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On Stabilization of String–Nonlinear Oscillator Interaction

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In the present paper, we consider a system of equations that describes the interaction of a nonlinear oscillator with an infinite string. The main result is the stabilization: roughly speaking, each finite energy solution to the system tends to a stationary solution as $t \to +\infty$ (and similarly as $t \to -\infty$). The proof uses the description of a reversible system by an irreversible. The limit stationary solutions corresponding to $t = \pm\infty$ may be different and arbitrary. The result gives a mathematical model of transitions to stationary states in reversible systems; these transitions are similar to Bohr ones. Such transitions are impossible for finite-dimensional Hamiltonian systems and for linear autonomous Shrödinger equations. The paper contains the complete exposition and an extension of the author's recent results.

INTRODUCTION

Mathematically, the problem is to solve the wave equation

$$\mu\ddot{u}(x,t) = Tu''(x,t), \qquad x \in \mathbb{R}\backslash 0,\ t \in \mathbb{R}, \tag{1}$$

with the following splicing conditions:

$$u(0+,t) = u(0-,t), \qquad t \in \mathbb{R}, \tag{2}$$

$$m\ddot{y}(t) = F(y(t)) + T[u'(0+,t) - u'(0-,t)]. \tag{3}$$

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Here

$$y(t) \equiv u(0\pm, t), \mu, T > 0, \qquad m \geq 0; \qquad \dot{u} \equiv \frac{\partial u}{\partial t}, u' \equiv \frac{\partial u}{\partial x}, \text{ etc.}$$

Physically, the problem (1)–(3) describes small crosswise oscillations of an infinite string stretched parallel to the $0x$-axis; a ball of mass $m \geq 0$ is attached to the string at the point $x = 0$; μ is the line density of the string; T is its tension; $F(y)$ is an external (nonlinear) force field perpendicular to $0x$; and the field subjects the ball, see Fig. 1.

The objective of the present paper is to study asymptotics of solutions $u(x, t)$ for (1)–(3) as $t \to \pm\infty$. Our main result is Theorem 2.1. Roughly speaking, if the external force field $F(y)$ of the oscillator satisfies conditions (4)–(6) (see below), then any finite energy solution $u(x, t)$ to system (1)–(3) tends to some stationary solutions locally uniformly in x as $t \to \pm\infty$.

Such stabilization of all solutions to different stationary states is a typical behaviour for dissipative systems [1, 5, 12]. The system (1)–(3) is an infinite-dimensional Hamiltonian system without dissipation of the energy (see Theorem 1.1). However, there exists in the system a scattering of the waves at infinity, discovered initially in linear and nonlinear scattering theories. It plays the role of a dissipation and provides the stabilization. Namely, there are known results [7, 8, 10, 11, 13] on the "local energy decay" of all finite energy solutions to multidimensional linear and nonlinear scattering problems. Thus the results mean the stabilization to a single (zero) stationary solution. Note that for problems considered in these articles there exists only one stationary solution, and this solution is identically zero. Stabilization of solutions to nonlinear wave equations in the cases when there exist nonzero stationary solutions was not considered previously. However, the absence of stabilization to zero for some nonlinear wave equations was discovered in [9].

Note that the system (1)–(3) also is formally equivalent to the one-dimensional nonlinear wave equation with the simplest nonlinear term $\delta(x)F(u)$ concentrated at the single point $x = 0$. However, we consider the general case of an arbitrary function $F(u)$ satisfying assumptions

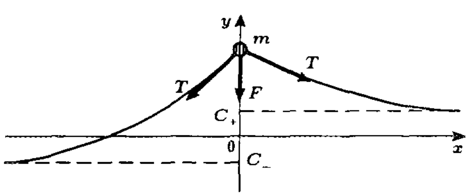

FIGURE 1

(4)–(6). With these assumptions, the system (1)–(3) may have (infinitely) many stationary solutions of finite energy. Stationary finite energy solutions to (1)–(3) are the functions $u(x) \equiv b$, where $F(b) = 0$.

We prove (Theorem 7.1) that for each two stationary solutions $u_{\pm}(x)$ of finite energy there exists a solution $u(x, t)$ of finite energy that connects them when the time varies from $-\infty$ to $+\infty$. Such a "transitivity" of transitions to stationary states is a purely nonlinear effect, which is impossible for linear autonomous equations similar to Schrödinger equations.

In the linear case when $F(y) = -ky$, the system (1)–(3) has a unique stationary (zero) solution. In this case the stabilization to zero was considered in [6].

Let us describe more precisely assumptions on external force $F(y)$ and solution $u(x, t)$.

Denote by $V(y) \equiv -\int F(y)\, dy$ the potential energy of the external field:

$$F(y) \equiv -V'(y), \qquad y \in \mathbb{R}. \tag{4}$$

We suppose that

$$F(y) \in C^1(\mathbb{R}), \qquad \text{hence } V(y) \in C^2(\mathbb{R}); \tag{5}$$

besides,

$$V(y) \to \infty \qquad \text{as } |y| \to \infty, \tag{6}$$

so

$$V(y) \geq V_0 \qquad \text{for all } y \in \mathbb{R} \tag{6'}$$

for a certain $V_0 \in \mathbb{R}$.

Let us introduce a class $\mathscr{E}$ of functions $u(x, t)$; we shall consider solutions of this class for the system (1)–(3). Roughly speaking, the class contains the functions of the Sobolev space $H^1_{\mathrm{loc}}(\mathbb{R}^2)$ that are constant for large $|x|$ for any fixed t.

DEFINITION 1. A function $u(x, t)$ belongs to $\mathscr{E}$ if

(1) $u \in C(\mathbb{R}^2)$,

(2) $\dot{u}, u' \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$, where the derivatives are understood in the sense of distributions,

(3) $\forall \tau > 0$ for a certain $A > 0$

$$u(x, t) = C_{\pm} \qquad \text{for } |t| < \tau \text{ and } \pm x > A, \tag{7}$$

where $C_{\pm} \in \mathbb{R}$ (see Fig. 1).

System (1)–(3) for the functions $u(x, t) \in \mathscr{C}$ is understood as follows.

The meaning of Eq. (2) for all $u \in \mathscr{C} \subset C(\mathbb{R}^2)$ is evident.

Equation (1) for $u \in C(\mathbb{R}^2)$ is understood in the sense of distributions in the region $(x, t) \in \mathbb{R}^2$, $x \neq 0$.

We now explain Eq. (3). It is known that for $u \in D'(\mathbb{R}^2)$ Eq. (1) is equivalent to the d'Alembert decomposition

$$u(x, t) = f_\pm(x - at) + g_\pm(x + at), \qquad \pm x > 0, \tag{8}$$

where $a = \sqrt{T/\mu}$, and $f_\pm$, $g_\pm \in D'(\mathbb{R})$. Then the condition $u \in C(\mathbb{R}^2)$ is equivalent to

$$f_\pm, g_\pm \in C(\mathbb{R}). \tag{9}$$

Further, (8) implies

$$\dot{u} = -af'_\pm(x - at) + ag'_\pm(x + at), \qquad u' = f'_\pm(x - at) + g'_\pm(x + at), \tag{10}$$

for $\pm x > 0$, where all the derivatives are understood in the sense of distributions. This implies the following remark.

