The techniques were developed in [2,3] for the Schrödinger equations in a more complicated spectral situation with presence of a nonzero eigenvalue in the linearized dynamics. In that case the transversal dynamics inherit the nonzero discrete spectrum. Now the decay for the transversal dynamics is obtained by the reduction to the Poincaré normal form, which makes obvious that the decay depends on the Fermi Golden Rule condition [17,24]. The condition states a strong interaction of the nonlinear term with the eigenfunctions of the continuous spectrum, which provides the dispersive energy radiation to infinity and the decay for the transversal dynamics. The extensions to higher dimensions were obtained in [5,6,28]. Tsai [31] developed the techniques in presence of an arbitrary finite number of discrete eigenvalues in the linearized dynamics.

- *Nonrelativistic Klein–Gordon equations* The asymptotics of type (1.12) were extended to the nonlinear three-dimensional Klein–Gordon equations with a potential [27], and for the translation invariant system of the three-dimensional Klein–Gordon equation coupled to a particle [12].
- *Wave front of three-dimensional Ginzburg–Landau equations* The asymptotic stability of wave front was proved for three-dimensional relativistic Ginzburg–Landau equations with initial data which differ from the wave front on a compact set [7]. The wave front is the solution which depends on one spatial variable, so it is not a finite energy soliton. The equation differs from the one-dimensional equation (1.1) by the additional two-dimensional Laplacian. The additional Laplacian improves the dispersive decay for the corresponding linearized Klein–Gordon equation in the continuous spectral space that provides the needed decay for the transversal dynamics.
- *Orbital stability of kinks* For one-dimensional relativistic nonlinear Ginzburg–Landau equations (1.1) the orbital stability of the kinks has been proved in [11].

The proving of the asymptotic stability of the kinks for relativistic equations remained an open problem until now. The main obstacle was the slow decay $\sim t^{-1/2}$ for the free one-dimensional Klein–Gordon equation (see the discussion in [7, Introduction]).

Let us comment on our approach. We follow the general strategy of [1–7, 12, 27, 30, 31]: linearization of the transversal equations and further Taylor expansion of the nonlinearity, the Poincaré normal forms and Fermi Golden Rule, etc. We develop, for relativistic equations, a general scheme which is common to almost all papers in this area: dispersive and $L^1 - L^\infty$ estimates for the linearized equation, virial estimates for the nonlinear equation and the method of majorants. However, the corresponding statements and their proofs in the context of relativistic equations are completely new.

Let us comment on our novel techniques.

(i) The "virial type" estimate (A.1) for the nonlinear wave equation (1.1) is a novel relativistic version of the bound [3, (1.2.5)] for the nonlinear Schrödinger equations.
(ii) We establish an appropriate relativistic version (3.11) of $L^1 \to L^\infty$ estimates.
(iii) We give the complete proof of the soliton asymptotics (1.12).
(iv) We construct examples of the potentials satisfying all our spectral conditions, including the Fermi Golden Rule.

Our paper is organized as follows. In Section 2 we formulate the main theorem. The linearization at the kink is carried out in Section 3. In Section 4 we derive the dynamical equations for the "discrete" and "continuous" components of the solution. In Section 5 we transform the dynamical equations to a Poincare "normal form". We apply the method of majorants in Section 6. In Section 7 we obtain the soliton asymptotics (1.12). In the Appendices we prove some key estimates and construct examples of the potentials.

2. Main results

We consider the Cauchy problem for (1.4), which we write as

$$\dot{Y}(t) = \mathcal{F}(Y(t)), \quad t \in \mathbb{R}: \ Y(0) = Y_0. \tag{2.1}$$

Here $Y(t) = (\psi(t), \pi(t))$, $Y_0 = (\psi_0, \pi_0)$. We will consider only odd states $Y = (\psi, \pi)$. The space of the odd states is invariant with respect to (2.1) since $F(\psi)$ is odd according to **U1**.

Let us introduce a suitable phase space. For $\sigma \in \mathbb{R}$, and $l = 0, 1, 2, \ldots, p \geqq 1$, denote by $W^{p,l}_\sigma$ the weighted Sobolev space of odd functions with finite norm

$$\|\psi\|_{W^{p,l}_\sigma} = \sum_{k=0}^{l} \|(1+|x|)^\sigma \psi^{(k)}\|_{L^p} < \infty.$$

Denote $H^l_\sigma := W^{2,l}_\sigma$, and $H^0_\sigma = L^2_\sigma$.

Definition 2.1. (i) $E_\sigma := H^1_\sigma \oplus L^2_\sigma$ is the space of odd states $Y = (\psi, \pi)$ with finite norm

$$\|Y\|_{E_\sigma} = \|\psi\|_{H^1_\sigma} + \|\pi\|_{L^2_\sigma}. \tag{2.2}$$

(ii) The phase space $\mathcal{E} := S + E$, where $E = E_0$ and $S = (s(x), 0)$. The metric in $\mathcal{E}$ is defined as

$$\rho_{\mathcal{E}}(Y_1, Y_2) = \|Y_1 - Y_2\|_E, \quad Y_1, Y_2 \in \mathcal{E}. \tag{2.3}$$

(iii) $W := W^{1,2}_0 \oplus W^{1,1}_0$ is the space of odd states $Y = (\psi, \pi)$ with finite norm

$$\|Y\|_W = \|\psi\|_{W^{1,2}_0} + \|\pi\|_{W^{1,1}_0}. \tag{2.4}$$

Obviously, the Hamilton functional (1.5) is continuous on the phase space $\mathcal{E}$. The existence and uniqueness of the solutions to the Cauchy problem (2.1) follows by methods [16,21,29]:

Proposition 2.2. (i) *For any $Y_0 \in \mathcal{E}$ there exists the unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to* (2.1).
(ii) *For every $t \in \mathbb{R}$, the map $U(t): Y_0 \mapsto Y(t)$ is continuous in $\mathcal{E}$.*
(iii) *The energy is conserved, that is*

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y_0), \quad t \in \mathbb{R}. \tag{2.5}$$

The main result of our paper is the following theorem

Theorem 2.3. *Let the potential U satisfy* **U1–U3** *with $k=7$, and let $Y(t)$ be the solution to the Cauchy problem* (2.1) *with any initial state $Y_0 \in \mathcal{E}$ which is sufficiently close to the kink:*

$$Y_0 = S + X_0, \quad d_0 := \|X_0\|_{E_\sigma \cap W} \ll 1, \tag{2.6}$$

where $\sigma > 5/2$. *Then the asymptotics hold*

$$Y(x,t) = (s(x), 0) + W_0(t)\Phi_\pm + r_\pm(x,t), \quad t \to \pm\infty, \tag{2.7}$$

where $\Phi_\pm \in E$, *and* $W_0(t) = e^{A_0 t}$ *is the dynamical group of the free Klein–Gordon equation (see* (3.13)*), while*

$$\|r_\pm(t)\|_E = \mathcal{O}(|t|^{-1/3}). \tag{2.8}$$

It suffices to prove the asymptotics (2.7) for $t \to +\infty$ since (1.4) is time reversible.

3. Linearization at the kink

3.1. Linearized equation

We linearize (1.4) at the kink $S(x)$, splitting the solution as the sum

$$Y(t) = S + X(t), \tag{3.1}$$

where $Y = (\psi, \pi)$ and $X = (\Psi, \Pi)$. We substitute (3.1) to (1.4) and using (1.6) obtain that

$$\dot{X}(t) = AX(t) + \mathcal{N}(X(t)), \quad t \in \mathbb{R} \tag{3.2}$$

where the linear operator A reads

$$A = \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix} \tag{3.3}$$

with

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - F'(s) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + m^2 + V(x), \tag{3.4}$$

and

$$V(x) = -F'(s(x)) - m^2 = U''(s(x)) - m^2. \tag{3.5}$$

$\mathcal{N}(X)$ is given by

$$\mathcal{N}(x, X) = \begin{pmatrix} 0 \\ N(x,\Psi) \end{pmatrix}, \quad N(x,\Psi) = F(s(x)+\Psi) - F(s(x)) - F'(s(x))\Psi. \tag{3.6}$$

3.2. Spectrum of linearized equation

Let us consider the eigenvalue problem for operator A:

$$A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \Lambda\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

From the first equation we have $u_2 = \Lambda u_1$. Then the second equation implies that

$$\left(H + \Lambda^2\right) u_1 = 0. \tag{3.7}$$

By **U2** operator A has two purely imaginary eigenvalues $\Lambda = \pm \mathrm{i}\mu$, where $\mu = \sqrt{\lambda_1}$. The corresponding eigenvectors

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \varphi_{\lambda_1} \\ \mathrm{i}\mu\varphi_{\lambda_1} \end{pmatrix}, \quad \overline{u} = \begin{pmatrix} \varphi_{\lambda_1} \\ -\mathrm{i}\mu\varphi_{\lambda_1} \end{pmatrix}.$$

where we choose φ_{λ_1} to be a real function. This is possible since H is a differential operator with real coefficients. The continuous spectrum of A coincides with $\mathcal{C} := (-\mathrm{i}\infty, -\mathrm{i}m] \cup [\mathrm{i}m, \mathrm{i}\infty)$. The edge points $\Lambda = \pm\mathrm{i}m$ are neither eigenvalues nor resonances for A, by condition **U2**.

3.3. Decay for linearized dynamics

We consider the linearized equation

$$\dot{X}(t) = AX(t), \quad t \in \mathbb{R}. \tag{3.8}$$

Let $\langle\cdot,\cdot\rangle$ be the scalar product in $L^2(\mathbb{R}, \mathbb{C}^2)$. Denote by P^d the symplectic projector onto the eigenspace E^d generated by u and $\overline{u}$:

$$P^d X = \frac{\langle X, ju\rangle}{\langle u, ju\rangle}u + \frac{\langle X, j\overline{u}\rangle}{\langle \overline{u}, j\overline{u}\rangle}\overline{u}, \quad X \in E_\sigma, \quad \sigma \in \mathbb{R}, \quad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.9}$$

Denote by $P^c = 1 - P^d$ the projector onto the continuous spectrum of A, and by E^c the continuous spectral subspace.

Next, decay estimates will play the key role in our proofs. The first estimate follows by Theorem 3.9 of [14] since the condition of type [14, (3.1)] holds in our case.

Proposition 3.1. *Let* **U2** *hold, and $\sigma > 5/2$. Then for any $X \in E_\sigma$ the bound holds*

$$\|e^{At}P^cX\|_{E_{-\sigma}} \leqq C(1+t)^{-3/2}\|X\|_{E_\sigma}, \quad t \in \mathbb{R}. \tag{3.10}$$

Corollary 3.2. *For $X \in E_\sigma \cap W$ with $\sigma > 5/2$ the bound holds*

$$\|(e^{At}P^cX)_1\|_{L^\infty} \leqq C(1+t)^{-1/2}(\|X\|_W + \|X\|_{E_\sigma}), \quad t \in \mathbb{R}. \tag{3.11}$$

Here $(\cdot)_1$ stands for the first component of the corresponding vector function.

