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On asymptotic completeness for scattering in the nonlinear lamb system

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We establish an asymptotic completeness for the scattering in a nonlinear Lamb system with a nontrivial set of stationary states. We consider the nonlinear system of the string coupled to a nonlinear oscillator of dimension $d \geq 1$. The scattering data consist of a stationary point of the oscillator and an asymptotic wave of finite energy. We describe the set of possible scattering data: (i) for the asymptotic wave with compact support, any stationary point is possible, and (ii) for one-dimensional oscillator and a nondegenerate stationary point, any finite energy asymptotic wave is possible. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3081428]

I. INTRODUCTION

In this paper we consider the asymptotic completeness in the nonlinear Lamb system of the string coupled to the nonlinear oscillator with the force function $F(y)$, $y \in \mathbb{R}^d$, in the case of zero oscillator mass $m=0$,

$$\begin{cases} \ddot{u}(x,t) = u''(x,t), & x \in \mathbb{R} \setminus \{0\} \\ 0 = F(y(t)) + u'(0+,t) - u'(0-,t); & y(t) := u(0,t), \end{cases} \tag{1.1}$$

where $\dot{u} := \partial u / \partial t$, $u' := \partial u / \partial x$. The solutions $u(x,t)$ take the values in $\mathbb{R}^d$ with $d \geq 1$.

The system (1.1) has been introduced originally by Lamb[14] in the linear case when $F(y) = -\omega^2 y$. The Lamb system with general nonlinear $F(y)$ and the oscillator mass $m \geq 0$ has been considered in Ref. 9 where the questions of irreversibility and nonrecurrence were discussed. The system was studied further in Ref. 10 where the global attraction to stationary states has been established for the first time, and in Ref. 4 where metastable regimes were studied for the stochastic Lamb system.

The Lamb system (1.1) is used in all the papers cited above as an example of simplest nontrivial nonlinear time reversible conservative system allowing an effective analysis of various questions. In present paper, we study an asymptotic completenes in the nonlinear scattering for the Lamb system with a nontrivial attractor. This is first result of such kind. We consider the Cauchy problem for the system (1.1) with the initial conditions

$$u|_{t=0} = u_0(x); \quad \dot{u}|_{t=0} = v_0(x). \tag{1.2}$$

Denote by $\|\cdot\|$ the norm in the Hilbert space $L^2(\mathbb{R}, \mathbb{R}^d)$.

Definition 1.1: *The phase space $\mathcal{E}$ of finite energy states for the system (1.1) is the Hilbert*

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space of the pars $(u(x),v(x))\in C(\mathbb{R},\mathbb{R}^d)\oplus L^2(\mathbb{R},\mathbb{R}^d)$ *with* $u'(x)\in L^2(\mathbb{R},\mathbb{R}^d)$ *and the global energy norm,*

$$\|(u,v)\|_{\mathcal{E}}=\|u'\|+|u(0)|+\|v\|. \tag{1.3}$$

The Cauchy problem (1.1) and (1.2) can be written in the form

$$\dot{Y}(t)=\mathbf{F}(Y(t)),\quad t\in\mathbb{R};\quad Y(0)=Y_0, \tag{1.4}$$

where $Y(t):=(u(\cdot,t),\dot{u}(\cdot,t))$ and $Y_0=(u_0,v_0)$. In Refs. 11 and 15, the scattering asymptotics have been proven,

$$Y(t)\sim S_\pm+W(t)\Psi_\pm,\quad t\to\pm\infty, \tag{1.5}$$

where $S_\pm=(s_\pm,0)$ are the limit stationary states with $s_\pm\in Z:=\{s\in\mathbb{R}:F(s)=0\}$, $W(t)$ is the dynamical group of the free wave equation, and $\Psi_\pm\in\mathcal{E}$ are the corresponding asymptotic states. The asymptotics (1.5) hold in the norm of the Hilbert phase space $\mathcal{E}$ if the following limits exist:

$$u_0^+:=\lim_{x\to+\infty}u_0(x),\quad u_0^-:=\lim_{x\to-\infty}u_0(x),\quad \bar{v}_0:=\int_{-\infty}^{\infty}v_0(y)dy. \tag{1.6}$$

Definition 1.2: *The scattering data* (s_+,Ψ_+) *of the solution* $Y(t)$ *at* $t\to\infty$ *consist of the limit stationary state* $s_+\in Z$ *and the asymptotic state* $\Psi_+(x)\in\mathcal{E}$.