Remark 1. Condition (2) of Definition 1 is equivalent to

$$f'_\pm, g'_\pm \in L^2_{\text{loc}}(\mathbb{R}). \tag{11}$$

DEFINITION 2. In Eq. (3), for $u \in \mathscr{C}$ satisfying (1), put

$$u'(0\pm, t) \equiv f'_\pm(-at) + g'_\pm(at) \in L^2_{\text{loc}}(\mathbb{R}_t), \tag{12}$$

while the derivative $\ddot{y}(t)$ of $y(t) \equiv u(0\pm, t) \in C(\mathbb{R})$ (or of $\dot{y}(t) \in L^2_{\text{loc}}(\mathbb{R})$) is understood in the sense of distributions.

Note that the functions $f_\pm$ and $g_\pm$ in (8) are unique up to an additive constant. Hence definition (12) is unambiguous.

Remark 2. Let $m \neq 0$. Then from (3), (5), (12) it follows that

$$\ddot{y}(t) \equiv \ddot{u}(0\pm, t) \in L^2_{\text{loc}}(\mathbb{R}) \Rightarrow y(t) \in C^1(\mathbb{R}) \tag{13}$$

for any solution $u \in \mathscr{C}$ of system (1)–(3).

To set up the Cauchy problem for the system (1)–(3), we need the following definition.

DEFINITION 3. The derivatives $\dot{u}(\cdot, t)$ and $u'(\cdot, t)$ for solutions $u \in \mathscr{C}$ of system (1)–(3) are defined by (10) $\forall t \in \mathbb{R}$.

Then by virtue of (11) and (7), (10) implies

$$\dot{u}(\cdot, t), u'(\cdot, t) \in L^2(\mathbb{R}), \qquad \forall t \in \mathbb{R}. \tag{14}$$

This fact yields the finiteness of energy for solutions $u \in \mathscr{C}$ of system (1)–(3) (see (1.4) below).

1. SOLVABILITY OF THE CAUCHY PROBLEM FOR (1)–(3)

Theorems 1.1 and 1.2 below describe all the solutions of (1)–(3) of the class $\mathscr{C}$ for $m > 0$ and $m = 0$, respectively, and show that there are sufficiently many of them.

1.1. First we consider the case $m > 0$. In this case we study the Cauchy problem for system (1)–(3) with the following initial conditions:

$$u|_{t=0} = u_0(x), \dot{u}|_{t=0} = u_1(x), x \in \mathbb{R}; \qquad \dot{y}|_{t=0} = y_1. \tag{1.1}$$

Remark 1.1. The meaning of these conditions for solutions $u \in \mathscr{C}$ of (1)–(3) is explained by Definition 3 and (13).

Let us introduce the phase space E for the system (1)–(3) so that $\mathscr{C}$ is the corresponding space of trajectories.

DEFINITION 1.1. E is the set of triples $(u_0(x), u_1(x), y_1)$, where

(1) $y_1 \in \mathbb{R}$,
(2) $u_1(x) \in L^2(\mathbb{R})$,
(3) $u_0(x) \in C(\mathbb{R})$, $u_0'(x) \in L^2(\mathbb{R})$,
(4) for a certain $A \geq 0$,

$$u_0(x) = C_\pm \text{ and } u_1(x) = 0 \qquad \text{for } \pm x > A, \tag{1.2}$$

where $C_\pm \in \mathbb{R}$.

The space E is a normed space with the norm

$$\|(u_0, u_1, y_1)\|_E = \|u_0'\| + \|u_1\| + |C_-| + |C_+| + |y_1|, \tag{1.3}$$

where $\|\cdot\|$ is the norm in $L^2(\mathbb{R})$.

THEOREM 1.1. (1) *Let $m > 0$. Then for any triple $(u_0, u_1, y_1) \in E$ the class $\mathscr{E}$ contains a unique solution u of* (1)–(3), (1.1).

(2) *$\forall t \in \mathbb{R}$ the map S_t taking (u_0, u_1, y_1) to $(u(\cdot, t), \dot{u}(\cdot, t), \dot{y}(t))$ is a continuous map from E to E.*

(3) *The energy conservation law is valid for any solution u of* (1)–(3) *of the class $\mathscr{E}$:*

$$\int_{-\infty}^{+\infty}\left[\mu\frac{(\dot{u}(x,t))^2}{2} + T\frac{(u'(x,t))^2}{2}\right]dx + m\frac{\dot{y}^2(t)}{2} + V(y(t)) = \text{const.}, \qquad t \in \mathbb{R}, \quad (1.4)$$

where $y(t) \equiv u(0\pm, t)$ as in (3).

Proof. (1) First we prove the uniqueness of the solution $u \in \mathscr{E}$ of (1)–(3), (1.1), provided such a solution exists. Simultaneously we get a method for constructing a solution; thus, in fact, the existence will be proved as well.

According to the d'Alembert method, we substitute decomposition (8) into conditions (1.1). We get a well-known formula for $f_\pm(z)$ and $g_\pm(z)$ in the region $\pm z > 0$:

$$\begin{aligned} f_\pm(z) &= \frac{u_0(z)}{2} - \frac{1}{2a}\int_0^z u_1(y)\,dy + C'_\pm, \qquad \pm z > 0,\\ g_\pm(z) &= \frac{u_0(z)}{2} + \frac{1}{2a}\int_0^z u_1(y)\,dy - C'_\pm, \qquad \pm z > 0, \end{aligned} \quad (1.5)$$

where $C'_\pm$ are arbitrary constants. Since $f_\pm$ and $g_\pm$ in (8) are defined up to a constant, we may assume that $C'_\pm = 0$.

Remark 1.2. Since $(u_0, u_1, y_1) \in E$, it follows from (1.5) that

$$f_\pm(z), g_\pm(z) \in C(\overline{\mathbb{R}}_\pm), \qquad \text{and} \qquad f'_\pm(z), g'_\pm(z) \in L^2(\mathbb{R}_\pm), \quad (1.6)$$

where $\mathbb{R}_\pm \equiv \{x \in \mathbb{R}: \pm x > 0\}$.

By (1.5), the usual d'Alembert formula (see [1]) is valid for $|x| \geq a|t|$,

$$u(x,t) = \frac{u_0(x-at) + u_0(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} u_1(y)\,dy. \quad (1.7)$$

Thus the solution $u(x, t)$ in the region $|x| \geq a|t|$ is defined uniquely. It remains to prove the uniqueness in the region $|x| < a|t|$.

Consider the case $t > 0$. Let $|x| < at$; the unknown functions in (8) are $f_+(x - at)$ for $x > 0$ and $g_-(x + at)$ for $x < 0$. Therefore, the unknown functions are $f_+(z)$ for $z < 0$ and $g_-(z)$ for $z > 0$ ($f_+(z)$ for $z > 0$ and $g_-(z)$ for $z < 0$ are defined in (1.5)). To find these unknown functions, we use the splicing conditions (2), (3) for $t > 0$: we substitute (8) in (2), (3), and by (12) we get

$$f_+(-at) + g_+(at) = f_-(-at) + g_-(at) \equiv y(t), \qquad t > 0, \tag{1.8}$$

$$\begin{aligned} m\ddot{y}(t) = F(y(t)) + T[f'_+(-at) + g'_+(at) \qquad t > 0. \\ - f'_-(-at) - g'_-(at)], \end{aligned} \tag{1.9}$$

This is a system of two equations with two unknown functions f_+ and g_-. By (1.8) we express f_+ and g_- by means of y, and after taking derivatives in t we get

$$-af'_+(-at) = \dot{y}(t) - ag'_+(at), \; ag'_-(at) = \dot{y}(t) + af'_-(-at), \qquad t > 0. \tag{1.10}$$

Substitute these formulae in (1.9) to obtain

$$m\ddot{y} = F(y(t)) + 2T\left[g'_+(at) - f'_-(-at) - \frac{1}{a}\dot{y}(t)\right], \qquad t > 0. \tag{1.11}$$

Now we find $y(t)$ for $t > 0$ uniquely, then we find f_+ and g_- uniquely by (1.8).