Proof. We apply the projector P^c to both sides of (3.8):

$$P^c\dot{X} = AP^cX = A_0P^cX - \mathcal{V}P^cX, \tag{3.12}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ \frac{d^2}{dx^2} - m^2 & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}. \tag{3.13}$$

Then the Duhamel representation gives,

$$e^{At}P^cX = e^{A_0t}P^cX - \int_0^t e^{A_0(t-\tau)}\mathcal{V}e^{A\tau}P^cX\,d\tau, \quad t > 0. \tag{3.14}$$

Applying estimate (265) from [22], the Hölder inequality and Proposition 3.1 we obtain

$$\begin{aligned}
&\|(e^{At}P^cX)_1\|_{L^\infty} \\
&\leqq C(1+t)^{-1/2}\|P^cX\|_W + C\int_0^t (1+t-\tau)^{-1/2}\|V(e^{A\tau}P^cX)_1\|_{W_0^{1,1}}\,d\tau \\
&\leqq C(1+t)^{-1/2}\|X\|_W + C\int_0^t (1+t-\tau)^{-1/2}\|e^{A\tau}P^cX\|_{E_{-\sigma}}\,d\tau \\
&\leqq C(1+t)^{-1/2}\|X\|_W + C\int_0^t (1+t-\tau)^{-1/2}(1+\tau)^{-3/2}\|X\|_{E_\sigma}\,d\tau \\
&\leqq C(1+t)^{-1/2}(\|X\|_W + \|X\|_{E_\sigma}).
\end{aligned}$$

□

Proposition 3.3. *For $\sigma > 5/2$ the bound holds*

$$\|e^{At}(A \mp 2i\mu - 0)^{-1}P^cX\|_{E_{-\sigma}} \leqq C(1+t)^{-3/2}\|X\|_{E_\sigma}, \quad t > 0. \tag{3.15}$$

We will prove the proposition in Appendix B.

4. Decomposition of dynamics

We decompose the solution to (2.1) as $Y(t) = S + X(t)$, where $X(t) = w(t) + f(t)$ with $w(t) = z(t)u + \overline{z}(t)\overline{u} \in E^d$ and $f(t) \in E^c$.

Lemma 4.1. *Let $Y(t) = S + w(t) + f(t)$ be a solution to (2.1). Then functions $z(t)$ and $f(t)$ satisfy the equations*

$$(\dot{z} - i\mu z)\langle u, ju\rangle = \langle \mathcal{N}, ju\rangle, \tag{4.1}$$

$$\dot{f} = Af + P^c\mathcal{N} \tag{4.2}$$

with $\mathcal{N}$ defined in (3.6).

Proof. Applying P^d to (3.2), we obtain

$$\dot{z}u + \dot{\overline{z}}\,\overline{u} = Aw + P^d\mathcal{N}. \tag{4.3}$$

Using $\langle \overline{u}, ju\rangle = 0$ and $Aw = \mathrm{i}\mu(zu - \overline{z}\,\overline{u})$, we get (4.1), after taking the scalar product of (4.3) with ju since $(P^d)^* j = jP^d$. Applying P^c to (3.2), we obtain (4.2) since P^c commutes with A. □

Remark 4.2. In the remaining part of the paper we will prove the asymptotics

$$\|f(t)\|_{E_{-\sigma}} \sim t^{-1}, \quad z(t) \sim t^{-1/2}, \quad \|f_1(t)\|_{L^\infty} \sim t^{-1/2}, \quad t \to \infty. \tag{4.4}$$

To justify these asymptotics, we will single out leading terms in the right-hand side of (4.1)–(4.2). Namely, we shall expand the expressions for $\dot{z}$ up to terms of the order $\mathcal{O}(t^{-3/2})$, and for $\dot{f}$ up to $\mathcal{O}(t^{-1})$ keeping in mind asymptotics (4.4). This choice allows us to obtain the uniform bounds using the method of majorants.

We expand $N(x, \Psi)$ from (3.6) in the Taylor series

$$N(x, \Psi) = N_2(x, \Psi) + \cdots + N_{12}(x, \Psi) + N_R(x, \Psi), \tag{4.5}$$

where

$$N_j(x, \Psi) = \frac{F^{(j)}(s(x))}{j!}\Psi^j, \quad j = 2, \ldots, 12 \tag{4.6}$$

and N_R is the remainder. Condition **U1** implies that $F(\psi) = -m^2(\psi \mp a) + \mathcal{O}(|\psi \mp a|^{13})$. Hence, $N_j(x, \Psi)$, $j \leqq 12$ decrease exponentially as $|x| \to \infty$ by (1.8), while for N_R we have

$$|N_R| = \mathcal{R}(|\Psi|)|\Psi|^{13} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})|\Psi|^{13}, \tag{4.7}$$

where $\mathcal{R}(A)$ is a general notation for a positive function which remains bounded as A is sufficiently small.

Denote $\mathcal{N}_2[X_1, X_2] = (0, N_2[\Psi_1, \Psi_2])$ and $\mathcal{N}_3[X_1, X_2, X_3] = (0, N_3[\Psi_1, \Psi_2, \Psi_3])$ where

$$N_2[\Psi_1, \Psi_2] = \frac{F''(s)}{2}\Psi_1\Psi_2, \quad N_3[\Psi_1, \Psi_2, \Psi_3] = \frac{F'''(s)}{6}\Psi_1\Psi_2\Psi_3. \tag{4.8}$$

4.1. Leading term in $\dot{z}$

Let us rewrite (4.1) in the form:

$$\dot{z} - \mathrm{i}\mu z = \frac{\langle \mathcal{N}, ju\rangle}{\langle u, ju\rangle} = \frac{\langle \mathcal{N}_2[w, w] + 2\mathcal{N}_2[w, f] + \mathcal{N}_3[w, w, w], ju\rangle}{\langle u, ju\rangle} + Z_R, \tag{4.9}$$

where

$$|Z_R| = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^2 + \|f\|_{E_{-\sigma}})^2. \tag{4.10}$$

Note that

$$\mathcal{N}_2[w,w] = (z^2 + 2z\overline{z} + \overline{z}^2)\mathcal{N}_2[u,u],$$
$$\mathcal{N}_3[w,w,w] = (z^3 + 3z^2\overline{z} + 3z\overline{z}^2 + \overline{z}^3)\mathcal{N}_3[u,u,u]. \tag{4.11}$$

Hence, (4.9) reads

$$\dot{z} = \mathrm{i}\mu z + Z_2(z^2 + 2z\overline{z} + \overline{z}^2) + Z_3(z^3 + 3z^2\overline{z} + 3z\overline{z}^2 + \overline{z}^3) + (z + \overline{z})\langle f, jZ_1'\rangle + Z_R, \tag{4.12}$$

where

$$Z_2 = \frac{\langle \mathcal{N}_2[u,u], ju\rangle}{\langle u, ju\rangle}, \quad Z_3 = \frac{\langle \mathcal{N}_3[u,u,u], ju\rangle}{\langle u, ju\rangle}, \quad Z_1' = 2\frac{\mathcal{N}_2[u,u]}{\langle u, ju\rangle}. \tag{4.13}$$

4.2. Leading term in $\dot{f}$

We now turn to (4.2), which we rewrite in the form

$$\dot{f} = Af + P^c\mathcal{N} = Af + P^c\mathcal{N}_2[w,w] + F_R. \tag{4.14}$$

The remainder $F_R = F_R(x,t)$ reads

$$F_R = P^c(\mathcal{N}(X) - \mathcal{N}_2[w,w]) = (1 - P^d)(\mathcal{N}(X) - \mathcal{N}_2[w,w])$$
$$= F_{\mathrm{I}} + F_{\mathrm{II}} + F_{\mathrm{III}}, \tag{4.15}$$

where

$$F_{\mathrm{I}} = -P^d(\mathcal{N}(X) - \mathcal{N}_2[w,w]), \quad F_{\mathrm{II}} = \mathcal{N}(X) - \mathcal{N}_2[w,w] - \mathcal{N}_R,$$
$$F_{\mathrm{III}} = \mathcal{N}_R = (0, N_R). \tag{4.16}$$

For $F_{\mathrm{I}} + F_{\mathrm{II}}$ the bound holds

$$\|F_{\mathrm{I}} + F_{\mathrm{II}}\|_{E_\sigma \cap W} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^3 + |z|\|f\|_{E_{-\sigma}} + \|f_1\|_{L^\infty}\|f\|_{E_{-\sigma}}). \tag{4.17}$$

Indeed, F_{I} admits the estimate by (3.9), since the function $u(x)$ decays exponentially. Further,

$$(F_{\mathrm{II}})_1 = 0, \quad (F_{\mathrm{II}})_2 = 2N_2(w_1, f_1) + N_2(f_1, f_1) + N_3(\Psi) + \cdots + N_{12}(\Psi),$$

and each summand contains an exponentially decreasing factor by **U1**, (1.8) and (4.6).

Similarly, we obtain

$$\|P^c\mathcal{N}_2[w,w]\|_{E_\sigma \cap W} \leqq C|z|^2. \tag{4.18}$$

It remains to estimate the term F_{III}.

Lemma 4.3. *The bounds hold*

$$\|F_{\mathrm{III}}\|_{E_{5/2+\nu}} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(1+t)^{4+\nu}(|z|^{12} + \|f_1\|^{12}_{L^\infty}), \quad 0 < \nu < 1/2, \tag{4.19}$$

$$\|F_{\rm III}\|_W = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^{11} + \|f_1\|_{L^\infty}^{11}). \tag{4.20}$$

Proof. Step (i) Bound (4.7) implies

$$\|N_R\|_{L^2_{5/2+\nu}} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^{12} + \|f_1\|_{L^\infty}^{12})\|\Psi\|_{L^2_{5/2+\nu}}.$$

We will prove in Appendix A that

$$\|\Psi(t)\|_{L^2_{5/2+\nu}} \leqq C(d_0)(1+t)^{4+\nu}. \tag{4.21}$$

Then (4.19) follows.

Step (ii) By the Cauchy formula,

$$N_R(x,t) = \frac{\Psi^{13}(x,t)}{(12)!}\int_0^1 (1-\rho)^{12} F^{(13)}(s+\rho\Psi(x,t))\,\mathrm{d}\rho. \tag{4.22}$$

Therefore,

$$\begin{aligned}\|N_R\|_{L^1} &= \mathcal{R}(|z| + \|f_1\|_{L^\infty}) \\ &\qquad \int |\Psi|^{13}\,\mathrm{d}x = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z| + \|f_1\|_{L^\infty})^{11}\|\Psi\|_{L^2}^2 \\ &= \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^{11} + \|f_1\|_{L^\infty}^{11})\end{aligned}$$

since $\|\Psi(t)\|_{L^2} \leqq C(d_0)$ by the results of [11]. Differentiating (4.22) in x, we obtain

$$\begin{aligned}N_R' = \frac{\Psi^{13}}{(13)!}&\int_0^1 (1-\rho)^{12}(s' + \rho\Psi')F^{(14)}(s+\rho\Psi)\,\mathrm{d}\rho \\ &+\frac{\Psi^{12}\Psi'}{(12)!}\int_0^1 (1-\rho)^{12}F^{(13)}(s+\rho\Psi)\,\mathrm{d}\rho.\end{aligned}$$

Hence,

$$\|N_R'\|_{L^1} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^{11} + \|f_1\|_{L^\infty}^{11})$$

since $\int |\Psi(x)\Psi'(x)|\,\mathrm{d}x \leqq \|\Psi\|_{L^2}\|\Psi'\|_{L^2} \leqq C(d_0)$. Then (4.20) follows. □

5. Poincare normal forms

Our goal is to transform the equations for z and f to a "normal form" removing the "nonresonant terms".