Here we study the asymptotic completeness which is one of main problem of the scattering theory. Namely, we find which scattering data (s_+,Ψ_+) are possible for the solutions with the initial states $Y_0\in\mathcal{E}$.

(a) First of all, the asymptotic state $\Psi_+=(\Psi_0,\Psi_1)$ necessarily satisfies the identity

$$\Psi_0^++\Psi_0^-+\bar{\Psi}_1=0, \tag{1.7}$$

where $\Psi_0^+=\lim_{x\to+\infty}\Psi_0(x)$, $\Psi_0^-=\lim_{x\to-\infty}\Psi_0(x)$, and $\bar{\Psi}_1=\int_{-\infty}^{\infty}\Psi_1(y)dy$.

(b) The pair (s_+,Ψ_+) with any $s_+\in Z$ is possible if the asymptotic state $\Psi_+(x)\in\mathcal{E}$ satisfies the identity (1.7) and has a compact support.

(c) For the one-dimensional oscillator (i.e., for $d=1$), the pair (s_+,Ψ_+) with a fixed $s_+\in Z$ is possible with each asymptotic states $\Psi_+(x)\in\mathcal{E}$ satisfying (1.7) if $F'(s_+)\neq 0$.

The same results hold for the scattering data (s_-,Ψ_-) at $t\to-\infty$.

We study the asymptotic completeness for the model Lamb system (1.1) which is nonlinear time reversible conservative system similarly to the coupled Schrödinger-Maxwell and Dirac-Maxwell equations. Note that the validity of the asymptotics (1.5) for the Lamb system has been established in Refs. 11 and 15 (we recall the result in Sec. II B).

The asymptotics of type (1.5) with $S_\pm=0$ were studied in scattering theory for the linear and nonlinear wave, Schrödinger and Klein–Gordon equations by many authors (see, e.g., Refs. 17, 16, and 20). In this case, the asymptotic completeness means a suitable description of the set of the corresponding asymptotic states $\Psi_\pm$.

First scattering asymptotics (1.5) with a nontrivial set of stationary states has been proved in Ref. 11. Now the problem of the "nonlinear asymptotic completeness" consists in an appropriate description of all possible scattering data $(S_\pm,\Psi_\pm)$ which is given for the first time in the present paper.

The scattering asymptotics similar to (1.5) were proven in Refs. 1–3 and 18 for the nonlinear Schrödinger and Klein–Gordon equations, and in Ref. 8 for the Klein–Gordon equation coupled to a particle. However, all the results concern the solutions with the initial states *sufficiently close to the solitary manifold*. In Refs. 13 and 12 similar asymptotics were proven for all finite energy solutions to three dimensional wave and Maxwell equations coupled to a particle. The asymptotic completeness for these equations is an open problem since the results[12,13] rely on the Wiener Tauberian theorem, which does not allow to specify the rate of the convergence in the asymptotics (1.5). Our results for the Lamb system (1.1) rely on an exact characterization of the rate obtained in Sec. VI from the inverse reduced ordinary differential equation (3.4).

The paper is organized as follows. In Sec. II we introduce basic notations, we recall some statements and constructions from Refs. 10, 11, and 15, and prove (a). In Secs. III and IV we reformulate the asymptotic completeness in terms of the solution to the inverse reduced equation for the oscillator. In Sec. V we prove the results (b). In Sec. VI we prove the results (c) for unstable ($F'(s_+)>0$) and stable ($F'(s_+)<0$) stationary points.

II. SCATTERING FOR THE LAMB SYSTEM

A. Existence of dynamics

We recall the construction[10,11] of the solution to the Cauchy problem (1.4) with the initial conditions $Y_0=(u_0,v_0)\in\mathcal{E}$. We assume that

$$F(u)=-\nabla V(u),\quad V(u)\in C^2(\mathbb{R}^d,\mathbb{R}),\quad \text{and}\quad V(u)\to+\infty,\quad |u|\to\infty. \tag{2.1}$$

Then the system (1.1) is formally Hamiltonian with the phase space $\mathcal{E}$ and the Hamilton functional

$$\mathcal{H}(u,v)=\frac{1}{2}\int[|v(x)|^2+|u'(x)|^2]dx+V(u(0)),\quad (u,v)\in\mathcal{E}. \tag{2.2}$$