By (1.6), the functions in (1.11) known from (1.5) will certainly satisfy the conditions $f'_-(-at), g'_+(at) \in L^2(\mathbb{R}_+)$ and $F(\cdot) \in C^1(\mathbb{R})$. On the other hand, by (13) the assumed existence of the solution $u \in \mathscr{E}$ implies $\ddot{y}(t) \in L^2_{\text{loc}}(\overline{\mathbb{R}}_+)$. By Definition 2, the derivative $\ddot{y}$ in (1.11) is understood in the sense of distributions. Hence by the Lebesgue theorem we conclude that Eq. (1.11) is equivalent to the same identity for almost all $t > 0$. Hence for any fixed initial data $y(0+)$ and $\dot{y}(0+)$ the solution $y(t)$ of Eq. (1.11) is unique on a certain interval $t \in [0, \varepsilon)$, where $\varepsilon > 0$. This can be proved easily by the contraction mapping principle if we rewrite (1.11) in the equivalent integral form

$$\begin{aligned} my(t) = \int_0^t \left(\int_0^s F(y(\tau))\,d\tau\right) ds + \frac{2T}{a}\int_0^t [g_+(as) + f_-(-as) - y(s)]\,ds \\ + C_0 + C_1 t, \qquad t > 0, \end{aligned} \tag{1.11'}$$

where

$$C_0 = my(0+) \qquad \text{and}$$

$$C_1 = -\frac{2T}{a}[g_+(0+) + f_-(0-) - y(0+)] + m\dot{y}(0+).$$

Thus $y(t)$ is determined uniquely on $[0, \varepsilon)$ for a certain $\varepsilon > 0$. Since $y \in C^1(\overline{\mathbb{R}}_+)$ by (13), $y(t)$ is determined uniquely for all $t > 0$ for any fixed $y(0+)$ and $\dot{y}(0+)$.

It remains to describe exactly the choice of $y(0+)$ and $\dot{y}(0+)$. First, form (1.1) we get

$$\dot{y}(0+) = y_1. \tag{1.12}$$

Next, since u is continuous for $|x| = at$, by (1.6) we see that $f_+(x - at)$ is continuous for $x = at$, and $g_-(x + at)$ is continuous for $x = -at$. Hence by (1.5) we get

$$\begin{cases} f_+(0-) = f_+(0+) \equiv \dfrac{u_0(0)}{2}, \\ g_-(0+) = g_-(0-) \equiv \dfrac{u_0(0)}{2}. \end{cases} \tag{1.13}$$

These two conditions together are equivalent to

$$y(0+) = f_\pm(0-) + g_\pm(0+) = u_0(0). \tag{1.13$'$}$$

As a result, $y(0+)$ and $\dot{y}(0+)$ are determined uniquely, hence $y(t)$ is determined uniquely for $t > 0$ as well. Then f_+ and g_- are determined uniquely by (1.8). So the uniqueness of the solution $u \in \mathscr{E}$ is proved for $t > 0$. The uniqueness for $t \leq 0$ can be proved in the same way.

We now prove the existence of a solution $u \in \mathscr{E}$ of (1)–(3), (1.1). First of all we define u in the region $|x| \geq a|t|$ by the d'Alembert formula (1.7). Then by (1.2) we see that u satisfies condition (7) of Definition 1 (of the class $\mathscr{E}$). Conditions (1) and (2) of this definition for $|x| \geq a|t|$ follow from (1.6).

We now construct a solution in the region $t > 0$, $|x| < at$: we define u by formula (8), where f_+ and g_- are determined from (1.8) with the help of the function $y(t)$. Here $y(t)$ is the solution of (1.11) with initial conditions (1.12), (1.13$'$).

LEMMA 1.1. *For any* $y(0+)$ *and* $\dot{y}(0+)$, *Eq.* (1.11) *has a solution for all* $t > 0$, *moreover,*

$$|y(t)| + |\dot{y}(t)| \leq A, \tag{1.14}$$

where A is bounded for bounded $\|(u_0, u_1, y_1)\|_E$.

Proof. Let us get an a priori estimate for $y(t)$. To do this we multiply (1.11) by $\dot{y}(t)$ for almost all $t > 0$, and by (4) we obtain

$$\frac{d}{dt} m \frac{\dot{y}^2(t)}{2} = -\frac{d}{dt} V(y(t)) + 2T\left[g'_+(at) - f'_-(-at) - \frac{1}{a}\dot{y}(t) \right]\dot{y}(t) \tag{1.15}$$

for almost all $t > 0$. Let us integrate this equality. By (1.6), the Lebesgue theorem, and the Cauchy inequality, we obtain

$$m\frac{\dot{y}^2(t)}{2} + V(y(t)) \leq B, \qquad t > 0, \tag{1.15'}$$

where B is bounded for bounded $|y_1|$, $\|g'_+\|_+$, and $\|f'_-\|_-$; here $\|\cdot\|_\pm$ is the norm in $L^2(\mathbb{R}_\pm)$. Finally, by (6′) we obtain

$$\dot{y}^2(t) \leq C, \qquad t > 0,$$

where C is bounded if the norm $\|(u_0, u_1, y_1)\|_E$ is bounded. This estimate implies the existence and uniqueness of the global solution of (1.11) for any $y(0+)$ and $\dot{y}(0+)$, as well as estimate (1.14) and the following fact:

$$\dot{y}(t) \in C(\overline{\mathbb{R}}_+). \tag{1.16}$$

The lemma is proved.

Thus we have defined f_+ and g_- by (1.8) in terms of the function $y(t)$ we have constructed. Then from (1.16), (1.6), and (1.10) it follows that

$$f'_+(z) \in L^2_{\text{loc}}(\overline{\mathbb{R}}_-), \qquad g'_-(z) \in L^2_{\text{loc}}(\overline{\mathbb{R}}_+). \tag{1.17}$$

Hence the solution $u(x, t)$ defined by (8) for $t > 0$ satisfies conditions (1), (2) of Definition 1 in the region $|x| < at$. From (1.13′) and (1.13), it follows that $u(x, t)$ satisfies conditions (1), (2) of Definition 1 for all $t > 0$ as well.

We construct the solution u in the region $t < 0$ in a similar way. From (1.7) it follows that the constructed function u is continuous for $t = 0$; hence $u \in \mathscr{E}$.