5.1. Normal form for $\dot{f}$

We rewrite (4.14) in a more detailed form as

$$\dot{f} = Af + (z^2 + 2z\overline{z} + \overline{z}^2)F_2 + F_R, \quad F_2 = P^c\mathcal{N}_2[u, u]. \tag{5.1}$$

Now we extract the term of order $z^2 \sim t^{-1}$ (see Remark 4.2). For this purpose we expand f as

$$f = h + k + g, \tag{5.2}$$

where

$$g(t) = -\mathrm{e}^{At}k(0), \quad k = a_{20}z^2 + 2a_{11}z\overline{z} + a_{02}\overline{z}^2 \tag{5.3}$$

with some $a_{ji} \equiv a_{ij}(x)$ satisfying $a_{ij}(x) = \overline{a}_{ij}(x)$. Note that $h(0) = f(0)$.

Lemma 5.1. *There exist $a_{ij} \in H^s_{-\sigma}$ with any $s > 0$ such that the equation for h reads*

$$\dot{h} = Ah + H_R, \tag{5.4}$$

where $H_R = F_R + H_I$, with $H_I = \sum a_{ij}(x)\mathcal{R}(|z| + \|f\|_{E_{-\sigma}})|z|(|z| + \|f\|_{E_{-\sigma}})^2$.

Proof. Substituting (5.3) into (5.1), we get

$$\begin{aligned}
\dot{h} &= \dot{f} - (2a_{20}z + 2a_{11}\overline{z})\dot{z} - (2a_{11}z + 2a_{02}\overline{z})\dot{\overline{z}} - \dot{g} \\
&= Af + (z^2 + 2z\overline{z} + \overline{z}^2)F_2 + F_R \\
&\quad -(2a_{20}z + 2a_{11}\overline{z})(\mathrm{i}\mu z + \mathcal{R}(|z| + \|f\|_{E_{-\sigma}})(|z| + \|f\|_{E_{-\sigma}})^2) \\
&\quad -(2a_{11}z + 2a_{02}\overline{z})(-\mathrm{i}\mu\overline{z} + \mathcal{R}(|z| + \|f\|_{E_{-\sigma}})(|z| + \|f\|_{E_{-\sigma}})^2) - Ag.
\end{aligned}$$

On the other hand, (5.4) means that $\dot{h} = A(f - a_{20}z^2 - 2a_{11}z\overline{z} - a_{02}\overline{z}^2 - g) + H_R$.

Equating the coefficients of the quadratic powers of z, we get

$$F_2 - 2\mathrm{i}\mu a_{20} = -Aa_{20}, \quad F_2 = -Aa_{11}, \quad F_2 + 2\mathrm{i}\mu a_{02} = -Aa_{02},$$

and

$$H_R = F_R + \sum a_{ij}\mathcal{R}(|z| + \|f\|_{E_{-\sigma}})|z|(|z| + \|f\|_{E_{-\sigma}})^2.$$

Notice that $F_2 \in E^c$ is a smooth, exponentially decreasing function. Hence, there exists a solution a_{11} in the form

$$a_{11} = -A^{-1}F_2, \tag{5.5}$$

where A^{-1} stands for the regular part of the resolvent $R(\lambda) = (A - \lambda)^{-1}$ at $\lambda = 0$, since the singular part of $R(\lambda)F_2$ vanishes for $F_2 \in E^c$. The function a_{11} is exponentially decreasing at infinity.

For a_{20} and a_{02} we choose the following inverse operators:

$$a_{20} = -(A - 2\mathrm{i}\mu - 0)^{-1}F_2, \quad a_{02} = \overline{a}_{20} = -(A + 2\mathrm{i}\mu - 0)^{-1}F_2. \tag{5.6}$$

This choice is motivated by Lemma 3.3. The remainder H_I can be written as

$$H_I = \sum_m (A - 2\mathrm{i}\mu m - 0)^{-1} C_m, \quad m \in \{-1, 0, 1\} \tag{5.7}$$

with $C_m \in E^c$, satisfying the estimate

$$\|C_m\|_{E_\sigma} = \mathcal{R}(|z| + \|f\|_{E_{-\sigma}})|z|(|z| + \|f\|_{E_{-\sigma}})^2. \tag{5.8}$$

□

5.2. *Normal form for $\dot z$*

Substituting (5.2) into (4.12) and putting the contribution of $f = h + k + g$ into Z_R, we obtain

$$\begin{aligned}\dot z = \mathrm{i}\mu z + Z_2(z^2 + 2z\bar z + \bar z^2) + Z_3(z^3 + 3z^2\bar z + 3z\bar z^2 + \bar z^3)\\ + Z'_{30}z^3 + Z'_{21}z^2\bar z + Z'_{12}z\bar z^2 + Z'_{03}\bar z^3 + \tilde Z_R,\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}&Z'_{30} = \langle a_{20}, jZ'_1\rangle, \quad Z'_{21} = \langle a_{11} + a_{20}, jZ'_1\rangle, \quad Z'_{03} = \langle a_{02}, jZ'_1\rangle,\\ &Z'_{12} = \langle a_{02} + a_{11}, jZ'_1\rangle\end{aligned} \tag{5.10}$$

by (5.2)–(5.3). We are specially interested in resonance term $Z'_{21}z^2\bar z = Z'_{21}|z|^2 z$. Formulas (4.13), (5.5), (5.6) imply

$$\begin{aligned}Z'_{21} = &-\left\langle A^{-1}P^c\mathcal{N}_2[u,u], 2j\frac{\mathcal{N}_2[u,u]}{\langle u, ju\rangle}\right\rangle\\ &-\left\langle (A - 2\mathrm{i}\mu - 0)^{-1}P^c\mathcal{N}_2[u,u], 2j\frac{\mathcal{N}_2[u,u]}{\langle u, ju\rangle}\right\rangle.\end{aligned} \tag{5.11}$$

For the $\langle u, ju\rangle$ we get

$$\langle u, ju\rangle = \mathrm{i}\delta, \quad \textit{with } \delta > 0. \tag{5.12}$$

Lemma 5.2. *Let non-degeneracy condition* **U3** *holds. Then*

$$\operatorname{Re} Z'_{21} < 0. \tag{5.13}$$

Proof. Coefficient $\langle A^{-1}P^c j\mathcal{N}_2[u,u], 2\mathcal{N}_2[u,u]\rangle$ that appears in (5.11) is real since operator $A^{-1}2P^c j$ is selfadjoint. Hence (5.12) implies that $\operatorname{Re} Z'_{21}$ reduces to

$$\begin{aligned}\operatorname{Re} Z'_{21} &= \operatorname{Re} 2\frac{\langle (A - 2\mathrm{i}\mu - 0)^{-1}P^c\mathcal{N}_2[u,u], j\mathcal{N}_2[u,u]\rangle}{\mathrm{i}\delta}\\ &= \frac{2}{\delta}\operatorname{Im}\langle R(2\mathrm{i}\mu + 0)P^c\mathcal{N}_2[u,u], j\mathcal{N}_2[u,u]\rangle.\end{aligned}$$

Since P^c commutes with $R(2\mathrm{i}\mu+0)$, then $R(2\mathrm{i}\mu+0)P^c = P^c R(2\mathrm{i}\mu+0)P^c$. We have also that $(P^c)^* j = jP^c$, hence

$$\operatorname{Re} Z'_{21} = \frac{2}{\delta}\operatorname{Im}\langle R(2\mathrm{i}\mu+0)\alpha,\, j\alpha\rangle$$

with $\alpha = P^c\mathcal{N}_2[u,u]$. Now we use the representation (see [3], Formula (2.1.9))

$$\begin{aligned}
\langle R(2\mathrm{i}\mu+0)\alpha,\, j\alpha\rangle &= \frac{1}{\mathrm{i}}\int_b^\infty \theta(\lambda)\,\mathrm{d}\lambda\left(\frac{\langle\alpha,\, ju(\mathrm{i}\lambda)\rangle\langle u(\mathrm{i}\lambda),\, j\alpha\rangle}{\mathrm{i}\lambda-2\mathrm{i}\mu-0}\right.\\
&\qquad\left.+\frac{\langle\alpha,\, j\overline{u}(\mathrm{i}\lambda)\rangle\langle\overline{u}(\mathrm{i}\lambda),\, j\alpha\rangle}{-\mathrm{i}\lambda-2\mathrm{i}\mu-0}\right)\\
&= \int_b^\infty \theta(\lambda)\,\mathrm{d}\lambda\left(\frac{\langle u(\mathrm{i}\lambda),\, j\alpha\rangle\overline{\langle u(\mathrm{i}\lambda),\, j\alpha\rangle}}{\lambda-2\mu+\mathrm{i}0}\right.\\
&\qquad\left.+\frac{\langle\overline{u}(\mathrm{i}\lambda),\, j\alpha\rangle\overline{\langle\overline{u}(\mathrm{i}\lambda),\, j\alpha\rangle}}{\lambda+2\mu-\mathrm{i}0}\right).
\end{aligned}\tag{5.14}$$

Since $\dfrac{1}{\nu+\mathrm{i}0} = \mathrm{p.v.}\dfrac{1}{\nu} - \mathrm{i}\pi\delta(\nu)$, where p.v. is the Cauchy principal value, we have

$$\begin{aligned}
\langle R(2\mathrm{i}\mu+0)\alpha,\, j\alpha\rangle &= \int_{\sqrt{2}}^\infty \theta(\lambda)\,\mathrm{d}\lambda\left(\frac{|\langle u(\mathrm{i}\lambda),\, j\alpha\rangle|^2}{\lambda-2\mu}+\frac{|\langle\overline{u}(\mathrm{i}\lambda),\, j\alpha\rangle|^2}{\lambda+2\mu}\right)\\
&\quad -\mathrm{i}\pi\theta(2\mu)|\langle u(2\mathrm{i}\mu),\, j\alpha\rangle|^2.
\end{aligned}\tag{5.15}$$

The integral term in (5.15) is real. Thus,

$$\operatorname{Im}\langle R_T(2\mathrm{i}\mu_T+0)\alpha,\, j\alpha\rangle = -\pi\theta(2\mu)|\langle u(2\mathrm{i}\mu),\, j\alpha\rangle|^2 < 0$$

since $\theta(2\mu) > 0$, and condition **U3** implies that

$$\begin{aligned}
\langle u(2\mathrm{i}\mu),\, j\alpha\rangle &= \langle u(2\mathrm{i}\mu),\, jP^c\mathcal{N}_2[u,u]\rangle = \langle u(2\mathrm{i}\mu),\, j\mathcal{N}_2[u,u]\rangle\\
&= -\int u_1(2\mathrm{i}\mu)(x)N_2[u,u](x)\,\mathrm{d}x\\
&= -\frac{1}{2}\int \varphi_{4\lambda_1}(x)F''(s(x))\varphi_{\lambda_1}^2(x)\,\mathrm{d}x \neq 0.
\end{aligned}$$

□

Now we estimate $\tilde{Z}_R$.