We construct unique solution $u(x,t)$ such that $Y(t)=(u(\cdot,t),\dot{u}(\cdot,t))\in C(\mathbb{R},\mathcal{E})$. The solution admits the d'Alembert representation

$$u(x,t)=f_+(x-t)+g_+(x+t),\qquad \pm x>0, \tag{2.3}$$

where $f_\pm(z),g_\pm(z)$ for $\pm z>0$ are defined by the d'Alembert formulas

$$f_\pm(z):=\frac{u_0(z)}{2}-\frac{1}{2}\int_0^z v_0(y)dy,\quad g_\pm(z):=\frac{u_0(z)}{2}+\frac{1}{2}\int_0^z v_0(y)dy,\qquad \pm z>0. \tag{2.4}$$

These formulas imply that

$$f'_\pm(z),g'_\pm(z)\in L^2(\mathbb{R}^\pm,\mathbb{R}^d) \tag{2.5}$$

since $(u_0,v_0)\in\mathcal{E}$. The "outgoing waves" $f_+(z)$ for $z<0$ and $g_-(z)$ for $z>0$ are given by

$$f_+(-t):=y(t)-g_+(t),\quad g_-(t):=y(t)-f_-(-t),\quad t>0 \tag{2.6}$$

since $y(t):=u(0,t)=f_+(-t)+g_+(t)=f_-(-t)+g_-(t)$. Hence,

$$u(x,t)=\left\{\begin{array}{ll} y(t-x)+g_+(x+t)-g_+(t-x), & 0<x<t\\ y(t+x)+f_-(x-t)-f_-(-x-t), & -t<x<0\end{array}\right| \qquad t>0. \tag{2.7}$$

Finally, the function $y(t)$ can be determined from the Cauchy problem for the "reduced equation,"

$$2\dot{y}(t)=F(y(t))+2\dot{w}_{\rm in}(t),\quad t>0,\quad y(0)=u_0(0), \tag{2.8}$$

where $w_{\rm in}(t)=g_+(t)+f_-(-t)$ for $t>0$ is the "incident wave." Note that $\dot{w}_{\rm in}\in L^2(\mathbb{R}^+)$ by (2.5), hence the Cauchy problem (2.8) admits a unique solution for all $t>0$, and the *a priori* bound holds,

$$\sup_{t>0}|y(t)|+\int_0^\infty|\dot{y}(t)|^2dt\le B<\infty. \tag{2.9}$$

These arguments imply that the Cauchy problem (1.4) admits a unique solution $Y(t)=(u(x,t),\dot{u}(x,t))\in C(\mathbb{R},\mathcal{E})$ for any $Y_0\in\mathcal{E}$, where $u(x,t)$ is defined by (2.3), (2.4), and (2.7).

The following lemma plays a crucial role in our constructions below.

Lemma 2.1: (i) *Similarly to the "direct" reduced Eq. (2.8), the "inverse" one holds,*

$$2\dot{y}(t) = -F(y(t)) + 2\dot{w}_{\mathrm{out}}(t), \quad t > 0, \quad \dot{w}_{\mathrm{out}} \in L^2(\mathbb{R}^+), \tag{2.10}$$

where the function w_{out} is the sum of the outgoing waves at the point $x=0$: $w_{\mathrm{out}}(t)=f_+(-t)+g_-(t)$ for $t>0$. (ii) *$Y(t)=(u(x,t),\dot{u}(x,t))\in C(\mathbb{R},\mathcal{E})$ is a solution to (1.4), if $u(x,t)$ is defined by (2.3), (2.4), and (2.7), where $y(t)$ is a solution to (2.10) with $\dot{y}\in L^2(\mathbb{R}^+)$.*

Let us note that the lemma follows from (2.8) and (2.9) by the change in variable $t\to -t$ when the ingoing waves become the outgoing waves.

B. Scattering asymptotics

The stationary states $S(x)=(s(x),0)\in\mathcal{E}$ for (1.4) are evidently determined: the set $\mathcal{S}$ of all stationary states $S\in\mathcal{E}$ is given by $\mathcal{S}=\{S_z=(z,0):z\in Z\}$, where $Z=\{z\in\mathbb{R}^d:F(z)=0\}$. Let us denote by $W(t)$ the dynamical group of free wave equation corresponding to $F(u)\equiv 0$.

Definition 2.2: *$\mathcal{E}_\infty$ is the space of $(u,v)\in\mathcal{E}$ such that the limits (1.6) exist.*

The following theorem is proved in Ref. 11, Theorem 4.5, part (ii) (b) and Ref. 15.