It remains to check that the function u is a solution of (1)–(3), (1.1). We obtain by (8) that u satisfies Eq. (1). From (1.7) and (1.12) it follows that u fits initial conditions (1.1). By the construction of f_+ and g_- we see that u satisfies (2) for $t > 0$, similarly for $t < 0$. Equation (1.11) for $t > 0$, a similar equation for $t < 0$, the equality $\dot{y}(0+) = y_1 = \dot{y}(0-)$, and the inclusion $\ddot{y}(t) \in L^2_{loc}(\overline{\mathbb{R}}_\pm)$ (the latter follows from (1.11), (1.16), (1.6)) all together imply (3). Hence the first assertion of Theorem 1.1 is proved.

(2) The second assertion of Theorem 1.1 follows from the above construction as well. It suffices to prove the following lemma. The lemma states that solutions of (1.11), (1.16) depend continuously on initial data in (1.1).

LEMMA 1.2. *Let $y_1(t)$ and $y_2(t)$ be the solutions of (1.11) with the initial data (u_0^i, u_1^i, y_1^i), $i = 1, 2$, respectively. Then for any $\mathcal{T} > 0$ there exists a constant $C_{\mathcal{T}}$ such that $C_{\mathcal{T}}$ is bounded for bounded $\|(u_0^i, u_1^i, y_1^i)\|_E$ and*

$$\sup_{[0,\mathcal{T}]} |y_1(t) - y_2(t)| \le C_{\mathcal{T}} \|(u_0^1 - u_0^2, u_1^1 - u_1^2, y_1^1 - y_1^2)\|_E. \tag{1.18}$$

To prove this estimate we subtract Eq. (1.11′) for y_1 from the same equation for y_2 and use the Gronwall inequality. We also use the a priori estimate (1.14).

Subtracting Eq. (1.11) for y_1 from the same equation for y_2 and using estimates (1.18) and (1.14) for y_1, y_2 in the right-hand side, we get

$$\|\ddot{y}_1(t) - \ddot{y}_2(t)\|_{L^2(0,\mathcal{T})} \le C_{\mathcal{T}}^1 \|(u_0^1 - u_0^2, u_1^1 - u_1^2, y_1^1 - y_1^2)\|_E \qquad \forall \mathcal{T} > 0, \tag{1.19}$$

where $C_{\mathcal{T}}^1$ is bounded for bounded $\|(u_0^i, u_1^i, y_1^i)\|_E$, $i = 1, 2$.

From this estimate it follows that the operator S_t from E to E is continuous.

(3) First we prove the energy conservation law (1.4) in the simpler case when $(u_0, u_1, y_1) \in E$ and $u_0(x) \in C^2(\mathbb{R}\backslash 0)$, $u_1(x) \in C^1(\mathbb{R}\backslash 0)$, and, moreover, the limits $u_0(0\pm)$, $u_0'(0\pm)$, $u_1(0\pm)$ exist.

Remark 1.3. The above smoother initial data form an everywhere dense set in the space E with the norm (1.3).

From the above construction of the solution $u(x, t)$ it follows that $u \in C(\mathbb{R}^2)$ for these initial conditions, and all the first and second classical partial derivatives of u exist and are locally bounded for $x \neq 0$ and $x \neq \pm at$. Moreover, for any $t \in \mathbb{R}$ the left-hand and right-hand limits of $\dot{u}(x, t)$ and $u'(x, t)$ as $x \to 0\pm$ and $x \to at \pm 0$ exist, and

$$(\dot{u} + au')\big|_{x=at-0} = (\dot{u} + au')\big|_{x=at+0} \qquad \forall t \neq 0. \tag{1.20}$$

This follows from the d'Alembert decomposition (8). Indeed, both sides of (1.20) vanish for the functions $f_{\pm}(x - at)$, while the functions $g_{\pm}(x + at)$ are continuously differentiable for $x = at \neq 0$. Similary,

$$(\dot{u} - au')\big|_{x=-at-0} = (\dot{u} - au')\big|_{x=-at+0} \qquad \forall t \neq 0. \tag{1.21}$$

Consider the "energy integral"

$$I(t) \equiv \int_{-\infty}^{+\infty} \left[\mu \frac{(\dot{u}(x,t))^2}{2} + T\frac{(u'(x,t))^2}{2} \right] dx \tag{1.22}$$

for $t > 0$. We express it as the sum of the integrals over the intervals $(-\infty, -at)$, $(-at, 0)$, $(0, at)$, and (at, ∞). Then we differentiate each of these summands in t and get

$$I'(t) = \Gamma_-(x,t)\Big|_{x=-at+0}^{x=-at-0} + Tu'\dot{u}\Big|_{x=0+}^{x=0-} + \Gamma_+(x,t)\Big|_{x=at+0}^{x=at-0}, \tag{1.23}$$

where

$$\Gamma_{\pm}(x,t) = \pm a \left[\mu\frac{\dot{u}^2}{2} + T\frac{u'^2}{2} \right] + Tu'\dot{u} = \pm\frac{\mu a}{2}[\dot{u} \pm au']^2.$$

Here we use the fact that Eq. (1.1) is valid for u in the classical sense, provided $x \neq 0$ and $x \neq \pm at$. Relations (1.20) and (1.21) imply

$$\Gamma_{\pm}\Big|_{x=\pm at+0}^{x=\pm at-0} = 0. \tag{1.24}$$

Finally, we get by (3)

$$Tu'\dot{u}\Bigg|_{x=0+}^{x=0-} = -\dot{y}(m\ddot{y} + V'(y)) = -\frac{d}{dt}\left[m\frac{\dot{y}^2}{2} + V(y(t)) \right], \qquad t > 0. \tag{1.25}$$

So by (1.23) we have

$$I'(t) + \frac{d}{dt}\left[m\frac{\dot{y}^2}{2} + V(y(t)) \right] = 0, \qquad t > 0. \tag{1.26}$$

Hence, (1.4) is true for $t > 0$. By taking the limit as $t \to 0+$ we get (1.4) for $t \geq 0$ as well. For $t \leq 0$ we can prove (1.4) in a similar way. Thus the equality (1.4) is proved for an everywhere dense set of initial data belonging to E. It remains to use the second assertion of Theorem 1.1.

1.2. Now consider the case $m = 0$. In this case we need not take into account the condition $\dot{y}(0) = y_1$ in (2.1). So the phase space E is replaced by E_0.

DEFINITION 1.2. E_0 is the set of pairs $(u_0(x), u_1(x))$, where the functions u_0 and u_1 are the same as in Definition 1.1.

The norm in the space E_0 is similar to (1.3):

$$\|(u_0, u_1)\|_{E_0} = \|u_0'\| + \|u_1\| + |C_-| + |C_+|. \tag{1.27}$$

E_0 can be identified with the subspace of E such that $y_1 \equiv 0$.

THEOREM 1.2. (1) *Let $m = 0$. Then for any pair $(u_0, u_1) \in E_0$ there exists a unique solution $u \in \mathscr{E}$ of* (1)–(3) *with initial data*

$$u\big|_{t=0} = u_0(x), \dot{u}\big|_{t=0} = u_1(x), \qquad x \in \mathbb{R}. \tag{1.28}$$

(2) *For any $t \in \mathbb{R}$ the operator S_t^0 that maps (u_0, u_1) to $(\dot{u}(\cdot, t))$ is continuous from E_0 to E_0.*

(3) *The energy conservation law* (1.4) *(with $m = 0$) is true for solutions u of* (1)–(3) *belonging to $\mathscr{E}$.*

The proof of this theorem coincides with that of Theorem 1.1, except for the investigation of (1.11) leading to estimate (1.19).

