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Attractors of non-linear Hamiltonian one-dimensional wave equations

A. I. Komech

Abstract. A theory is constructed for attractors of all finite-energy solutions of conservative one-dimensional wave equations on the whole real line. The attractor of a non-degenerate (that is, generic) equation is the set of all stationary solutions. Each finite-energy solution converges as $t \to \pm\infty$ to this attractor in the Fréchet topology determined by local energy seminorms. The attraction is caused by energy dissipation at infinity. Our results provide a mathematical model of Bohr transitions ('quantum jumps') between stationary states in quantum systems.

Contents

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§0.1. Introduction

The aim of this paper is to study the large-time asymptotics of all finite-energy solutions of non-linear Hamiltonian wave equations and systems.

0.1.1. Bohr transitions to stationary states. The exceptional role of stationary states in many phenomena described by non-linear Hamiltonian wave equations is very surprising. The persistent recurrence of stationary states suggests that each solution $Y(t)$ of these equations tends to some stationary limit as $t \to \infty$:

$$Y(t) \to S_{\pm}, \qquad t \to \pm\infty. \tag{0.1.1}$$

For brevity, we refer to this behaviour of the system as *stabilization*. This means that the set $\mathcal{S}$ of all stationary states is a point attractor of the corresponding equation:

$$Y(t) \to \mathcal{S}, \qquad t \to \pm\infty. \tag{0.1.2}$$

Problems of this sort have been known in diverse fields for a long time. Examples include the radiative friction problem ([23], [15]) in classical electrodynamics as well as problems related to Bohr transitions between stationary states [4], de Broglie's wave-particle duality, and stability and classification of elementary particles in quantum theory. Furthermore, according to Schrödinger only quantum transitions can be responsible for the gene reproduction paradox [52].

Although quantum physics postulates transitions (0.1.1) on the basis of experimental evidence, it has never been proved that they follow from the dynamic equations. If there were such a proof, then quantum stationary states could be defined intrinsically as points belonging to the attractor of the dynamic equations. The main problem is that the convergence (0.1.1) is apparently paradoxical and inconsistent with the fact that Hamiltonian equations are reversible and conservative. Furthermore, such a convergence is impossible for linear autonomous equations.

The present paper deals mainly with a proof that in principle such convergence is possible for, and even typical of, non-linear Hamiltonian wave equations of the form

$$\ddot{u}(x,t) = u''(x,t) + f(x, u(x,t)), \qquad x \in \mathbb{R}. \tag{0.1.3}$$

We prove the convergence (0.1.1), (0.1.2) for a non-linearity $f(x,u) = \delta(x)F(u)$ concentrated at a single point of a string (the Lamb system) in Chapter I, and for a non-linearity $f(x,u) = \sum \delta(x - x_k)F_k(u)$ concentrated at finitely many points in Chapter II. The same results are obtained in Chapter III for general one-dimensional non-linear wave equations and systems with spatially localized non-linearities:

$$f(x,u) = 0, \qquad |x| \geqslant \text{const}\,. \tag{0.1.4}$$

One of the main achievements in this research is the choice of an appropriate topology in which the convergence (0.1.1), (0.1.2) is possible. For example, the convergence (0.1.1) is impossible in the energy metric in general because of energy conservation. We prove (0.1.1), (0.1.2) in the Fréchet topology determined by local energy seminorms. This topology is apparently the best one for describing the attraction to an attractor in the case of Hamiltonian equations.

The main result of this investigation is the discovery of what causes the attraction (0.1.1), (0.1.2). The cause is energy dissipation at infinity. This energy dissipation was originally revealed in linear and non-linear scattering theory ([8], [17]–[19], [22], [26], [42]–[45], [53], [57], [61]–[64]; see the surveys [49] and [58]) for the case in which the attractor $\mathcal{S}$ consists of the single point 0. Then (0.1.1) and (0.1.2) are equivalent to the well-known local-energy decay property.

When combined with perturbation theory, linearization is a powerful tool for studying non-linear problems. However, problems related to large-time behaviour cannot be solved in the framework of the linearization approach unless the trajectory remains close to some known solution. In particular, this pertains to radiative friction, Bohr transitions, and many other problems. To study the large-time behaviour of non-linear Hamiltonian equations, we develop methods of non-linear scattering theory without resorting to perturbation theory.

We restrict ourselves to the study of finite-energy solutions. Infinite-energy solutions, for which the large-time asymptotics depends crucially on the behaviour of the initial data at infinity, are not considered here.

By now these results have been generalized to systems describing the interaction of a classical charged particle in three-dimensional space with a scalar wave field ([38], [36], [35]) or a Maxwell electromagnetic field ([37]; see the survey [34]).

We note that the attraction to an attractor consisting of stationary states is a well-studied property of dissipative systems ([1], [21], [40], [51], [60]) which substantially differ from Hamiltonian equations in that for dissipative systems, solutions converge to the attractor in the global energy metric, but only as $t \to +\infty$.

Our results imply that the asymptotic behaviour (0.1.1), (0.1.2) is typical of 'non-degenerate' (that is, generic) Hamiltonian equations. On the other hand, numerous examples, as well as numerical evidence, show that the large-time behaviour can be completely different for some 'exceptional' classes of equations. For example, G-invariant equations with some Lie symmetry group G (see [20]) are exceptional, and the asymptotics (0.1.1), (0.1.2) does not hold for such G-invariant equations in general. Indeed, such equations in general have solutions in the form of 'solitary waves' [20] $\exp(i\Omega t)\Psi(x)$, where Ω is an element of the corresponding Lie algebra. The asymptotics (0.1.1) does not hold in general for a solitary wave if $\Omega \neq 0$. Our results concerning the stabilization (0.1.1) correspond to the trivial symmetry group $G = \{e\}$.

The existence of non-trivial solitary waves for scalar $U(1)$-invariant non-linear Hamiltonian wave equations was proved in [3]. For $U(1)$-invariant non-linear Dirac, Maxwell–Dirac, and Klein–Gordon–Dirac Hamiltonian systems, the corresponding result was obtained in [13], [14].

For a number of non-degenerate $U(1)$-invariant systems, solitary waves form an attractor in the Fréchet topology ([5]–[7], [48], [55], [56]) but this has yet to be proved for the Maxwell–Dirac system.

Remark. The Yang–Mills system describing strong interaction is $SU(2)$-invariant [67] or $SU(3)$-invariant [16], and the system describing electroweak interaction is $SU(2)\times U(1)$-invariant [65]. Our results for the trivial symmetry group $G = \{e\}$, as well as the results of [5]–[7], [48], [55], [56] for $G = U(1)$, suggest that the attractor of 'non-degenerate' G-invariant Hamiltonian equations consists of 'solitary waves'

$\exp(i\Omega t)\Psi(x)$. This conjecture is also corroborated by well-known experimental results concerning the relationship between the classification of elementary particles and that of Lie algebras of the symmetry groups of the corresponding equations ([16], [46]).

0.1.2. Notation and definitions. We consider some classes of one-dimensional equations of the form

$$\begin{cases} \ddot{u}(x,t) = u''(x,t) + f(x, u(x,t)), \quad x \in \mathbb{R}, \\ u\big|_{t=0} = u^0(x), \quad \dot{u}\big|_{t=0} = v^0(x). \end{cases} \tag{0.1.5}$$

Here and in the following, all derivatives are treated in the sense of distributions, $u \in \mathbb{R}^d$, $d \geqslant 1$, and

$$f(x,u) = -\nabla_u V(x,u), \tag{0.1.6}$$

where $V(x,u)$ is a real-valued function referred to as a *potential.* Then (0.1.5) is a formally Hamiltonian system with Hamiltonian

$$\mathcal{H}(u, \dot{u}) = \int \left(\frac{|\dot{u}|^2}{2} + \frac{|u'|^2}{2} + V(x,u) \right) dx. \tag{0.1.7}$$

Our results pertain to one-dimensional equations (0.1.5) with a spatially localized non-linearity (see (0.1.4)). We set $Y(t) = (u(\,\cdot\,,t), \dot{u}(\,\cdot\,,t))$ and $Y^0 = (u^0, v^0)$ and rewrite problem (0.1.5) in the form

$$\dot{Y}(t) = \mathcal{V}(Y(t)), \qquad t \in \mathbb{R}, \quad Y(0) = Y^0. \tag{0.1.8}$$

We introduce the phase space of finite-energy states for problem (0.1.8). Let $|\cdot|$ be the norm in $L^2(\mathbb{R}, \mathbb{R}^d)$ and let $|\cdot|_R$, $R > 0$ be the norm in $L^2(B_R, \mathbb{R}^d)$, where B_R is the interval $[-R, R]$.

Definition 0.1.1. i) $\mathcal{E}$ is the Hilbert space of pairs $(u(x), v(x)) \in C(\mathbb{R}, \mathbb{R}^d) \oplus L^2(\mathbb{R}, \mathbb{R}^d)$ with finite norm

$$\|(u,v)\|_{\mathcal{E}} = |u'| + |u(0)| + |v|. \tag{0.1.9}$$

ii) $\mathcal{E}_F$ is the space $\mathcal{E}$ equipped with the topology defined by the local energy seminorms

$$\|(u,v)\|_R = |u'|_R + |u|_R + |v|_R, \qquad R > 0. \tag{0.1.10}$$

By $\xrightarrow{\mathcal{E}_F}$ we denote the convergence in $\mathcal{E}_F$.

Remark. We note that $\mathcal{E}_F$ is a countably normed linear topological space. Hence the topology of $\mathcal{E}_F$ is metrizable. This means that the convergence $Y_k \xrightarrow{\mathcal{E}_F} Y$ is equivalent to the convergence with respect to some metric $\rho(\,\cdot\,,\cdot\,)$ in $\mathcal{E}_F$. (The metric is obviously not unique.) Furthermore, $\mathcal{E}_F$ is separable.

For brevity, we refer to separable metrizable spaces as Fréchet spaces even if they are not complete. In particular, $\mathcal{E}_F$ is a Fréchet space in this sense. In such spaces, a subset is compact if and only if it is sequentially compact.

We note that the Hamiltonian $\mathcal{H}$ is continuous on $\mathcal{E}$ but is not continuous in the Fréchet topology of $\mathcal{E}_F$ in all problems considered in the present paper.

Definition 0.1.2. $\mathcal{S} = \{S \in \mathcal{E} : \mathcal{V}(S) = 0\}$ is the set of stationary states of problem (0.1.8).

In each chapter, we consider some class of equations of the form (0.1.5) distinguished by appropriate conditions imposed on the non-linearity $f(x,u)$. For each initial state $Y^0 \in \mathcal{E}$ we establish the existence and uniqueness of a solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ and the conservation of energy:

$$\mathcal{H}(Y(t)) = \text{const}, \qquad t \in \mathbb{R}. \tag{E}$$

Our main results imply large-time behaviour of the following two types.

Relaxation. *For each initial state $Y_0 \in \mathcal{E}$ the orbit $O(Y) = \{Y(t) : t \in \mathbb{R}\}$ is precompact in $\mathcal{E}_F$ and satisfies*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S} \quad \textit{as} \quad t \to \pm\infty. \tag{0.1.11}$$

By definition, this means that for each neighbourhood $\mathcal{O}(\mathcal{S})$ of $\mathcal{S}$ in $\mathcal{E}_F$ there is a $T > 0$ such that $Y(t) \in \mathcal{O}(\mathcal{S})$ for $t > T$.

Stabilization. *There are stationary states $S_\pm \in \mathcal{S}$ depending on the solution $Y(t)$ such that*

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm \quad \textit{as} \quad t \to \pm\infty. \tag{0.1.12}$$

Remark. Let $\rho(\,\cdot\,,\cdot\,)$ be an arbitrary metric defining the Fréchet topology of $\mathcal{E}_F$. Then the convergence (0.1.11) is equivalent to convergence in this metric:

$$\inf_{S \in \mathcal{S}} \rho(Y(t), S) \to 0 \quad \text{as} \quad t \to \pm\infty \quad \forall\, R > 0, \tag{0.1.13}$$

since the orbit $O(Y)$ is precompact. This is also equivalent to convergence in each of the seminorms:

$$\inf_{S \in \mathcal{S}} \|Y(t) - S\|_R \to 0 \quad \text{as} \quad t \to \pm\infty \quad \forall\, R > 0. \tag{0.1.14}$$

The relation (0.1.11) implies (0.1.12) if the set $\mathcal{S}$ is finite. The implication (0.1.11) $\Longrightarrow$ (0.1.12) can be formalized with the help of the following definition. Let $\mathcal{T}$ be a subset of a topological space $\mathcal{F}$.

Definition 0.1.3. The set $\mathcal{T}$ is called a *trapping subset* of $\mathcal{F}$ if for each continuous curve $Y(t) \in C(\mathbb{R}, \mathcal{F})$ whose orbit $O(Y)$ is precompact in $\mathcal{F}$ it follows from the convergence $Y(t) \xrightarrow{\mathcal{F}} \mathcal{T}$ as $t \to \infty$ that $Y(t)$ converges in $\mathcal{F}$ to some point $T \in \mathcal{T}$ as $t \to \infty$.

For example, a closed subset $Z \subset \mathbb{R}$ is trapping in $\mathbb{R}$ if and only if

$$Z \text{ contains no non-empty interval } (c_1, c_2). \tag{$T1$}$$

(Condition $(T1)$ holds if and only if $\mathbb{R} \backslash Z$ is dense in $\mathbb{R}$.) In particular, every Cantor subset of $\mathbb{R}$ is a trapping subset of $\mathbb{R}$.

0.1.3. Energy dissipation at infinity. At present, there is no reasonably general description of the typical large-time behaviour of solutions of non-linear wave equations. Nor is there a universal method for the analysis of large-time behaviour. That is why we study model problems in which possible types of large-time behaviour can be examined. We consider three types of such problems in the subsequent chapters.

Let us briefly describe the main results. We establish large-time asymptotics (0.1.11) and (0.1.12) for the following three types **M1**–**M3** of one-dimensional wave equations (0.1.5).

M1 In Chapter I, we consider a singular non-linearity $f(x,u)=\delta(x)F(u)$, concentrated at the single point $x=0$. We assume that $d \geqslant 1$,

$$F(u) \in C^1(\mathbb{R}^d,\mathbb{R}^d), \quad F(u) = -\nabla V(u), \quad \text{and} \quad V(u) \to +\infty \text{ as } |u| \to \infty. \tag{0.1.15}$$

Then (0.1.11) holds for all solutions $Y(t) \in C(\mathbb{R},\mathcal{E})$. If, moreover, $Z=\{z \in \mathbb{R}^d : F(z)=0\}$ is a trapping subset of $\mathbb{R}^d$, then (0.1.12) also holds. For any two stationary states $S_\pm \in \mathcal{S}$, there are solutions $Y(t) \in C(\mathbb{R},\mathcal{E})$ satisfying (0.1.12).

M2 In Chapter II, we consider a singular non-linearity of the form

$$f(x,u) = \sum_{k=1}^{N} \delta(x-x_k)F_k(u),$$

concentrated at finitely many points. We assume that $d \geqslant 1$ and that the following conditions hold:

$$F_k \in C^1(\mathbb{R}^d,\mathbb{R}^d), \quad F_k(u) = -\nabla V_k(u); \tag{0.1.16}$$

$$\inf_{u\in\mathbb{R}^d} \min_k V_k(u) > -\infty \quad \text{and} \quad \max_k V_k(u) \to +\infty \quad \text{as} \quad |u| \to \infty. \tag{0.1.17}$$

Then (0.1.11) holds for all solutions $Y(t) \in C(\mathbb{R},\mathcal{E})$. If, moreover, $d=1$ and all F_k are real-analytic functions on $\mathbb{R}$, then (0.1.12) also holds.

M3 In Chapter III, we consider continuous non-linearities $f(x,u)$ such that $f(x,u)=0$ for $|x|>a$ with some $a>0$. We assume that $d \geqslant 1$ and the following conditions hold:

$$f(x,u) = \chi(x)F(u), \quad F \in C^1(\mathbb{R}^d,\mathbb{R}^d), \quad \chi(x) \in C(\mathbb{R}); \tag{0.1.18}$$

$$F(u) = -\nabla V(u) \quad \text{and} \quad V(u) \to +\infty \quad \text{as} \quad |u| \to \infty; \tag{0.1.19}$$

$$\chi(x) \geqslant 0, \quad \chi(x) \not\equiv 0 \quad \text{and} \quad \chi(x) = 0 \quad \text{for} \quad |x| \geqslant a. \tag{0.1.20}$$

Then (0.1.11) holds for all solutions $Y(t) \in C(\mathbb{R},\mathcal{E})$. If, moreover, $d=1$ and F is a real analytic function on $\mathbb{R}$, then (0.1.12) is also valid.

Remarks. i) The system **M1** was introduced by Lamb [41] and was also considered in [24].

ii) **M1** is a special case of **M2**, and **M2** is formally a special case of **M3**. However, we consider these systems separately for the following reasons. First, we deal with a modified model **M1**, and for **M1** (respectively, **M2**) we obtain sharper results than for **M2** (respectively, **M3**). Second, the methods used in the analysis of **M2** (respectively, **M3**) are a natural generalization of those used in the analysis of **M1** (respectively, **M2**).

0.1.4. Research methods. All the results of the present paper are based on a study of the energy dissipation at infinity. In each of the problems one can obtain a lower bound for the energy dissipated at infinity via the corresponding 'dissipation integral'. On the other hand, this energy is *a priori* bounded. This results in an estimate of the dissipation integral that ensures the convergence (0.1.11).

For the Hamiltonian systems in question we develop methods of global analysis of systems with dissipation ([1], [21], [40], [51], [60]) and apply the following general scheme.

I. The dissipation integral is bounded.
II. The time derivatives of the solution decay.
III. The solution has an integral representation.
IV. The trajectory is attracted in $\mathcal{E}_F$ to a set that is compact in $\mathcal{E}_F$.
V. Each ω-limit point (in $\mathcal{E}_F$) of the trajectory is a stationary state.
VI. $\mathcal{S}$ is a trapping subset of $\mathcal{E}_F$.

However, implementation of this scheme is different for different systems. Step IV for the system **M2** is based on the notion of 'relaxation at infinity' of a function $f(t)$, $t>0$; this notion is a weakened form of the convergence $f'(t)\to 0$ as $t\to\infty$. Roughly speaking, relaxation at infinity means that $\max_{0\leqslant\tau\leqslant T}|f(t+\tau)-f(t)|\to 0$ as $t\to\infty$ for each $T>0$. This notion has a number of convenient properties (the properties **R0**–**R8** in § 2.6.1) which give the desired result when used systematically.

For the system **M3** step IV follows from an analysis of the Goursat problem for the non-linear wave equation. We first only manage to obtain convergence to a compact set $\mathcal{A}$ 'in the mean':

$$\int_0^\infty \rho_R^2(t)\,dt<\infty, \tag{0.1.21}$$

where $\rho_R(t)=\inf_{S\in\mathcal{A}}\|Y(t)-S\|_R$. From this we infer that actually $\rho_R(t)\to 0$ as $t\to\pm\infty$. To this end, we use the compactness of $\mathcal{A}$, the uniform continuity of the dynamics near a compact set, and reasoning similar to the Borel–Cantelli lemmas.

On the other hand, in all the systems **M1**–**M3** step V follows from the universal Lemma 1.5.3 due to Egorov [10], and step VI is obtained from the following simple universal criterion for $\mathcal{S}\subset\mathcal{E}_F$ to be a trapping subset.

Lemma 0.1.4. *Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are Fréchet spaces, $I\colon\mathcal{F}_1\to\mathcal{F}_2$ is a continuous map, $\mathcal{T}_2$ is a trapping subset of $\mathcal{F}_2$, $I\mathcal{T}_1\subset\mathcal{T}_2$, and $I\colon\mathcal{T}_1\to\mathcal{T}_2$ is injective. Then $\mathcal{T}_1$ is a trapping subset of $\mathcal{F}_1$.*

We explicitly construct such a map I from $\mathcal{F}_1=\mathcal{E}_F$ to $\mathcal{F}_2=\mathbb{R}$ for each of the systems **M1**–**M3**.