Lemma 5.3. *The bound holds*

$$|\tilde{Z}_R| = \mathcal{R}_1(|z|+\|f\|_{L^\infty})\left[(|z|^2+\|f\|_{E_{-\sigma}})^2+|z|\|g\|_{E_{-\sigma}}+|z|\|h\|_{E_{-\sigma}}\right].\tag{5.16}$$

Proof. The remainder $\tilde{Z}_R$ is given by

$$\tilde{Z}_R = Z_R + (z + \overline{z})\langle f - k, jZ_1' \rangle,$$

where Z_R satisfies (4.10). Since $f - k = g + h$ then $|\langle f - k, Z_1' \rangle| \leqq C(\|g\|_{E_{-\sigma}} + \|h\|_{E_{-\sigma}})$.

Hence, (5.16) follows. □

Now we can apply the Poincaré method of normal coordinates to (5.9).

Lemma 5.4. (cf. [3, Proposition 4.9]) *There exist coefficients c_{ij} such that function $z_1(t)$ defined by*

$$z_1 = z + c_{20}z^2 + c_{11}z\overline{z} + c_{02}\overline{z}^2 + c_{30}z^3 + c_{12}z\overline{z}^2 + c_{03}\overline{z}^3 \tag{5.17}$$

satisfies an equation of the form

$$\dot{z}_1 = i\mu z_1 + iK|z_1|^2 z_1 + \hat{Z}_R, \tag{5.18}$$

where $\hat{Z}_R$ satisfies estimates of the same type as $\tilde{Z}_R$, and

$$\text{Re } iK = \text{Re } Z_{21}' < 0. \tag{5.19}$$

Proof. Substituting z_1 in (5.9) and equating the coefficients, we get, in particular,

$$c_{20} = \frac{\mathrm{i}}{\mu} Z_2, \quad c_{11} = -\frac{2\mathrm{i}}{\mu} Z_2, \quad c_{02} = -\frac{\mathrm{i}}{3\mu} Z_2, \tag{5.20}$$

and

$$\mathrm{i}K = 3Z_3 + Z_{21}' + (4c_{20} - c_{11} - 2c_{20})Z_2. \tag{5.21}$$

Since coefficients Z_2 and Z_3 defined in (4.13) are purely imaginary then (5.19) follows. □

It is easier to deal with $y = |z_1|^2$ rather than z_1 because y decreases at infinity while z_1 is oscillating. Multiplying (5.18) by $\overline{z}_1$ and taking the real part, we obtain

$$\dot{y} = 2\,\text{Re}(\mathrm{i}K)y^2 + Y_R, \tag{5.22}$$

where

$$|Y_R| = \mathcal{R}_1(|z| + \|f\|_{L^\infty})|z| \left[(|z|^2 + \|f\|_{E_{-\sigma}})^2 + |z|\|g\|_{E_{-\sigma}} + |z|\|h\|_{E_{-\sigma}} \right]. \tag{5.23}$$

5.3. Summary of normal forms

We summarize the main formulas of Sections 5.1–5.2. First we recall that

$$f = k + g + h,$$

where k and g are defined in (5.3). The equations satisfied by f and h are respectively (see (4.14) and (5.4))

$$\dot{f} = Af + \tilde{F}_R, \tag{5.24}$$

$$\dot{h} = Ah + H_R. \tag{5.25}$$

Here $\tilde{F}_R = P^c\mathcal{N}_2[w,w] + F_R$, $F_R = F_{\mathrm{I}} + F_{\mathrm{II}} + F_{\mathrm{III}}$, $H_R = F_R + H_I$. The remainders F_{I}, F_{II}, $P^c\mathcal{N}_2[w,w]$ and F_{III} are estimated in (4.17)–(4.19), (4.20). The remainder H_I is estimated in (5.7) and (5.8). Note, that

$$\|f\|_{E_{-\sigma}} \leqq C(\|g\|_{E_{-\sigma}} + |z|^2 + \|h\|_{E_{-\sigma}}). \tag{5.26}$$

The second equation describes the evolution of z_1 from (5.18):

$$\dot{z}_1 = \mathrm{i}\mu z_1 + \mathrm{i}K|z_1|^2 z_1 + \hat{Z}_R, \tag{5.27}$$

where the remainder $\hat{Z}_R$ admits (5.16). The fourth equation is the dynamics for $y = |z_1|^2$

$$\dot{y} = 2\,\mathrm{Re}(\mathrm{i}K)y^2 + Y_R, \tag{5.28}$$

where the remainder Y_R admits (5.23). Here $\mathrm{Re}\,\mathrm{i}K < 0$ by Lemma 5.2.

6. Majorants

We define the 'majorants'

$$\mathcal{M}_1(T) = \max_{0\leqq t\leqq T} |z(t)| \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{-1/2}, \tag{6.1}$$

$$\mathcal{M}_2(T) = \max_{0\leqq t\leqq T} \|f_1(t)\|_{L^\infty} \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{-1/2} \log^{-1}(2+\varepsilon t), \tag{6.2}$$

$$\mathcal{M}_3(T) = \max_{0\leqq t\leqq T} \|h(t)\|_{E_{-5/2-\nu}} \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{-3/2} \log^{-1}(2+\varepsilon t), \tag{6.3}$$

and denote by $\mathcal{M}$ the three-dimensional vector $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$. The goal of this section is to prove that all these majorants are bounded uniformly in T for sufficiently small $\varepsilon > 0$.

6.1. Bound for g

Lemma 6.1. *Function $g(t)$ defined in (5.3) admits the bound*

$$\|g(t)\|_{E_{-\sigma}} \leqq c|z(0)|^2 \frac{1}{(1+t)^{3/2}} \leqq c\frac{\varepsilon}{(1+t)^{3/2}}, \quad \sigma > 5/2. \tag{6.4}$$

Proof. By (5.3), we have $g = -e^{At}k(0)$ and $k(0) = a_{20}z^2(0) + a_{11}z(0)\overline{z}(0) + a_{02}\overline{z}^2(0)$ with a_{ij} defined in (5.5)–(5.6). Then (6.4) follows by Lemma 3.3. □

6.2. Bounds for remainders

Here we rewrite bounds for remainders in terms of majorants.

Lemma 6.2. *The remainder Y_R defined in (5.23) admits the estimate*

$$|Y_R| = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\frac{\varepsilon^{5/2}}{(1+\varepsilon t)^2\sqrt{\varepsilon t}}\log(2+\varepsilon t)(1+|\mathcal{M}|)^5. \tag{6.5}$$

Proof. Using the equality $f = k + g + h$ and estimate (5.23), we obtain

$$\begin{aligned}
|Y_R| &= \mathcal{R}_2(|z|+\|f\|_{L^\infty})|z|\left[(|z|^2+\|g\|_{E_{-\sigma}}+\|h\|_{E_{-\sigma}})^2+|z|(\|g\|_{E_{-\sigma}}+\|h\|_{E_{-\sigma}})\right]\\
&= \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{1/2}\mathcal{M}_1\left[\left(\frac{\varepsilon}{1+\varepsilon t}\mathcal{M}_1^2+\frac{\varepsilon}{(1+t)^{3/2}}+\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}\right.\right.\\
&\quad\left.\times\log(2+\varepsilon t)\mathcal{M}_3\right)^2+\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{1/2}\mathcal{M}_1\left(\frac{\varepsilon}{(1+t)^{3/2}}+\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}\right.\\
&\quad\left.\left.\times\log(2+\varepsilon t)\mathcal{M}_3\right)\right].
\end{aligned}$$

Hence, (6.2) follows. □

Now let us turn to the remainders $F_R = F_{\mathrm{I}}+F_{\mathrm{II}}+F_{\mathrm{III}}$, $\tilde{F}_R = P^c\mathcal{N}_2[w,w]+F_R$, and $H_R = F_R + H_I$ in equations (5.24) and (5.25) for f and h respectively.

Lemma 6.3. *For $0 < \nu < 1/2$ the remainder F_R admits the bound*

$$\begin{aligned}
\|F_R\|_{E_{5/2+\nu}} = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}\log(2+\varepsilon t)\Big((\mathcal{M}_1+\mathcal{M}_2)(1+\mathcal{M}_1^2)\\
+\varepsilon^{1/2-\nu}(1+|\mathcal{M}|)^{12}\Big).
\end{aligned} \tag{6.6}$$

Proof. Step (i) Applying (4.17) with $\sigma = 5/2 + \nu$ and (5.26) we obtain

$$\|F_{\mathrm{I}} + F_{\mathrm{II}}\|_{E_\sigma} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^3 + |z|\|f\|_{E_{-\sigma}} + \|f_1\|_{L^\infty}\|f\|_{E_{-\sigma}})$$
$$= \mathcal{R}(|z| + \|f_1\|_{L^\infty})\Big[|z|^3 + |z|\|g\|_{E_{-\sigma}} + |z|\|h\|_{E_{-\sigma}} + \|f_1\|_{L^\infty}(|z|^2 + \|g\|_{E_{-\sigma}} + \|h\|_{E_{-\sigma}})\Big]$$
$$= \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Bigg(\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\mathcal{M}_1^3 + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\frac{\varepsilon}{(1+t)^{3/2}}\mathcal{M}_1 + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{2}\log(2+\varepsilon t)\mathcal{M}_1\mathcal{M}_3 + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\log(2+\varepsilon t)\mathcal{M}_2 \times\Big[\frac{\varepsilon}{1+\varepsilon t}\mathcal{M}_1^2 + \frac{\varepsilon}{(1+t)^{3/2}} + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\log(2+\varepsilon t)\mathcal{M}_3\Big]\Bigg)$$

which implies (6.6) for $F_{\mathrm{I}} + F_{\mathrm{II}}$.

Step (ii) Further, by (4.19)

$$\|F_{\mathrm{III}}\|_{E_{5/2+\nu}} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(1+t)^{4+\nu}(|z|^{12} + \|f_1\|_{L^\infty}^{12})$$
$$= \mathcal{R}(\varepsilon^{1/2}\mathcal{M})(1+t)^{4+\nu}\log^{12}(2+\varepsilon t)\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{6}(M_1^{12} + M_2^{12}), \quad (6.7)$$

and then (6.6) for F_{III} follows.