We point out the necessary corrections in (1.11)–(1.19) in the case $m = 0$.

When $m = 0$, Eq. (1.11) has the form

$$\dot{y}(t) = \frac{a}{2T}F(y(t)) + a[g_+'(at) - f_-'(-at)] \qquad \text{for almost all } t > 0. \tag{1.29}$$

For any $y(0+)$ there exists a unique solution of (1.29) on a certain interval $t \in [0, \varepsilon)$, where $\varepsilon > 0$. This can be proved by the contraction mapping principle. Since (1.13′) is valid for $y(0+)$, the uniqueness of $y(t)$ and of the solution $u \in \mathscr{E}$ is proved.

To construct a global solution we take into account that the a priori estimate (1.15′) holds with $m = 0$. Then from (6) it follows that $y(t)$ is bounded for $t > 0$.

Since $y(0+)$ fits condition (1.13′), $y(t)$ is determined uniquely for all $t > 0$. We construct $u(x, t)$ by means of $y(t)$, then $u(x, t) \in \mathscr{E}$ and conditions (1)–(3), (1.28) are satisfied.

So the first assertion of Theorem 1.2 is proved.

(2) If $m = 0$, we need the following estimate instead of (1.18):

$$\max_{[0,\mathcal{T}]}|y_1(t) - y_2(t)| \leq C_{\mathcal{T}}\|(u_0^1 - u_0^2, u_1^1 - u_1^2)\|_{E_0} \qquad \forall \mathcal{T} > 0, \quad (1.30)$$

where $C_{\mathcal{T}}$ is bounded for bounded $\|(u_0^i, u_1^i)\|_{E_0}$, $i = 1, 2$. We get (1.30) from (1.29) by the aid of the Gronwall inequality and the a priori estimate (1.15′). Relations (1.30), (1.29), (1.15′) imply the estimate

$$\|\dot{y}_1(t) - \dot{y}_2(t)\|_{L^2(0,\mathcal{T})} \leq C_{\mathcal{T}}^0\|u_0^1 - u_0^2, u_1^1 - u_1^2))\|_{E_0} \qquad \forall \mathcal{T} > 0, \quad (1.31)$$

where $C_{\mathcal{T}}^0$ is bounded for bounded $\|(u_0^i, u_1^i)\|_{E_0}$, $i = 1, 2$.

Equation (1.31) implies the second assertion of Theorem 1.2.

The third assertion of Theorem 1.2 follows from that of Theorem 1.1

Theorem 1.2 is proved.

2. Stationary Solutions and Stabilization

It is easy to find all the stationary solutions of (1)–(3), i.e., the solutions $u \in \mathscr{E}$ which do not depend on time: $u(x, t) \equiv u(x)$. Indeed, substitute the latter function in (1) to get $0 = u''(x)$. Hence

$$u(x) = a_{\pm}x + b_{\pm} \qquad \text{for } \pm x > 0. \qquad (2.1)$$

It follows from (7) that $a_{\pm} = 0$, and $b_+ = b_-$, since $u(x)$ is continuous. Finally,

$$u(x) \equiv \text{const} = b, \qquad x \in \mathbb{R}. \qquad (2.2)$$

Then (3) implies

$$0 = F(b). \qquad (2.3)$$

Conversely, (2.3) implies that $u(x) \equiv b$ is a stationary solution for (1)–(3) of class $\mathscr{E}$.

Corollary 2.1. *There is a bijection between the set of all stationary solutions for* (1)–(3) *of class* $\mathscr{E}$ *and the set of all zeros of the function* F.

Definition 2.1. Let S be the set of all zeros of the function F: $S = \{b \in \mathbb{R}: F(b) = 0\}$.

The main result of the present paper is the following.

THEOREM 2.1. *Let $u(x, t)$ be an arbitrary solution for* (1)–(3) *of class* $\mathscr{E}$. *Then there exists $b_+ \in S$ such that for any $B > 0$*

$$u(x, t) \rightrightarrows b_+ \qquad \text{for } |x| < B \text{ as } t \to +\infty. \tag{2.4}$$

Moreover, for any $\alpha, \beta \in \mathbb{N} \cup \{0\}$ such that $0 < \alpha + \beta \le 2$, we have

$$\partial_x^\alpha \partial_t^\beta u(x, t) \rightrightarrows 0 \qquad \text{for } |x| < B \text{ as } t \to +\infty. \tag{2.5}$$

Similar statements are true for $t \to -\infty$.

The main idea of the proof is the following. By (1.2) and (1.5), for a certain $A > 0$ (A depends on the initial data $u(x, 0)$, $\dot{u}(x, 0)$) we have

$$f_-(x) = C^- \text{ for } x < -A, \qquad g_+(x) = C^+ \text{ for } x > A. \tag{2.6}$$

Then (1.11) implies

$$m\ddot{y}(t) = F(y(t)) - \frac{2T}{a}\dot{y}(t) \qquad \text{for } at > A. \tag{2.7}$$

Since $\alpha = \sqrt{T/\mu}$, we rewrite (2.7) as

$$m\ddot{y}(t) = F(y(t)) - 2\sqrt{T\mu m}\dot{y}(t), \qquad t > A/a \equiv t_0. \tag{2.8}$$

The proof of Theorem 2.1 is based on examining this equation. By (1.8) and (2.6) we get

$$y(t) = f_+(-at) + C^+ = C^- + g_-(at) \qquad \text{for } at > A. \tag{2.9}$$

From (8) and (2.6) it follows that

$$u(x, t) = \begin{cases} f_+(x - at) + C^+, x > 0, \\ C^- + g_-(x + at), x < 0 \end{cases} \qquad \text{for } at > A. \tag{2.10}$$

Then

$$u(x, t) = y(t - |x|/a) \qquad \text{for } at > A + |x|. \tag{2.11}$$

We call (2.8) the *reduced equation* of the oscillator.

In view of (2.11), Theorem 2.1 follows from the following lemma. We suppose that $F(y)$ fits conditions (4)–(6), μ, $T > 0$, and $m \ge 0$.

LEMMA 2.1. (1) *For any solution of the reduced equation* (2.8), *there exists a number* $b_+ \in \mathbb{R}$ *such that*

$$y(t) \to b_+ \qquad \text{as } t \to +\infty; \tag{2.12}$$

(2) *moreover*, $F(b_+) = 0$; *and*

(3) $\dot{y}(t) \to 0$, $\ddot{y}(t) \to 0$ *as* $t \to +\infty$.

Remark 2.1. (1) The description of the reversible system (1)–(3) by means of irreversible equations (2.7), (1.11) seems to be paradoxical. The solution of the paradox is the following: together with (1.11), a similar equation with the "negative" friction holds, but with "outgoing" waves on the right-hand side instead of "ingoing" waves involving the right-hand side of (1.11). Hence, these equations transfer one to the other under the inversion of time. Roughly speaking, the irreversible equation (1.11) is related to the reversible system (1)–(3) by a "covariant" way with respect to the time inversion.