0.1.5. Comments. We establish that all finite-energy solutions of the Hamiltonian systems in this paper converge as $t\to\pm\infty$ to an attractor, which possibly consists of infinitely many points. This is a generalization of known results for dissipative systems ([1], [21], [40], [51], [60]), but there are several fundamental differences.

I. In general, the convergence in dissipative systems occurs only as $t \to +\infty$.

II. The cause of the convergence is quite different. There is energy absorption in dissipative systems, but in Hamiltonian systems the energy is conserved, and the role of absorption is played by energy dissipation at infinity.

III. The convergence fails in general for Hamiltonian wave equations in bounded domains, since the waves are reflected by the boundary. That is why we consider (0.1.5) in the entire space.

IV. For dissipative systems the convergence (0.1.11), (0.1.12) holds in the (global) energy metric of the corresponding phase space $\mathcal{E}$. For Hamiltonian systems the convergence (0.1.12) in the metric of $\mathcal{E}$ is in general impossible by virtue of the energy conservation law. Indeed, if $\|Y(t) - S_\pm\|_\mathcal{E} \to 0$ as $t \to \pm\infty$, then the relation (E) implies that $\mathcal{H}(S_\pm) = \mathcal{H}(Y(t))$, since the Hamiltonian $\mathcal{H}$ is continuous on $\mathcal{E}$ in all our problems. Hence the convergence in $\mathcal{E}$ of all finite-energy solutions would mean that $\mathcal{H}(\mathcal{E}) \subset \mathcal{H}(\mathcal{S})$. However, this is impossible for any non-trivial Hamiltonian system if $\mathcal{S}$ is discrete. Likewise, the convergence (0.1.12) of all solutions is impossible for any non-trivial finite-dimensional Hamiltonian system.

V. For a dissipative system the flow consists of compact maps of $\mathcal{E}$, while for a Hamiltonian system the dynamic maps are only bounded, that is, compact in the weak topology. This is the main difference, which complicates the proof of the asymptotics (0.1.11), (0.1.12) and necessitates the use of the Fréchet topology in the study of Hamiltonian systems.

The results of Chapter I show that all the conditions imposed on the system **M1** are necessary for the validity of the main results. The conditions imposed on **M2** are also close to being necessary, which is illustrated by numerous examples in Chapter 2. The same pertains to the conditions imposed on **M3** in Chapter III.

The convergence (0.1.12) implies a transition

$$S_- \mapsto S_+ \tag{0.1.22}$$

as time varies from $-\infty$ to $+\infty$. This provides a mathematical model of Bohr transitions between stationary states in quantum systems. For the Lamb system **M1**, there are transitions (0.1.22) between two arbitrary stationary states $S_\pm \in \mathcal{S}$ (Lemma 1.2.4).

In all cases considered in the paper, the convergence (0.1.12) implies the inequality

$$\mathcal{H}(S_\pm) \leqslant \mathcal{H}(Y(t)) \equiv \mathcal{H}(Y^0), \qquad t \in \mathbb{R}, \tag{0.1.23}$$

by analogy with a well-known property of weak convergence in Hilbert and Banach spaces. Examples show that strict inequality in (0.1.23) is possible; this can be called *energy dissipation at infinity.*

The results of the paper show that the asymptotic behaviour (0.1.11), (0.1.12) is valid for 'generic' systems and may be violated for certain 'exceptional' systems. Specifically, (0.1.12) holds if $\mathcal{S}$ is a trapping subset, that is, is discrete in some sense. On the other hand, many examples show that the large-time behaviour of solutions for some 'narrow' classes of equations (0.1.5) can be quite different from (0.1.11) and (0.1.12). For example, this is the case for 'degenerate' systems **M1** by Lemma 1.2.3 and for the G-invariant equations considered in [20].

Chapters I and II contain a generalization of the author's results [27]–[32]; the results of Chapter III were first published in [33].

0.1.6. Open problems. The existence of stationary states of the form $(\Phi(x), \exp(i\omega t)\psi(x))$ was recently proved in [13], [14] for the non-linear Dirac, Maxwell–Dirac, and Klein–Gordon–Dirac systems. However, the large-time convergence of type (0.1.11) or (0.1.12) to these stationary states is an open problem.

The large-time asymptotics (0.1.11), (0.1.12) has yet to be proved for relativistically invariant equations (0.1.3), that is, for the case in which the non-linearity $f(x,u) = F(u)$ is independent of x. If the asymptotics (0.1.12) is valid for solutions of this type, then they also have soliton-like asymptotics. This follows from the relativistic invariance of the equation.

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CHAPTER I

THE LAMB SYSTEM: A STRING WITH A NON-LINEAR OSCILLATOR

We prove convergence to stationary states for a system of equations describing the interaction of an infinite string with a non-linear oscillator. This system was considered by Lamb for the case of a linear oscillator [41]. The system is formally equivalent to the one-dimensional wave equation with a non-linearity $\delta(x)F(u)$ concentrated at the point $x = 0$. The limit stationary states corresponding to $t = \pm\infty$ may be distinct and arbitrary. The problem is reduced to studying an ordinary non-linear equation for the oscillator. This chapter contains a generalization of the author's results in [27]–[30].

§1.1. Introduction

We establish convergence of type (0.1.11), (0.1.12) to stationary states for solutions $u(x,t)$ of the system

$$\begin{aligned} &\ddot{u}(x,t) = u''(x,t), \quad x \in \mathbb{R} \setminus \{0\}, \\ &m\ddot{y}(t) = F(y(t)) + u'(0+,\,t) - u'(0-,\,t), \quad y(t) \equiv u(0,t). \end{aligned} \tag{1.1.1}$$

The solutions $u(x,t)$ take values in $\mathbb{R}^d$, $d \geqslant 1$. We consider the Cauchy problem for the system (1.1) with initial conditions

$$u\big|_{t=0} = u^0(x), \quad \dot{u}\big|_{t=0} = v^0(x), \quad \dot{y}|_{t=0} = p^0. \tag{1.1.2}$$

The last condition in (1.1.2) is needed only if $m > 0$. In the following, we consider the case $m > 0$ and sometimes comment on the difference between this and the case $m = 0$. From the viewpoint of physics, problem (1.1.1) describes small transverse vibrations of an infinite string parallel to the axis $0x$ with a particle of mass $m \geqslant 0$ attached to the string at the point $x = 0$; $F(y)$ is an external (non-linear) force field acting on the particle in the direction perpendicular to the axis $0x$ (Fig. 1).

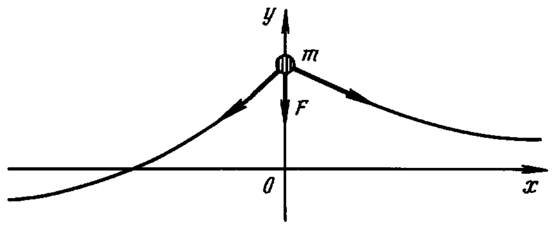

Figure 1

The system (1.1.1) is formally equivalent to the one-dimensional non-linear wave equation with non-linearity $\delta(x)F(u)$ concentrated at the point $x = 0$ (and with mass m concentrated at the same point):

$$(1 + m\delta(x))\ddot{u}(x,\, t) = u''(x,\, t) + \delta(x)F(u(x,\, t)), \qquad (x,\, t) \in \mathbb{R}^2. \tag{1.1.3}$$

In the linear case, where $F(y) = -ky$, $k > 0$, the system (1.1.1) has the unique stationary finite-energy solution $s(x) \equiv 0$. In this case stabilization to zero was considered for the first time by Lamb [41].

Let us introduce the phase space of finite-energy states for the system (1.1.1) with $m > 0$ (respectively, $m = 0$). By L^2 we denote the Hilbert space $L^2(\mathbb{R}, \mathbb{R}^d)$ with norm $|\cdot|$.

Definition 1.1.1. i) $\mathcal{E}$ (respectively, $\mathcal{E}^0$) is the Hilbert space of triples $(u(x), v(x), p) \in C(\mathbb{R}, \mathbb{R}^d) \oplus L^2 \oplus \mathbb{R}^d$ (respectively, pairs $(u(x), v(x)) \in C(\mathbb{R}, \mathbb{R}^d) \oplus L^2$) such that $u'(x) \in L^2$, with norm

$$\begin{gathered} \|(u, v, p)\|_{\mathcal{E}} = \|u'\| + |u(0)| + \|v\| + |p| \\ (\|(u, v)\|_{\mathcal{E}^0} = \|u'\| + |u(0)| + \|v\|). \end{gathered} \tag{1.1.4}$$

ii) $\mathcal{E}_F$ is the space $\mathcal{E}$ equipped with the Fréchet topology defined by the seminorms

$$\|(u, v, p)\|_R \equiv |u'|_R + |u(0)| + |v|_R + |p|, \qquad R > 0, \tag{1.1.5}$$

where $|\cdot|_R$ is the norm in $L^2(-R, R; \mathbb{R}^d)$.

We assume that conditions (0.1.15) are satisfied. The system (1.1.1) is formally a Hamiltonian system with phase space $\mathcal{E}$ and Hamiltonian

$$\mathcal{H}(u, v, p) = \frac{1}{2}\int_{\mathbb{R}} \left[|v(x)|^2 + |u'(x)|^2\right] dx + m\frac{|p|^2}{2} + V(u(0)) \tag{1.1.6}$$

for $(u, v, p) \in \mathcal{E}$. Let us consider solutions $u(x, t)$ such that

$$Y(t) = (u(\,\cdot\,, t), \dot{u}(\,\cdot\,, t), \dot{y}(t)) \in C(\mathbb{R}, \mathcal{E}), \qquad y(t) \equiv u(0, t),$$

and rewrite the Cauchy problem (1.1.1), (1.1.2) in the form

$$\dot{Y}(t) = \mathcal{V}(Y(t)) \quad \text{for} \quad t \in \mathbb{R}, \quad Y(0) = Y^0, \tag{1.1.7}$$

where $Y^0 = (u^0, v^0, p^0)$. We discuss the statement of this Cauchy problem for functions $Y(t) \in C(\mathbb{R}, \mathcal{E})$. First, $u \in C(\mathbb{R}^2, \mathbb{R}^d)$, since $Y(t) \in C(\mathbb{R}, \mathcal{E})$. It follows that the first equation in (1.1.1) is equivalent to the d'Alembert formula

$$u(x,t) = f_\pm(x-t) + g_\pm(x+t), \qquad \pm x > 0, \tag{1.1.8}$$

where

$$f_\pm, g_\pm \in C(\mathbb{R}, \mathbb{R}^d). \tag{1.1.9}$$

Consequently,

$$\begin{aligned} \dot{u}(x,t) &= -f'_\pm(x-t) + g'_\pm(x+t), \\ u'(x,t) &= f'_\pm(x-t) + g'_\pm(x+t), \end{aligned} \qquad \text{for } \pm x > 0, \tag{1.1.10}$$

where all derivatives are understood in the sense of distributions. The assumption $Y(t) \in C(\mathbb{R}, \mathcal{E})$ implies that

$$f'_\pm, g'_\pm \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d). \tag{1.1.11}$$

Let us discuss the meaning of the second equation in (1.1.1).

Definition 1.1.2. In the second equation in (1.1.1), we set

$$u'(0\pm, t) \equiv f'_\pm(-t) + g'_\pm(t) \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d), \tag{1.1.12}$$

where the derivative $\ddot{y}(t)$ of the function $y(t) \equiv u(0\pm, t) \in C(\mathbb{R}, \mathbb{R}^d)$ (or of $\dot{y}(t) \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d)$) is understood in the sense of distributions.

We note that the functions $f_\pm$ and $g_\pm$ in (1.1.8) are determined uniquely up to an additive constant. Hence definition (1.1.12) is unambiguous.

One can readily find stationary states $S = (s(x), 0, 0) \in \mathcal{E}$ for (1.1.7). For each $c \in \mathbb{R}^d$ we introduce the constant function

$$s_c(x) = c \quad \text{for} \quad x \in \mathbb{R}. \tag{1.1.13}$$

Then the set $\mathcal{S}$ of all stationary states $S \in \mathcal{E}$ has the form

$$\mathcal{S} = \{S_z = (s_z(x), 0, 0) : z \in Z\}, \tag{1.1.14}$$

where $Z = \{z \in \mathbb{R}^d : F(z) = 0\}$. We say that the system (1.1.1) and the potential $V(u)$ are 'non-degenerate' if Z is a trapping subset of $\mathbb{R}^d$. For $d = 1$, this means that condition $(T1)$ holds (see the Introduction), which is also equivalent to the condition

$$F(u) \not\equiv 0 \text{ on every non-empty interval } c_1 < u < c_2. \tag{ND}$$

The main result of this chapter is that $\mathcal{S}$ is a point attractor of the system (1.1.1) in the Fréchet topology of $\mathcal{E}_F$.

§1.2. Main results

We start from a construction of the dynamics.

Proposition 1.2.1. *Suppose that $m > 0$ in the system* (1.1.1), $d \geqslant 1$, *and condition* (0.1.15) *is valid. Then the following assertions hold.*

i) *For every $Y^0 \in \mathcal{E}$ the Cauchy problem* (1.1.7) *has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.*
ii) *The map $W_t \colon Y^0 \mapsto Y(t)$ is continuous in $\mathcal{E}$ and $\mathcal{E}_F$ for every $t \in \mathbb{R}$.*
iii) *The energy conservation law* (E) *holds.*

Remark. For $m = 0$ the last condition in (1.1.2) is omitted. Then Proposition 1.2.1 remains valid with $\mathcal{E}$ replaced by $\mathcal{E}^0$.

Our main result is given by the following theorem.

Theorem 1.2.2. *Suppose that the assumptions of Proposition* 1.2.1 *hold and the initial state Y^0 belongs to $\mathcal{E}$. Then the following assertions hold.*

i) *The orbit $O(Y)$ of the solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ of the Cauchy problem* (1.1.7) *is precompact in $\mathcal{E}_F$, and*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S} \quad as \quad t \to \pm\infty. \tag{1.2.1}$$

ii) *In addition, suppose that Z is a trapping subset of $\mathbb{R}^d$. Then there are stationary states $S_\pm \in \mathcal{S}$, depending on the solution $Y(t)$, such that*

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm \quad as \quad t \to \pm\infty. \tag{1.2.2}$$

iii) *Let $d = 1$. Then* (1.2.2) *holds for any set Z if*

$$u^0(x) = C_\pm, \quad v^0(x) = 0 \quad for \quad \pm x > r^0 \tag{1.2.3}$$

for some $r^0 \geqslant 0$ and $C_\pm \in \mathbb{R}$.

Remarks. i) It suffices to verify (1.2.1) and (1.2.2) only for the case $t \to +\infty$, since the system (1.1.1) is reversible.

ii) By Fatou's lemma (0.1.23) follows from the convergence (1.2.2) in view of (1.1.6) and (0.1.15).

iii) A similar theorem is valid for $m = 0$.

The convergence (1.2.2) is a typical property of 'non-degenerate' systems with a trapping set Z. The question arises as to whether 'degenerate' systems (1.1.1) whose sets of stationary states contain certain continuous components exhibit more complicated large-time behaviour. One can expect that the convergence (1.2.1) to these components is combined with 'damped wanderings' along these components. The following lemma describes an example of this more complicated behaviour by showing that for $d = 1$ condition (ND) is necessary for the convergence (1.2.2) of all solutions $Y(t) \in C(\mathbb{R}, \mathcal{E})$.

Lemma 1.2.3. *Suppose that* $d = 1$ *and all assumptions of Proposition* 1.2.1 *are satisfied, but condition* (ND) *is violated. Then there are solutions* $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *of the system* (1.1.7) *for which condition* (1.2.2) *is violated.*

Remarks. i) Theorem 1.2.2, i) implies (1.2.1) for the solutions $Y(t)$ in Lemma 1.2.3. Hence these solutions describe damped wanderings along the attractor $\mathcal{S}$.

ii) It follows from Theorem 1.2.2, iii) that condition (1.2.3) is violated for the initial data $Y(0)$ of all solutions described in Lemma 1.2.3.

Next, there is the question of the relationship between the limit stationary states $S_\pm$ of solutions of the system (1.1.7) as $t \to \pm\infty$. The following lemma shows that the limit stationary states in (1.2.2) can be arbitrary.

Lemma 1.2.4. *Let* $d \geqslant 1$ *and let* $F(y) \in C(\mathbb{R}^d, \mathbb{R}^d)$ *in the system* (1.1.1). *Then for two arbitrary stationary states* $S_\pm \in \mathcal{S}$ *there are solutions* $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *of the system* (1.1.7) *connecting* $S_\pm$ *in the sense of* (1.2.2).

Remark. Lemma 1.2.4 means that there is no exclusion principle in the system (1.1.1). In other words, this is a system with non-trivial 'Bohr' transitions between distinct stationary states $S_+ \neq S_-$. Such transitions are a purely non-linear phenomenon, which is impossible in the linear autonomous Schrödinger and Dirac equations.

Example. As a trivial example, we consider (1.1.1) with $d \geqslant 1$, $m = 0$, and $F(y) \equiv 0$. In this case, (1.1.1) coincides with the linear d'Alembert equation, and neither of conditions (0.1.15) and (ND) is satisfied. Accordingly, the orbit $O(Y)$ of a solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ is not precompact in $\mathcal{E}_F$ in the general case, and the convergence (1.2.2) does not hold. At the same time, we have the convergence (1.2.1), and the assertions of Theorem 1.2.2, iii), as well as Lemmas 1.2.3 and 1.2.4, remain valid. This readily follows from the d'Alembert formula for the solution of the Cauchy problem.

In §1.3 we derive the cited equation for the non-linear oscillator and prove Proposition 1.2.1. In § 1.4 we construct the large-time asymptotics of solutions of the reduced equation and give typical examples.

In §1.5 we prove Theorem 1.2.2 and Lemmas 1.2.3 and 1.2.4.

§1.3. Existence of dynamics and *a priori* estimates

Here we prove Proposition 1.2.1.

1.3.1. Existence of dynamics. First, we prove uniqueness of the solution $Y(t) = (u(\,\cdot\,, t), \dot{u}(\,\cdot\,, t), \dot{y}(t)) \in C(\mathbb{R}, \mathcal{E})$, existence being assumed. The proof also includes an algorithm for constructing the solution, and so we essentially prove existence as well.

Uniqueness of the solution. The initial conditions (1.1.2) and the d'Alembert formula (1.1.8) imply the following well-known formulae for $f_\pm(z)$ and $g_\pm(z)$ in the domain $\pm z > 0$:

$$\begin{aligned} f_\pm(z) &= \frac{u^0(z)}{2} - \frac{1}{2}\int_0^z v^0(y)\,dy + C_\pm, \qquad \pm z > 0, \\ g_\pm(z) &= \frac{u^0(z)}{2} + \frac{1}{2}\int_0^z v^0(y)\,dy - C_\pm, \qquad \pm z > 0, \end{aligned} \tag{1.3.1}$$

where the $C_\pm$ are arbitrary constants. Since $f_\pm$ and $g_\pm$ in (1.1.8) are determined only up to a constant, we can assume that $C_\pm = 0$. It follows from (1.3.1) that

$$f'_\pm(z), g'_\pm(z) \in L^2(\mathbb{R}_\pm, \mathbb{R}^d), \tag{1.3.2}$$

since $(u^0, v^0, p^0) \in \mathcal{E}$, where $\mathbb{R}_\pm \equiv \{x \in \mathbb{R} : \pm x > 0\}$. From (1.3.1) we obtain the usual d'Alembert formula for $|x| \geqslant |t|$:

$$u(x,t) = \frac{u^0(x-t) + u^0(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} v^0(y)\,dy. \tag{1.3.3}$$

Thus, the solution $u(x,t)$ is uniquely determined in the domain $|x| \geqslant |t|$. It remains to prove that the solution is unique in the domain $|x| < |t|$.