Theorem 2.3: *Let the assumptions (2.1) and (1.6) hold, the set Z be a discrete subset in $\mathbb{R}^d$, and initial state $Y_0\in\mathcal{E}_\infty$.*

(i) *For the corresponding solution $Y(t)\in C(\mathbb{R},\mathcal{E})$ to the Cauchy problem (1.4), the scattering asymptotics hold,*

$$Y(t) = S_+ + W(t)\Psi_+ + r_+(t), \quad t \ge 0, \tag{2.11}$$

with some limit stationary state $S_+\in\mathcal{S}$ and asymptotic state $\Psi_+\in\mathcal{E}_\infty$. The remainder is small in the global energy norm,

$$\|r_+(t)\|_{\mathcal{E}} \to 0, \quad t \to \infty. \tag{2.12}$$

(ii) *The dispersive wave $W(t)\Psi_+$ converges to zero in local energy seminorms, i.e.,*

$$\|W(t)\Psi_+\|_{\mathcal{E},R} \to 0, \quad t\to\infty, \quad \forall R>0. \tag{2.13}$$

Here $\|(u,v)\|_{\mathcal{E},R} := \|u'\|_R + |u(0)| + \|v\|_R$, where $\|\cdot\|_R$ stands for the norm in $L^2(-R,R;\mathbb{R}^d)$.

(iii) *$W(t)\Psi_+$ admits the representation $W(t)\Psi_+=(\mathcal{W}_{\mathrm{out}}(x,t),\dot{\mathcal{W}}_{\mathrm{out}}(x,t))$, where*

$$\mathcal{W}_{\mathrm{out}}(x,t) = C_0 + f_+(x-t) + g_-(x+t), \quad C_0 := \frac{u_0^+ + u_0^- + \bar{v}_0}{2} - 2s_+. \tag{2.14}$$

Remark 2.4: (i) *Similar asymptotics hold for $t\to -\infty$.* (ii) *Representations (2.14) and (2.10) imply that*

$$\mathcal{W}_{\mathrm{out}}(0,t) = C_0 + w_{\mathrm{out}}(t), \quad t>0.$$

Thus,

$$\dot{\mathcal{W}}_{\mathrm{out}}(0,t) = \dot{w}_{\mathrm{out}}(t), \quad t>0. \tag{2.15}$$

The following lemma on relaxation for the reduced equation plays a crucial role in the proof of Theorem 2.3.

Lemma 2.5: *[See Refs.* 15 and 11*, Lemma 4.9, part (ii)]. Let all assumptions of Theorem 2.3 hold. Then there exists an $s_+\in Z$ such that*

$$y(t) \to s_+, \quad t\to\infty, \tag{2.16}$$

where $y(t)$ is the solution to the Cauchy problem (2.8).

Definition 2.6: *Let $Y(t) \in C(\mathbb{R}, \mathcal{E})$ be the solution to (1.4) with $Y(0)=Y_0 \in \mathcal{E}_\infty$. Let us set*

$$W_+ Y_0 = (\Psi_+, s_+) \in \mathcal{E}_\infty \times Z, \tag{2.17}$$

where Ψ_+ is defined by (2.11) and $s_+ = S_+(0)$. The map $W_+ : \mathcal{E}_\infty \to \mathcal{E}_\infty \times Z$ is called the wave operator, and (s_+, Ψ_+) -the scattering data, corresponding to Y_0.

C. Expression of the asymptotic states

First let us express the asymptotic states in initial data and the function $y(t)$. Substituting the expressions (2.4) and (2.6) into (2.14), we obtain that the asymptotic state $\Psi_+ = (\Psi_0, \Psi_1) \in \mathcal{E}_\infty$ is expressed in the initial data $(u_0, v_0) \in \mathcal{E}_\infty$ by the formulas

$$\Psi_0(x) = C_0 + \begin{cases} y(x) + \dfrac{u_0(x) - u_0(-x)}{2} - \dfrac{1}{2}\displaystyle\int_{-x}^{x} v_0(y)dy, & x \geq 0 \\ y(-x) + \dfrac{u_0(x) - u_0(-x)}{2} + \dfrac{1}{2}\displaystyle\int_{-x}^{x} v_0(y)dy, & x \leq 0, \end{cases} \tag{2.18}$$

$$\Psi_1(x) = \begin{cases} y'(x) - \dfrac{u_0'(x) - u_0'(-x)}{2} + \dfrac{v_0(x) - v_0(-x)}{2}, & x > 0 \\ y'(-x) + \dfrac{u_0'(x) - u_0'(-x)}{2} + \dfrac{v_0(x) - v_0(-x)}{2}, & x < 0, \end{cases} \tag{2.19}$$

where C_0 is given by (2.14). Further, let us justify some properties of the asymptotic states.

