Dispersive Estimates for 1D Discrete Schrödinger and Klein-Gordon Equations

A.I. KOMECH\textsuperscript{1}, E.A. KOPYLOVA\textsuperscript{2}, M. KUNZE\textsuperscript{3}

Abstract

We derive the long-time asymptotics for solutions of the discrete 1D Schrödinger and Klein-Gordon equations.

Keywords: discrete Schrödinger and Klein-Gordon equations, lattice, Cauchy problem, long-time asymptotics.

2000 Mathematics Subject Classification: 39A11, 35L10.

1 Introduction

In this paper, we establish the long-time behavior of the solutions to the discrete Schrödinger and Klein-Gordon equations in one space dimension. We extend a general strategy introduced by Vainberg [12], Jensen-Kato [6], and Murata [8], which concerns the wave, Klein-Gordon, and Schrödinger equations, to the discrete case. Namely, we establish the Puiseux expansion for a resolvent of a stationary problem. Then the long-time asymptotics can be obtained by means of the (inverse) Fourier-Laplace transform.

We adopt the general scheme of [8] and make all constructions for the concrete case in detail. We restrict ourselves to a “nonsingular case”, in the sense of [8], where the truncated resolvent is bounded at the ends of the continuous spectrum; this holds for a generic potential. It is just this case which allows us to get the desired time decay of order $\sim t^{-3/2}$, as is desirable for applications to scattering problems.

\textsuperscript{1}Faculty of Mathematics Vienna University, A-1090 Vienna, Austria
\textsuperscript{2}Wolfgang Pauli Institute Vienna University, A-1090 Vienna, Austria
\textsuperscript{3}Universität Duisburg-Essen, Fachbereich Mathematik, D-45117 Essen, Germany
First we consider the 1D discrete version of the Schrödinger equation

\[
\begin{align*}
\left\{ \begin{array}{l}
i \dot{\psi}(x,t) = H\psi(x,t) := (-\Delta + V(x)) \psi(x,t) \\
\psi\big|_{t=0} = \psi_0
\end{array} \right. \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}.
\end{align*}
\] (1.1)

Here \(\Delta\) stands for the difference Laplacian in \(\mathbb{Z}\), defined by

\[
\Delta \psi(x) = \psi(x+1) - 2\psi(x) + \psi(x-1), \quad x \in \mathbb{Z},
\]

for functions \(\psi : \mathbb{Z} \to \mathbb{C}\). Denote by \(S\) the set of real functions on the lattice \(\mathbb{Z}\) with a finite support. For the potential \(V\) we assume that \(V \in S\). If we apply the Fourier-Laplace transform

\[
\tilde{\psi}(x,\omega) = \int_{-\infty}^{\infty} e^{i\omega t} \psi(x,t) \, dt,
\]

\(\text{Im} \, \omega > 0\), to (1.1), then the stationary equation

\[
(H - \omega) \tilde{\psi}(\omega) = -i\psi_0, \quad \text{Im} \, \omega > 0,
\] (1.2)

is obtained. Here \(\tilde{\psi}(\omega) := \tilde{\psi}(\cdot, \omega)\). Note that the integral converges, since \(\|\psi(\cdot, t)\|_2 = \text{const}\) by charge conservation. Hence we get as the solution

\[
\tilde{\psi}(\omega) = -i R(\omega)\psi_0,
\] (1.3)

where \(R(\omega) = (H - \omega)^{-1}\) is the resolvent of the Schrödinger operator \(H\).

We are going to use the function spaces which are the discrete version of the Agmon spaces [1]. These are the weighted Hilbert spaces \(l^2_\sigma = l^2_\sigma(\mathbb{Z})\) with the norm

\[
\|u\|_{l^2_\sigma} = \|(1 + x^2)^{\sigma/2} u\|_2, \quad \sigma \in \mathbb{R}.
\]

Let us denote

\[
B(\sigma, \sigma') = \mathcal{L}(l^2_\sigma, l^2_\sigma'), \quad \mathbf{B}(\sigma, \sigma') = \mathcal{L}(l^2_\sigma \oplus l^2_\sigma, l^2_\sigma' \oplus l^2_\sigma'),
\]

the space of bounded linear operators from \(l^2_\sigma\) to \(l^2_\sigma'\) and from \(l^2_\sigma \oplus l^2_\sigma\) to \(l^2_\sigma' \oplus l^2_\sigma'\), respectively. Concerning further notation, we write \(K = \text{Op}(K(x, y))\) for the operator with kernel \(K(x, y)\), i.e.,

\[
(Ku)(x) = \sum_{y \in \mathbb{Z}} K(x, y)u(y), \quad x \in \mathbb{Z}.
\]