Let us consider the case $t > 0$. Then in the domain $|x| < t$ the unknown functions in (1.1.8) are $f_+(x-t)$ for $0 < x < t$ and $g_-(x+t)$ for $-t < x < 0$. Thus, we must find $f_+(z)$ for $z < 0$ and $g_-(z)$ for $z > 0$. To find these unknown functions, we derive an ordinary non-linear differential equation for $y(t) \equiv u(0,t)$.

Lemma 1.3.1. *For each solution $Y(t) = (u(\,\cdot\,,t), \dot u(\,\cdot\,,t), \dot y(t)) \in C(\mathbb{R}, \mathcal{E})$ of the system* (1.1.7), *the function $y(t) \equiv u(0,t)$ is a solution of the 'reduced' equation*

$$m\ddot y(t) = F(y(t)) - 2\dot y(t) + 2\dot w(t), \qquad t \in \mathbb{R}, \tag{1.3.4}$$

where $w(t)$ is the sum at $x = 0$ of the 'incoming' waves in (1.1.8):

$$w(t) \equiv g_+(t) + f_-(-t), \qquad t \in \mathbb{R}, \tag{1.3.5}$$

$$\dot w(t) \in L^2(0, \infty). \tag{1.3.6}$$

Proof. The formulae (1.3.4) and (1.3.5) are obtained from the second equation in the system (1.1.1) by substituting of the expressions for $u(x,t)$ via $y(t) \equiv u(0,t)$ and the incoming waves:

$$u(x,t) \equiv \begin{cases} y(t-x) + g_+(x+t) - g_+(t-x) & \text{for } x > 0, \\ y(t+x) + f_-(x-t) - f_-(-x-t) & \text{for } x < 0. \end{cases} \tag{1.3.7}$$

In turn, these formulae follow from the expressions for the outgoing waves $f_+(x-t)$ and $g_-(x+t)$ via the incoming waves $g_+(x+t)$ and $f_-(x-t)$:

$$y(t) \equiv f_+(-t) + g_+(t) \equiv f_-(-t) + g_-(t), \qquad t \in \mathbb{R}. \tag{1.3.8}$$

Finally, (1.3.6) follows from (1.3.5) and (1.3.2).

Remarks. i) Equation (1.3.4) contains energy dissipation due to the string-oscillator interaction. The description of the reversible system (1.1.1) via the 'irreversible' equation (1.3.4) is seemingly paradoxical. The explanation is as follows: along with (1.3.4) one has a similar equation with 'negative' friction but with the 'outgoing' waves $w(t)$ instead of the 'incoming' waves on the right-hand side in (1.3.4).

Hence these equations exchange their roles, that is, pass into each other, under time inversion. Thus, the relationship between (1.3.4) and the reversible system (1.1.1) is 'covariant' with respect to time inversion.

ii) To obtain the asymptotics of solutions of (1.1.1) as $t \to +\infty$, one must use (1.3.4) with positive friction and incoming waves. The point is that the incoming waves, unlike the outgoing waves, are directly determined by the initial data.

iii) The relationship between (1.3.4) and (1.1.1) was studied from a slightly different viewpoint in [24]. For the case of the linear oscillator $F(y) \equiv -ky$, $k > 0$, (1.1.1) and (1.3.4) were considered for the first time by Lamb [41].

According to (1.3.2), the functions occurring in (1.3.4) and known from (1.3.1) satisfy the conditions $f'_-(-t), g'_+(t) \in L^2(\mathbb{R}_+, \mathbb{R}^d)$. On the other hand, it follows by (1.1.11) from the assumption on the existence of a solution that $\ddot{y}(t) \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^d)$. By Definition 1.1.2, the derivative $\ddot{y}$ in (1.3.4) is understood in the sense of distributions. Consequently, $y \in C^1(\overline{\mathbb{R}}_+, \mathbb{R}^d)$, and by the Lebesgue theorem, (1.3.4) is equivalent to the same identity for almost all $t > 0$. Hence for any given initial data $y(0+)$ and $\dot{y}(0+)$, there is a unique solution $y(t)$ of (1.3.4) on some interval $t \in [0, \varepsilon)$, where $\varepsilon > 0$, since $F(\,\cdot\,) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. One can readily derive this from the contraction mapping principle by rewriting (1.3.4) in the equivalent integral form

$$my(t) = \int_0^t \left(\int_0^s F(y(\tau))\, d\tau \right) ds + 2 \int_0^t [w(s) - y(s)]\, ds + C_0 + C_1 t, \quad t > 0, \quad (1.3.9)$$

where

$$C_0 = my(0+) \quad \text{and} \quad C_1 = -2[w(0+) - y(0+)] + m\dot{y}(0+).$$

Thus, $y(t)$ is uniquely determined on $[0, \varepsilon)$ for some $\varepsilon > 0$. Since $y \in C^1(\overline{\mathbb{R}}_+, \mathbb{R}^d)$, it follows that $y(t)$ is also uniquely determined for all $t > 0$ for any given $y(0+)$ and $\dot{y}(0+)$.

It remains to specify $y(0+)$ and $\dot{y}(0+)$. First, it follows from (1.1.2) that

$$\dot{y}(0+) = p^0. \quad (1.3.10)$$

Next, since $u(x, t)$ is continuous for $|x| = t$, it follows from (1.3.2) that $f_+(x - t)$ is continuous for $x = t$ and $g_-(x + t)$ is continuous for $x = -t$. Consequently, in view of (1.3.1) we have

$$\begin{cases} f_+(0-) = f_+(0+) \equiv \dfrac{u^0(0)}{2}\,, \\ g_-(0+) = g_-(0-) \equiv \dfrac{u^0(0)}{2}\,. \end{cases} \quad (1.3.11)$$

These conditions together are equivalent to

$$y(0+) = f_\pm(0-) + g_\pm(0+) = u^0(0). \quad (1.3.12)$$

Thus, both $y(0+)$ and $\dot{y}(0+)$, and hence $y(t)$ for $t \geqslant 0$, are determined uniquely. Then f_+ and g_- are determined uniquely by (1.3.8). Thus we have proved the uniqueness of the solution $u(x, t)$ for $t \geqslant 0$. The proof for $t \leqslant 0$ is similar.

Existence of solutions. Let us prove the existence of a solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ of (1.1.7). First, we define $u(x,t)$ in the domain $|x| \geqslant |t|$ by the d'Alembert formula (1.3.3). In view of (1.3.2), $u(x,t)$ then satisfies the definition of the class $C(\mathbb{R}, \mathcal{E})$ of solutions in the domain $|x| > |t|$.

We now construct the solution in the domain $t > 0$, $|x| < t$ as follows. We define $u(x,t)$ by the formula (1.1.8), where f_+ and g_- are found from (1.3.8) with given $y(t)$. Here $y(t)$ is the solution of (1.3.4) with the initial conditions (1.3.10), (1.3.12).

Lemma 1.3.2. *For any $y(0+)$ and $\dot{y}(0+)$, (1.3.4) has a unique solution for all $t > 0$, and*

$$\sup_{t>0} |y^{(k)}(t)| \leqslant y^k, \qquad k = 0, 1, \tag{1.3.13}$$

where the y^k are bounded for bounded $\|(u^0, v^0, p^0)\|_{\mathcal{E}}$.

Proof. Let us derive an *a priori* estimate for $y(t)$. To this end, we multiply (1.3.4) by $\dot{y}(t)$ for almost all $t > 0$ and get, using $F(y) = -\nabla V(y)$, that

$$m \frac{d}{dt} \frac{\dot{y}^2(t)}{2} = -\frac{d}{dt} V(y(t)) + 2[\dot{w}(t) - \dot{y}(t)]\dot{y}(t) \tag{1.3.14}$$

for almost all $t > 0$. By integrating this, we obtain from (1.3.6) and the Cauchy–Bunyakovskii–Schwarz inequality that

$$m\frac{\dot{y}^2(t)}{2} + V(y(t)) \leqslant B, \qquad t > 0, \tag{1.3.15}$$

where B is bounded for bounded $\|(u^0, v^0, p^0)\|_{\mathcal{E}}$ by virtue of (1.3.2). Consequently, (0.1.15) implies (1.3.13) with $k = 1$.

It follows from this estimate that there is a unique global solution of (1.3.4) for any $y(0+)$ and $\dot{y}(0+)$, the estimate (1.1.13) holds, and the inclusion

$$\dot{y}(t) \in C(\overline{\mathbb{R}}_+, \mathbb{R}^d) \tag{1.3.16}$$

is valid.

Thus, we determine f_+ and g_- from (1.3.8) with the function $y(t)$ thus constructed. Then it follows from (1.3.16), (1.3.2), and (1.3.8) that

$$f_+'(z) \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}}_-, \mathbb{R}^d), \qquad g_-'(z) \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^d). \tag{1.3.17}$$

Hence the function $u(x,t)$ defined for $t > 0$ by (1.1.8) satisfies the definition of the class $C(\mathbb{R}, \mathcal{E})$ of solutions in the domain $|x| < t$. Consequently, it follows from (1.3.12) and (1.3.11) that for $t > 0$ the function $Y(t) = (u(x,t), \dot{u}(x,t), \dot{y}(t))$ is the restriction to the domain $t > 0$ of some function belonging to $C(\mathbb{R}, \mathcal{E})$.

The solution $u(x,t)$ in the domain $t < 0$ can be constructed in a similar way. It follows from (1.3.3) that the function $Y(t)$ thus constructed is continuous at $t = 0$; consequently, $Y(t) \in C(\mathbb{R}, \mathcal{E})$.

It remains to verify that $u(x,t)$ is a solution of the problem (1.1.1), (1.1.2). It follows from (1.1.8) that $u(x,t)$ satisfies the first equation in (1.1.1). The initial conditions (1.1.2) for u follow from (1.3.3) and (1.3.10). It follows from the construction of f_+ and g_- that $u(x,t)$ satisfies the second equation in (1.1.1) for $t > 0$ (and likewise for $t < 0$). The second equation in (1.1.1) follows from (1.3.4) for $t > 0$, a similar equation for $t < 0$, the equalities $\dot{y}(0+) = p^0 = \dot{y}(0-)$, and the inclusion $\ddot{y}(t) \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}}_\pm, \mathbb{R}^d)$. (The last inclusion follows from (1.3.4), (1.3.16), and (1.3.2).) Thus we have proved the first assertion of Proposition 1.2.1.

1.3.2. Continuity of the dynamics. The second assertion of Proposition 1.2.1 can also be derived from the above construction. To this end, it suffices to prove the following lemma on the continuous dependence of solutions of (1.3.4) on the initial data in (1.1.2).

Lemma 1.3.3. *Let* $y_1(t)$ *and* $y_2(t)$ *be the solutions of* (1.3.4) *with initial data* (u_i^0, v_i^0, p_i^0), $i = 1, 2$, *respectively. Then for each* $T > 0$ *there is a constant* C_T, *which is bounded for bounded* $\|(u_i^0, v_i^0, p_i^0)\|_{\mathcal{E}}$, *such that*

$$\sup_{[0,T]} |y_2(t) - y_1(t)| \leqslant C_T \|(u_2^0 - u_1^0, v_2^0 - v_1^0, p_2^0 - p_1^0)\|_{\mathcal{E}}, \tag{1.3.18}$$

$$\sup_{[0,T]} |\dot{y}_2(t) - \dot{y}_1(t)| \leqslant C_T \|(u_2^0 - u_1^0, v_2^0 - v_1^0, p_2^0 - p_1^0)\|_{\mathcal{E}}. \tag{1.3.19}$$

Proof. To prove the estimate (1.3.18), it suffices to subtract (1.3.9) for y_1 from the same equation for y_2 and apply the Gronwall inequality. We also use the *a priori* estimate (1.3.13). By differentiating (1.3.9), we obtain the estimate (1.3.19).

This lemma, together with the representation (1.3.7), implies the continuity of the operator W_t in $\mathcal{E}$. It follows that W_t is also continuous in $\mathcal{E}_F$, since $u(x,t)$ depends only on the initial data $(u^0(y), v^0(y), p^0)$ with $|y - x| < |t|$ by virtue of (1.3.7) and (1.3.9).

1.3.3. Conservation of energy. First, we prove the energy conservation law (1.1.3) for sufficiently smooth initial data. Namely, we assume that $u^0(x) \in C^2(\mathbb{R} \setminus \{0\}, \mathbb{R}^d)$ and $v^0(x) \in C^1(\mathbb{R} \setminus \{0\}, \mathbb{R}^d)$, the limits $u^0(0\pm)$, $(u^0)'(0\pm)$, and $v^0(0\pm)$ exist, and condition (1.2.3) is satisfied with some constant r^0. We note that such sufficiently smooth initial data form a dense subset of the Banach space $\mathcal{E}$ with norm (1.1.4).

It follows from the above construction of $u(x,t)$ that for such initial data one has $u \in C(\mathbb{R}^2, \mathbb{R}^d)$ and all first and second partial derivatives of $u(x,t)$ exist in the usual sense and are locally bounded for $x \neq 0$ and $x \neq \pm t$. Furthermore, for all $t \in \mathbb{R}$ there are one-sided limits of $\dot{u}(x,t)$ and $u'(x,t)$ as $x \to 0\pm$ and $x \to t \pm 0$, and moreover,

$$(\dot{u} + u')\big|_{x=t-0} = (\dot{u} + u')\big|_{x=t+0} \quad \forall\, t \neq 0. \tag{1.3.20}$$

This follows from the d'Alembert formula (1.1.8). Indeed, both sides of (1.3.20) are zero for $f_\pm(x - t)$, and the functions $g_\pm(x + t)$ are continuously differentiable for $x = t \neq 0$. Similarly,

$$(\dot{u} - u')\big|_{x=-t-0} = (\dot{u} - u')\big|_{x=-t+0} \quad \forall\, t \neq 0. \tag{1.3.21}$$

We consider the 'energy integral'

$$I(t) \equiv \int_{-\infty}^{+\infty} \left[\frac{|\dot{u}(x,t)|^2}{2} + \frac{|u'(x,t)|^2}{2} \right] dx \tag{1.3.22}$$

for $t > 0$. Let us split it into the sum of the integrals over $(-\infty, -t)$, $(-t, 0)$, $(0, t)$, and (t, ∞). Then by differentiating each of these integrals with respect to t we obtain

$$I'(t) = \Gamma_-(x,t)\Big|_{x=-t+0}^{x=-t-0} + u'\dot{u}\Big|_{x=0+}^{x=0-} + \Gamma_+(x,t)\Big|_{x=t+0}^{x=t-0}, \tag{1.3.23}$$

where

$$\Gamma_\pm(x,t) = \pm\left[\frac{|\dot u(x,t)|^2}{2} + \frac{|u'(x,t)|^2}{2}\right] + u'(x,t)\dot u(x,t) = \pm\frac{1}{2}|\dot u(x,t) \pm u'(x,t)|^2.$$

Here we have used the fact that (1.1.1) is satisfied for $u(x,t)$ in the usual sense for $x \neq 0$ and $x \neq \pm t$. It follows from (1.3.20) and (1.3.21) that

$$\Gamma_\pm\Big|_{x=\pm t+0}^{x=\pm t-0} = 0. \tag{1.3.24}$$

Finally, from (1.1.1) we obtain

$$u'\dot u\Big|_{x=0+}^{x=0-} = -\dot y(m\ddot y + V'(y)) = -\frac{d}{dt}\left[m\,\frac{\dot y^2}{2} + V(y(t))\right], \qquad t > 0. \tag{1.3.25}$$

Hence it follows from (1.3.23) that

$$I'(t) + \frac{d}{dt}\left[m\,\frac{\dot y^2}{2} + V(y(t))\right] = 0, \qquad t > 0. \tag{1.3.26}$$

This implies (E) for $t > 0$. By passing to the limit as $t \to 0+$, we obtain (E) for $t \geqslant 0$. For $t \leqslant 0$, the identity (E) can be proved in a similar way. Thus we have proved (E) on a dense set of initial data in $\mathcal{E}$. It remains to use Proposition 1.2.1, ii).

§1.4. Relaxation for the reduced equation

In the next section we shall derive Theorem 1.2.2 from the representation (1.3.7) and the following relaxation lemma for the reduced equation. Let $\mathcal{Z} = \{(z,0) \in \mathbb{R}^{2d} : z \in Z\}$.

Lemma 1.4.1. *Suppose that the assumptions of Theorem* 1.2.2 *are satisfied. Then the following assertions hold.*

i) *The dissipation integral is finite*:

$$\int_0^\infty |\dot y(t)|^2\,dt < \infty. \tag{1.4.1}$$

ii) *For each solution* $y(t)$ *of* (1.3.4),

$$(y(t), \dot y(t)) \to \mathcal{Z} \quad \textit{as} \quad t \to \infty. \tag{1.4.2}$$

iii) *In addition, suppose that* Z *is a trapping subset of* $\mathbb{R}^d$. *Then there is a point* $(z,0) \in \mathcal{Z}$ *such that*

$$(y(t), \dot y(t)) \to (z,0) \quad \textit{as} \quad t \to \infty. \tag{1.4.3}$$

iv) *Moreover, if* $d = 1$ *and* $\dot w(t) \equiv 0$ *for* $t > t^0$, *then* (1.4.3) *holds for an arbitrary set* Z.

Before proving this lemma, we give some typical examples for $d = 1$.

1.4.1. Examples. For simplicity we assume that the initial data satisfy condition (1.2.3). Then it follows from (1.3.5) that $\dot{w}(t) \equiv 0$ for $t > r^0$, and (1.3.4) is autonomous for large t.

Non-degenerate potentials. First, we consider the case $m > 0$. Then the phase curves of the reduced equation (1.3.4) on the phase plane $(y, \dot{y})$ are described by the system

$$\begin{cases} \dot{y}(t) = v(t), \\ m\dot{v}(t) = F(y(t)) - 2\nu v(t), \end{cases} \qquad t > r^0. \tag{1.4.4}$$

Here $\nu = 1$ for the system (1.1.1) and $\nu = \sqrt{\mu T}$ in a more general case, where μ is the string mass per unit length and T is the string tension. We treat this system as a perturbation of the system

$$\begin{cases} \dot{y} = v, \\ m\dot{v} = F(y) \end{cases} \tag{1.4.5}$$

with $\nu = 0$, which corresponds to a free oscillator not attached to the string. Let us establish some simple relationships between the phase portraits of these two systems.

A. The stationary points of these systems coincide.

B. The vertical component $\dot{v}$ of the phase velocity vector in the system (1.4.4) is smaller than in (1.4.5) in the upper half-plane $v > 0$ and is greater than in (1.4.5) in the lower half-plane $v < 0$. The horizontal components of these vectors are the same (Fig. 2).

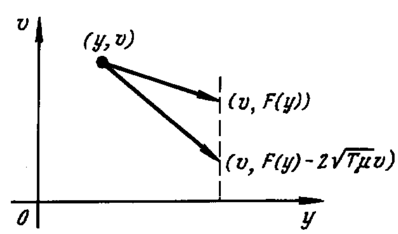

Figure 2

C. Hence the phase curves of (1.4.4) intersect those of (1.4.5) downwards in the half-plane $v > 0$ and upwards in the half-plane $v < 0$.

Let us consider the typical example

$$V(y) = \frac{y^4}{4} - \frac{y^2}{2}, \qquad y \in \mathbb{R}, \tag{1.4.6}$$

of a non-degenerate potential. This potential satisfies the conditions (0.1.15). Furthermore, (1.4.5) has the following phase curves:

- closed curves corresponding to periodic solutions;
- two separatrices entering the point $(0, 0)$ and issuing from the same point;
- three stationary points, namely, a saddle at $(0, 0)$ and two centres at $(\pm 1, 0)$ (Fig. 3).