(2) For the proof of the asymptotics of solutions to the system (1)–(3) as $t \to +\infty$ it is necessary to use precisely the equation (1.11) with positive friction and with ingoing waves. Indeed, we obtain the ingoing waves directly from initial data, unlike the outgoing waves.[1]

Remark 2.2. Theorem 2.1 shows that the set

$$\mathfrak{A} \equiv \{(b, 0, 0) \in E : b \in S\} \tag{2.13}$$

is a "point" attractor for the system (1)–(3) in the sense of convergence (2.4), (2.5). In [1, 5, 12] attractors for nonlinear dissipative systems are constructed. System (1)–(3) has no dissipation of energy according to the energy conservation law (1.4). Therefore the results [1, 5, 12] cannot be applied directly to prove (2.4). On the other hand, the reduced equation (2.8) involves the dissipative term and thus we can apply the results cited. It follows from them that for any solution $y(t)$ of (2.8) the point $(y(t), \dot{y}(t))$ tends to the set of stationary points of (2.8) in the phase plane $(y, \dot{y})$. Note that (2.12) contains more information in the cases, when the set S is not discrete.

Remark 2.3. Note that unlike the cases considered in [1, 5, 12], trajectories of (1)–(3) do not tend to the attractor (2.14) in the phase space metric (the metric (1.3) of the space E in our case). Roughly speaking the convergence occurs in the weak topology on E only.

[1] Professor J. Lebowitz kindly told me that the connection between Eq. (1.11) and the system (1)–(3) was also discussed from a similar viewpoint in [2].

Remark 2.4. The proof of Lemma 2.1 is trivial in the case $m = 0$. Indeed, Eq. (2.8) transforms to

$$2\sqrt{T\mu}\dot{y}(t) = -V'(y).$$

Hence the function $V(y(t))$ decreases along the trajectories. By (6), any trajectory is bounded for $t > 0$. Since $\dot{y}(t)$ does not change its sign, any trajectory $y(t)$ is monotone. Thus Lemma 2.1 is proved for $m = 0$, as well as Theorem 2.1.

Taking into account Remark 2.4, we assume in what follows $m > 0$ while proving Lemma 2.1 and Theorem 2.1.

3. Phase Portrait of the Reduced Equation: Examples

First consider the case $m > 0$. In the phase plane $(y, \dot{y})$ the orbits of the reduced equation (2.8) are determined by the following system:

$$\begin{cases} \dot{y}(t) = v(t), \\ m\dot{v}(t) = F(y(t)) - 2\sqrt{T\mu}v(t), \qquad t > t_0. \end{cases} \tag{3.1}$$

We consider this system as a perturbation of the system with $T = 0$:

$$\begin{cases} \dot{y} = v, \\ m\dot{v} = F(y). \end{cases} \tag{3.2}$$

Let us establish some simple relationships between phase portraits of these two systems.

Remark 3.1. (1) It is clear that these systems have the same stationary points.

(2) The vertical component of the phase velocity vector (i.e., $\dot{v}$) of (3.1) is less than that of (3.2) if $v > 0$ and is greater if $v < 0$. The horizontal components of these vectors are equal; see Fig. 2.

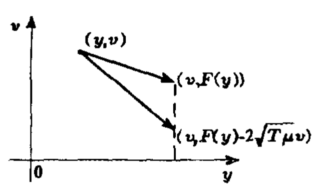

Figure 2

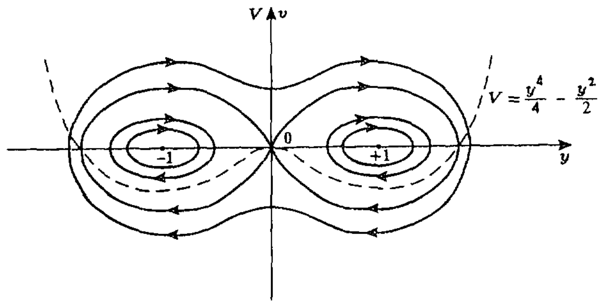

FIGURE 3

(3) Hence the orbits of (3.1) intersects those of (3.2) from the top in the halfplane $v > 0$ and from the bottom in the halfplane $v < 0$.

Consider a typical example of a potential

$$V(y) = \frac{y^4}{4} - \frac{y^2}{2}, \qquad y \in \mathbb{R}. \tag{3.3}$$

It satisfies conditions (5), (6). First let $m > 0$. Then the orbits of (3.2) are the following:

- closed curves corresponding to periodic solutions,
- two separatrices both leaving and entering the point (0, 0),
- three stationary points: a saddle at the point (0, 0) and two centers at the points (± 1, 0); see Fig. 3.

Taking into account item (3) of Remark 3.1, we see that for the system (3.1) with potential (3.3):

- the points (± 1, 0) are stable foci for small $T\mu > 0$ (stable nodes for large $T\mu > 0$), and
- the point (0, 0) is a saddle for all $T\mu \geq 0$; see Fig. 4.

Now let us consider the case $m = 0$. For the potential (3.3), the orbits of (2.8) are the rays $y < -1$, $y > 1$, the intervals $(-1, 0)$, $(0, +1)$, and the three stationary points (stable points ± 1 and unstable point 0); see Fig. 5.

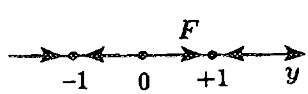

FIGURE 4

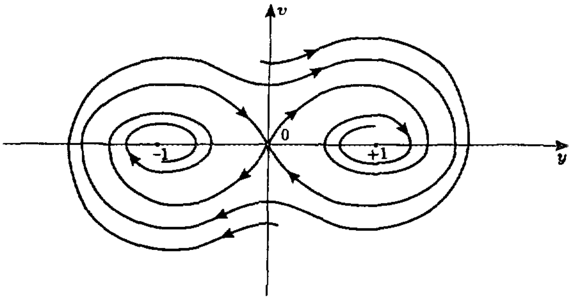

FIGURE 5

Remark 3.2. For the examples (potential (3.3) for $m > 0$ and $m = 0$) the conditions of Lemma 2.1 are satisfied, so its statements are true. In particular, all the orbits enter some stationary point as $t \to +\infty$. Note that this limit point can be unstable, e.g. the saddle (0, 0) on Fig. 4 (here two separatrices enter the saddle).

4. Dissipation Integral and Stabilization for Solutions of the Reduced Equation

It follows from (5) that any solution $y(t)$ of the reduced equation (2.8) belongs to $\mathbb{C}^3$ for $at > A$. Let us multiply (2.8) by $\dot{y}(t)$ and integrate over t from $t_0 \equiv A/a$ to $t > t_0$. Since $F(y) = -V'(y)$, we obtain

$$m\frac{\dot{y}^2(t)}{2} + V(y(t)) = E_0 - 2\sqrt{T\mu}\int_{t_0}^{t} \dot{y}^2(\tau)d\tau, \qquad t > t_0, \tag{4.1}$$

where

$$E_0 \equiv m\frac{\dot{y}^2(t_0)}{2} + V(y(t_0))$$

is the oscillator's energy at the moment t_0.

Remark 4.1. We see by (4.1) that the oscillator's Hamiltonian $H \equiv m(\dot{y}^2(t)/2 + V(y(t))$ decreases along any trajectory of system (3.1) and is constant just for the stationary solution. In terms of [1] we can say that H is a "Lyapunov function" of system (3.1).