Lemma 2.7: *(Reference 15) Let all assumptions of Theorem 2.3 hold, and $\Psi_+ = (\Psi_0, \Psi_1)$ be the asymptotic state from (2.11).*

(i) *$\Psi_+ \in \mathcal{E}_\infty$, i.e., there exist the finite limits*

$$\Psi_0^+ = \lim_{x \to +\infty} \Psi_0(x), \quad \Psi_0^- = \lim_{x \to -\infty} \Psi_0(x), \quad \bar{\Psi}_1 = \int_{-\infty}^{\infty} \Psi_1(y)dy. \tag{2.20}$$

(ii) *The following identity holds:*

$$\Psi_0^+ + \Psi_0^- + \bar{\Psi}_1 = 0. \tag{2.21}$$

Proof: The existence of the limits (2.20) follows from (2.18) and (2.19) by (1.6) and (2.16). The identity (2.21) follows from the d'Alembert formula,

$$\mathcal{W}_{\text{out}}(x,t) = \frac{\Psi_0(x-t) + \Psi_0(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \Psi_1(y)dy \tag{2.22}$$

since $W(t)\Psi_+ = (\mathcal{W}_{\text{out}}(x,t), \dot{\mathcal{W}}_{\text{out}}(x,t)) \to 0$ in $\mathcal{E}_F$ as $t \to \infty$ by (2.13). ■

The lemma implies that

$$W_+ \mathcal{E}_\infty \subset \mathcal{E}_\infty^+ \times Z, \quad \text{where} \quad \mathcal{E}_\infty^+ := \{\Psi^+ \in \mathcal{E}_\infty : (2.20) - (2.21) \quad \text{hold}\}. \tag{2.23}$$

III. RECONSTRUCTION OF INITIAL CONDITIONS

In the following sections we prove the asymptotic completeness. Namely, we fix an asymptotic state Ψ_+, and we want to construct the trajectory $Y(t)$ of (1.4) such that the asymptotics (2.11) hold. Here, we start with the reconstruction of Y_0 via Ψ_+ and $y(t)$. For $\Psi_+ = (\Psi_0, \Psi_1) \in \mathcal{E}_\infty^+$, let us introduce the function

$$S(t) := \mathcal{W}_{\text{out}}(0,t) = \frac{\Psi_0(t)+\Psi_0(-t)}{2} + \frac{1}{2}\int_{-t}^{t}\Psi_1(y)dy, \quad t \in \mathbb{R}. \tag{3.1}$$

Then

$$\dot{S}(t) = \dot{w}_{\text{out}}(t) \in L^2(\mathbb{R}), \quad S^{+} := \lim_{t\to\infty} S(t) = 0 \tag{3.2}$$

by (2.15), (2.10), and (2.21).

Lemma 3.1: *Let $Y(t)\in C(\mathbb{R},\mathcal{E})$ be a solution of (1.4) with $Y(0)=Y_0\in\mathcal{E}_\infty$, and (2.17) hold.*

(i) *The initial conditions are expressed in Ψ_+ and $y(t)=u(0,t)$ by*

$$u_0(x) = \Psi_0(x) + \begin{cases} y(x)-S(x), & x\ge 0 \\ y(-x)-S(-x), & x\le 0 \end{cases} \qquad v_0(x) = \Psi_1(x) + \begin{cases} y'(x)-S'(x), & x>0 \\ y'(-x)-S'(-x), & x<0. \end{cases} \tag{3.3}$$

(ii) *The function $y(t)$ satisfies the following conditions:*

$$2\dot{y}(t) = -F(y(t)) + 2\dot{S}(t), \quad t>0; \quad \dot{y}\in L^2(\mathbb{R}_+), \quad y(t)\to s_+, \quad t\to+\infty. \tag{3.4}$$

Proof:

(i) Differentiating (2.18), and using (2.19), we obtain

$$\Psi_0'(x)+\Psi_1(x) = 2y'(x)+u_0'(-x)-v_0(-x),$$

$$\Psi_0'(x)-\Psi_1(x) = u_0'(x)-v_0(x), \quad \text{for } x>0, \tag{3.5}$$

$$\Psi_0'(x)-\Psi_1(x) = -2y'(-x)+u_0'(-x)+v_0(-x),$$

$$\Psi_0'(x)+\Psi_1(x) = u_0'(x)+v_0(x), \quad \text{for } x<0. \tag{3.6}$$

Hence, we know $u_0'(x)$ and $v_0(x)$ for $x\neq 0$. Integrating $u_0'(x)$, we obtain (3.3) since $S(0)=\Psi_0(0)$ by (3.1) and $u_0(0)=y(0)$ by (2.8).