Lemma 6.4. *For $0 < \nu < 1/2$ the remainder $\tilde F_R$ admits the bound*

$$\|\tilde F_R\|_{E_{5/2+\nu}\cap W} = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\frac{\varepsilon}{1+\varepsilon t}\Big(M_1^2 + \varepsilon^{1/2}(1+|\mathcal{M}|)^{12}\Big). \quad (6.8)$$

For F_{I} and F_{II} the bound follows from (4.17). Further, by (4.20)

$$\|F_{\mathrm{III}}\|_W = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^{11} + \|f_1\|_{L^\infty}^{11})$$
$$= \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{5/2}\log^{11}(2+\varepsilon t)(M_1^{11} + M_2^{11}),$$

which together with (6.7) implies (6.8) for F_{III}. Finally, (4.18) implies

$$\|P^c\mathcal{N}_2[w,w]\|_{E_\sigma\cap W} = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\frac{\varepsilon}{1+\varepsilon t}\mathcal{M}_1^2,$$

and then (6.8) follows. □

The term H_I is represented by (5.7) with C_m estimated in (5.8). For C_m we now obtain

Lemma 6.5. *For $m = 0, \pm 1$, the bounds hold*

$$\|C_m\|_{E_\sigma} = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\Big(\mathcal{M}_1^3 + \varepsilon^{1/2}(1+|\mathcal{M}|)^3\Big). \quad (6.9)$$

Proof. Estimate (5.8) implies

$$\|C_m\|_{E_\sigma} = \mathcal{R}(|z| + \|f\|_{E_{-\sigma}})|z|(|z| + \|g\|_{E_{-\sigma}} + \|h\|_{E_{-\sigma}})^2$$
$$= \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{1/2}\mathcal{M}_1\left(\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{1/2}\mathcal{M}_1 + \frac{\varepsilon}{(1+t)^{3/2}}\right.$$
$$\left. + \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}\log(2+\varepsilon t)\mathcal{M}_2\right)^2,$$

which implies (6.9). □

6.3. Initial conditions

We assume the smallness of the initial condition:

$$|z(0)| \leqq \varepsilon^{1/2}, \quad \|f(0)\|_{E_\sigma} = \|h(0)\|_{E_\sigma} \leqq \varepsilon^{3/2}h_0,$$
$$\|f(0)\|_{E_\sigma \cap W} \leqq \varepsilon^{1/2}f_0, \tag{6.10}$$

where h_0, f_0 are some fixed constant, and $\varepsilon > 0$ is sufficiently small by (2.6). Equation (5.17) implies $|z_1|^2 \leqq |z|^2 + \mathcal{R}(|z|)|z|^3$. Therefore

$$y_0 = y(0) = |z_1(0)|^2 \leqq \varepsilon + C(|z(0)|)\varepsilon^{3/2}. \tag{6.11}$$

6.4. Estimates via majorants

This section is devoted to studying equations (5.24), (5.25), (5.28) for f, h and y under assumptions (6.10) on initial data and estimates (6.5)-(6.9) of the remainders.

First we consider equation (5.28) for y which is of Ricatti type.

Lemma 6.6. [3, Proposition 5.6] *The solution to* (5.28) *with an initial condition and a remainder satisfying* (6.11) *and* (6.5)*, respectively, admits the bound:*

$$\left|y - \frac{y_0}{1 + 2y_0 \operatorname{Im} Kt}\right| \leqq \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\frac{\varepsilon^{5/2}}{(1+\varepsilon t)^2\sqrt{\varepsilon t}}\log(2+\varepsilon t)(1+|\mathcal{M}|)^5. \tag{6.12}$$

Corollary 6.7. *The majorant $\mathcal{M}_1$ satisfies*

$$\mathcal{M}_1^2 = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\left(1 + \varepsilon^{1/2}(1+|\mathcal{M}|)^5\right). \tag{6.13}$$

Proof. Bounds (6.11) and (6.12) imply

$$y \leqq \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\left[\frac{\varepsilon}{1+\varepsilon t} + \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}\log(2+\varepsilon t)(1+|\mathcal{M}|)^5\right].$$

Using that $|z|^2 \leqq y + \mathcal{R}(|z|)|z|^3$, we get

$$|z|^2 \leqq \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\left[\frac{\varepsilon}{1+\varepsilon t} + \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}\log(2+\varepsilon t)(1+|\mathcal{M}|)^5\right.$$
$$\left. + \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}\mathcal{M}_1^3\right].$$

Hence, (6.13) follows. □

Second, we consider Equation (5.24) for f.

Lemma 6.8. *The solution to* (5.24) *admits the bound*

$$\|f_1\|_{L^\infty} \leqq C\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{1/2}\log(2+\varepsilon t)\Big(f_0+\mathcal{R}(\varepsilon^{1/2}\mathcal{M})\\ (\mathcal{M}_1^2+\varepsilon^{1/2}(1+|\mathcal{M}|)^{12}\Big). \tag{6.14}$$

Proof. We have

$$f(t)=\mathrm{e}^{At}f(0)+\int_0^t \mathrm{e}^{A(t-\tau)}\tilde{F}_R(\tau)\,\mathrm{d}\tau.$$

Using the the bounds (3.11) and the estimates (6.8), (6.10), we obtain

$$\begin{aligned}\|f_1\|_{L^\infty} &\leqq \frac{C}{(1+t)^{1/2}}\|f(0)\|_{E_\sigma\cap W}+\int_0^t \frac{C}{(1+(t-\tau))^{1/2}}\|\tilde{F}_R(\tau)\|_{E_\sigma\cap W}\,\mathrm{d}\tau\\ &\leqq C\left[f_0\left(\frac{\varepsilon}{1+t}\right)^{1/2}+\mathcal{R}(\varepsilon^{1/2}\mathcal{M})(\mathcal{M}_1^2+\varepsilon^{1/2}(1+|\mathcal{M}|)^{12}\right.\\ &\qquad\left.\times\int_0^t\frac{\mathrm{d}\tau}{(t-\tau)^{1/2}}\frac{\varepsilon}{1+\varepsilon\tau}\,\mathrm{d}\tau\right].\end{aligned}$$

Hence, (6.14) follows. □

Corollary 6.9.

$$\mathcal{M}_2=\mathcal{R}(\varepsilon^{1/2}\mathcal{M})\left(M_1^2+\varepsilon^{1/2}(1+|\mathcal{M}|)^{12}\right). \tag{6.15}$$

Finally, we consider Equation (5.25) for h.

Lemma 6.10. *The solution to* (5.25) *admits the bound*

$$\|h\|_{E_{-\sigma}} \leqq C\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}\log(2+\varepsilon t)\Big(h_0+\mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big[(1+\mathcal{M}_1^2+\mathcal{M}_2)\\ \times(1+\mathcal{M}_1^2)+\varepsilon^{1/2-\nu}(1+|\mathcal{M}|)^{12}\Big]\Big). \tag{6.16}$$

Proof. We have

$$h=\mathrm{e}^{At}h(0)+\int_0^t \mathrm{e}^{A(t-\tau)}H_R(\tau)\,\mathrm{d}\tau.$$

Bounds (6.10), (6.6), (6.9), Proposition 3.1 and Corollary 3.3 imply

$$
\begin{aligned}
\|h\|_{E_{-\sigma}} \leqq{} & \frac{C}{(1+t)^{3/2}}\|h(0)\|_{E_\sigma} + \int_0^t \frac{C}{(1+(t-\tau))^{3/2}}\Bigg(\|F_R(\tau)\|_{E_\sigma} \\
& + \sum_m \|C_m(\tau)\|_{E_\sigma}\Bigg)\,\mathrm{d}\tau \\
\leqq{} & C\left[h_0\left(\frac{\varepsilon}{1+t}\right)^{\frac{3}{2}} + \mathcal{R}(\varepsilon^{\frac{1}{2}}\mathcal{M})\left[(\mathcal{M}_1+\mathcal{M}_2)(1+\mathcal{M}_1^2)\right.\right. \\
& \left. +\varepsilon^{\frac{1}{2}-\nu}(1+|\mathcal{M}|)^{12}\right]\int_0^t \frac{\log(2+\varepsilon t)\,\mathrm{d}\tau}{(1+(t-\tau))^{\frac{3}{2}}}\left(\frac{\varepsilon}{1+\varepsilon\tau}\right)^{\frac{3}{2}} \\
& + \sum_m \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\left(\mathcal{M}_1^3 + \varepsilon^{1/2}(1+|\mathcal{M}|)^3\right)\int_0^t \frac{\mathrm{d}\tau}{(1+(t-\tau))^{3/2}} \\
& \left.\left(\frac{\varepsilon}{1+\varepsilon\tau}\right)^{3/2}\right],
\end{aligned}
$$

which implies (6.16). □

Corollary 6.11.

$$
\mathcal{M}_3 = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\left[(1+\mathcal{M}_1^2+\mathcal{M}_2)(1+\mathcal{M}_1^2) + \varepsilon^{1/2-\nu}(1+|\mathcal{M}|)^{12}\right]. \quad (6.17)
$$

6.5. Uniform bounds for majorants

The aim of this section is to prove that if ε is sufficiently small, all the $\mathcal{M}_i$ are bounded uniformly in T and ε.

Lemma 6.12. *For ε sufficiently small, there exists a constant M independent of T and ε, such that,*

$$
|\mathcal{M}(T)| \leqq M. \quad (6.18)
$$

Proof. Combining (6.13), (6.15), and (6.17) we obtain

$$
\mathcal{M}^2 \leqq \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\left[(1+\mathcal{M}_1^2+\mathcal{M}_2)^4 + \varepsilon^{1/2-\nu}(1+|\mathcal{M}|)^{24}\right].
$$

Replacing $\mathcal{M}_1^2$ and $\mathcal{M}_2$ by its bound (6.13) and (6.15), we get

$$
\mathcal{M}^2 \leqq \mathcal{R}(\varepsilon^{1/2}\mathcal{M})(1+\varepsilon^{1/2-\nu}F(\mathcal{M})),
$$

where $F(\mathcal{M})$ is an appropriate function. The last inequality implies that $\mathcal{M}$ is bounded uniformly in ε, since $\mathcal{M}(0)$ is small and $\mathcal{M}(t)$ is continuous. □

Corollary 6.13. *For $t > 0$ and $\sigma > 5/2$ the bounds hold*

$$|z(t)| \leqq M\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{1/2}, \tag{6.19}$$

$$\|f_1\|_{L^\infty} \leqq M\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{1/2}\log(1+\varepsilon t), \tag{6.20}$$

$$\|h\|_{E_{-\sigma}} \leqq M\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}\log(1+\varepsilon t), \tag{6.21}$$

$$\|f\|_{E_{-\sigma}} \leqq M\left(\frac{\varepsilon}{1+\varepsilon t}\right). \tag{6.22}$$

Thus we have proved the following result:

Theorem 6.14. *Let the conditions of Theorem 2.3 hold. Then*

(i) *for ε small enough, one can write the solution of (2.1) in the form*

$$Y(x,t) = s(x) + (z(t) + \overline{z}(t))u + f(x,t), \tag{6.23}$$

(ii) *in addition, for all $t > 0$, there exists a constant $M > 0$ such that*

$$|z(t)| \leqq M\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{1/2}, \quad \|f\|_{E_{-\sigma}} \leqq M\left(\frac{\varepsilon}{1+\varepsilon t}\right), \quad \sigma > 5/2. \tag{6.24}$$