Remark 4.2. We see by (2.8) that the string's tension creates "friction" for the oscillator with coefficient $2\sqrt{T\mu}$. According to (4.1), the oscillator loses energy; by the conservation law (1.4) it is transformed to the energy of the string. Thus we can say that the friction in (2.8) is an "emissive friction."

Remark 4.3. At the moment $t_0 \equiv A/a$ the waves $f_-(x - at)$ and $g_+(x + at)$ falling on the oscillator at the point $x = 0$ become constant (we can assume that the constant is equal to zero).

We now come to a principal point: let us use the condition (6′) for the potential of the oscillator. From (4.1) and (6′) it follows that

$$2\sqrt{T\mu}\int_{t_0}^{t} \dot{y}^2(\tau)\,d\tau \leq E_0 - V_0 \qquad \text{for } t \geq t_0. \tag{4.2}$$

We proceed to the limit as $t \to +\infty$ and obtain that the following "dissipation integral" is finite (compare [1]):

$$J \equiv 2\sqrt{T\mu}\int_{t_0}^{\infty} \dot{y}^2(\tau)\,d\tau < \infty. \tag{4.3}$$

This is the main analytic instrument to prove Lemma 2.1.

COROLLARY 4.1. *It follows from* (4.1) *and* (4.3) *that*

$$m\frac{\dot{y}^2(t)}{2} + V(y(t)) \to E_0 - J \qquad \text{as } t \to +\infty. \tag{4.4}$$

Let us sketch the proof of Lemma 2.1. First we prove (2.12). Further, by (4.4) we obtain

$$m\frac{\dot{y}^2(t)}{2} \to E_0 - J - V(b_+) \qquad \text{as } t \to +\infty. \tag{4.5}$$

So $\dot{y}(t)$ tends to a limit as $t \to +\infty$ and by (2.12) the limit equals zero. By (2.8) we have

$$m\ddot{y}(t) \to F(b_+) \qquad \text{as } t \to +\infty. \tag{4.6}$$

Again by (2.12), the limit $F(b_+)$ equals zero (if we differentiate (2.8) in t, we also obtain $y^{(3)}(t) \to 0$). So it is sufficient to prove (2.12) alone. The proof is contained in Sections 5, 6; see below. The proofs are different in the cases of degenerate and nondegenerate potentials $V(y)$.

DEFINITION 4.1. A potential $V(y)$ is called nondegenerate if

$$V(y) \not\equiv \text{const on each nonempty interval } b_1 < y < b_2, \tag{4.7}$$

and degenerate otherwise.

5. STABILIZATION IN THE CASE OF A NON-DEGENERATE POTENTIAL

Let $V(y)$ be a nondegenerate potential. We know that it is sufficient to prove (2.12). Let us assume the converse: $y(t)$ has no limit as $t \to +\infty$. Note that by (4.1)

$$V(y(t)) \leq E_0 \qquad \text{for } t > t_0. \tag{5.1}$$

Then from (6) it follows that for a certain $B < \infty$,

$$|y(t)| \leq B \qquad \text{for } t > t_0. \tag{5.2}$$

PROPOSITION 5.1. *Suppose $y(t)$ has no limit as $t \to +\infty$. Then there exist points $b^{\pm} \in [-B, B]$, $b^- < b^+$, and sequences $t_k^{\pm} \to +\infty$ such that*

$$y(t_k^{\pm}) = b^{\pm}, \qquad k = 1, 2, 3, \ldots, \tag{5.3}$$

and

$$\dot{y}(t) \neq 0 \qquad \text{if } b \equiv y(t) \in [b^-, b^+],\ t > t_0. \tag{5.4}$$

Proof. We need only check (5.4). It follows from (5.3) that for any $b \in (b^-, b^+)$ there exists a sequence $t_k \to \infty$ such that $t_k > t_0$ and

$$y(t_k) = b. \tag{5.5}$$

By (4.1) we have for $t_k > t > t_0$

$$m\frac{\dot{y}^2(t_k)}{2} - m\frac{\dot{y}^2(t)}{2} = -2\sqrt{T\mu}\int_t^{t_k} \dot{y}^2(\tau)\, d\tau < 0. \tag{5.6}$$

Indeed, suppose the last integral equals zero. Then $\dot{y}^2(\tau) \equiv 0$ for $\tau \in (t, t_k)$. By the uniqueness theorem for (3.1), we have $y(\tau) \equiv \text{const}$ for $\tau \in (t, \infty)$. This contradicts our assumption. At last, by (5.6) we get

$$|\dot{y}(t_k)| < |\dot{y}(t)|, \tag{5.7}$$

which implies (5.4).

Let us use Propositions 5.1 to obtain (2.12). By (5.4) we can assume that $t_k^- < t_k^+$, and the function $y(t)$ is monotone increasing on every segment $\Delta_k \equiv [t_k^-, t_k^+]$. Hence $y|_{\Delta_k}$ has the inverse function $t_k(y)$ on the segment $\beta \equiv [b^-, b^+]$. Let us define the following functions:

$$v_k(y) \equiv \dot{y}(t_k(y)) \qquad \text{for } y \in \beta. \tag{5.8}$$

By (5.7) we have for any k,

$$0 < v_{k+1}(y) < v_k(y) \tag{5.9}$$

for $y \in \beta$ (see Fig. 6).
Then

$$v_k(y) \to v_\infty(y) \qquad \forall y \in \beta \text{ as } k \to \infty. \tag{5.10}$$

Besides, since the dissipation integral (4.3) is finite, we obtain

$$\int_{b^-}^{b^+} v_k(y)\, dy = \int_{t_k^-}^{t_k^+} \dot{y}^2(t)\, dt \to 0 \qquad \text{as } k \to \infty. \tag{5.11}$$

From (5.9)–(5.11) and the Lebesgue theorem, it follows that

$$\int_{b^-}^{b^+} v_\infty(y)\, dy = 0. \tag{5.12}$$

Since $v_\infty(y) \geq 0$ for all $y \in \beta$, we get

$$v_\infty(y) = 0 \qquad \text{for almost all } y \in \beta. \tag{5.13}$$

On the other hand, by (4.3) we can rewrite (4.1) as

$$m\frac{v_k^2(y)}{2} + V(y) = E_0 - 2\sqrt{T\mu}\int_{t_0}^{t_k(y)} \dot{y}^2(\tau)\, d\tau \underset{k\to\infty}{\longrightarrow} E_0 - J \qquad \forall y \in \beta. \tag{5.14}$$

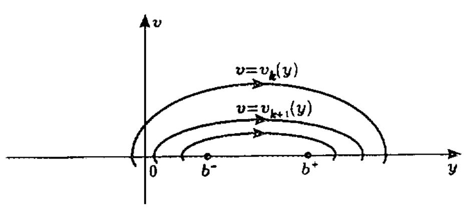

FIGURE 6

Then from (5.13) and (5.10) it follows that

$$V(y) \equiv E_0 - J \qquad \text{for almost all } y \in \beta. \tag{5.15}$$

But $V(y)$ is a continuous function, so (5.15) is true for all $y \in \beta$. This contradicts the fact that the potential V is nondegenerate.

So we have proved (2.12) and Lemma 2.1, and thus Theorem 2.1 in the case of nondegenerate potential.