We prove below that the continuous spectrum of the operator \(H\) coincides with the interval \([0, 4]\). Then our main results are as follows. For a generic
potential $V \in \mathcal{S}$ (see Definition 5.1) satisfying the condition $\sum_{x \in \mathbb{Z}} V(x) \neq 0$, we derive the Puiseux expansion for the resolvent at the singular spectral points $\mu = 0$ and $\mu = 4$ as

$$R(\mu + \omega) = R_0^\mu + R_1^\mu \omega^{1/2} + R_2^\mu \omega + R_3^\mu \omega^{3/2} + \ldots + \mathcal{O}(|\omega|^{N/2}), \ \omega \to 0. \ (1.4)$$

This expansion is valid in the norm $B(\sigma, -\sigma)$ with a $\sigma$ depending on $N$. Then taking the inverse Fourier-Laplace transform of (1.3), it follows that for $\sigma > 7/2$

$$\left\| e^{-itH} - \sum_{j=1}^{n} e^{-it\omega_j P_j} \right\|_{B(\sigma, -\sigma)} = \mathcal{O}(t^{-3/2}), \ t \to \infty. \ (1.5)$$

Here $P_j$ are the orthogonal projections in $l^2$ onto the eigenspaces of $H$, corresponding to the discrete eigenvalues $\omega_j \in \mathbb{R}$.

For the proof, we first calculate an explicit formula for the resolvent of the free equation in the case where $V = 0$. This formula allows us to construct the expansion of the type (1.4) for the free resolvent. Then we prove (1.4) for $V \neq 0$, developing the Fredholm alternative arguments similar to [6], [8]. Finally, Lemma 10.2 of Jensen-Kato [6] plays a crucial role in verifying the decay (1.5).

We also obtain similar results for the discrete Klein-Gordon equation

$$\begin{cases}
\ddot{\psi}(x, t) = (\Delta - m^2 - V(x)) \psi(x, t) \\
\psi|_{t=0} = \psi_0, \ \dot{\psi}|_{t=0} = \pi_0
\end{cases} \quad x \in \mathbb{Z}, \ t \in \mathbb{R}. \ (1.6)$$

Set $\Psi(t) \equiv (\psi(\cdot, t), \dot{\psi}(\cdot, t))$, $\Psi_0 \equiv (\psi_0, \pi_0)$. Then (1.6) takes the form

$$i \dot{\Psi}(t) = H \Psi(t), \ t \in \mathbb{R}; \ \Psi(0) = \Psi_0, \ (1.7)$$

where

$$H = \begin{pmatrix}
0 & i \\
i(\Delta - m^2 - V) & 0
\end{pmatrix}$$

The resolvent $R(\omega) = (H - \omega)^{-1}$ of the operator $H$ can be expressed in terms of the resolvent $R(\omega)$, and this expression yields the corresponding properties of $R(\omega)$. In particular, we derive the asymptotic expansion of the type (1.4) for $R(\omega)$, and also the long-time asymptotics of the type (1.5) for the solution.

Let us comment on previous results in this direction. Eskina [3] and Shaban–Vainberg [10] considered the difference Schrödinger equation in dimensions $n \geq 1$. They proved the limiting absorption principle and applied
it to the Sommerfeld radiation condition. However, [3, 10] do not concern the asymptotic expansion of \( R(\omega) \) and the long-time asymptotics of the type (1.5).

The asymptotic expansion of the resolvent and the asymptotics (1.5) for continuous hyperbolic equations were obtained in [7], [11], [12], [13], and for Schrödinger equation in [4], [5], [6], [8]; also see [9] for an up-to-date review and many references concerning dispersive properties of solutions to the continuous Schrödinger equation in various norms. For the discrete Schrödinger and Klein-Gordon equations, the asymptotic expansion (1.4) and long-time asymptotics (1.5) seem to be obtained for the first time in the present paper.