(ii) Equation (3.4) follows from (2.10) and (3.2). Other conditions in (3.4) follow from (2.9) and (2.16).

■

Remark 3.2: *For any given $\Psi_+=(\Psi_0,\Psi_1)\in\mathcal{E}_\infty^+$ and $y(t)\in C(\overline{\mathbb{R}^+})$, the formulas (3.3) imply (2.18) and (2.19) with $C_0=\Psi_0(0)-y(0)$.*

Proof: Differentiating (3.3), we obtain (3.5) and (3.6), and then (2.18) and (2.19) follow by integration.

IV. CHARACTERIZATION OF THE ASYMPTOTIC STATES

In this section we prove that the conditions (2.20) and (2.21) are sufficient for the existence of the dynamics with the scattering asymptotics (2.11) provided that the inverse reduced differential equation (3.4) has an appropriate solution.

Lemma 4.1: *Let $(\Psi_+,s+)\in\mathcal{E}_\infty^+\times Z$, and the following condition hold: there exists a trajectory $y(t)$ satisfying (3.4), with $S(t)$ given by (3.1). Then there exists $Y_0\in\mathcal{E}_\infty$ such that (2.17) hold.*

Proof: Let us define u_0 and v_0 by (3.3) and $u(x,t)$ by (2.3), (2.4), and (2.7). Then $Y(t):=(u(x,t),\dot{u}(x,t))\in C(\mathbb{R},\mathcal{E})$, and $Y(t)$ is a solution to Cauchy problem (1.4) by Lemma 2.1, part (ii).

It remains to prove (2.17). First, $Y_0=(u_0,v_0)\in\mathcal{E}_\infty$ by (3.3) and (3.4). Hence, by Theorem 2.3 and Lemma 2.7 $Y(t)$ satisfies asymptotics of type (2.11): $Y(t)=S_+^*+W(t)\Psi_+^*+r_+^*(t)$, where

$$S_+^*=(s_+^*,0), \quad \Psi_+^*\in\mathcal{E}_\infty^+, \quad \|r_+^*(t)\|_{\mathcal{E}}\to 0, \quad t\to\infty. \tag{4.1}$$

It remains to check that $s_+^*=s_+$ and $\Psi_+^*=\Psi_+$. First, $u(0,t)=y(t)$ by (2.7), hence $s_+^*=\lim_{t\to\infty}y(t)$ $=s_+$ by (3.4). Second, Ψ_+^* is expressed by formulas (2.18) and (2.19) with the constant C_0 $=((u_0^*)^+ +(u_0^*)^- +\overline{v_0^*})/2-2s_+^*$ by (2.14). On the other hand, Ψ_+ also is expressed by the same formulas, with $C_0=\Psi_0(0)-y(0)$ by Remark 3.2. Therefore, $\Psi_1^*=\Psi_1$ and $\Psi_0^*(x)-\Psi_0(x)=\text{const}$, $x\in\mathbb{R}$. Finally, the constant is zero since the identity (2.21) holds (i) for $(\Psi_0,\Psi_1)\in\mathcal{E}_\infty^+$ by definition and (ii) for (Ψ_0^*,Ψ_1^*) by Lemma 2.7. ∎

Remark 4.2: (i) The condition (2.21) plays an essential role in our proof. (ii) We choose the inverse reduced equation (3.4) for the characterization of the asymptotic states since the term $\dot{S}(t)$ is expressed in the scattering data Ψ_+ by (3.1).

V. COMPACTLY SUPPORTED ASYMPTOTIC STATES

In this section we prove the asymptotic completeness for the asymptotic states with compact support.

Lemma 5.1: *Let the function $\Psi_+(x)=(\Psi_0(x),\Psi_1(x))\in\mathcal{E}_\infty^+$ has a compact support, $d\geq 1$, and the force function F satisfy conditions (2.1). Then for arbitrary $s_+\in Z$ there exists $Y_0\in\mathcal{E}_\infty$ such that (2.17) hold.*

Proof: According to Lemma 4.1, it suffices to prove that there exists a trajectory $y(t)$ satisfying (3.4).