7. Soliton asymptotics

7.1. Long time behavior of $z(t)$

We start with Equation (5.18) for z_1. By (5.16) the remainder $\widehat{Z}_R$ satisfies

$$\widehat{Z}_R = \mathcal{R}(\varepsilon^{1/2}M)\frac{\varepsilon^2\log(2+\varepsilon t)}{(1+\varepsilon t)^{3/2}\sqrt{\varepsilon t}}(1+M^4) \leqq \frac{C\varepsilon^2\log(2+\varepsilon t)}{(1+\varepsilon t)^{3/2}\sqrt{\varepsilon t}}.$$

On the other hand, (6.11) and (6.12) imply

$$\left|y - \frac{y_0}{1+2\operatorname{Im}Ky_0t}\right| \leqq C\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2}\log(2+\varepsilon t)$$

with $|y_0 - \varepsilon| \leqq C\varepsilon^{3/2}$. With estimate (6.19) for $|z|$ and (obviously) the same one for $|z_1|$, we have

$$\dot{z}_1 = \mathrm{i}\mu z_1 + \mathrm{i}K\frac{y_0}{1+2\operatorname{Im}Ky_0t}z_1 + Z_1 \tag{7.1}$$

with

$$|Z_1| \leqq \frac{C\varepsilon^2\log(2+\varepsilon t)}{(1+\varepsilon t)^{3/2}\sqrt{\varepsilon t}}. \tag{7.2}$$

Since $y_0 = \varepsilon + \mathcal{O}(\varepsilon^{3/2})$, we have that $2\,\mathrm{Im}\,K y_0 = k\varepsilon$. We also denote $\rho = \frac{\mathrm{Re}\,K}{\mathrm{Im}\,K}$. The solution z_1 of (7.1) is written in the form

$$
\begin{aligned}
z_1 &= \frac{\mathrm{e}^{\mathrm{i}\mu t}}{(1+k\varepsilon t)^{1/2-\mathrm{i}\rho}}\left[z_1(0) + \int_0^t \mathrm{e}^{-\mathrm{i}\mu s}(1+k\varepsilon s)^{1/2-\mathrm{i}\rho} Z_1(s)\,\mathrm{d}s\right] \\
&= z_{L\infty}\frac{\mathrm{e}^{\mathrm{i}\mu t}}{(1+k\varepsilon t)^{1/2-\mathrm{i}\rho}} + z_R,
\end{aligned}
$$

where

$$
\begin{aligned}
z_\infty &= z_1(0) + \int_0^\infty \mathrm{e}^{-\mu s}(1+k\varepsilon s)^{1/2-\mathrm{i}\rho} Z_1(s)\,\mathrm{d}s, \\
z_R &= -\int_t^\infty \mathrm{e}^{\mathrm{i}\mu t}\left(\frac{1+k\varepsilon s}{1+k\varepsilon t}\right)^{1/2-\mathrm{i}\rho} Z_1(s)\,\mathrm{d}s.
\end{aligned}
$$

From (7.2) it follows that $|z_R| \leqq \dfrac{C\varepsilon\log(2+\varepsilon t)}{(1+\varepsilon t)}$. Therefore $z_1(t)$ satisfies the estimate

$$
z_1(t) = z_\infty\frac{\mathrm{e}^{\mathrm{i}\mu t}}{(1+k\varepsilon t)^{1/2-\mathrm{i}\rho}} + \mathcal{O}\left(\frac{\varepsilon}{1+\varepsilon t}\log(2+\varepsilon t)\right). \tag{7.3}
$$

Here $z_\infty = z_1(0) + \mathcal{O}(\varepsilon)$, $z = z_1 + \mathcal{O}(\dfrac{\varepsilon}{1+\varepsilon t})$, and $|z(0)| = \varepsilon^{1/2}$. Thus $|z_\infty| = \varepsilon^{1/2} + \mathcal{O}(\varepsilon)$. Hence,

$$
z(t) = z_\infty\frac{\mathrm{e}^{\mathrm{i}\mu t}}{(1+k\varepsilon t)^{1/2-\mathrm{i}\rho}} + \mathcal{O}\left(\frac{\varepsilon}{1+\varepsilon t}\log(2+\varepsilon t)\right). \tag{7.4}
$$

7.2. Asymptotic completeness

Here we prove our main Theorem 2.3. We have obtained solution $Y(x, y)$ to (2.1) in the form

$$
Y = S + w + f.
$$

We include w into the remainder $r_\pm$ from (2.7) since $z(t) \sim t^{-1/2}$ by (7.4). It remains to extract the dispersive wave $W(t)\Phi_\pm$ from f. We rewrite (4.14) as

$$
\begin{cases} \dot f_1 = f_2 + Q_1 \\ \dot f_2 = f_1'' - m^2 f_1 + Q_2 \end{cases}, \tag{7.5}
$$

where

$$
\begin{aligned}
Q_1 &= (P^c\mathcal{N})_1 = -(P^d\mathcal{N})_1 = -\frac{1}{\mathrm{i}\delta}\langle N, u_1\rangle u_1 + \frac{1}{\mathrm{i}\delta}\langle N, u_1\rangle u_1 = 0 \\
Q_2 &= (P^c\mathcal{N})_2 = (P^c\mathcal{N}_2[w, w])_2 + (F_R)_2 - V f_1
\end{aligned}
$$

by (3.9) and (5.12). Then

$$\begin{aligned} f(t) &= W_0(t) f(0) + \int_0^t W_0(t-\tau) Q(\tau)\,\mathrm{d}\tau \\ &= W_0(t)\left(f(0) + \int_0^\infty W_0(-\tau) Q(\tau)\,\mathrm{d}\tau \right) \\ &\quad - \int_t^\infty W_0(t-\tau) Q(\tau)\,\mathrm{d}\tau = W_0(t)\phi_+ + r_+(t). \end{aligned} \tag{7.6}$$

Here $Q(t) := (0, Q_2(t))$. Equation (7.6) implies asymptotics of type (2.7) and (2.8), if all the integrals converge. To complete the proof it remains to prove the following proposition.

Proposition 7.1. *The bound holds*

$$\|r_+(t)\|_E = \mathcal{O}(t^{-1/3}), \quad t \to \infty. \tag{7.7}$$

Proof. To check (7.7), we should obtain an appropriate decay for $Q_2(t)$.
Step (i) According to (4.15), (4.17), (4.19), (6.19), (6.20), and (6.22), we have

$$\|(F_R)_2\|_{L^2} = \mathcal{O}(t^{-3/2}\log t). \tag{7.8}$$

Further, (3.9), (4.11), and (4.13) imply

$$\begin{aligned} (P^c \mathcal{N}_2[w,w])_2 &= N_2[w,w] - (P^d \mathcal{N}_2[w,w])_2 \\ &= (z^2 + 2z\overline{z} + \overline{z}^2)\,(N_2[u,u] - 2\mathrm{i}\mu u_1 Z_2). \end{aligned}$$

Hence, from (5.2) and (5.3) it follows that

$$Q_2 = q_{20} z^2 + 2q_{11} z\overline{z} + q_{02}\overline{z}^2 + Q_{2R} \tag{7.9}$$

with

$$q_{ij} = N_2[u_1,u_1] - 2iZ_2\mu u_1 - Va_{ij,1}, \quad Q_{2R} = (F_R)_2 - V(f_1 - k_1), \tag{7.10}$$

where $a_{ij,1}$ and k_1 are the first components of vector-functions a_{ij} and k from (5.3). By (1.10), (5.2), (6.4) and (6.21) we have

$$\|V(f_1 - k_1)\|_{L^2} = \mathcal{O}(t^{-3/2}\log t), \quad t \to \infty.$$

The last bound and (B.5) imply that

$$\|Q_{2R}\|_{L^2} = \mathcal{O}(t^{-3/2}\log t), \quad t \to \infty. \tag{7.11}$$

Therefore, the term Q_{2R} gives the contribution of order $\mathcal{O}(t^{-1/2}\log t)$ to $r_+(t)$.
Step (ii) It remains to estimate the contribution to $r_+(t)$ of the quadratic term from (7.9). Functions $q_{ij}(x)$ are smooth with exponential decay at infinity since $a_{ij} \in H^s_{-\sigma}$ with any $s > 0$ by Lemma 5.1. On the other hand, time decay of functions $z^2(t), z(t)\overline{z}(t), \overline{z}^2(t)$ is very slow like $\mathcal{O}(t^{-1})$. Therefore, the integral

representing the contribution of the quadratic term to $r_+(t)$ does not converge absolutely. Fortunately, we may define the integral as

$$\int_t^\infty W(t-\tau)(q_{20}z^2 + 2q_{11}z\overline{z} + q_{02}\overline{z}^2)\,\mathrm{d}\tau$$
$$:= \lim_{T\to\infty}\int_t^T W(t-\tau)(q_{20}z^2 + 2q_{11}z\overline{z} + q_{02}\overline{z}^2)\,\mathrm{d}\tau.$$

We prove below the convergence of the integral with the values in E and the decay rate $\mathcal{O}(t^{-1/3})$.

First we estimate the contribution of $q_{11}(x)z\overline{z}$. Note that (7.4) implies the asymptotics $z\overline{z} \sim (1+k\varepsilon t)^{-1}$.

Lemma 7.2. *Let $q(x) \in L^2(\mathbb{R})$. Then*

$$I(t) := \left\| \int_t^\infty W_0(-\tau)\begin{pmatrix}0\\ q\end{pmatrix}\frac{\mathrm{d}\tau}{1+\tau}\right\|_E = \mathcal{O}(t^{-1}), \quad t\to\infty. \tag{7.12}$$

Proof. Denote $\omega = \omega(\xi) = \sqrt{\xi^2+m^2}$. Then

$$I(t) = \left\| \int_t^\infty \begin{pmatrix} -\sin\omega\tau\ \hat{q}(\xi)\\ -\cos\omega\tau\ \hat{q}(\xi)\end{pmatrix}\frac{\mathrm{d}\tau}{1+\tau}\right\|_{L^2\oplus L^2} \leqq \frac{C}{1+t}\|\hat{q}(\xi)/\omega(\xi)\|_{L^2} \tag{7.13}$$

since the partial integration implies that

$$\left|\int_t^\infty \frac{\mathrm{e}^{\mathrm{i}\omega\tau}}{1+\tau}\,\mathrm{d}\tau\right| = \left|\int_t^\infty \frac{d\mathrm{e}^{\mathrm{i}\omega\tau}}{\mathrm{i}\omega(1+\tau)}\,\mathrm{d}\tau\right| \leqq \left|\frac{\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega(1+t)}\right| + \left|\int_t^\infty \frac{\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega(1+\tau)^2}\,\mathrm{d}\tau\right|$$
$$\leqq \frac{C}{\omega(1+t)}. \tag{7.14}$$

□

Next we estimate the contribution of $q_{20}(x)z^2$ and $q_{02}(x)\overline{z}^2$ (see [3, Proposition 6.5]). By (7.4) we have $z^2 \sim \mathrm{e}^{2\mathrm{i}\mu\tau}/(1+k\varepsilon t)^{1-2\mathrm{i}\rho}$ and $\overline{z}^2 \sim \mathrm{e}^{-2\mathrm{i}\mu\tau}/(1+k\varepsilon t)^{1+2\mathrm{i}\rho}$.