The paper is organized as follows. In Section 2 we obtain an explicit formula for the free resolvent. In Section 3 we derive the asymptotic expansion of the free resolvent. The limiting absorption principle for the perturbed resolvent is proved in Section 4. In Sections 5 and 6 we get the Puiseux expansion of the perturbed resolvent. In Section 7 we prove the long-time asymptotics (1.5). In Section 8 we extend the results to the discrete Klein-Gordon equation. Finally, in an appendix we illustrate the presence of a discrete spectrum for potentials which are supported at one or two points.

2 The free resolvent

We start with an investigation of the unperturbed problem for equation (1.1) with \( V = 0 \). The discrete Fourier transform of \( u : \mathbb{Z} \to \mathbb{C} \) is defined by the formula

\[
\hat{u}(\theta) = \sum_{x \in \mathbb{Z}} u(x) e^{i\theta x}, \quad \theta \in T := \mathbb{R}/2\pi \mathbb{Z}.
\]

After taking the Fourier transform, the operator \( H_0 = -\Delta \) becomes the operator of multiplication by \( \phi(\theta) = 2 - 2 \cos \theta \):

\[
-\hat{\Delta} \hat{u}(\theta) = \phi(\theta) \hat{u}(\theta).
\]

Thus, the operator \( H_0 \) is selfadjoint and its spectrum is absolutely continuous. It coincides with the range of the function \( \phi \), that is \( \text{Spec} \, H_0 = [0, 4] \). Denote by \( R_0(\omega) = (H_0 - \omega)^{-1} \) the resolvent of the difference Laplacian. Then the kernel of the resolvent \( R_0(\omega) = (H_0 - \omega)^{-1} \) reads as

\[
R_0(\omega, x, y) = \frac{1}{2\pi} \int_{T} \frac{e^{-i\theta(x-y)}}{\phi(\theta) - \omega} \, d\theta, \quad \omega \in \mathbb{C} \setminus [0, 4]. \tag{2.1}
\]

Let us calculate an explicit formula for \( R_0(\omega, x, y) \) using the Cauchy residue theorem.
Lemma 2.1. For $\omega \in \mathbb{C} \setminus [0, 4]$ the resolvent is given by

$$R_0(\omega, x, y) = -i \frac{e^{-i \theta(\omega)|x-y|}}{2 \sin \theta(\omega)}, \ x, y \in \mathbb{Z}, \ (2.2)$$

where $\theta(\omega)$ is the unique solution of the equation

$$2 - 2 \cos \theta = \omega \quad (2.3)$$

in the domain $D := \{-\pi \leq \Re \theta \leq \pi, \ \Im \theta < 0\}$.

Proof. First let us assume that $x - y \geq 0$. Denote by $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ the path indicated in Fig. 1, where

- $\Gamma_1 : \ \Re \theta = -\pi, \ \Im \theta \in [-\infty, 0]$,
- $\Gamma_2 : \ \Im \theta = 0, \ \Re \theta \in [-\pi, 0]$,
- $\Gamma_3 : \ \Im \theta = 0, \ \Re \theta \in [0, \pi]$,
- $\Gamma_4 : \ \Re \theta = \pi, \ \Im \theta \in [0, -\infty]$.

The map $\theta \mapsto \phi(\theta) = 2 - 2 \cos \theta$ transforms the paths $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ to the (oriented) intervals of the real axis $(\infty, 4], [4, 0], [0, 4], [4, \infty)$ respectively. Note, that the path $\Gamma_c : \Re \theta = 0, -\infty < \Im \theta \leq 0$ is mapped onto the interval $(-\infty, 0)$ and the region $D$ is transformed to the complex plane with the cut $[0, 4]$. Hence, there exists a unique solution $\theta(\omega)$ of the equation $\phi(\theta) = \omega, \ \omega \notin [0, 4]$, in the domain $D$.

Figure 1: Conformal mapping $\phi(\theta)$
Therefore the integrand in (2.1) has one simple pole at the point $\theta(\omega)$, and from the Cauchy residue theorem it follows that

$$R_0(\omega, x, y) = \frac{1}{2\pi} \int \frac{e^{-i\theta(x-y)}}{\phi(\theta) - \omega} \, d\theta = -i \text{res}_{\theta(\omega)} \left( \frac{e^{-i\theta(x-y)}}{\phi(\theta) - \omega} \right).$$

This implies (2.2) for $x - y \geq 0$. If $x - y \leq 0$, we choose a similar path in the upper half-plane $\text{Im}\, \theta > 0$ and get the same formula (2.2).