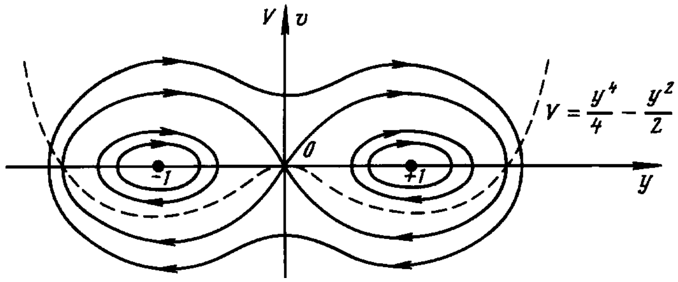

Figure 3

By the property **C**, we find that (1.4.4) with the potential (1.4.6) has:

- stable foci at the points $(\pm 1, 0)$ for small $\nu > 0$ (stable nodes for large $\nu > 0$);
- a saddle at the point $(0, 0)$ (Fig. 4).

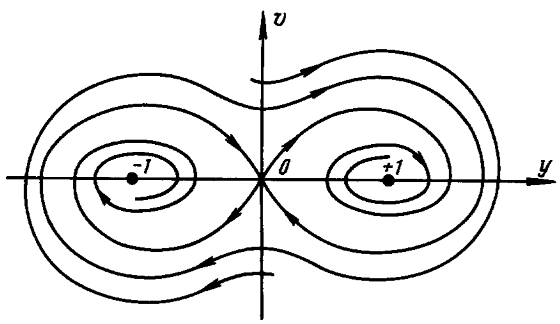

Figure 4

Now let us consider the case $m = 0$. Then the phase curves of (1.4.4) with the potential (1.4.6) are the rays $y < -1$ and $y > 1$, the intervals $(-1, 0)$ and $(0, +1)$, and three stationary points (the stable points ± 1 and the unstable point 0) (Fig. 5).

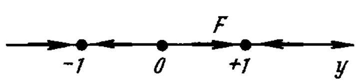

Figure 5

Remark. In this example (the potential (1.4.6) for $m > 0$), the assumptions of Lemma 1.4.1 are satisfied and the assertions of this lemma hold. In particular, each phase curve enters some stationary point as $t \to +\infty$. We note that the limit point may be unstable, like the saddle point $(0, 0)$ in Fig. 4, which has two separatrices entering it.

Degenerate potentials. The function

$$V(y) \equiv \begin{cases} k\dfrac{(y-b_1)^2}{2}, & y \leqslant b_1, \\ 0, & b_1 \leqslant y \leqslant b_2, \\ k\dfrac{(y-b_2)^2}{2}, & y \geqslant b_2, \end{cases} \tag{1.4.7}$$

where $k > 0$ and $b_1 < b_2$, is an example of a degenerate potential. In the strip $b_1 \leqslant y \leqslant b_2$ the phase curves are described by the system (1.4.4) with $F \equiv 0$, that is,

$$\begin{cases} \dot{y} = v \\ m\dot{v} = -2\nu v \end{cases} \quad \Rightarrow \quad \frac{dv}{dy} = -\alpha \equiv -\frac{2\nu}{m}. \tag{1.4.8}$$

Hence the equations of the phase curves in the strip $b_1 \leqslant y \leqslant b_2$ have the form

$$v = -\alpha y + \text{const}. \tag{1.4.9}$$

In particular, the phase curves for $\nu = 0$ are segments parallel to the y-axis.

From the preceding we obtain the phase portraits of (1.4.4) with the potential (1.4.7).

1) The phase portrait for $\nu = 0$ is shown in Fig. 6.

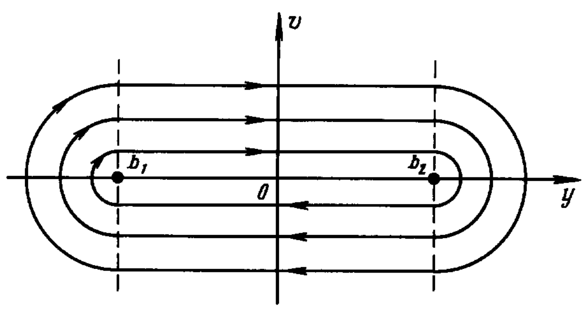

Figure 6

We see from the figure that ii) and iii) of Lemma 1.4.1 fail in this case. The oscillator performs periodic oscillations as $t \to +\infty$.

2) The phase portrait for small $\nu > 0$ is shown in Fig. 7.

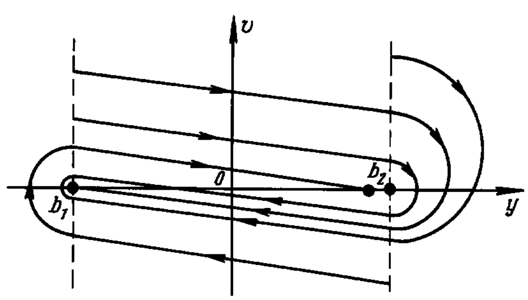

Figure 7

We see from the figure that in this case (as well as for large $\nu > 0$) all solutions have limits as $t \to +\infty$, which agrees with Lemma 1.4.1, iv). We note that the potential (1.4.7) does not satisfy condition (ND).

1.4.2. Proof of the relaxation property for the reduced equation. Let us prove Lemma 1.4.1.

i) Equation (1.3.4) implies the energy identity

$$\frac{d}{dt}\left[m\,\frac{|\dot y(t)|^2}{2} + V(y(t))\right] = -2|\dot y(t)|^2 + 2\dot w(t)\dot y(t), \qquad t \in \mathbb{R}. \tag{1.4.10}$$

The right-hand side is majorized by the function $-\varepsilon\dot y^2(t) + C_\varepsilon \dot w^2(t)$ with some $\varepsilon, C_\varepsilon > 0$; integrating, we obtain

$$\frac{\dot y^2(t)}{2} + V(y(t)) + \varepsilon\int_0^t \dot y^2(\tau)\,d\tau \leqslant C_0 + C_\varepsilon\int_0^t \dot w^2(\tau)\,d\tau \quad \text{for} \quad t > 0. \tag{1.4.11}$$

Hence from (0.1.15) and (1.3.6) we obtain (1.4.1).

ii) It follows from (1.3.13) that the trajectory $(y(t), \dot y(t))$ is bounded. Hence it remains to prove that $(\overline{y}, \overline{p}) \in \mathcal{Z}$ if $(y(t_k), \dot y(t_k)) \to (\overline{y}, \overline{p})$ for some sequence $t_k \to \infty$. The proof is by contradiction. Suppose that $(\overline{y}, \overline{p}) \notin \mathcal{Z}$. First, we analyze the case $\overline{p} \neq 0$. It follows from (1.4.11) that the right-hand side of (1.4.10) is integrable on $(0, \infty)$. Consequently, the limit

$$E_\infty \equiv \lim_{t\to+\infty}\left[m\,\frac{|\dot y(t)|^2}{2} + V(y(t))\right] \tag{1.4.12}$$

exists. Then $\overline{p} \neq 0$ implies that $V(\overline{y}) < E_\infty$. Thus, $V(y) < E_\infty - h$ in a small neighbourhood U of the limit point $(\overline{y}, \overline{p})$ for some $h > 0$. Hence it follows from (1.4.12) that

$$m\,\frac{|\dot y(t)|^2}{2} \geqslant h > 0 \quad \text{for} \quad (y(t), \dot y(t)) \in U \tag{1.4.13}$$

for large t. The trajectory $(y(t), \dot y(t))$ visits U infinitely often. We introduce the entry and exit times

$$\tau_k^- = \inf\{t \in [0, t_k] : (y(s), \dot y(s)) \in U \text{ for all } s \in (t, t_k]\}, \tag{1.4.14}$$

$$\tau_k^+ = \sup\{t \in [t_k, \infty) : (y(s), \dot y(s)) \in U \text{ for all } s \in (t_k, t]\}. \tag{1.4.15}$$

Passing to a subsequence, we can assume that $t_{k+1} - t_k > 2$ for each $k = 1, 2, \ldots$; we set $t_k^- = \max(\tau_k^-, t_k - 1)$ and $t_k^+ = \min(\tau_k^+, t_k + 1)$. Then the intervals $[t_k^-, t_k^+]$ are disjoint and $(y(t), \dot y(t)) \in U$ for $t \in [t_k^-, t_k^+]$. The interval $[t_k^-, t_k^+]$ will be called the interval of the kth visit. For large k, the point $(y(t_k), \dot y(t_k))$ is very close to $(\overline{y}, \overline{p})$, and then the duration of the visit satisfies $t_k^+ - t_k^- \geqslant T = \min(1, C/y^1) > 0$ by virtue of (1.3.13) with $k = 1$. Hence it follows from (1.4.13) that the contribution of each visit to the dissipation integral (1.4.1) admits a lower bound of the form

$$\frac{m}{2}\int_{t_k^-}^{t_k^+} |\dot y(t)|^2\,dt \geqslant hT > 0. \tag{1.4.16}$$

However, the number of visits is infinite, and hence (1.4.1) cannot hold; this is a contradiction proving that $\overline{p}=0$.

Finally, $\nabla V(\overline{y})=0$, which follows from a similar analysis of visits. Indeed, if $\nabla V(\overline{y})\neq 0$, then the contribution of each visit to the dissipation integral (1.4.1) admits a lower bound similar to (1.4.16). This follows from (1.3.4) in view of (1.4.1).

iii) The relation (1.4.2) implies (1.4.3), since Z is a trapping subset of $\mathbb{R}$ and hence $\mathcal{Z}$ is a trapping subset of $\mathbb{R}^{2d}$.

iv) If $\dot{w}(t)\equiv 0$ for $t>t^0$, then (1.3.4) acquires the form

$$m\ddot{y}(t)=F(y(t))-2\dot{y}(t),\qquad t>t^0. \tag{1.4.17}$$

Let us prove (1.4.3) by contradiction. If $y(t)$ has no limit, then

$$y_-=\liminf_{t\to+\infty}y(t)<y_+=\limsup_{t\to+\infty}y(t).$$

Hence it follows from (1.4.12) and the relation $\dot{y}(t)\to 0$ that $V(y)=E_\infty$ for $y_-\leqslant y\leqslant y_+$. Consequently,

$$m\ddot{y}(t)=-2\dot{y}(t)\quad\text{if}\quad y_-<y(t)<y_+, \tag{1.4.18}$$

and the phase curves in the strip $y_-<y<y_+$ of the plane $y,\dot{y}$ are the lines

$$\dot{y}=-\frac{2}{m}y+\text{const}\,. \tag{1.4.19}$$

Hence (1.4.3) follows from the fact that $\dot{y}(t)\to 0$ (see Fig. 7).

§1.5. Large-time asymptotics

1.5.1. Transitions to stationary states. Here we prove Theorem 1.2.2.

A compact attracting set. First, we construct a compact attracting set $\mathcal{A}$ for the trajectory $Y(t)$ in question.

Definition 1.5.1. $\mathcal{A}=\{S_z : z\in\mathbb{R}^d,\ |z|\leqslant y^0\}$, where the S_z are defined in (1.1.14) and y^0 in (1.3.13).

The set $\mathcal{A}$ is compact in $\mathcal{E}_F$, since it is homeomorphic to a compact set in $\mathbb{R}^d$.

Lemma 1.5.2. *Let all the hypotheses of Theorem* 1.2.2 *be satisfied. Then* $Y(t)\xrightarrow{\mathcal{E}_F}\mathcal{A}$ *as* $t\to\pm\infty$.

Proof. It suffices to verify that

$$\|Y(t)-S_{y(t)}\|_R=|u'(\,\cdot\,,t)|_R+|\dot{u}(\,\cdot\,,t)|_R+|\dot{y}(t)|\to 0\quad\text{as}\quad t\to\infty \tag{1.5.1}$$

for each $R>0$ (see (0.1.14)). Here the seminorms $|\cdot|_R$ converge to zero as $t\to\infty$ by virtue of (1.3.7), (1.3.2), and (1.4.1). Hence, (1.4.2) completes the proof.

Proof of Theorem 1.2.2. i) It follows from Lemma 1.5.2 that the orbit $O(Y)$ is precompact in $\mathcal{E}_F$. Hence the lemma below implies (1.2.1). Let $\Omega(Y)$ be the ω-limit set of $Y(t)$ in the Fréchet topology of $\mathcal{E}_F$: $\overline{Y}\in\Omega(Y)$ if and only if $Y(t_k)\xrightarrow{\mathcal{E}_F}\overline{Y}$ for some sequence $t_k\to\pm\infty$.

Lemma 1.5.3. *$\Omega(Y)$ is contained in $\mathcal{S}$.*

Proof. We have $\Omega(Y) \subset \mathcal{A}$, since $\mathcal{A}$ is an attracting set. Furthermore, the set $\Omega(Y)$ is invariant with respect to W_t, $t \in \mathbb{R}$, by virtue of the continuity of W_t in $\mathcal{E}_F$. Hence for each $\overline{Y} \in \Omega(Y)$ there is a C^2 curve $t \mapsto z(t) \in \mathbb{R}^d$ such that $W_t\overline{Y} = S_{z(t)}$. This implies that $S_{z(t)}$ is a solution of (1.1.7). But then $\dot{z}(t) = 0$, that is, $z(t) \equiv z$ and $\overline{Y} = S_z \in S$.

ii) The representation (1.3.7), (1.3.2), with Lemma 1.4.1, i), iii), implies (1.2.2).

iii) It follows from (1.2.3) that $\dot{w}(t) \equiv 0$ for $t > r^0$. Then, by (1.2.3) and the representation (1.3.7),

$$u(x,t) = y(t - |x|) \quad \text{for} \quad t - |x| > r^0. \tag{1.5.2}$$

Hence Lemma 1.4.1 implies (1.2.2).

1.5.2. Damped wanderings. Let us derive Lemma 1.2.3 from formulae (1.3.4)–(1.3.7). Suppose that (ND) fails to hold. Without loss of generality, we can assume that

$$F(z) \equiv 0 \quad \text{for} \quad -1 \leqslant z \leqslant 1. \tag{1.5.3}$$

We consider the cases $m = 0$ and $m \neq 0$ separately.

I. If $m = 0$, then we can take $y(t)$ to be an arbitrary function with values in $[-1, 1]$ such that $\dot{y}(t) \in L^2(\mathbb{R})$, and set $u(x,t) = y(t - x)$. Then $Y(t) = (u(\,\cdot\,,t), \dot{u}(\,\cdot\,,t)) \in C(\mathbb{R}, \mathcal{E}^0)$ is a solution of (1.1.1). We take, say,

$$y(t) = \sin\log(1 + t^2) \quad \text{for} \quad t \in \mathbb{R}. \tag{1.5.4}$$

Then (1.2.2) obviously fails.

II. If $m \neq 0$, then we again take $y(t)$ in the form (1.5.4), find $\dot{w}(t)$ from (1.3.4), and set $g_+(t) \equiv w(t)$ and $f_-(t) \equiv 0$. Then (1.3.4) holds, and $u(x,t) = y(t - |x|)$ by (1.3.7). Consequently, $Y(t) = (u(\,\cdot\,,t), \dot{u}(\,\cdot\,,t), \dot{y}(t)) \in C(\mathbb{R}, \mathcal{E})$ is a solution of (1.1.1), and (1.2.2) obviously fails to be true.

1.5.3. Transitivity. Let us prove Lemma 1.2.4. Let $S_\pm = (s_\pm(x), 0, 0)$, where $s_\pm(x) \equiv z_\pm \in Z$. One can construct transitions $S_- \to S_+$ in various ways. We choose the simplest way. Let us construct a solution $Y(t) = (u(\,\cdot\,,t), \dot{u}(\,\cdot\,,t), \dot{y}(t)) \in C(\mathbb{R}, \mathcal{E})$ of (1.1.7) such that

$$y(t) \equiv u(0\pm, t) = \begin{cases} z_- & \text{for } t \leqslant -1, \\ z_+ & \text{for } t \geqslant 1. \end{cases} \tag{1.5.5}$$

We extend $y(t)$ arbitrarily to the interval $t \in (-1, 1)$ taking care that $y(t) \in C^2(\mathbb{R}, \mathbb{R}^d)$. Then we set $g_+ \equiv z_-$ and find f_- from (1.3.4):

$$m\ddot{y}(t) = F(y(t)) + 2\big(f_-'(-t) - \dot{y}(t)\big), \qquad t \in \mathbb{R}. \tag{1.5.6}$$

Then $f_-'(z) \in C(\mathbb{R}, \mathbb{R}^d)$. Since $F(z_\pm) = 0$, it follows that

$$f_-'(-t) \equiv 0 \quad \text{for} \quad t \leqslant -1 \quad \text{and for} \quad t \geqslant 1. \tag{1.5.7}$$

We can determine f_- uniquely, say, by requiring that

$$f_-(-t) \equiv z_- \quad \text{for} \quad t \leqslant -1. \tag{1.5.8}$$

Then the outgoing waves g_- and f_+ are determined from (1.3.8). Since $y(t)$, $f_-(-t)$, and $g_+(t)$ are constant for large $|t|$, so are $f_+(-t)$ and $g_-(t)$. Hence for the function $u(x,t)$ defined in (1.3.7), the trajectory $Y(t) = (u(\,\cdot\,,t), \dot{u}(\,\cdot\,,t), \dot{y}(t)) \in C(\mathbb{R}, \mathcal{E})$ is a solution of (1.1.7), and (1.2.2) holds.

Remarks. i) The solution thus constructed describes the following situation. The oscillator is at the stationary point z_- for $t \leqslant -1$; then the wave $f_-(x-t)$ comes to the oscillator and takes it to the state z_+ by the time $t = 1$; furthermore, for $t > -1$ this wave generates two outgoing waves, $g_-(x+t)$ for $x < 0$ and $f_+(x-t)$ for $x > 0$. These outgoing waves propagate along the strip $-1 < t - |x| < 1$.

ii) From the viewpoint of physics, the inequality $z_+ \neq z_-$ implies that the oscillator absorbs radiation if $V(z_+) > V(z_-)$ or emits radiation if $V(z_+) < V(z_-)$.

CHAPTER II

A STRING WITH FINITELY MANY NON-LINEAR OSCILLATORS

We consider a system of equations describing the vibrations of an infinite string coupled with finitely many non-linear oscillators. We prove convergence to stationary states as $t \to \pm\infty$. In [31] and [32] this result was obtained under certain additional restrictions imposed on the initial data.