Lemma 7.3. *Let $q(x) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Then*

$$\left\| \int_t^\infty W_0(-\tau)\begin{pmatrix}0\\ q\end{pmatrix}\frac{e^{\pm 2i\mu\tau}\,\mathrm{d}\tau}{(1+\tau)^{1\mp 2i\rho}}\right\|_E = \mathcal{O}(t^{-1/3}), \quad t\to\infty. \tag{7.15}$$

Proof. We consider, for example, the integral with $\mathrm{e}^{-2\mathrm{i}\mu\tau}$ and omit for simplicity the factor $(1+t)^{2\mathrm{i}\rho}$, since with the factor the proof is similar. Let us represent $\sin\omega\tau$ and $\cos\omega\tau$ as a linear combination of $\mathrm{e}^{\mathrm{i}\omega\tau}$ and $\mathrm{e}^{-\mathrm{i}\omega\tau}$. The contribution of "nonresonant" terms with the $\mathrm{e}^{-\mathrm{i}\omega\tau}$ to (7.15) is $\mathcal{O}(t^{-1})$, similarly to (7.13) and (7.14). It remains to prove that

$$I(t) = \left\| \int_t^\infty \frac{\mathrm{e}^{\mathrm{i}(\omega-2\mu)\tau}\hat{q}(\xi)\,\mathrm{d}\tau}{1+\tau}\right\|_{L^2} = \mathcal{O}(t^{-1/3}). \tag{7.16}$$

For a fixed $\beta > 0$, let us denote

$$\chi_\tau(\xi) = \begin{cases} 1, & |\omega(\xi) - 2\mu| \leqq 1/\tau^\beta \\ 0, & |\omega(\xi) - 2\mu| > 1/\tau^\beta \end{cases}.$$

Then

$$I(t) \leqq \left\| \int_t^\infty \frac{e^{i(2\omega-\mu)\tau} \chi_\tau(\xi)\hat{q}(\xi)\, d\tau}{1+\tau} \right\|_{L^2} + \left\| \int_t^\infty \frac{e^{i(2\omega-\mu)\tau}(1-\chi_\tau(\xi))\hat{q}(\xi)\, d\tau}{1+\tau} \right\|_{L^2}$$
$$= I_1(t) + I_2(t).$$

Since $\hat{q}(\xi)$ is bounded function, and $\|\chi_\tau\|^2 \leqq 1/\tau^\beta$, we have

$$I_1(t) \leqq C\|\hat{q}\|_{L^\infty}/(1+t)^{\beta/2}.$$

On the other hand, partial integration implies that

$$I_2(t) = \left\| \int_t^\infty \frac{(1-\chi_\tau(\xi))\hat{q}(\xi)\, de^{i(2\omega-\mu)\tau}}{(2\omega-\mu)(1+\tau)} \right\|_{L^2} \leqq \frac{Ct^\beta}{1+t}\|\hat{q}\|_{L^2} + C\int_t^\infty \frac{\tau^\beta\, d\tau}{(1+\tau)^2}$$
$$\|\hat{q}\|_{L^2}$$
$$\leqq \frac{C\|\hat{q}\|_{L^2}}{(1+t)^{1-\beta}}.$$

Equating $\beta/2 = 1-\beta$, we get $\beta = 2/3$. $\square$

Proposition 7.1 is proved. $\square$

Appendix A. Virial type estimates

Here we prove weighted estimate (4.21). Let us recall that we split the solution $Y(t) = (\psi(\cdot,t), \pi(\cdot,t)) = S + X(t)$, and denote $X(t) = (\Psi(t), \Pi(t))$, $X_0 = (\Psi_0, \Pi_0) := (\Psi(0), \Pi(0))$.

Proposition 1.1. *Let condition* **U1** *hold, and let X_0 satisfy* (2.6) *with $\sigma = 5/2+\nu$. Then the bound holds*

$$\|\Psi(t)\|_{L^2_{5/2+\nu}} \leqq C(d_0)(1+t)^{4+\nu}, \quad t > 0. \tag{A.1}$$

We will deduce the proposition from the following two lemmas. Denote

$$e(x,t) = \frac{|\pi(x,t)|^2}{2} + \frac{|\psi'(x,t)|^2}{2} + U(\psi(x,t)).$$

Lemma 1.2. *For the solution $\psi(x,t)$ to* (1.1) *the local energy estimate holds*

$$\int_a^b e(x,t)\, dx \leqq \int_{a-t}^{b+t} e(x,0)\, dx, \quad a < b, \quad t > 0. \tag{A.2}$$

Proof. The estimate follows by standard arguments: multiplication (1.1) by $\dot\psi(x,t)$ and integration over trapezium $ABCD$, where $A=(a-t,0)$, $B=(a,t)$, $C=(b,t)$, $D=(b+t,0)$. Then (A.2) follows by partial integration using that $U(\psi)\geqq 0$. □

Lemma 1.3. *For any* $\sigma\geqq 0$

$$\int(1+|x|^\sigma)e(x,t)\,\mathrm{d}x\leqq C(\sigma)(1+t)^{\sigma+1}\int(1+|x|^\sigma)e(x,0)\,\mathrm{d}x. \tag{A.3}$$

Proof. By (A.2)

$$\int(1+|x|^\sigma)\left(\int_{x-1}^{x}e(y,t)\,\mathrm{d}y\right)\mathrm{d}x\leqq\int(1+|x|^\sigma)\left(\int_{x-1-t}^{x+t}e(y,0)\,\mathrm{d}y\right)\mathrm{d}x.$$

Hence,

$$\int e(y,t)\left(\int_{y}^{y+1}(1+|x|^\sigma)\,\mathrm{d}x\right)\mathrm{d}y\leqq\int e(y,0)\left(\int_{y-t}^{y+t+1}(1+|x|^\sigma)\,\mathrm{d}x\right)\mathrm{d}y.$$

and then (A.3) follows. □

Proof of Proposition 1.1. First we verify that

$$U_0:=\int(1+|x|^{5+2\nu})U(\psi_0(x))\,\mathrm{d}x<\infty,\quad \psi_0(x)=\psi(x,0). \tag{A.4}$$

Indeed, $\psi_0(x)=s(x)+\Psi_0(x)$ is bounded since $\Psi_0\in H^1(\mathbb{R})$. Hence by **U1**

$$|U(\psi_0(x))|\leqq C(d_0)(\psi_0(x)\pm a)^2\leqq C(d_0)\left((s(x)\pm a)^2+\Psi_0(x)^2\right),$$

and then (A.4) follows by (2.6). Now (A.3) with $\sigma=5+2\nu$ and (2.6), (A.4) imply that

$$\begin{aligned}\|\Psi(t)\|^2_{L^2_{5/2+\nu}}&=\int(1+|x|^{5+2\nu})\left(\int_0^t\dot\Psi(x,s)\,\mathrm{d}s-\Psi_0(x)\right)^2\mathrm{d}x\\&\leqq 2\int(1+|x|^{5+2\nu})\Psi_0^2(x)\,\mathrm{d}x+2t\int(1+|x|^{5+2\nu})\,\mathrm{d}x\int_0^t\pi^2(x,s)\,\mathrm{d}s\\&\leqq 2d_0^2+2t\left[\|X_0\|^2_{E_{5/2+\nu}}+U_0\right]\int_0^t(1+s)^{6+2\nu}\,\mathrm{d}s\leqq C(d_0)(1+t)^{8+2\nu}.\end{aligned}$$

Appendix B. Proof of Proposition 3.3

First we prove the following lemma. Denote by B a Banach space with the norm $\|\cdot\|$.

Lemma 2.1. *Let $L(\nu) \in B$, $\nu \in \mathbb{R}$, and*

$$K(t) = \int \zeta(\nu) e^{i\nu t} Q(\nu)\,\mathrm{d}\nu, \quad Q(\nu) := \frac{L(\nu) - L(\nu_0)}{\nu - \nu_0}, \tag{B.1}$$

where $\zeta \in C_0^\infty(\mathbb{R})$, and

$$M_k := \sup_{\nu \in \mathrm{supp}\,\zeta} \|\partial_\nu^k L(\nu)\| < \infty, \quad k = 0, 1, 2. \tag{B.2}$$

Then

$$\|K(t)\| = \mathcal{O}(t^{-3/2}), \quad t \to \infty. \tag{B.3}$$

Proof. We take $\varphi \in C_0^\infty(\mathbb{R})$ and split $\zeta = \zeta_{1t} + \zeta_{2t}$, where

$$\zeta_{1t}(\nu) := \zeta(\nu)\varphi((\nu - \nu_0)\sqrt{t}), \quad \zeta_{2t}(\nu) := \zeta(\nu)[1 - \varphi((\nu - \nu_0)\sqrt{t})]. \tag{B.4}$$

Then

$$K(t) = \int \zeta_{1t}(\nu) \mathrm{e}^{\mathrm{i}\nu t} Q(\nu)\,\mathrm{d}\nu + \int \zeta_{2t}(\nu) \mathrm{e}^{\mathrm{i}\nu t} Q(\nu)\,\mathrm{d}\nu = K_1(t) + K_2(t).$$

Step (i) Integrating twice by parts, we obtain

$$\begin{aligned} K_1(t) = &-\frac{1}{it} \int_{|\nu - \nu_0| < \frac{1}{\sqrt{t}}} \zeta_{1t} \mathrm{e}^{\mathrm{i}\nu t} Q'(\nu)\,\mathrm{d}\nu - \frac{1}{t^2} \int_{|\nu - \nu_0| < \frac{1}{\sqrt{t}}} \zeta''_{1t} \mathrm{e}^{\mathrm{i}\nu t} Q(\nu)\,\mathrm{d}\nu \\ &-\frac{1}{t^2} \int_{|\nu - \nu_0| < \frac{1}{\sqrt{t}}} \zeta'_{1t} \mathrm{e}^{\mathrm{i}\nu t} Q'(\nu)\,\mathrm{d}\nu. \end{aligned}$$

Since $|\partial_\nu^k \zeta_{jt}(\nu)| \leqq C(k) t^{k/2}$, $j = 1, 2$, and

$$\|Q(\nu)\| = \frac{\left\| \int_{\nu_0}^{\nu} L'(r)\,\mathrm{d}r \right\|}{|\nu - \nu_0|} \leqq M_1, \quad \|Q'(\nu)\| = \frac{\left\| \int_{\nu_0}^{\nu} [\int_r^{\nu} L''(s)\,\mathrm{d}s]\,\mathrm{d}r \right\|}{|\nu - \nu_0|^2} \leqq \frac{1}{2} M_2, \tag{B.5}$$

then

$$\|K_1(t)\|_{\mathcal{L}(E_\sigma, E_{-\sigma})} \leqq C_1 t^{-3/2}.$$

Step (ii) Integrating three times by parts, we obtain

$$\begin{aligned} K_2(t) = &-\frac{1}{t^2} \int \mathrm{e}^{\mathrm{i}\nu t} \zeta_{2t} Q''(\nu)\,\mathrm{d}\nu - \frac{2}{t^2} \int \mathrm{e}^{\mathrm{i}\nu t} \zeta'_{2t} Q'(\nu)\,\mathrm{d}\nu + \frac{1}{it^3} \int \mathrm{e}^{\mathrm{i}\nu t} \zeta'''_{2t} Q(\nu)\,\mathrm{d}\nu \\ &+ \frac{1}{it^3} \int \mathrm{e}^{\mathrm{i}\nu t} \zeta''_{2t} Q'(\nu)\,\mathrm{d}\nu \\ = &\ K_{21}(t) + K_{22}(t) + K_{23}(t) + K_{24}(t). \end{aligned}$$