First, (3.1) implies that supp $\dot{S}(t)$ is a compact set since supp Ψ_0 and supp Ψ_1 are compact. Let $T>0$ be such that $\dot{S}(t)=0$ for $t\geq T$.

Second, let us construct a solution to Eq. (3.4) for $0\leq t\leq T$ with the "initial condition" $y(T)=s_+$, and then set $y(t):=s_+$ for $t\geq T$. It suffices to prove *a priori* estimate. Multiplying Eq. (3.4) by $\dot{y}(t)$ and using (2.1), we obtain that

$$V'(y(t))\dot{y}(t)=2|\dot{y}(t)|^2-2\dot{S}(t)\dot{y}(t), \quad 0<t<T. \tag{5.1}$$

Integrating and using the initial condition, we get

$$V(y(t))=V(s_+)+2\int_t^T\dot{S}(\tau)\dot{y}(\tau)d\tau-2\int_t^T|\dot{y}(\tau)|^2d\tau, \quad 0\leq t\leq T. \tag{5.2}$$

Using the Young inequality, we estimate the second term in the right hand side as

$$2\left|\int_t^T\dot{S}(\tau)\dot{y}(\tau)d\tau\right|\leq\int_t^T|\dot{S}(\tau)|^2d\tau+\int_t^T|\dot{y}(\tau)|^2d\tau.$$

Hence,

$$V(y(t))+\int_t^T|\dot{y}(\tau)|^2d\tau\leq V(s_+)+\int_t^T|\dot{S}(\tau)|^2d\tau\leq B, \quad t\in[0,T]. \tag{5.3}$$

Therefore, $y(t)$ is bounded for $t\in[0,T]$ by (2.1). ∎

VI. ONE-DIMENSIONAL OSCILLATOR

In this section we consider the case $d=1$ and prove the asymptotic completeness for all finite energy asymptotic states and *nondegenerate* stationary states s_+ with

$$F'(s_+)\neq 0. \tag{6.1}$$

Theorem 6.1: *Let* $d=1$, $\Psi_+(x)=(\Psi_0(x),\Psi_1(x))\in\mathcal{E}_\infty^+$, *and (6.1) hold. Then there exists* $Y_0 \in\mathcal{E}_\infty$ *such that (2.17) hold.*

Proof: According to Lemma 4.1, it suffices to prove that there exists a trajectory $y(t)$ satisfying (3.4). We can assume that $s_+=0$, hence it suffices to construct a solution $y(t)$ to the inverse reduced equation (3.4) satisfying

$$y(t)\to 0,\quad t\to\infty,\quad \dot{y}\in L^2(0,\infty). \tag{6.2}$$

Let us reduce the vector field $F(y)$ in a small neighborhood of the stationary point $s_+=0$ to a canonical linear form αz where $\alpha=F'(0)$. Namely, (6.1) implies by the Grobman–Hartman theorem[5] that for some $\delta>0$ there exists a diffeomorphism $z=\phi(y)$ transforming the vector field $F(y)$, $y\in(-\delta,\delta)$ into αz. In our case the theorem is trivial, and the function $\phi(y)\in C^1(-\delta,\delta)$ can be found as the solution to the differential equation

$$\phi'(y)F(y)=\alpha\phi(y). \tag{6.3}$$

Obviously, $\phi(0)=0$, and ϕ is a diffeomorphism of $(-\delta,\delta)$ onto a neighborhood of the point $z=0$ containing an interval $(-\varepsilon,\varepsilon)$ with a small $\varepsilon>0$. Hence, Eq. (3.4) for $y(t)$ with $|y(t)|<\delta$ is equivalent to the equation

$$\dot{z}(t)=-\frac{\alpha}{2}z(t)+\phi'(y(t))\dot{S}(t),\quad t\in\mathcal{T}(z), \tag{6.4}$$

for the function $z(t):=\phi(y(t))$, where $\alpha\neq 0$ by (6.1) and $\mathcal{T}(z):=\{t>0:|z(t)|<\varepsilon\}$. Let us consider Eq. (6.4) with an initial value $z(T)\in(-\varepsilon,\varepsilon)$ at sufficiently large $T>0$. Denoting $y(t):=\phi^{-1}(z(t))$ for $|z(t)|<\varepsilon$, we obtain the equivalent identity

$$z(t)=\int_T^t e^{-\alpha/2(t-s)}\phi'(y(s))\dot{S}(s)ds+z(T)e^{-\alpha/2(t-T)} \tag{6.5}$$

for t close to T.