3 Puiseux expansion of the free resolvent

The free resolvent $R_0(\omega)$ is an analytic function with values in $B(0, 0)$ for $\omega \in \mathbb{C} \setminus [0, 4]$. This follows directly from the formula (2.2) since $\text{Im}\, \theta(\omega) < 0$, and the kernel (2.2) decays exponentially. For $\omega \in (0, 4)$, the decay fails due to $\text{Im}\, \theta(\omega) = 0$, whereas for $\omega = 0$ and $\omega = 4$ the kernel does not exist since then $\sin \theta(\omega) = 0$. Nevertheless, for the free resolvent the following limiting absorption principle holds.

**Lemma 3.1.** For $\sigma > 1/2$ the following limit exists as $\varepsilon \to 0^+$:

$$R_0(\omega \pm i\varepsilon) \xrightarrow{B(\sigma, -\sigma)} R_0(\omega \pm i0), \quad \omega \in (0, 4). \quad (3.1)$$

**Proof.** $R_0(\omega)$ is the operator with the kernel $R_0(\omega, x, y)$. If $\sigma > 1/2$ and $\omega \notin \{0, 4\}$, then the formula (2.2) implies that this is a Hilbert-Schmidt operator in the space $B(\sigma, -\sigma)$. For $\omega \in (0, 4)$ and $x, y \in \mathbb{Z}$, there exists the pointwise limit

$$R_0(\omega \pm i\varepsilon, x, y) \to R_0(\omega \pm i0, x, y), \quad \varepsilon \to 0^+.$$

Moreover, $|R_0(\omega \pm i\varepsilon, x, y)| \leq C(\omega)$. Therefore,

$$\sum_{x,y \in \mathbb{Z}} (1 + x^2)^{-\sigma}|R_0(\omega \pm i\varepsilon, x, y) - R_0(\omega \pm i0, x, y)|^2(1 + y^2)^{-\sigma} \to 0$$

as $\varepsilon \to 0^+$ by the Lebesgue dominated convergence theorem. Hence the Hilbert-Schmidt norm of the difference $R_0(\omega \pm i\varepsilon) - R_0(\omega \pm i0)$ converges to zero, and (3.1) is proved.

**Remark 3.1.** Note that

$$R_0(\omega - i0, x, y) = \overline{R_0(\omega + i0, x, y)}, \quad \omega \in (0, 4). \quad (3.2)$$

This is a consequence of the relation $\overline{\theta(\omega)} = -\theta(\overline{\omega})$ for $\omega \in \mathbb{C} \setminus [0, 4]$. 

Further, we need more information on the behavior of \( R_0(\omega) \) near \( \omega = 0 \) and \( \omega = 4 \). Without loss of generality we consider only the case \( \omega = 0 \). By means of Taylor expansion we obtain from (2.3) that

\[
\frac{1}{\sin \theta(\omega)} = \left( \omega - \frac{\omega^2}{4} \right)^{-1/2} = -\frac{1}{\sqrt{\omega}}(1 + \frac{\omega}{8} + \frac{3\omega^2}{128} + \ldots), \quad \omega \to 0,
\]

where \( \text{Im} \sqrt{\omega} > 0 \). This choice of the branch provides \( \text{Im} \theta(\omega) < 0 \) that corresponds to the exponentially decay of the kernel (2.2). Similarly,

\[
e^{-i\theta(\omega)} = \cos \theta(\omega) - i \sin \theta(\omega) = 1 - \frac{\omega}{2} + i\sqrt{\omega}(1 - \frac{\omega}{8} - \frac{\omega^2}{128} - \ldots), \quad \omega \to 0.
\]

Therefore, we get the formal expansion

\[
R_0(\omega, x, y) \sim \sum_{j=-1}^{\infty} \omega^{j/2} R_j^0(x, y), \quad \omega \to 0,
\]

(3.3)

where \( R_{-1}^0(x, y) = \frac{i}{2}, R_0^0(x, y) = -\frac{1}{2} |x - y|, \) and \( R_j^0(x, y) = \sum_{k=0}^{j+1} c_{kj} |x - y|^k \)

for \( j \in \mathbb{N} \), with suitable coefficients \( c_{kj} \in \mathbb{C} \).