Using the bounds from Step (i) we obtain

$$\|K_{2j}(t)\|_{\mathcal{L}(E_\sigma, E_{-\sigma})} \leqq C_2 t^{-3/2}, \quad j = 2, 3, 4.$$

To estimate $K_{21}(t)$, note that $\zeta_{2t}(\nu) = 0$ for $|\nu - \nu_0| \leqq \frac{1}{2\sqrt{t}}$ and

$$|Q''(\nu)| = \frac{1}{|\nu - \nu_0|^3} \left| L''(\nu)(\nu - \nu_0)^2 - 2\int_{\nu_0}^{\nu} \left[\int_r^{\nu} L''(s)\,\mathrm{d}s \right] \mathrm{d}r \right| \leqq \frac{CM_2}{|\nu - \nu_0|}.$$

Therefore,

$$\|K_{21}(t)\|_{\mathcal{L}(E_\sigma, E_{-\sigma})} \leqq Ct^{-3/2}.$$

□

Proof of Proposition 3.3. We apply the Laplace representation

$$\mathrm{e}^{At}(A - 2\mathrm{i}\mu - 0)^{-1} = -\frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{\lambda t} R(\lambda + 0)\,\mathrm{d}\lambda\, R(2\mathrm{i}\mu + 0).$$

Using the Hilbert identity for the resolvent

$$R(\lambda_1)R(\lambda_2) = \frac{1}{\lambda_1 - \lambda_2}[R(\lambda_1) - R(\lambda_2)], \quad \operatorname{Re}\lambda_1, \quad \operatorname{Re}\lambda_2 > 0$$

for $\lambda_1 = \lambda + 0$ and $\lambda_2 = 2\mathrm{i}\mu + 0$, we obtain

$$\begin{aligned}\mathrm{e}^{At}(A - 2\mathrm{i}\mu - 0)^{-1} &= -\frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{\lambda t} \frac{R(\lambda + 0) - R(2\mathrm{i}\mu + 0)}{\lambda - 2\mathrm{i}\mu}\,\mathrm{d}\lambda \\ &= P_1(t) + P_2(t) + P_3(t),\end{aligned}$$

where

$$\begin{aligned}
P_1(t) &= -\frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{\lambda t} \zeta(\lambda) \frac{R(\lambda + 0) - R(2\mathrm{i}\mu + 0)}{\lambda - 2\mathrm{i}\mu}\,\mathrm{d}\lambda, \\
P_2(t) &= -\frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}_+ \cup \mathcal{C}_-} \mathrm{e}^{\lambda t} (1 - \zeta(\lambda)) \frac{R(\lambda + 0) - R(2\mathrm{i}\mu + 0)}{\lambda - 2\mathrm{i}\mu}\,\mathrm{d}\lambda, \\
P_2(t) &= -\frac{1}{2\pi\mathrm{i}} \int_{(-\mathrm{i}\infty, \mathrm{i}\infty) \setminus (\mathcal{C}_+ \cup \mathcal{C}_-)} \mathrm{e}^{\lambda t} (1 - \zeta(\lambda)) \frac{R(\lambda + 0) - R(2\mathrm{i}\mu + 0)}{\lambda - 2\mathrm{i}\mu}\,\mathrm{d}\lambda,
\end{aligned}$$

where $\zeta(\lambda) \in C_0^\infty(\mathrm{i}\mathbb{R})$, $\zeta(\lambda) = 1$ for $|\lambda - 2\mathrm{i}\mu| < \delta/2$ and $\zeta(\lambda) = 0$ for $|\lambda - 2\mathrm{i}\mu| > \delta$, with $0 < \delta < 2\mu - \sqrt{2}$. Applying Lemma 2.1 with $B = \mathcal{L}(E_\sigma, E_{-\sigma})$, and $L(\nu) = R(\lambda + 0)$ we obtain

$$\|P_1(t)\|_{\mathcal{L}(E_\sigma, E_{-\sigma})} = \mathcal{O}(t^{-3/2}), \quad t \to \infty, \quad \sigma > 5/2.$$

Since Proposition 3.1 imply (B.2) for $L(\nu) = R(\mathrm{i}\nu + 0)$. Proposition 3.1 also yields

$$\|P_2(t)\|_{\mathcal{L}(E_\sigma, E_{-\sigma})} = \mathcal{O}(t^{-3/2}), \quad t \to \infty.$$

Here the choice of the sign in $A - 2\mathrm{i}\mu - 0$ plays a crucial role. Further, the integrand in $K_3(t)$ is an analytic function of $\lambda \neq 0, \pm\mathrm{i}\mu$ with the values in $\mathcal{L}(E_\sigma, E_{-\sigma})$ for $\sigma \geqq 0$. At $\lambda = 0, \pm\mathrm{i}\mu$ the integrand has poles of finite order. However, all the Laurent coefficients vanish when applied to $P_c h$. Hence, integrating by parts twice, we obtain

$$\|P_3(t)P^c h\|_{E_{-\sigma}} \leqq c(1 + t)^{-2}\|h\|_{E_\sigma},$$

completing the proof. □

Appendix C. Examples

We construct examples of $U(\psi)$ satisfying **U1–U3**. We will construct $U(\psi)$ by small perturbations of the cubic Ginzburg–Landau potential $U_0(\psi) := (1-\psi^2)^2/4$. For $U(\psi) = U_0(\psi)$

$$s(x) = s_0(x) := \tanh \frac{x}{\sqrt{2}}, \quad V(x) = V_0(x) = U_0''(s_0(x)) - 2 = -3\cosh^{-2}\frac{x}{\sqrt{2}}. \tag{C.6}$$

Let us consider the corresponding Schrödinger operator

$$H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2 + V_0(x) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2 - \frac{3}{\cosh^2(x/\sqrt{2})}$$

restricted to odd functions. The continuous spectrum of H_0 coincides with $[2, \infty)$. It is well known (see [10, pp. 64–65]) that

(i) The discrete spectrum of H_0 consists of one point $\lambda_0 = 3/2$.
(ii) The edge point $\lambda = 2$ is not eigenvalue nor resonance.

Hence, the condition **U2** holds for U_0. The non-degeneracy condition **U3** reads

$$\int \phi_6(x) \frac{\sinh^3(x/\sqrt{2})}{\cosh^5(x/\sqrt{2})}\,\mathrm{d}x \neq 0, \tag{C.7}$$

where $\phi_6(x)$ is a nonzero odd solution to $H_0\phi_6(x) = 6\psi_6(x)$. Numerical calculation [15] demonstrate the validity of (C.7) and hence **U3** holds. Further, $U_0(\psi)$ satisfies (1.2) with $a = 1$ and $m^2 = 2$. However, $U_0(\psi)$ does not satisfy (1.3) since $U_0'''(\pm 1) = \pm 6$, $U_0^{(4)}(\pm 1) = 6$.

Therefore we will construct a small perturbation U_0. Namely, for an appropriate fixed $C > 0$, and any sufficiently small $\delta > 0$, there exists $U(\psi)$ satisfying (1.3) such that

$$U(\psi) = U_0(\psi) \text{ for } ||\psi| - 1| > \delta, \qquad \sup_{\psi\in\mathbb{R}, k=0,1,2} |U^{(k)}(\psi) - U_0^{(k)}(\psi)| \leqq C\delta,$$
$$\sup_{\psi\in\mathbb{R}} |U'''(\psi) - U_0'''(\psi)| \leqq C. \tag{C.8}$$

For example, let us set

$$U(\psi) = U_0(\psi) - \left[\frac{1}{4}(|\psi| - 1)^4 + (|\psi| - 1)^3\right]\chi_\delta(|\psi| - 1),$$

where $\chi_\delta(z) = \chi(z/\delta)$, $\chi(z) \in C_0^\infty(\mathbb{R})$, $\chi(z) = 1$ for $|z| < 1/2$, and $\chi(z) = 0$ for $|z| > 1$. Then (C.8) holds, and

$$U(\psi) = (|\psi| - 1)^2 \text{ for } ||\psi| - 1| < \delta/2, \quad \text{and} \quad U(\psi) = U_0(\psi) \text{ for } ||\psi| - 1| > \delta.$$

Hence, $U(\psi)$ satisfies **U1**. It remains to prove that $U(\psi)$ satisfies **U2** and **U3**.

Denote $\mathcal{S} = \{x \in \mathbb{R} : ||s(x)| - 1|, \ ||s_0(x)| - 1| < \delta\}$. Then $s(x) = s_0(x)$ and $V(x) = V_0(x)$ for $x \in \mathbb{R} \setminus \mathcal{S}$. For $x \in \mathcal{S}$, using (C.8), we obtain

$$\sup_{x\in\mathcal{S}} |V(x) - V_0(x)| \leqq \sup_{x\in\mathcal{S}} |U''(s(x)) - U''(s_0(x))| + \sup_{x\in\mathcal{S}} |U''(s_0(x)) - U_0''(s_0(x))|$$
$$= \sup_{||\phi|-1|<\delta} |U'''(\phi)| \sup_{x\in\mathcal{S}} |s_0(x) - s(x)| + \mathcal{O}(\delta) = \mathcal{O}(\delta)$$

since $\sup_{x\in\mathcal{S}} |s_0(x) - s(x)| \leqq 2\delta$. Hence

$$\sup_{x\in\mathbb{R}} |V(x) - V_0(x)| = \mathcal{O}(\delta). \tag{C.9}$$

Let us verify the uniform decay of $V(x)$ for small $\delta > 0$. We consider the case $x \geqq 0$ (the case $x \leqq 0$ can be considered similarly). Note that $U(\psi) \geqq (\psi - 1)^2/4$ for $0 \leqq \psi < 1$. Using the identity $\int_0^{s(x)} \mathrm{d}s/\sqrt{2U(s)} = x$ we obtain for $x > 0$ and $0 \leqq s(x) < 1$ that

$$x \leqq \int_0^{s(x)} \frac{\sqrt{2}\,\mathrm{d}s}{\sqrt{(1-s)^2}} = \int_0^{s(x)} \frac{\sqrt{2}\,\mathrm{d}s}{1-s} = -\sqrt{2}\,\ln(1 - s(x)).$$

Hence, $1 - s(x) \leqq \mathrm{e}^{-x/\sqrt{2}}$ for $x \geqq 0$, and then $|1 - |s(x)|| \leqq \mathrm{e}^{-|x|/\sqrt{2}}$. Therefore

$$|V(x)| \leqq C\mathrm{e}^{-|x|/\sqrt{2}}, \quad x \in \mathbb{R}. \tag{C.10}$$

Finally, (C.9)–(C.10) imply that **U2** and **U3** hold for $U(\psi)$ for sufficiently small $\delta > 0$, since they hold for $U_0(\psi)$.

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