§2.1. Introduction

We establish the convergence of the type (0.1.11), (0.1.12) to stationary states for solutions of the system

$$\begin{cases} \ddot{u}(x,t) = u''(x,t), \quad x \in \mathbb{R} \setminus Q, \quad t \in \mathbb{R}, \\ u(x_k+0,t) = u(x_k-0,t), \quad t \in \mathbb{R}, \quad k = 1,\dots,N, \\ 0 = F_k(u(x_k,t)) + u'(x_k+0,t) - u'(x_k-0,t). \end{cases} \tag{2.1.1}$$

Here $Q = \{x_1,\dots,x_N\}$ is a finite set of N points $x_i \in \mathbb{R}$. For $N = 1$, the system coincides with the Lamb system (1.1.1) with $m = 0$. The solutions $u(x,t)$ take values in $\mathbb{R}^d$, $d \geqslant 1$. We consider the Cauchy problem for the system (2.1.1) with initial conditions

$$u\big|_{t=0} = u^0(x), \quad \dot{u}\big|_{t=0} = v^0(x), \quad x \in \mathbb{R}. \tag{2.1.2}$$

The system (2.1.1) is formally equivalent to the non-linear wave equation

$$\ddot{u}(x,t) = u''(x,t) + \sum_{k=1}^{N} \delta(x-x_k) F_k(u(x,t)), \qquad (x,t) \in \mathbb{R}^2, \tag{2.1.3}$$

with non-linearity $\sum_{k=1}^{N} \delta(x-x_k) F_k(u)$ concentrated on the set Q (see (1.1.3)). From the viewpoint of physics, (2.1.1) describes small transverse vibrations

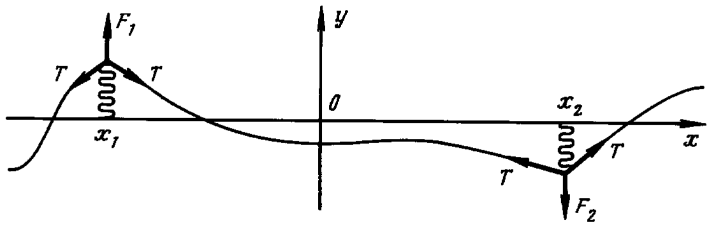

Figure 8

of the string, which is subjected to forces F_k applied at the points x_k and perpendicular to the string. For example, $F_k = -Ry_k$ if a linear spring of stiffness R is attached to the string at x_k (Fig. 8). In general, the functions $F_k(y_k)$ are non-linear.

We introduce the 'configuration space' $\mathcal{Q}$ and the phase space $\mathcal{E}$ of finite-energy states of (2.1.1), which coincides with the space $\mathcal{E}^0$ introduced in Chapter I. By L^2 we denote the Hilbert space $L^2(\mathbb{R}, \mathbb{R}^d)$ with the norm $|\cdot|$, and by $|\cdot|_R$ the norm in $L^2(-R, R; \mathbb{R}^d)$, $R > 0$.

Definition 2.1.1. i) $\mathcal{Q}$ is the Hilbert space $\{u(x) \in C(\mathbb{R}, \mathbb{R}^d) : u'(x) \in L^2\}$, with norm

$$\|u\|_{\mathcal{Q}} = |u'| + |u(0)|. \tag{2.1.4}$$

ii) $\mathcal{E} = \mathcal{Q} \oplus L^2$ is the Hilbert space of pairs $(u(x), v(x))$, with norm

$$\|(u, v)\|_{\mathcal{E}} = \|u\|_{\mathcal{Q}} + |v|. \tag{2.1.5}$$

iii) $\mathcal{E}_F$ is the space $\mathcal{E}$, equipped with the Fréchet topology defined by the seminorms

$$\|(u, v)\|_R \equiv |u'|_R + |u(0)| + |v|_R, \qquad R > 0. \tag{2.1.6}$$

We assume that conditions (0.1.16)–(0.1.17) hold. Then (2.1.1) is formally a Hamiltonian system with phase space $\mathcal{E}$ and Hamiltonian

$$\mathcal{H}(u, v) = \frac{1}{2} \int_{\mathbb{R}} [|v(x)|^2 + |u'(x)|^2]\, dx + \sum_{k=1}^{N} V_k(u(x_k)) \tag{2.1.7}$$

for $(u, v) \in \mathcal{E}$. We consider solutions $u(x, t)$ such that $Y(t) = (u(\cdot, t), \dot{u}(\cdot, t)) \in C(\mathbb{R}, \mathcal{E})$ and we rewrite the Cauchy problem (2.1.1)–(2.1.2) in the form

$$\dot{Y}(t) = \mathcal{V}(Y(t)) \quad \text{for} \quad t \in \mathbb{R}, \quad Y(0) = Y^0, \tag{2.1.8}$$

where $Y^0 = (u^0, v^0)$. Let us discuss the statement of this Cauchy problem for functions $Y(t) \in C(\mathbb{R}, \mathcal{E})$. The second equation in (2.1.1) makes sense and is satisfied automatically, since $u \in C(\mathbb{R}^2, \mathbb{R}^d)$, which follows from the fact that $Y(t) \in C(\mathbb{R}, \mathcal{E})$. The meaning of the third equation in (2.1.1) for $N = 1$ was explained in Chapter I. Let us apply these constructions to (2.1.1). Equation (2.1.1) is treated in the sense of distributions in the domain $x \in \mathbb{R} \setminus Q$, $t \in \mathbb{R}$. Hence it is equivalent to the d'Alembert formula for all $k = 1, \dots, N+1$:

$$u(x,t) = f_k(t-x) + g_k(t+x), \qquad x \in \Delta_k, \quad t \in \mathbb{R}, \tag{2.1.9}$$

where $f_k, g_k \in C(\mathbb{R}, \mathbb{R}^d)$ since $u \in C(\mathbb{R}^2, \mathbb{R}^d)$. It follows that

$$\dot{u}(x,t) = f_k'(t-x) + g_k'(t+x), \qquad u'(x,t) = -f_k'(t-x) + g_k'(t+x) \tag{2.1.10}$$

for all $k = 1, \dots, N$ and almost all $(x,t) \in \Delta_k \times \mathbb{R}$, where all derivatives are in the sense of distributions. It follows from the condition $Y(t) \in C(\mathbb{R}, \mathcal{E})$ that

$$f_k'(\,\cdot\,), g_k'(\,\cdot\,) \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d) \quad \forall\, k = 1, \dots, N+1. \tag{2.1.11}$$

Now we discuss the third equation in the system (2.1.1).

Definition 2.1.2. In the third equation in (2.1.1),

$$\begin{aligned} u'(x_k - 0, t) &\equiv -f_k'(t - x_k) + g_k'(t + x_k) \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d), \\ u'(x_k + 0, t) &\equiv -f_{k+1}'(t - x_k) + g_{k+1}'(t + x_k) \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d) \end{aligned} \tag{2.1.12}$$

for all $k = 1, \dots, N$.

We note that the functions f_k and g_k in (2.1.9) are determined up to a constant. Hence Definition 2.1.12 is unambiguous.

§2.2. Main results

We start from a construction of the dynamics.

Proposition 2.2.1. *Suppose that $d \geqslant 1$ and conditions* (0.1.16)–(0.1.17) *hold. Then*:

i) *for each $Y^0 \in \mathcal{E}$ the Cauchy problem* (2.1.8) *has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$*;
ii) *the map $W_t \colon Y^0 \mapsto Y(t)$ is continuous in $\mathcal{E}$ and $\mathcal{E}_F$ for each $t \in \mathbb{R}$*;
iii) *the energy conservation law* (E) *holds.*

By $\mathcal{S}$ we denote the set of stationary states $S = (s(x), 0) \in \mathcal{E}$ of the system (2.1.8). The functions $s(x)$ are linear in x on each interval $\Delta_k \equiv (x_{k-1}, x_k)$, $k = 2, \dots, N$. They are constant on $\Delta_1 \equiv (-\infty, x_1)$ and $\Delta_{N+1} \equiv (x_N, +\infty)$. The following proposition gives a criterion for $\mathcal{S}$ to be a non-empty discrete trapping subset of $\mathcal{E}_F$.

Proposition 2.2.2. *Suppose that conditions* (0.1.16)–(0.1.17) *hold and, moreover, $d = 1$ and all the functions $F_k(y)$, $k = 1, \dots, N$, are real-analytic on $\mathbb{R}$. Then $\mathcal{S}$ is a discrete trapping subset of $\mathcal{E}_F$, and the set $\{S \in \mathcal{S} : \mathcal{H}(S) \leqslant h\}$ is finite for each $h \in \mathbb{R}$.*

The main result of this chapter implies that $\mathcal{S}$ is a point attractor of the system (2.1.8) in the Fréchet topology of the space $\mathcal{E}_F$.

Theorem 2.2.3. *Suppose that all the assumptions of Proposition* 2.2.2 *are satisfied and the initial state* Y^0 *belongs to* $\mathcal{E}$. *Then*:

i) *the orbit* $O(Y)$ *of the solution* $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *of the Cauchy problem* (2.1.8) *is precompact in* $\mathcal{E}_F$, *and*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S} \quad as \quad t \to \pm\infty; \tag{2.2.1}$$

ii) *if, moreover,* $d = 1$ *and all the functions* $F_k(y_k)$ *are real-analytic on* $\mathbb{R}$, *then there are stationary states* $S_\pm \in \mathcal{S}$, *depending on the solution* $Y(t)$, *such that*

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm \quad as \quad t \to \pm\infty. \tag{2.2.2}$$

In the following we set $d = 1$, since all proofs remain valid without change for $d \geqslant 1$.

Remarks. i) Assertion ii) of this theorem follows from i) in view of Proposition 2.2.2.

ii) By Fatou's lemma it follows from the convergence (2.2.2) in conjunction with (2.1.7) and (0.1.16), (0.1.17) that (0.1.23) holds.

§2.3. Existence of dynamics and *a priori* estimates

For $N = 1$, Proposition 2.2.2 was proved in § 1.2. For $N \geqslant 1$ the proof is similar. The solution is constructed by the d'Alembert method, that is, by using the representation (2.1.9). However, for $N > 1$ we must find the waves repeatedly reflected by the points x_k, $k = 1, \dots, N$. The non-linear equations for the reflected waves were considered in detail in § 1.2. Let us show that the energy conservation law (E) implies the following *a priori* estimate.

Corollary 2.3.1. *Let conditions* (0.1.16)–(0.1.17) *be satisfied. Then for each solution* $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *of* (2.1.8), *all the functions* $y_k(t) \equiv u(x_k, t)$, $t \in \mathbb{R}$, *are bounded*:

$$\max_{1 \leqslant k \leqslant N} |y_k(t)| \leqslant B \quad for \quad t \in \mathbb{R}, \tag{2.3.1}$$

where $B < \infty$ *depends on the solution.*

We actually prove a stronger assertion. Let $y_k = y_k(u) = u(x_k)$ and $\overline{y} = \overline{y}(u) = (y_1, \dots, y_N)$ for $u \in \mathcal{Q}$. We define a potential energy functional $\mathcal{U}$ on $\mathcal{Q}$ by setting

$$\mathcal{U}(u) \equiv \mathcal{H}(u, 0) = \frac{1}{2} \int_{-\infty}^{\infty} |u'(x)|^2 \, dx + \sum_{k=1}^{N} V_k(u(x_k)), \qquad y_k \equiv u(x_k) \tag{2.3.2}$$

for $u \in \mathcal{Q}$.

Lemma 2.3.2. *Let conditions* (0.1.16)–(0.1.17) *be satisfied. Then*

$$\mathcal{U}(u) \to \infty \quad as \quad |\overline{y}(u)| \to \infty. \tag{2.3.3}$$

Proof. Let us show that

$$\sup_{\mathcal{U}(u) \leqslant E} |\overline{y}(u)| < \infty \tag{2.3.4}$$

for any bounded constant E. Indeed, it follows from (2.3.2) and (0.1.17) that

$$\sup_{\mathcal{U}(u)\leqslant E} \int_{-\infty}^{\infty} |u'(x)|^2\,dx = D < \infty. \tag{2.3.5}$$

Hence from the Cauchy–Bunyakovskii–Schwarz inequality we obtain

$$\sup_{\mathcal{U}(u)\leqslant E} |y_k - y_j| = \sup_{\mathcal{U}(u)\leqslant E} \left| \int_{x_k}^{x_j} u'(x)\,dx \right| \leqslant |x_k - x_j|^{1/2} D^{1/2} \tag{2.3.6}$$

for all k, j. In conjunction with (0.1.17), this implies (2.3.4).

Corollary 2.3.1 follows from Lemma 2.3.2 and (0.1.17).

Remark. It follows from conditions (0.1.16) that the Hamiltonian $\mathcal{H}$ is Fréchet differentiable on $\mathcal{E}$. If $u''\big|_{\Delta_k} \in L^2(\Delta_k)$ for all $k = 1, \dots, N+1$, then the limits $u'(x_k \pm 0)$ exist for all $k = 1, \dots, N$ and

$$\frac{\delta \mathcal{H}}{\delta v(x)} = v(x),$$
$$\frac{\delta \mathcal{H}}{\delta u(x)} = -u''(x) + \sum_{k=1}^{N} (-[u'(x_k+0) - u'(x_k-0)] + \nabla V_k(y_k))\delta(x - x_k),$$

where $y_k \equiv u(x_k)$. Hence (2.1.1) is a formally Hamiltonian system:

$$\dot{u} = \frac{\delta \mathcal{H}}{\delta v}\,, \qquad \dot{v} = -\frac{\delta \mathcal{H}}{\delta u}\,. \tag{2.3.7}$$

More precisely, if we assume that $\dot{v}(\,\cdot\,, t)$ is a regular distribution, then the second equation in (2.3.7) implies the third equation in (2.1.1).

§2.4. Stationary states

Let us prove Proposition 2.2.2. To find all stationary solutions, we substitute $u(x,t) = s(x)$ into (2.1.1). Then it follows from the first equation in (2.1.1) that $u''(x) = 0$ for $x \in \mathbb{R} \setminus Q$, and so

$$s(x) = a_k x + b_k \quad \text{for} \quad x \in \Delta_k, \quad k = 1, \dots, N+1. \tag{2.4.1}$$

It follows from the condition $s' \in L^2(\mathbb{R})$ that

$$a_1 = a_{N+1} = 0. \tag{2.4.2}$$

By substituting (2.4.1) in the second and third equations in (2.1.1), we obtain

$$\begin{cases} a_k x_k + b_k = a_{k+1} x_k + b_{k+1} \ (\equiv y_k), \\ 0 = F_k(y_k) + a_{k+1} - a_k \end{cases} \tag{2.4.3}$$

for all $k = 1, \dots, N$. It follows from (2.4.2) that the function (2.4.1) is uniquely determined by its values y_k at the points x_k, $k = 1, \dots, N$:

$$a_k = \frac{y_k - y_{k-1}}{l_k}, \quad b_k = y_k - a_k x_k, \quad k = 1, \dots, N; \qquad b_{N+1} = y_N. \tag{2.4.4}$$

Here $y_0 \equiv y_1$, $y_{N+1} \equiv y_N$, $l_k = x_k - x_{k-1}$, and $l_1 \equiv l_{N+1} \equiv 1$ (for example). For the unknowns y_k, the system (2.4.3) is equivalent to

$$F_k(y_k) + \frac{y_{k+1} - y_k}{l_{k+1}} - \frac{y_k - y_{k-1}}{l_k} = 0, \qquad k = 1, \dots, N. \tag{2.4.5}$$

Let $\mathcal{S}_N$ be the set of all real solutions $(y_1, \dots, y_N) \in \mathbb{R}^N$ of (2.4.5).

Remark. Since $s(x) \in C^2(\overline{\Delta}_k)$ for all $k = 1, \dots, N+1$, it follows that the formulae (2.3.7) are valid for the stationary states $(s(x), 0)$ of (2.1.8). Consequently, for stationary states (2.1.8) is equivalent to the equation

$$\frac{\delta \mathcal{U}}{\delta u}(s) = 0. \tag{2.4.6}$$

Proof of Proposition 2.2.2. On $\mathbb{R}^N$ we define the function

$$\mathcal{U}_N(y_1, \dots, y_N) = \mathcal{U}(s), \tag{2.4.7}$$

where $s = s(x)$ is the function (2.4.1) with a_k and b_k defined in (2.4.4). Then it follows from (2.3.2) that

$$\mathcal{U}_N(y_1, \dots, y_N) = \sum_{k=1}^{N} V_k(y_k) + \frac{1}{2} \sum_{k=2}^{N} \left| \frac{y_k - y_{k-1}}{l_k} \right|^2 l_k. \tag{2.4.8}$$

The above remark suggests that (2.4.5) is equivalent to

$$\frac{\partial \mathcal{U}_N}{\partial y_k}(y_1, \dots, y_N) = 0, \qquad k = 1, \dots, N. \tag{2.4.9}$$

This equivalence can readily be verified by a straightforward computation.

Remark. It follows from (2.3.3) that

$$\mathcal{U}_N(y_1, \dots, y_N) \to \infty \quad \text{as} \quad |(y_1, \dots, y_N)| \to \infty. \tag{2.4.10}$$

Consequently, $\mathcal{U}_N$ attains its minimum at some point $(y_1, \dots, y_N) \in \mathbb{R}^N$, so (2.4.9) holds and $\mathcal{S} \neq \varnothing$.

We take $y_0(\lambda) = y_1(\lambda) = \lambda \in \mathbb{R}$. Then we can uniquely determine $y_2(\lambda), \dots, y_N(\lambda), y_{N+1}(\lambda)$ by the formulae (2.4.5) with $k = 1, \dots, N$. Hence the continuous map $I \colon \mathcal{E}_F \to \mathbb{R}$ defined by the formula $I(u(x), v(x)) = u(y_1)$ is injective on $\mathcal{S}$. Thus Proposition 2.2.2 is a consequence of Lemma 0.1.4 and the following lemma.

Lemma 2.4.1. *$Z = I\mathcal{S}$ is a discrete set in $\mathbb{R}$.*

Proof. All the functions $y_k(\lambda)$ are real-analytic on $\mathbb{R}$, $k = 2, \dots, N+1$. The sequence $\{y_k(\lambda) : k = 2, \dots, N+1\}$ defines a stationary solution $s_\lambda(x)$ of the form (2.4.4), (2.4.1) if and only if $a_{N+1} = 0$, that is, $y_{N+1} = y_N$. We obtain the following equation for λ:

$$y_{N+1}(\lambda) = y_N(\lambda). \tag{2.4.11}$$

Both sides of this equation are real-analytic functions of $\lambda \in \mathbb{R}$. Thus, either the set Z of all solutions of (2.4.11) is discrete in $\mathbb{R}$, or $Z = \mathbb{R}$. Let us show that the case $Z = \mathbb{R}$ is impossible by virtue of (0.1.16) and (0.1.17) even if the F_k are not real-analytic.

Lemma 2.4.2. *Let $F_k \in C^1(\mathbb{R})$ for all i, and let the V_k satisfy conditions* (0.1.16) *and* (0.1.17). *Then $Z \neq \mathbb{R}$.*

Proof. Suppose the contrary: $Z = \mathbb{R}$. Then

$$\mathcal{U}_N(y_1(\lambda), \dots, y_N(\lambda)) = \text{const} \quad \text{for} \quad \lambda \in \mathbb{R}. \tag{2.4.12}$$

Indeed, since $F_k \in C^1(\mathbb{R})$, it follows that $y_k(\lambda) \in C^1(\mathbb{R})$ for all $k = 1, \dots, N$. Hence it follows from (2.4.9) that

$$\frac{d}{d\lambda}\mathcal{U}_N(y_1(\lambda), \dots, y_N(\lambda)) = \sum_{k=1}^{N} \frac{\partial \mathcal{U}_N}{\partial y_k} \frac{dy_k}{d\lambda}(\lambda) = 0, \qquad \lambda \in \mathbb{R}. \tag{2.4.13}$$

On the other hand, it follows from (2.4.8) that

$$\mathcal{U}_N(y_1(\lambda), \dots, y_N(\lambda)) = \sum_{k=1}^{N} V_k(y_k(\lambda)) + \frac{1}{2}\sum_{k=2}^{N} \left| \frac{y_k(\lambda) - y_{k-1}(\lambda)}{l_k} \right|^2 l_k. \tag{2.4.14}$$

Consequently, the second sum on the right-hand side in (2.4.14) is bounded for $\lambda \in \mathbb{R}$ by virtue of (2.4.12) and (0.1.16). Hence

$$y_k(\lambda) \to \infty \quad \text{as} \quad y_1 = \lambda \to \infty \quad \forall\, k = 1, \dots, N. \tag{2.4.15}$$

But then the first sum on the right-hand side in (2.4.8) tends to infinity as $\lambda \to \infty$ in view of (0.1.17). Hence

$$\mathcal{U}_N(y_1(\lambda), \dots, y_N(\lambda)) \to \infty \quad \text{as} \quad \lambda \to \infty, \tag{2.4.16}$$

which contradicts (2.4.12).