Further we consider two cases separately: $\alpha>0$ and $\alpha<0$. For $\alpha>0$ we will show that all the solutions $y(t):=\phi^{-1}(z(t))$ satisfy (6.2) provided $|z(T)|$ is sufficiently small. For $\alpha<0$ we will construct at least one solution $y(t):=\phi^{-1}(z(t))$ satisfying (6.2).

A. Unstable stationary point: $\alpha>0$

In this case, the solution (6.5) admits an extension to all $t>T$ if T is sufficiently large. Indeed, (6.5) implies the bound

$$|z(t)|\le C(\alpha)B\|\dot{S}\|_{L^2(T,\infty)}+|z(T)|e^{-\alpha/2(t-T)},\quad t>T,\quad |z(t)|<\varepsilon, \tag{6.6}$$

where $\alpha>0$ and $B=\sup_{|y|\le\delta}|\phi'(y)|<\infty$. Therefore, $|z(t)|\le\varepsilon$ for all $t>T_\varepsilon$ if T_ε is sufficiently large since $\|\dot{S}\|_{L^2(t,\infty)}\to 0$, as $t\to\infty$ by (3.4). Hence, formula (6.5) holds with $T=T_\varepsilon$ for all $t>T_\varepsilon$. Then (6.5) and (6.6) also hold for any $T>T_\varepsilon$ that implies $|z(t)|\le C(\alpha)B\|\dot{S}\|_{L^2(2T,\infty)}+\varepsilon e^{-\alpha T/2}$, $t>2T$. Hence, $z(t)\to 0$ as $t\to\infty$, and

$$y(t)\to 0,\quad t\to\infty. \tag{6.7}$$

Similarly to (5.2),

$$V(y(T_\varepsilon))=V(y(t))+2\int_{T_\varepsilon}^t \dot{S}\dot{y}d\tau-2\int_T^t|\dot{y}|^2d\tau,\quad t>T_\varepsilon. \tag{6.8}$$

Using (6.7), we obtain that

$$\int_{T_\varepsilon}^t |\dot{y}|^2 d\tau \le \int_{T_\varepsilon}^t |\dot{S}\dot{y}| d\tau + \text{const}, \quad t > T_\varepsilon,$$

which implies that $\dot{y} \in L^2(T_\varepsilon, \infty)$.

Finally, the extension of the solution $y(t)$ to $t \in [0, T_\varepsilon]$ can be done as in SecV.

B. Stable stationary point: $\alpha < 0$

In this case, we consider the solutions $z = z_T$ to (6.5) with $z_T(T) = 0$ and estimate it for $t < T$ as

$$|z_T(t)| \le B \int_t^T e^{\alpha(s-t)/2} |\dot{S}(s)| ds \le CB \|\dot{S}\|_{L^2(t,\infty)}, \quad t < T, \tag{6.9}$$

since $\alpha < 0$. Therefore, $|z(t)| \le \varepsilon$ for $t \in [T_\varepsilon, T]$ if $T \ge T_\varepsilon > 0$ and T_ε is sufficiently large. Then Eq. (6.4) holds for all $t \in [T_\varepsilon, T]$, since $|z(t)| \le \varepsilon$.

Lemma 6.2: *For any $T^* > T_\varepsilon$, the family of functions z_T, with $T > T^*$, restricted to $t \in [T_\varepsilon, T^*]$ is precompact in $C[T_\varepsilon, T^*]$.*

Proof: First, the family is uniformly bounded. Second, the equicontinuity follows from Eq. (6.4). ■

The lemma implies that there exists a sequence $T^k \to \infty$, $k \to \infty$, such that

$$z_{T^k}(t) \to z(t), \quad k \to \infty, \quad t > T_\varepsilon, \tag{6.10}$$

and the convergence is uniform for bounded t. Hence, $z(t)$ satisfies Eq. (6.4) for all $t > T_\varepsilon$. Finally, the first equation (6.2) holds since (6.9) and (6.10) imply that

$$|z(t)| \le CB \|\dot{S}\|_{L^2(t,\infty)}, \quad t > 0. \tag{6.11}$$

Now the second condition (6.2) follows from (6.8). The extension of the solution $y(t)$ to $t \in [0, T_\varepsilon]$ can be done as in Sec. V.

Theorem 6.1 is proven. ■

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