6. Stabilization in the Case of a Degenerate Potential: Examples

Prior to proving Lemma 2.1 in the general case of degenerate potential, let us consider a typical example. The example is provided by the following potential:

$$V(y) \equiv \begin{cases} k\dfrac{(y-b^-)^2}{2}, & y \le b^-, \\ 0, & b^- \le y \le b^+, \\ k\dfrac{(y-b^+)^2}{2}, & b^+ \le y, \end{cases} \tag{6.1}$$

where $k > 0$ and $b^- < b^+$.

In the region $b^- \le y \le b^+$, the phase curves are determined by Eq. (2.8) with $F \equiv 0$, i.e., by

$$m\ddot{y} = -2\sqrt{T\mu}\,\dot{y}.$$

So

$$y = C_1 + C_2 e^{-\alpha t}, \qquad \alpha \equiv \frac{2\sqrt{T\mu}}{m}.$$

The corresponding system is

$$\begin{cases} \dot{y} = v, \\ m\dot{v} = -2\sqrt{T\mu v} \end{cases} \Rightarrow \frac{dv}{dy} \equiv -\alpha. \tag{6.2}$$

Then the orbits in the region $b^- \le y \le b^+$ are defined by

$$v = -\alpha y + \text{const}. \tag{6.3}$$

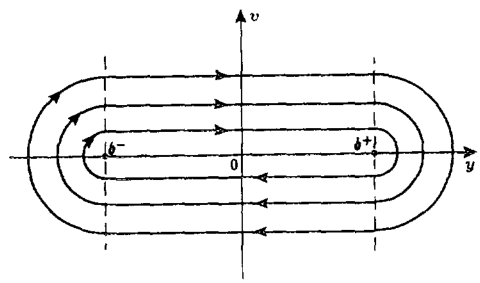

FIGURE 7

In particular, for $T = 0$ the orbits are line segments parallel to the y-axis.

Taking the above into account, we obtain the phase portraits of the system (3.1) for the potential (6.1).

(1) For $T = 0$; we have Fig. 7.

We see from the picture that in case (1) both the condition $T > 0$ and the statement of Lemma 2.1 are not true. The oscillator moves periodically as $t \to +\infty$.

(2) For $0 < T\mu < km$; we have Fig. 8.

In case (2) (as well as for $T\mu \geq km$) the statement of Lemma 2.1 is true: all the solutions have limits as $t \to +\infty$. Note that potential (6.1) does not fit condition (5) but fits (6).

We now complete the proof of Lemma 2.1 in the general case.

Let us assume the converse, as in Section 5 (see the Proof of Proposition 1), and find a nonempty interval (b^-, b^+) on which $V(y) \equiv \text{const}$. From (5.10) and (5.13) it follows that for almost all $y_0 \in (b^-, b^+)$,

$$v_k(y_0) \to 0 \qquad \text{as } k \to \infty. \tag{6.4}$$

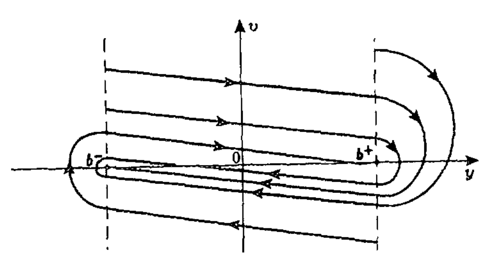

FIGURE 8

The point $(y_0, v_k(y_0))$ of the phase plane (y, v) belongs to an orbit of type (6.3); the equation of the orbit is

$$v - v_k(y_0) = -\alpha(y - y_0). \tag{6.5}$$

This line intersects the interval (b^-, b^+) of the $0y$-axis at a point b_+ if $v_k(y_0)$ is sufficiently small ($v_k(y_0) > 0$ according to (5.9)):

$$0 < v_k(y_0) < \alpha(b^+ - y_0). \tag{6.6}$$

By (6.4), this condition is satisfied for large k. Then (2.12) is true; this contradicts our assumption.

Lemma 2.1 and Theorem 2.1 are proved.

7. Transitivity of the String-Oscillator System

By Theorem 2.1, any solution u of (1)–(3) of class $\mathscr{C}$ provides the transition of the string-oscillator system from a state $b_- \in S$ to a state $b_+ \in S$ in the following sense:

$$u(x, t) \to b_\pm \qquad \text{as } t \to \pm\infty, \tag{7.1}$$

where the convergence is understood in the sense of (2.4), (2.5).

Let us show that the transition exists for any $b_\pm \in S$.

Theorem 7.1. *For any $b_\pm \in S$ there exists a solution u of* (1)–(3) *of class $\mathscr{C}$ such that b_- and b_+ are connected in the sense* (7.1).

Proof. It is possible to provide the transition $b_- \to b_+$ in different ways. We choose one of them, which is possibly the most obvious. Namely, we construct a solution $u \in \mathscr{C}$ of (1)–(3) such that

$$y(t) \equiv u(0\pm, t) = \begin{cases} b_- & \text{for } t \le -1, \\ b_+ & \text{for } t \ge 1. \end{cases} \tag{7.2}$$

To do this, we extend $y(t)$ for $t \in (-1, 1)$ arbitrarily so that $y \in C^\infty(\mathbb{R})$ ($y \in C^3(\mathbb{R})$ is sufficient). Then we put $g_+ \equiv b_-$ and determine f_- by (1.11):

$$m\ddot{y}(t) = F(y(t)) + 2T(f'_-(-at) - \frac{1}{\alpha}\dot{y}(t)), \qquad t \in \mathbb{R}. \tag{7.3}$$

Then $f'_-(z) \in C^1(\mathbb{R})$. Since $F(b_\pm) = 0$, we get

$$f'_-(-at) \equiv 0 \quad \text{for } t \le -1 \text{ and for } t \ge 1. \tag{7.4}$$

To determine f_- uniquely, we may require that

$$f_-(-at) \equiv b_- \qquad \text{for } t \le -1. \tag{7.5}$$

Then the reflected waves g_- and f_+ are determined by (1.8).

Since $y(t), f_-(-at)$ and $g_+(at)$ are constant for large $|t|, f_+(-at), g_-(at)$ are also constant for large $|t|$. Thus solution (8) belongs to $\mathscr{E}$ and satisfies (7.1).

Remark 7.1. The constructed solution means that the oscillator is in the stationary point b_- for $t \le -1$; then the wave $f_-(x - at)$ falls on the oscillator and takes it to the state b_+ by $t = 1$; moreover, for $t > -1$ it generates a pair of reflected waves; $g_-(x + at)$ for $x < 0$ and $f_+(x - at)$ for $x > 0$.

Remark 7.2. Physically, the inequality $b_+ \neq b_-$ means the capture of radiation by the oscillator if $V(b_+) > V(b_-)$, or the emission of radiation by the oscillator if $V(b_+) < V(b_-)$. Note that for linear autonomous systems (e.g. the Schrödinger linear equation with potential) there are no transitions between bound states, i.e. different asymptotics of a solution as $t \to \pm\infty$. Indeed, Schrödinger equation is equivalent to a system of independent oscillators with equal amplitudes of bound states as $t \to +\infty$ and $t \to -\infty$. Thus, system (1)–(3) is an example of a system with a "nontrivial scattering of bound states."

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