Thus the set Z is discrete in $\mathbb{R}$, and hence $\mathcal{S}_N$ is discrete in $\mathbb{R}^N$. It remains to note that for stationary states $(s(x), 0) \in \mathcal{E}$ of (2.1.18) the condition $\mathcal{H}(s(x), 0) \leqslant h$ is equivalent to $\mathcal{U}_N(y_1, \dots, y_N) \leqslant h$. The set of such points $(y_1, \dots, y_N) \in \mathbb{R}^N$ is bounded in $\mathbb{R}^N$ by (2.4.10). Hence the intersection of this set with the discrete set $\mathcal{S}_N$ is finite for any $h \in \mathbb{R}$.

Remark. The equations (2.4.9) follow from (2.4.6) and (2.4.5). But the converse is not obvious. For $x \in \mathbb{R} \setminus Q$, (2.4.6) follows directly from (2.4.1). Roughly speaking, (2.4.9) ensures the validity of (2.4.6) also for $x \in Q$.

Example 0. Suppose that each potential $V_k(y)$ is a polynomial of even degree $p_k + 1 \geqslant 2$ with positive leading coefficient. Then all assumptions of Lemma 2.4.2 are satisfied. All the $F_k(y)$ are polynomials of degree $p_k \geqslant 1$. By (2.4.5), all functions $y_k(\lambda)$ are polynomials of degree $p_1 \cdots p_{k-1}$. Consequently, (2.4.11) has at most $\overline{p} \equiv p_1 \cdots p_N$ roots $\lambda \in \mathbb{R}$, and the set $\mathcal{S}$ consists of at most $\overline{p}$ points.

Remark. If the potentials V_k fail to satisfy condition (0.1.16), condition (0.1.17), or the analyticity condition, then $\mathcal{S}$ need not be discrete, as shown by the following examples.

Example 1. Let us omit condition (0.1.16). We consider (2.1.1) with $N = 2$, $x_1 = -1$, $x_2 = 1$, and

$$V_k(y) = -\frac{y^2}{2}, \qquad k = 1, 2. \tag{2.4.17}$$

Then $F_k(y) = y$ is a repulsive force with centre $y = 0$, and (2.1.1) has a continuum of solutions of the form

$$s_\lambda(x) = \begin{cases} \lambda, & x \leqslant -1, \\ -\lambda x, & -1 \leqslant x \leqslant 1, \\ -\lambda, & x \geqslant 1 \end{cases} \tag{2.4.18}$$

(see Fig. 9).

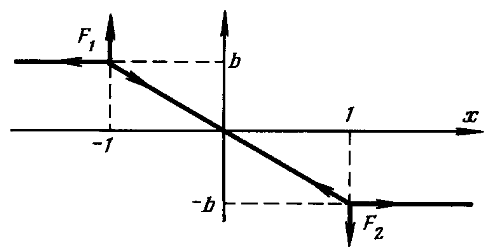

Figure 9

Here $y_1 = s_\lambda(-1) = \lambda$ is an arbitrary real number, that is, $Y_1 = \mathbb{R}$. The potentials $V_k(y)$ are real-analytic functions. Condition (0.1.17) is violated but can be ensured formally. Namely, let us add the elastic force $F_3(y) = -y$ with potential $V_3(y) = y^2/2$ at the point $x_3 = 0$. Then condition (0.1.17) is satisfied. The functions (2.4.18) remain stationary solutions of the new system involving three forces. Indeed, since $s_\lambda(0) = 0$ for all $\lambda \in \mathbb{R}$, it follows that the force F_3 is zero. Thus, condition (0.1.17) and the analyticity condition are satisfied, but $\mathcal{S}$ is not discrete.

Example 2. Let us omit condition (0.1.17). We take $V_k(y) \equiv \text{const}$ for all i; then $F_k(y) \equiv 0$. Consequently, $s_\lambda(x) \equiv \lambda$, $x \in \mathbb{R}$, is a stationary solution of (2.1.1). We see that $Y_1 = \mathbb{R}$, just as in Example 1. Condition (0.1.16) and the analyticity condition are satisfied, but $\mathcal{S}$ is not discrete.

Example 3. Let us omit the analyticity condition. We consider potentials $V_k(y)$ such that

i) $V_k(y) \in C^2(\mathbb{R})$ satisfies condition (0.1.16);
ii) $V_k(y) \to \infty$ as $|y| \to \infty$ for all $k = 1, \dots, N$, and

$$V_k(y) = C_{kn} \quad \text{for} \quad y \in I_n \equiv [2n, 2n+1], \quad n \in \mathbb{Z} \tag{2.4.19}$$

for some constants $C_{kn} \in \mathbb{R}$.

Clearly, such functions V_k exist and are not real-analytic. Furthermore, $F_k(y) \equiv 0$ for $y \in I_n$ and for all $n \in \mathbb{Z}$. Hence the functions $s_\lambda(x) \equiv \lambda$, $x \in \mathbb{R}$, are stationary solutions of (2.1.1) if $\lambda \in I_n$ for some $n \in \mathbb{Z}$. It follows that $\mathcal{S}$ is not discrete despite the fact that conditions (0.1.16) and (0.1.17) are satisfied. However, we note that $Y_1 \neq \mathbb{R}$ in accordance with Lemma 2.4.2.

§2.5. Large-time asymptotics

Here we prove Theorem 2.2.3.

A compact attracting set. First, we construct a compact attracting set $\mathcal{A}$ for the trajectory $Y(t)$ in question. This set consists of piecewise-linear functions. For $a = \{(a_k, b_k) \in \mathbb{R}^2 : k = 1, \dots, N+1\} \in (\mathbb{R}^2)^{N+1}$, we set (see (2.4.1))

$$u_a(x) = a_k x + b_k \quad \text{for} \quad x \in \Delta_k, \quad k = 1 \dots, N+1. \tag{2.5.1}$$

Let $A_{\mathcal{E}} = \{a \in (\mathbb{R}^2)^{N+1} : u_a(x_k - 0) = u_a(x_k + 0),\ k = 1, \dots, N;\ a_1 = a_{N+1} = 0\}$. Then $(u_a(x), 0) \in \mathcal{E}$ for each $a \in A_{\mathcal{E}}$.

Definition 2.5.1. We set $\mathcal{A} = \{S_a = (u_a(x), 0) \in \mathcal{E} : a \in A_{\mathcal{E}},\ |u(x_k)| \leqslant B$ for all $k = 1, \dots, N\}$, where B is the bound in (2.3.1).

The set $\mathcal{A}$ is compact in $\mathcal{E}_F$, since it is homeomorphic to a compact set in $(\mathbb{R}^2)^{N+1}$. In the next section, we shall prove the following lemma.

Lemma 2.5.2. *Let all assumptions of Theorem* 2.2.3 *be satisfied. Then* $Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{A}$ *as* $t \to \pm\infty$.

Proof of Theorem 2.2.3. It follows from Lemma 2.5.2 that the orbit $O(Y)$ is precompact in $\mathcal{E}_F$. Hence the following lemma implies (2.2.1).

Lemma 2.5.3. *The orbit* $\Omega(Y)$ *is contained in* $\mathcal{S}$.

Proof. We have $\Omega(Y) \subset \mathcal{A}$, since $\mathcal{A}$ is an attracting set. Furthermore, $\Omega(Y)$ is invariant with respect to the group W_t, $t \in \mathbb{R}$, since W_t is continuous in $\mathcal{E}_F$. It follows that for each $\overline{Y} \in \Omega(Y)$ there is a C^2 curve $t \mapsto a(t) \in A_{\mathcal{E}}$ such that $W_t\overline{Y} = S_{a(t)}$. This implies that $S_{a(t)}$ is a solution of (2.1.8). But then $\dot{a}(t) = 0$, that is, $a(t) \equiv a$ and $\overline{Y} = S_a \in \mathcal{S}$.

§2.6. Attraction to a compact set

Here we prove Lemma 2.5.2. It suffices to construct a function $a(t) \in C[0,\infty; A_{\mathcal{E}})$ such that

$$\|Y(t) - S_{a(t)}\|_R \to 0 \quad \text{as} \quad t \to +\infty \tag{2.6.1}$$

for each $R > 0$ (see (0.1.14)). Without loss of generality, we can assume that $x_1 = 0$. Then $S_a(x_1) = b_1$ and (2.6.1) takes the form

$$\int_{-R}^{R} |u'(x,t) - u_t'(x)|^2\,dx + \int_{-R}^{R} |\dot u(x,t)|^2\,dx + |y_1(t) - b_1(t)| \to 0 \quad \text{as} \quad t \to +\infty. \tag{2.6.2}$$

We set $b_1(t) = y_1(t)$ for $t > 0$; then in the notation (2.5.1), (2.6.2) acquires the form

$$\int_{-R}^{x_1} |u'(x,t)|^2\,dx + \sum_{2\leqslant k\leqslant N} \int_{x_{k-1}}^{x_k} |u'(x,t) - a_k(t)|^2\,dx$$
$$+ \int_{x_N}^{R} |u'(x,t)|^2\,dx + \int_{-R}^{R} |\dot u(x,t)|^2\,dx \to 0 \quad \text{as} \quad t \to +\infty \tag{2.6.3}$$

if $R > a$. It remains to verify the convergence in (2.6.3) with some $a_k(t)$.

Proposition 2.6.1. *For each $k = 1,\dots,N+1$ there is a function $c_k(t) \in C(\mathbb{R}_+)$ such that*

$$|f_k'(t+\,\cdot\,) - c_k(t)|_R + |g_k'(t+\,\cdot\,) + c_k(t)|_R \to 0 \quad \textit{as} \quad t \to +\infty \tag{2.6.4}$$

for every $R > 0$, and moreover, $c_1(t) = c_{N+1}(t) = 0$ for $t \in \mathbb{R}$.

This proposition, together with the d'Alembert formula (2.1.9), obviously implies the convergence in (2.6.3) with $a_k(t) = 2c_k(t)$.

2.6.1. Relaxation at infinity. Let us prove Proposition 2.6.1 by using an appropriate notion of relaxation describing large-time behaviour of the type (2.6.4). We introduce the standard Sobolev metric $\|\cdot\|_R$ of the space $H^1(-R,R)$:

$$\|y\|_R^2 \equiv |y(s)|_R^2 + |\dot y(s)|_R^2. \tag{2.6.5}$$

Definition 2.6.2. i) A function $z(t) \in L^2_{\text{loc}}(\mathbb{R}_+)$ *relaxes in* L^2 if there is a function $\overline{z}(t)$ such that

$$|z(t+\,\cdot\,) - \overline{z}(t)|_R^2 \to 0 \quad \text{as} \quad t \to +\infty \tag{2.6.6}$$

for each $R > 0$. In this case we write $z(t) \overset{L^2}{\sim} \overline{z}(t)$ as $t \to +\infty$.

ii) A function $y(t) \in H^1_{\text{loc}}(\mathbb{R}_+)$ *relaxes in* H^1 if there is a function $\overline{y}(t)$ such that

$$\|y(t+\,\cdot\,) - \overline{y}(t)\|_R \to 0 \quad \text{as} \quad t \to +\infty \tag{2.6.7}$$

for every $R > 0$. In this case we write $y(t) \overset{H^1}{\sim} \overline{y}(t)$ as $t \to +\infty$.

The following properties of relaxation are obvious.

R0. Without loss of generality one can set $\overline{y}(t) \equiv y(t)$ in (2.6.7).
R1. If a function $z(t)$ relaxes in H^1, then it also relaxes in L^2.
R2. For a function $z(t)$ to relax in L^2, it is sufficient that

$$\int_0^\infty |z(t)|^2 \, dt < \infty. \tag{2.6.8}$$

In this case one can set $\overline{z}(t) \equiv 0$; in other words, $z(t) \overset{L^2}{\sim} 0$ as $t \to +\infty$.
R3. For a function $y(t)$ to relax in H^1, it is sufficient that

$$\int_0^\infty |\dot{y}(t)|^2 \, dt < \infty. \tag{2.6.9}$$

Indeed, then it follows from the Cauchy–Bunyakovskii–Schwarz inequality that

$$|y(t+s) - y(t)| = \left| \int_t^{t+s} \dot{y}(\tau) \, d\tau \right| \leqslant R^{1/2} |\dot{y}^2(t+\,\cdot\,)|_R^{1/2} \to 0 \quad \text{as} \quad t \to +\infty \tag{2.6.10}$$

for $|s| \leqslant R$.
R4. If $y(t)$ relaxes in H^1, then the derivative $z(t) \equiv \dot{y}(t)$ relaxes in L^2 and $\dot{y}(t) \overset{L^2}{\sim} 0$ as $t \to +\infty$ by virtue of **R2**.
R5. If $z(t)$ relaxes in L^2, then the integral $y(t) = \displaystyle\int_{t+h_-}^{t+h_+} z(s) \, ds$ relaxes in H^1 for every $h_\pm \in \mathbb{R}$. Moreover, one can take

$$\overline{y}(t) \equiv (h_+ - h_-)\overline{z}(t). \tag{2.6.11}$$

R6. If $y(t) \sim \overline{y}(t)$ as $t \to +\infty$ in H^1 (respectively, L^2), then $y(t+h) \sim \overline{y}(t)$ in H^1 (respectively, L^2) for every $h \in \mathbb{R}$.
R7. The set of all functions $z(t)$ relaxing in L^2 (or H^1) is a linear space, and $z_1(t) + z_2(t) \sim \overline{z}_1(t) + \overline{z}_2(t)$ if $z_j(t) \sim \overline{z}_j(t)$, $j = 1, 2$.
R8. Let $F(\,\cdot\,) \in C^1(\mathbb{R})$ and $y(t) \in C_b(\mathbb{R}_+)$. Then $y(t) \overset{L^2}{\sim} \overline{y}(t)$ implies that $F(y(t)) \overset{L^2}{\sim} F(\overline{y}(t))$.

In the following section we prove that the Cauchy data

$$y_k(t) \equiv u(x_k, t) \quad \text{and} \quad z_k^\pm(t) \equiv u'(x_k \pm 0, t), \quad t \in \mathbb{R}, \quad k = 1, \dots, N, \tag{2.6.12}$$

of the solution $u(x,t)$ on the lines $x = x_k \pm 0$ relax.

Lemma 2.6.3. *All the functions $y_k(t)$, $k = 1, \dots, N$, relax in H^1, and all the functions $z_k^\pm(t)$, $k = 1, \dots, N$, relax in L^2. Moreover, $y_1, y_{N+1} \overset{H^1}{\sim} 0$ and $z_1, z_{N+1} \overset{L^2}{\sim} 0$ as $t \to +\infty$.*

This lemma, together with the d'Alembert formula (2.1.9) and relaxation properties **R0**–**R8**, enables us to prove Proposition 2.6.1.

Proof of Proposition 2.6.1. Let us prove (2.6.4) for $k \geqslant 2$ (the case $k = 1$ is completely similar to this one). The representation (2.1.9) results in the well-known d'Alembert formula

$$u(x,t) = \frac{y_k(t-(x-x_k)) + y_k(t+(x-x_k))}{2} + \frac{1}{2}\int_{t-(x-x_k)}^{t+(x-x_k)} z_k^+(s)\,ds \qquad (2.6.13)$$

for $x_k < x < x_{k+1}$. Consequently,

$$\begin{aligned} \dot u(x,t) &= \frac{\dot y_k(t-(x-x_k)) + \dot y_k(t+(x-x_k))}{2} \\ &\quad + \frac{z_k^+(t+(x-x_k)) - z_k^+(t-(x-x_k))}{2}, \\ u'(x,t) &= \frac{-\dot y_k(t-(x-x_k)) + \dot y_k(t+(x-x_k))}{2} \\ &\quad + \frac{z_k^+(t+(x-x_k)) + z_k^+(t-(x-x_k))}{2}. \end{aligned} \qquad (2.6.14)$$

Hence the relation (2.6.4) with $c_k(t) = -\overline{z_k^+}(t)/2$ follows from Lemma 2.6.3, **R7**, **R6**, and **R4**.

Remark. One can obtain (2.6.3) directly from (2.6.14) and Lemma 2.6.3. We derive (2.6.3) from Proposition 2.6.1 for simplicity.

It remains to prove Lemma 2.6.3. To this end, we analyze the energy dissipation at infinity.

2.6.2. Energy dissipation at infinity.

Lemma 2.6.4.

$$\int_0^\infty \big(|\dot y_1(t)|^2 + |z_1^-(t)|^2 + |\dot y_{N+1}(t)|^2 + |z_{N+1}^+(t)|^2\big)\,dt < \infty. \qquad (2.6.15)$$

Proof. It follows from the d'Alembert representation (2.1.9) with $k = 1$ and $k = N + 1$ that (2.6.15) is equivalent to

$$\int_0^\infty \big(|f_1'(t-x_1)|^2 + |g_1'(t+x_1)|^2 + |f_{N+1}'(t-x_N)|^2 + |g_{N+1}'(t+x_N)|^2\big)\,dt < \infty. \quad (2.6.16)$$

Here the integrals of f_1' and g_{N+1}' are bounded in view of the d'Alembert formulae

$$\begin{aligned} f_1(-x) &= \frac{u^0(x)}{2} - \frac{1}{2}\int_{x_1}^{x} v^0(s)\,ds \quad \text{for} \quad -x < x_1, \\ g_{N+1}(x) &= \frac{u^0(x)}{2} + \frac{1}{2}\int_{x_1}^{x} v^0(s)\,ds \quad \text{for} \quad x > x_N, \end{aligned}$$

since $(u^0, v^0) \in \mathcal{E}$. To prove (2.6.16) for g_1' and f_{N+1}', we introduce the following energy functional on the interval $\Delta = [x_1, x_N]$ for $Y = (u(x), v(x)) \in \mathcal{E}$:

$$\mathcal{H}_\Delta(Y) = \frac{1}{2}\int_{x_1}^{x_N} \big[|v(x)|^2 + |u'(x)|^2\big]\,dx + \sum_{k=1}^{N} V_k(y_k), \quad \text{where} \quad y_k = u(x_k). \qquad (2.6.17)$$

Let us consider the energy flux from Δ. It follows from (2.1.1) and (2.1.9) with $k = 1$ and $k = N + 1$ that

$$\frac{d}{dt}\mathcal{H}_\Delta(Y(t)) = \dot{u}u'\Big|_{x=x_1-0}^{x=x_N+0} = |f_1'(t - x_1)|^2 - |g_1'(t + x_1)|^2 + |g_{N+1}'(t + x_N)|^2 - |f_{N+1}'(t - x_N)|^2 \quad \text{for almost all} \quad t \in \mathbb{R}$$

for piecewise-smooth initial data (u^0, v^0). By integrating, we obtain the energy identity

$$\mathcal{H}_\Delta(Y(t)) + \int_0^t \big(|g_1'(s + x_1)|^2 + |f_{N+1}'(s - x_N)|^2\big)\, ds = \mathcal{H}_\Delta(Y(0)) + \int_0^t \big(|f_1'(s - x_1)|^2 + |g_{N+1}'(s + x_N)|^2\big)\, ds \quad \text{for} \quad t \in \mathbb{R}. \quad (2.6.18)$$

We see that the estimate (2.6.16) for g_1' and f_{N+1}' follows from the same estimate for f_1' and g_{N+1}', since

$$\inf_{Y \in \mathcal{E}} \mathcal{H}_\Delta(Y) > -\infty$$

by (0.1.16).

2.6.3. The relaxation lemma. Here we prove Lemma 2.6.3 by induction on k. First, let $k = 1$ or $k = N+1$. It follows from (2.6.15) that $y_1(t)$, $y_{N+1}(t)$ and $z_1^-(t)$, $z_{N+1}^+(t)$ relax by virtue of **R3** and **R2**, respectively. Consequently, the relaxation of $z_1^+(t)$ and $z_{N+1}^-(t)$ follows from **R7**, **R8**, and (2.3.1) in view of the third equation in (2.1.1) with $k = 1, N$, that is,

$$z_k^+(t) - z_k^-(t) = -F_k(y_k(t)), \qquad t \in \mathbb{R}. \quad (2.6.19)$$

Now let $k = 2$. We prove relaxation of $y_2(t)$ and $z_2^-(t)$. First, it follows from (2.6.13) with $k = 2$ that

$$y_2(t) = u(x_2, t) = \frac{y_1(t - l_2) + y_1(t + l_2)}{2} + \frac{1}{2}\int_{t-l_2}^{t+l_2} z_1^+(s)\, ds, \quad \text{where} \quad l_2 \equiv |x_2 - x_1|. \quad (2.6.20)$$

Hence relaxation of $y_2(t)$ in H^1 follows from **R5** and **R6**. Second, let us differentiate (2.6.13). We obtain

$$z_2^-(t) \equiv u'(x_2 - 0, t) = \frac{-\dot{y}_1(t - l_2) + \dot{y}_1(t + l_2)}{2} + \frac{z_1^+(t + l_2) + z_1^+(t - l_2)}{2}. \quad (2.6.21)$$

We see that relaxation of $z_2^-(t)$ in L^2 follows from **R2**, **R6**, and **R7**. The proof of Lemma 2.6.3 can be completed by induction.

CHAPTER III

A NON-LINEAR STRING WITH A SPATIALLY LOCALIZED NON-LINEARITY

In this chapter we establish convergence to stationary states for a one-dimensional non-linear wave equation with a non-linearity concentrated on a bounded interval. The result generalizes that of [33].

§3.1. Introduction and main results

We establish the convergence (0.1.11), (0.1.12) to stationary states for solutions of the Cauchy problem

$$\ddot{u}(x,t) = u''(x,t) + f(x,u(x,t)), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{3.1.1}$$

$$u\big|_{t=0} = u^0(x), \qquad \dot{u}\big|_{t=0} = v^0(x). \tag{3.1.2}$$

The solutions $u(x,t)$ take values in $\mathbb{R}^d$, $d \geqslant 1$. We assume that $f(x,u) = 0$ for $|x| \geqslant a$ with some $a > 0$. From the viewpoint of physics, (3.1.1) describes small transverse vibrations of a string interacting with a non-linear elastic medium on the interval $[-a,a]$. We use the configuration space $\mathcal{Q}$ and the phase spaces $\mathcal{E}$ and $\mathcal{E}_F$ introduced in Definition 2.1.1, and we consider general functions $f(x,u)$ satisfying (0.1.18)–(0.1.20). By $V(x,u) = \chi(x)V(u)$ we denote the potential of the non-linear force. Under these assumptions, (3.1.1) is formally a Hamiltonian system with phase space $\mathcal{E}$ and Hamiltonian

$$\mathcal{H}(u,v) = \int_{\mathbb{R}} \left[\frac{1}{2}|v(x)|^2 + \frac{1}{2}|u'(x)|^2 + V(x,u(x))\right] dx \quad \text{for} \quad (u,v) \in \mathcal{E}. \tag{3.1.3}$$

We consider solutions $u(x,t)$ for which $Y(t) = (u(\,\cdot\,,t), \dot{u}(\,\cdot\,,t)) \in C(\mathbb{R},\mathcal{E})$, and we rewrite the Cauchy problem (3.1.1)–(3.1.2) in the form

$$\dot{Y}(t) = \mathcal{V}(Y(t)) \quad \text{for} \quad t \in \mathbb{R}, \quad Y(0) = Y^0, \tag{3.1.4}$$

where $Y^0 = (u^0, v^0)$.

Proposition 3.1.1. *Suppose that* $d \geqslant 1$ *and conditions* (0.1.18)–(0.1.20) *are satisfied. Then*:

i) *for each* $Y^0 \in \mathcal{E}$ *the Cauchy problem* (3.1.4) *has a unique solution* $Y(t) \in C(\mathbb{R},\mathcal{E})$;

ii) *the map* $W_t \colon Y^0 \mapsto Y(t)$ *is continuous in* $\mathcal{E}$ *and* $\mathcal{E}_F$ *for all* $t \in \mathbb{R}$;

iii) *the energy conservation law* (E) *holds.*

Let $\mathcal{S}$ be the set of stationary states $S = (s(x),0) \in \mathcal{E}$ of the system (3.1.4). We set $\mathcal{S}^h = \{S \in \mathcal{S} : \mathcal{H}(S) \leqslant h\}$ for $h \in \mathbb{R}$. The set $\mathcal{S}^h$ is closed and bounded in $\mathcal{E}$ by virtue of (0.1.18)–(0.1.20) and (3.1.3):

$$\sup_{S \in \mathcal{S}^h} \|S\|_{\mathcal{E}} < \infty \quad \forall\, h \in \mathbb{R}. \tag{3.1.5}$$

Proposition 3.1.2. *Suppose that conditions* (0.1.18)–(0.1.20) *are satisfied*, $d=1$, *and* $F(u)$ *is a real-analytic function on* $\mathbb{R}$. *Then* $\mathcal{S}^h$ *is a finite set for every* $h \in \mathbb{R}$.

The main result of this chapter implies that the set $\mathcal{S}$ is a point attractor of the system (3.1.4) in the Fréchet topology of $\mathcal{E}_F$.

Theorem 3.1.3. *Suppose that the assumptions of Proposition* 3.1.1 *hold and the initial state* Y^0 *lies in* $\mathcal{E}$. *Then the following assertions hold.*

i) *The orbit* $O(Y)$ *of the solution* $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *of the Cauchy problem* (3.1.4) *is precompact in* $\mathcal{E}_F$, *and*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S} \quad as \quad t \to \pm\infty. \tag{3.1.6}$$

ii) *In addition, suppose that* $d=1$ *and* $F(u)$ *is a real-analytic function on* $\mathbb{R}$. *Then there are stationary states* $S_\pm \in \mathcal{S}$, *depending on the solution* $Y(t)$, *such that*

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm \quad as \quad t \to \pm\infty. \tag{3.1.7}$$

Remarks. i) By Fatou's lemma, it follows from the convergence (3.1.7) together with (0.1.18)–(0.1.20) and (3.1.3) that the relation (0.1.23) holds.

ii) For simplicity, we assume that $f(x,u) = \chi(x)F(u)$. All results of this chapter can readily be transferred to the case of non-linearities $f(x,u)$ that do not have this structure; one only needs to generalize conditions (0.1.18)–(0.1.20) appropriately.

§3.2. Existence of dynamics and *a priori* estimates

Let us derive Proposition 3.1.1 from the contraction mapping principle. Let W_t^0 be the dynamical group corresponding to the linear equation (3.1.1) with $f(x,u) \equiv 0$. Then the Cauchy problem (3.1.4) for $Y(t) \in C(\mathbb{R}, \mathcal{E})$ is equivalent to the integral equation

$$Y(t) = W_t^0 Y^0 + \int_0^t W_{t-\tau}^0 \bigl(0, f(\,\cdot\,, u(\,\cdot\,, \tau))\bigr)\, d\tau. \tag{3.2.1}$$

By the contraction mapping principle, there is a unique local solution $Y(t) \in C(-\varepsilon, \varepsilon; \mathcal{E})$ for some $\varepsilon > 0$. The continuity of W_t in $\mathcal{E}$ and $\mathcal{E}_F$ for small $|t|$ follows from this construction by virtue of the corresponding properties of the operators W_t^0.

To prove the energy conservation law, we assume momentarily that $u^0(x) \in C^2(\mathbb{R})$, $v^0(x) \in C^1(\mathbb{R})$, and

$$u^0(x) = v^0(x) = 0 \quad \text{for} \quad |x| \geqslant R^0. \tag{3.2.2}$$

Then it follows from the integral representation (3.2.1) that $u(x,t) \in C^2(\mathbb{R} \times (-\varepsilon, \varepsilon))$ and

$$u(x,t) = 0 \quad \text{for} \quad |x| \geqslant \overline{R} + |t|, \quad \overline{R} = \max(R^0, a). \tag{3.2.3}$$

Hence the energy conservation law (E) follows by standard integration by parts. Energy conservation for arbitrary $(u^0, v^0) \in \mathcal{E}$ follows by a standard continuity argument.

The energy conservation law (E), together with the existence of a local solution, implies the existence of a global solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ with all the above properties for all $t \in \mathbb{R}$.

However, we need a more subtle characterization of properties of the solutions.

Proposition 3.2.1. *Suppose that conditions* (0.1.18)–(0.1.20) *are satisfied. Then*:

i) *the mapping* $W_t\colon Y^0 \mapsto Y(t)$ *is Lipschitz continuous in* $\mathcal{E}_F$, *that is*,

$$\|W_tY_1 - W_tY_2\|_R \leqslant L_T\|Y_1 - Y_2\|_{R+T} \quad \textit{for} \quad |t| \leqslant T \tag{3.2.4}$$

for any $R, T > 0$, *where the constant* L_T *is bounded if the norms* $\|Y_1\|_{R+T}$ *and* $\|Y_2\|_{R+T}$ *are bounded*;

ii) *the a priori estimate*

$$|u(x,t)| \leqslant \alpha + \beta\sqrt{|x|} \quad \textit{for} \quad x \in \mathbb{R}, \quad t \in \mathbb{R} \tag{3.2.5}$$

holds, where the constants α *and* β *are bounded if the energy* $\mathcal{H}(Y^0)$ *is bounded*;

iii) $u(x,\,\cdot\,) \in C(\mathbb{R}, H^1_{\mathrm{loc}}(\mathbb{R}))$ *and* $u'(x,\,\cdot\,) \in C(\mathbb{R}, L^2_{\mathrm{loc}}(\mathbb{R}))$;

iv)

$$\int_t^{t+1} \big(|\dot u(x,s)|^2 + |u'(x,s)|^2 + |u(x,s)|^2\big)\,ds \leqslant e(x) < \infty \quad \textit{for} \quad t \in \mathbb{R} \tag{3.2.6}$$

for almost all $x \in \mathbb{R}$, *where* $e(x)$ *may depend on* x *and* $\mathcal{H}(Y^0)$ *and is independent of* $t \in \mathbb{R}$.

Proof. i) The Lipschitz continuity (3.2.4) for small $T > 0$ follows from the construction of W_t by the contraction mapping method in view of the corresponding properties of W_t^0. The generalization to arbitrary $T > 0$ is obvious.

ii) It follows from (E), (0.1.19), and (0.1.20) that

$$D = \sup_{t\in\mathbb{R}} \int |u'(x,t)|^2\,dx < \infty; \tag{3.2.7}$$

moreover, D is bounded if $\mathcal{H}(Y^0)$ is. Then it follows from the Cauchy–Bunyakovskii–Schwarz inequality that

$$|u(x,t) - u(x_0,t)| = \left|\int_{x_0}^x u'(y,t)\,dy\right| \leqslant \sqrt{D}\sqrt{|x - x_0|} \quad \text{for} \quad x, x_0, t \in \mathbb{R}. \tag{3.2.8}$$

We take an x_0 such that $\chi(x_0) > 0$. Then it follows from (E), (0.1.19), and (0.1.20) that

$$\sup_{t\in\mathbb{R}} |u(x_0,t)| < \infty.$$

Hence (3.2.8) implies (3.2.5).

iii) Let us use the integral representation (3.2.1). The first term on the right-hand side in (3.2.1) obviously possesses the desired properties, and the same is true of the integral, since $u(x,t) \in C(\mathbb{R}^2)$.

iv) The estimate (3.2.6) follows from (E), (3.2.5), and the integral representation of type (3.2.1)

$$Y(s) = W^0_{s-t}Y(t) + \int_0^{s-t} W^0_\theta\big(0, f(\,\cdot\,, u(\,\cdot\,, t+\theta))\big)\,d\theta, \tag{3.2.9}$$

which is valid since the solution is unique. Namely, estimates like (3.2.6) are valid for the first term on the right-hand side in (3.2.9), since the estimate (E) holds for $Y(t)$ uniformly in t. The same is true of the second term, in view of the estimates (3.2.5), which are uniform in t.

Remark. It follows from conditions (0.1.18)–(0.1.20) that the Hamiltonian $\mathcal{H}$ is Fréchet differentiable on $\mathcal{E}$ and

$$\frac{\delta\mathcal{H}}{\delta v(x)} = v(x), \qquad \frac{\delta\mathcal{H}}{\delta u(x)} = -u''(x) - f(x, u(x)). \tag{3.2.10}$$

Hence we can rewrite (3.1.1) in the Hamiltonian form

$$\dot{u} = \frac{\delta\mathcal{H}}{\delta v}\,, \qquad \dot{v} = -\frac{\delta\mathcal{H}}{\delta u}\,. \tag{3.2.11}$$

§3.3. Stationary states

Let us prove Proposition 3.1.2. To find all stationary states, we substitute $u(x,t) = s(x)$ in (3.1.1). Then from (0.1.18)–(0.1.20) we obtain

$$\begin{cases} s''(x) + f(x, s(x)) = 0 & \text{for } x \in [-a, a], \\ s(x) = s(\pm a) & \text{for } \pm x \geqslant a, \end{cases} \tag{3.3.1}$$

since $s'(x) \in L^2(\mathbb{R})$. Hence the continuous map $I\colon \mathcal{E}_F \to \mathbb{R}$ given by the formula $I(u(x), v(x)) = u(-a)$ is injective on $\mathcal{S}$. Now Proposition 3.1.2 is a consequence of the following assertion.

Lemma 3.3.1. *The set $Z^h = I\mathcal{S}^h$ is finite for each $h \in \mathbb{R}$.*

Proof. The set $\mathcal{S}^h$ is compact in $\mathcal{E}_F$ by (3.1.5) and (3.3.1). Hence $\mathcal{Z}^h$ is a closed bounded subset of $\mathbb{R}$. It remains to verify that $\mathcal{Z}^h$ has no limit points. Suppose the contrary: there is a sequence

$$z_k \in Z^h \quad \text{such that} \quad z_k \to \overline{z} \in Z^h \quad \text{as} \quad k \to \infty. \tag{3.3.2}$$

Let $s_\lambda(x)$ be a solution of the problem

$$\begin{cases} s_\lambda''(x) + f(x, s_\lambda(x)) = 0 \quad \text{for} \quad x \in [-a, a], \\ s_\lambda'(-a) = 0, s_\lambda(-a) = \lambda, \end{cases} \tag{3.3.3}$$

if a solution exists at all. By Λ we denote the set of all $\lambda \in \mathbb{R}$ such that $s_\lambda(x)$ exists. We extend $s_\lambda(x)$ by constants to $|x| > a$:

$$s_\lambda = s_\lambda(\pm a) \quad \text{for} \quad \pm x > a. \tag{3.3.4}$$

Then $S_\lambda = (s_\lambda(x), 0) \in \mathcal{E}$ for each $\lambda \in \Lambda$. We consider the map $T\colon \Lambda \to \mathbb{R}$ given by the formula

$$T\colon \lambda \mapsto s_\lambda'(a-0). \tag{3.3.5}$$

Then $Z^h = \{\lambda \in \Lambda : T(\lambda) = 0,\ \mathcal{H}((s_\lambda(x), 0)) \leqslant h\}$. The set Λ is open, and hence

$$\Lambda = \bigcup_1^\infty \Lambda_j, \qquad \Lambda_j = (\lambda_j^-, \lambda_j^+) \neq \varnothing, \quad \lambda_j^-, \lambda_j^+ \notin \Lambda. \tag{3.3.6}$$

Needless to say, $\overline{z} \in \Lambda_l$ for some l. We claim that

$$|\lambda_l^\pm| < \infty \quad \text{and} \quad \lambda_l^\pm \in \Lambda. \tag{3.3.7}$$

This contradicts (3.3.6) and completes the proof of Lemma 3.3.1.

The map $T\colon \Lambda \to \mathbb{R}$ is real-analytic, and $T(z) = 0$ for $z \in Z$. Hence $T(\lambda) = 0$ for all $\lambda \in \Lambda_l$ by virtue of (3.3.2), that is,

$$(s_\lambda(x), 0) \in \mathcal{S} \quad \forall\, \lambda \in \Lambda_l. \tag{3.3.8}$$

Let $\mathcal{U}$ be the potential energy functional on the configuration space $\mathcal{Q}$:

$$\mathcal{U}(u) \equiv \mathcal{H}(u, 0) = \int_{-\infty}^{\infty} \left(\frac{1}{2}|u'(x)|^2 + V(x, u(x))\right) dx \quad \text{for} \quad u \in \mathcal{Q}. \tag{3.3.9}$$

Then (3.3.1) is equivalent to the equation

$$\delta\mathcal{U}(s) = 0 \tag{3.3.10}$$

(which follows also from (3.2.11)), where $\delta\mathcal{U}$ is the Fréchet differential of $\mathcal{U}$ on $\mathcal{Q}$. Hence, by (3.3.8),

$$\frac{d}{d\lambda}\mathcal{U}(s_\lambda) = \left\langle \delta\mathcal{U}(s_\lambda), \frac{d}{d\lambda}s_\lambda \right\rangle = 0 \quad \text{for} \quad \lambda \in \Lambda_l, \tag{3.3.11}$$

and consequently the function $\lambda \mapsto \mathcal{U}(s_\lambda)$ is constant on Λ_l. By analogy with (3.2.5), this implies the *a priori* estimate

$$|s_\lambda(x)| \leqslant \alpha_1 + \beta_1\sqrt{a} \quad \text{for} \quad |x| \leqslant a \quad \text{and} \quad \lambda \in \Lambda_l \tag{3.3.12}$$

with some α_1 and β_1 independent of $\lambda \in \Lambda_l$. We now see that the interval Λ_l is bounded, since $s_\lambda(-a) = \lambda$. On the other hand, it follows from the uniform estimates (3.3.12) and from (3.3.3) that the set $\{s_\lambda(\,\cdot\,) : \lambda \in \Lambda_l\}$ of functions is precompact in $\mathcal{Q}$, and we obtain $\lambda_l^\pm \in \Lambda_l$.

The proof of Proposition 3.1.2 is complete.

§3.4. Large-time asymptotics

Here we prove Theorem 3.1.3.

3.4.1. A compact attracting set. Let us construct a compact attracting set $\mathcal{A}$ for the trajectory $Y(t)$ in question. By $\overline{\alpha}$ and $\overline{\beta}$ we denote some positive constants to be chosen later.

Definition 3.4.1. We set $\mathcal{A} = \mathcal{A}_{\overline{\alpha}\overline{\beta}} = \{S_\lambda = (s_\lambda(x), 0) \in \mathcal{E} : \lambda \in \Lambda \text{ and } |s_\lambda(x)| \leqslant \overline{\alpha} + \overline{\beta}\sqrt{x} \text{ for } |x| \leqslant a\}$.

The set $\mathcal{A}$ is compact in $\mathcal{E}_F$ by virtue of (3.3.1). In the next section, we shall prove the following lemma on attraction.

Lemma 3.4.2. *Let the assumptions of Theorem* 3.1.3 *be satisfied. Then*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{A} = \mathcal{A}_{\overline{\alpha}\overline{\beta}} \quad as \quad t \to \pm\infty \tag{3.4.1}$$

if the constants $\overline{\alpha}$ *and* $\overline{\beta}$ *are sufficiently large.*

3.4.2. Proof of Theorem 3.1.3. i) It follows from Lemma 3.4.2 that the orbit $O(Y)$ is precompact in $\mathcal{E}_F$. Hence the following lemma implies (3.1.6).

Lemma 3.4.3. *The orbit* $\Omega(Y)$ *is contained in* $\mathcal{S}$.

Proof. We have $\Omega(Y) \subset \mathcal{A}$, since $\mathcal{A}$ is an attracting set. Furthermore, $\Omega(Y)$ is invariant with respect to W_t, $t \in \mathbb{R}$, since W_t is continuous in $\mathcal{E}_F$. Hence for each $\overline{Y} \in \Omega(Y)$ there is a C^2 curve $t \mapsto \lambda(t) \in \mathbb{R}$ such that $W_t\overline{Y} = S_{\lambda(t)}$. This implies that $S_{\lambda(t)}$ is a solution of (3.2.1). But then $\dot{\lambda}(t) = 0$, that is, $\lambda(t) \equiv \lambda$ and $\overline{Y} = S_\lambda \in \mathcal{S}$.

ii) By analogy with (0.1.23), one can prove that $\Omega(Y) \subset \mathcal{S}^h$ for $h = \mathcal{H}(Y^0)$. Hence $Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}^h$ by (3.1.6). However, $\mathcal{S}^h$ is finite in view of Proposition 3.1.2. Hence (3.1.7) follows from the continuity of $Y(t)$.

§3.5. Attraction to a compact set

We derive Lemma 3.4.2 from the following lemma on 'attraction in the mean', which will be proved in the next section. For $R > 0$, we set

$$\rho_R(t) = \inf_{S \in \mathcal{A}} \|Y(t) - S\|_R \quad \text{for} \quad t \in \mathbb{R}. \tag{3.5.1}$$

Lemma 3.5.1. *For any* $R > 0$,

$$\int_0^\infty \rho_R^2(t)\, dt < \infty. \tag{3.5.2}$$

Let us choose an arbitrary metric $\rho(\cdot\,,\cdot)$ defining the topology $\mathcal{E}_F$ on $\mathcal{E}$. We prove (3.4.1) by contradiction. Suppose that there is an $\varepsilon > 0$ and a sequence $t_k \to \infty$ such that

$$\rho(Y(t_k), \mathcal{A}) \geqslant \varepsilon \quad \text{for all} \quad k = 1, 2, \dots. \tag{3.5.3}$$

Let us prove that this is impossible, thus completing the proof of Lemma 3.4.2. We can assume that $t_k + 1 < t_{k+1}$ for each k. Then it follows from (3.5.2) by Fatou's theorem that

$$\int_0^1 \sigma_R(\theta)\, d\theta < \infty, \quad \text{where} \quad \sigma_R(\theta) = \sum_1^\infty \rho_R^2(t_k + \theta). \tag{3.5.4}$$

Consequently, $\sigma_R(\theta) < \infty$ for each $\theta \in \Theta(R)$, where $\Theta(R) \subset [0,1]$, and moreover, $\displaystyle\int_{\Theta(R)} dx = 1$. Then

$$\rho_R(t_k+\theta) \to 0 \quad \text{as} \quad k \to \infty \quad \text{for} \quad \theta \in \Theta = \bigcap_{R\in\mathbb{N}}^{\infty} \Theta(R) \tag{3.5.5}$$

for each $R > 0$. Hence $Y(t_k+\theta) \xrightarrow{\mathcal{E}_F} \mathcal{A}$ as $k \to \infty$ for each $\theta \in \Theta \subset [0,1]$, and moreover, $\displaystyle\int_{\Theta} dx = 1$. Then for each $\theta \in \Theta$ it follows from the compactness of $\mathcal{A}$ in $\mathcal{E}_F$ that

$$Y(t_{k(\theta)}+\theta) \xrightarrow{\mathcal{E}_F} \overline{Y}(\theta) \in \mathcal{A} \quad \text{as} \quad k(\theta) \to \infty \quad \text{for} \quad \theta \in \Theta \tag{3.5.6}$$

for some sequence $k(\theta) \to \infty$. Since the map $W_{-\theta}$ is continuous in $\mathcal{E}_F$, we see that

$$Y(t_{k(\theta)}) \xrightarrow{\mathcal{E}_F} W_{-\theta}\overline{Y}(\theta) \quad \text{as} \quad k(\theta) \to \infty \quad \text{for} \quad \theta \in \Theta. \tag{3.5.7}$$

On the other hand, since $\mathcal{A}$ is compact in $\mathcal{E}_F$, there is a sequence $\theta_j \in \Theta$ such that $\theta_j \to 0$ as $j \to \infty$ and

$$\overline{Y}(\theta_j) \xrightarrow{\mathcal{E}_F} Y^* \in \mathcal{A} \quad \text{as} \quad j \to \infty. \tag{3.5.8}$$

Finally, since that maps $W_{-\theta}$, $\theta \in [0,1]$, satisfy the uniform Lipschitz condition (3.2.4) and $W_{-\theta_j}Y^* \xrightarrow{\mathcal{E}_F} Y^*$ as $j \to \infty$, we obtain

$$W_{-\theta_j}\overline{Y}(\theta_j) \xrightarrow{\mathcal{E}_F} Y^* \quad \text{as} \quad j \to \infty. \tag{3.5.9}$$

But this convergence, together with (3.5.7) for $\theta = \theta_j$, contradicts (3.5.3).

§3.6. Attraction in the mean

Here we prove Lemma 3.5.1. It suffices to construct a function $S_{\lambda(t)} \in \mathcal{A} = \mathcal{A}_{\overline{\alpha}\overline{\beta}}$ defined for $t \geqslant \overline{T}$, where $\overline{\alpha}$, $\overline{\beta}$, $\overline{T} > 0$ are sufficiently large, with the following property:

$$\int_{\overline{T}}^{\infty} \|Y(t) - S_{\lambda(t)}\|_R^2 \, dt < \infty \tag{3.6.1}$$

for each $R > 0$. We claim that one can take $\lambda(t) = u(-a,n)$ for $n \leqslant t < n+1$, $n = 0, 1, \dots$. We can replace the seminorm $\|\cdot\|_R$ in (2.1.6) by the equivalent seminorm with $|u(-a)|$ instead of $|u(0)|$. Then (3.6.1) with $R > a$ implies that

$$\int_{\overline{T}}^{\infty} \biggl(\int_{|x|<a} \bigl(|u'(x,t) - s'_{\lambda(t)}(x)|^2 + |\dot{u}(x,t)|^2\bigr)\, dx + |u(-a,t) - \lambda(t)|^2 \\ + \int_{a<|x|<R} \bigl(|u'(x,t)|^2 + |\dot{u}(x,t)|^2\bigr)\, dx \biggr) dt < \infty. \tag{3.6.2}$$

3.6.1. Energy dissipation at infinity. We have

$$\int_0^\infty \big(|\dot{y}_-(t)|^2 + |z_-(t)|^2 + |\dot{y}_+(t)|^2 + |z_+(t)|^2\big)\,dt < \infty, \tag{3.6.3}$$

where $y_\pm(t) = u(\pm a, t)$ and $z_\pm(t) = u'(\pm a, t)$. By analogy with (2.6.15), this follows from the d'Alembert representation

$$u(x,t) = f_\pm(t-x) + g_\pm(t+x), \qquad \pm x > a, \quad t \in \mathbb{R}, \tag{3.6.4}$$

and the finiteness of the energy flux to infinity. The proof uses the following energy functional on the interval $\Delta = [-a, a]$:

$$\mathcal{H}_\Delta(Y) = \int_\Delta \left[\frac{|v(x)|^2}{2} + \frac{|u'(x)|^2}{2} + V(x, u(x))\right] dx \tag{3.6.5}$$

for $Y = (u(x), v(x)) \in \mathcal{E}$.

3.6.2. The non-linear Goursat problem. We consider the following Goursat problem for the wave equation (3.1.1) with the Cauchy data on the lines $x = \mathrm{const}$:

$$\begin{cases} \ddot{u}(x,t) = u''(x,t) + f(x, u(x,t)), \\ u\big|_{x=r} = y(t), \quad u'\big|_{x=r} = z(t), \quad t \in \mathbb{R}. \end{cases} \tag{3.6.6}$$

Let us prove that the map $G_{r,x}\colon (y(\,\cdot\,), z(\,\cdot\,)) \mapsto (u(x,\,\cdot\,), u'(x,\,\cdot\,))$ is continuous. Using this continuity, we derive (3.6.2) from (3.6.3) in the next section.

Remark. Although our assumptions (0.1.19) and (0.1.20) ensure that the Cauchy problem (3.1.1), (3.1.2) is well-posed globally with respect to t, the Goursat problem (3.6.6) is not well-posed globally with respect to $x \in \mathbb{R}$ in general. However, the Goursat problem is well-posed locally with respect to x, which is sufficient for our purposes. To derive (3.6.1) from (3.6.3), we need only prove that the map $G_{b,x}$ is continuous for $b = -a$ and for x in a bounded interval $[-R, R]$. This continuity holds 'for large t' along the (global) solution $u(x,t)$ in question.

Let σ be an arbitrary interval of length $|\sigma|$ on $\mathbb{R}$.

Definition 3.6.1. $\mathcal{E}(\sigma)$ is the Hilbert space of functions $(y(t), z(t)) \in H^1(\sigma) \oplus L^2(\sigma)$ such that

$$\|(y,z)\|_{\mathcal{E}(\sigma)} = |\dot{y}|_\sigma + |y|_\sigma + |z|_\sigma < \infty, \tag{3.6.7}$$

where $|\cdot|_\sigma$ is the norm in $L^2(\sigma)$.

Definition 3.6.2. $\overline{\mathcal{E}}$ is the space of functions $(y(t), z(t)) \in H^2_{\mathrm{loc}}(\mathbb{R}) \oplus L^2(\mathbb{R})$ such that

$$\|(y,z)\|_{\overline{\mathcal{E}}} = \sup_{|\sigma| \geqslant 1} \frac{\|(y,z)\|_{\mathcal{E}(\sigma)}}{\sqrt{|\sigma|}} < \infty. \tag{3.6.8}$$

Remark. It follows from Proposition 3.2.1, iii) and iv) that $(u, u')|_{x=r} \in \overline{\mathcal{E}}$ for each $r \in \mathbb{R}$, and moreover, $\|(u,u')|_{x=r}\|_{\overline{\mathcal{E}}} \leqslant 2e(r)$.

We consider solutions $u(x,t)$ of the Goursat problem (3.6.6) with $(y,z) \in \overline{\mathcal{E}}$ such that $(u, u') \in C(r - \varepsilon, r + \varepsilon; \overline{\mathcal{E}})$ for some $\varepsilon > 0$. For such solutions the Goursat problem is equivalent to the following integral equation, which is similar to (3.2.1):

$$Z(x) = W^0_{x-r} Z_r + \int_r^x W^0_{x-s}\big(0, f(s, u(s,\,\cdot\,))\big)\,ds, \tag{3.6.9}$$

where $Z(x) = (u(x,\,\cdot\,), u'(x,\,\cdot\,))$ and $Z_r = (y(\,\cdot\,), z(\,\cdot\,))$.

Lemma 3.6.3. *Suppose that conditions* (0.1.18)–(0.1.20) *hold and* $Z_r \in \overline{\mathcal{E}}$. *Then the following assertions are valid.*

i) *The Goursat problem* (3.6.6) *has a unique solution* $Z(x) =: G_{r,x} Z_r \in C(r-\varepsilon, r+\varepsilon; \overline{\mathcal{E}})$ *with some* $\varepsilon > 0$.

ii) *The number* $\varepsilon = \varepsilon(R,B) > 0$ *in* i) *depends only on* R *and* B *for* $r \leqslant R$ *and* $\|Z_r\|_{\overline{\mathcal{E}}} \leqslant B$.

iii) *Let* $R, B > 0$ *be arbitrary. For any* $|r| \leqslant R$, $\|Z_r\|_{\overline{\mathcal{E}}} \leqslant B$, $|x-r| < \varepsilon(R,B)$, *and any interval* $\sigma \subset \mathbb{R}$, *the function* $Z(x,\,\cdot\,)|_\sigma$ *depends only on* $Z_r\big|_\Sigma$, *where* Σ *is the* δ*-neighbourhood of the interval* σ *in* $\mathbb{R}$, $\delta = |x-r|$.

iv) *For* $\|Z_r\|_{\overline{\mathcal{E}}} \leqslant B$ *the map* $G_{r,x}\colon Z_r\big|_\Sigma \mapsto Z(x,\,\cdot\,)\big|_\sigma$ *is a Lipschitz-continuous map* $\mathcal{E}(\Sigma) \to \mathcal{E}(\sigma)$:

$$\|G_{r,x} Z_r^1 - G_{r,x} Z_r^2\|_{\mathcal{E}(\sigma)} \leqslant L(R,B) \|Z_r^1 - Z_r^2\|_{\mathcal{E}(\Sigma)} \tag{3.6.10}$$

for $|r| \leqslant R$ *and* $\delta = |x-r| \leqslant \varepsilon(R,B)$ *for any* $Z_r^j \in \overline{\mathcal{E}}$, $j = 1,2$. *The Lipschitz constant* $L(R,B)$ *is independent of the interval* σ.

Proof. It follows from the contraction mapping principle that there is a unique solution $Z(x)$ of the problem (3.6.9) such that $Z(x) \in C(r-\varepsilon, r+\varepsilon; \mathcal{E}(\sigma))$ for any closed interval $\sigma \subset \mathbb{R}$. The point is that $\varepsilon = \varepsilon(R,B) > 0$ is independent of the position of σ, since we have uniform estimates for $\|Z_r\|_{\mathcal{E}(\sigma)}$ with bounded $|\sigma| \geqslant 1$ and since the problem is homogeneous with respect to t.

The properties iii) and iv) follow from the same properties of the Picard approximations in view of the corresponding properties of the operators W^0_{x-s}.

3.6.3. Proof of attraction in the mean. Let us derive (3.6.2) from (3.6.3) with the help of the estimates (3.6.10). We set $\lambda(t) = y_-(n) \equiv u(-a,n)$ for $n \leqslant t < n+1$, $n = 0, 1, \dots$.

Step 1. It follows from the estimates (3.6.3) and the d'Alembert representation (3.6.4) that the integral $\displaystyle\int_{\overline{T}}^\infty \int_{a<|x|<R} \cdots$ in (3.6.2) converges.

Step 2. The integral $\displaystyle\int_0^\infty |u(-a,t) - \lambda(t)|^2\,dt$ also converges, since it is equal to

$$\sum_{n=0}^\infty \int_n^{n+1} |y_-(t) - y_-(n)|^2\,dt \leqslant \int_0^\infty |\dot y_-(t)|^2\,dt < \infty.$$

Step 3. Let us verify that

$$\sum_{n=\overline{N}}^\infty \int_n^{n+1} \biggl(\int_{-a}^a \bigl(|u'(x,t) - s'_{\lambda(n)}(x)|^2 + |\dot u(x,t)|^2\bigr)\,dx \biggr) dt < \infty \tag{3.6.11}$$

for sufficiently large $\overline{N}$. Proposition 3.2.1, iv) implies that the solution $Z(x) = (u(x,\,\cdot\,), u'(x,\,\cdot\,)) = G_{-a,x}(y_-(\,\cdot\,), z_-(\,\cdot\,))$ of (3.6.9) satisfies

$$\|Z(r)\|_{\overline{\mathcal{E}}} \leqslant \overline{B} = 2e(a) \quad \text{for} \quad r \in [-a,a]. \tag{3.6.12}$$

On the other hand, for each $n = 0, 1, \dots$ the function

$$S_n(x) = (s_{\lambda(n)}(x), 0) = G_{-a,x}(y_-(n), 0)$$

is also a solution of (3.6.9). Hence Lipschitz continuity (Lemma 3.6.3, iv)) enables us to estimate the difference between these two solutions.

Lemma 3.6.4. *For all sufficiently large $n \geqslant \overline{N}$ there are solutions $S_n(x) = G_{-a,x}(y_-(n), 0)$ of* (3.6.9). *Moreover,*

$$\|Z(x)-S_n(x)\|^2_{\mathcal{E}([n,n+1])} \leqslant \overline{L} \int_{n-\delta}^{n+1+\delta} \big(|z_-(t)|^2+|\dot{y}_-(t)|^2\big)\, dt \quad \textit{for} \quad n \geqslant \overline{N} \tag{3.6.13}$$

for each $x = -a + \delta \in [-a, a]$.

We shall prove this lemma later. Bu summing (3.6.13) over $n \geqslant \overline{N}$ and by integrating with respect to $x \in [-a, a]$, we obtain (3.6.11) in view of (3.6.3).

Step 4. We claim that $S_n(x) \in \mathcal{A}_{\overline{\alpha}\overline{\beta}}$ for sufficiently large $\overline{\alpha}, \overline{\beta} > 0$. Indeed, it follows from (3.6.11) and the estimate (3.2.7) that

$$\overline{D} = \sup_{n \geqslant \overline{N}} \int |s'_{\lambda(n)}(x)|^2\, dx < \infty \tag{3.6.14}$$

for sufficiently large $\overline{N}$. Moreover, it follows from (3.2.5) with $x = -a$ that

$$\overline{d} = \sup_{n \geqslant 0} |s_{\lambda(n)}(-a)| < \infty. \tag{3.6.15}$$

Hence from (3.6.14), we obtain by analogy with (3.2.7) that

$$\sup_{n \geqslant 0} |s_{\lambda(n)}(x)| \leqslant \overline{\alpha} + \overline{\beta}\sqrt{x} \quad \text{for} \quad |x| \leqslant a \tag{3.6.16}$$

if $\overline{\alpha}$ and $\overline{\beta}$ are sufficiently large.

Proof of Lemma 3.6.4. We set $\overline{\varepsilon} = \varepsilon(a, \overline{B})$ and prove the existence of a solution $S_n(x) = G_{-a,x}(y_-(n), 0)$ and the estimate (3.6.13) for $-a + (k-1)\overline{\varepsilon} \leqslant x \leqslant -a + k\overline{\varepsilon}$ by induction on $k = 1, 2, \dots$ for $k \leqslant 2a/\overline{\varepsilon} + 1$.

First, let $k = 1$. On the interval $-a \leqslant x \leqslant -a + \overline{\varepsilon}$ the existence of a solution $S_n(x) = G_{-a,x}(y_-(n), 0)$ and the estimate (3.6.13) for all $n \geqslant 0$ follow readily from the inequality (3.6.10) with $r = -a$ for the two solutions $Z(x)$ and $S_n(x)$, since $Z(-a) - S_n(-a) = (y_-(\,\cdot\,) - y_-(n), z_-(\,\cdot\,))$ and

$$\|Z(-a) - S_n(-a)\|^2_{\mathcal{E}([n-\delta,n+1+\delta])} \leqslant C(\delta) \int_{n-\delta}^{n+1+\delta} \big(|z_-(t)|^2 + |\dot{y}_-(t)|^2\big)\, dt. \tag{3.6.17}$$

The estimate (3.6.10) can be applied to these solutions by virtue of (3.6.12) with $r = -a$ and a similar estimate for $S_n(-a) = (y_-(n), 0)$.

Now let $k = 2$. On the interval $-a + \overline{\varepsilon} \leqslant x \leqslant -a + 2\overline{\varepsilon}$ the existence of $S_n(x) = G_{-a,x}(y_-(n), 0) = G_{-a+\overline{\varepsilon},x} S_n(-a + \overline{\varepsilon})$ and the estimate (3.6.13) can be proved by repeated application of (3.6.10) with sufficiently large $n \geqslant N_1$ such that

$$\|S_n(-a + \overline{\varepsilon})\|_{\overline{\mathcal{E}}} \leqslant \overline{B} \quad \text{for} \quad n \geqslant N_1. \tag{3.6.18}$$

The existence of a number $N_1 < \infty$ with this property follows from (3.12) and the estimate (3.13) with $x = -a + \overline{\varepsilon}$, which has already been proved, since

$$\int_{n-\delta}^{n+1+\delta} \big(|z_-(t)|^2 + |\dot{y}_-(t)|^2\big)\, dt \to 0$$

as $n \to \infty$ by virtue of (3.6.3).

Induction on k completes the proof.

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Moscow State University
E-mail address: komech@sci.lpi.ac.ru

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