For the next result, cf. [6, Lemma 2.3].

**Lemma 3.2.** i) If \( \sigma > 1/2 + j + 1 \), then \( R_j^0 = \text{Op}(R_j^0(x, y)) \in B(\sigma, -\sigma) \).

ii) The asymptotics (3.3) hold in the operator sense:

\[
R_0(\omega) = \sum_{j=-1}^{N} \omega^{j/2} R_j^0 + r_N(\omega), \quad \omega \to 0,
\]

(3.4)

where \( \|r_N(\omega)\|_{B(\sigma, -\sigma)} = \mathcal{O}(|\omega|^{(N+1)/2}) \) with \( \sigma > 1/2 + N + 2 \).

iii) In the same sense, (3.4) can be differentiated \( N + 2 \) times in \( \omega \):

\[
(d/d\omega)^{N} R_0(\omega) = \sum_{j=-1}^{N} (d/d\omega)^{N} \omega^{j/2} R_j^0 + \tilde{r}_N(\omega), \quad \omega \to 0,
\]

where \( \|\tilde{r}_N(\omega)\|_{B(\sigma, -\sigma)} = \mathcal{O}(|\omega|^{(N+1)/2-\tau}) \) with the same \( \sigma > 1/2 + N + 2 \).

**Proof.** By Taylor expansion with remainders, it is possible to check that

\[
r_N(\omega, x, y) = \left( \sum_{k=0}^{N+2} b_k(\omega) |x - y|^k \right) \omega^{(N+1)/2},
\]

where all \( b_k(\omega) = \mathcal{O}(1) \). It remains to note that for \( k = 0, \ldots, N + 2 \) the kernels \( |x - y|^k \) define Hilbert-Schmidt operators in the spaces \( B(\sigma, -\sigma) \), provided that \( \sigma > 1/2 + N + 2 \); this is due to the fact that \( |x - y|^{2k} \leq C((1 + x^2)^k + (1 + y^2)^k) \).
4 The limiting absorption principle

Let $M < \infty$ be the number of points in the support of $V$. Then the rank of the operator of multiplication by $V$ equals $M$. Therefore we have the following result.

**Lemma 4.1.** i) $\text{Spec}_{\text{ess}} H = [0, 4]$.

ii) The spectrum of $H$, outside the interval $[0, 4]$, consists of real eigenvalues $\omega_j$, $j = 1, \ldots, n$, where $n \leq M$.

Unfortunately we do not know an example of a potential $V$ for which the discrete spectrum is empty. In the appendix we provide some illustration by showing that the discrete spectrum is nonempty, if the support of $V$ consists of one or two points.

In the next lemma we develop the results of [3], [10] for the 1D case and prove the limiting absorption principle in the sense of the operator convergence. It will be needed for the proof of the long-time asymptotics (1.5).

**Lemma 4.2.** Let $V \in S$ and $\sigma > 1/2$. Then the following limits exist as $\varepsilon \to 0^+$

$$R(\omega \pm i\varepsilon) \xrightarrow{B(\sigma, -\sigma)} R(\omega \pm i0), \quad \omega \in (0, 4).$$  \hfill (4.1)

**Proof.** Step i) First we verify that for $\omega \in (0, 4)$ the operator $1 + VR_0(\omega \pm i0)$ has only a trivial kernel; for instance, we consider the “+”-case. Let $h$ be a solution of

$$h + VR_0(\omega + i0)h = 0.$$  \hfill (4.2)

Note that $V(x) = 0$ for some $x \in \mathbb{Z}$ also yields $h(x) = 0$, i.e., $h \in S$. Now for $x \in \text{supp} V$, (4.2) implies

$$\sum_{y \in \mathbb{Z}} R_0(\omega + i0, x, y)h(y) = \frac{h(x)}{V(x)}.$$  \hfill (4.3)

Multiplying (4.3) by $\overline{h}(x)$ and taking the sum over $x \in \text{supp} V$, we get from (2.2) and Lemma 3.1,

$$\text{Im} \left[ \sum_{x,y \in \mathbb{Z}} \frac{e^{-i\theta_+|x-y|}}{2\sin \theta_+} h(y)\overline{h}(x) \right] = 0,$$  \hfill (4.4)

where $\theta_+ = \theta(\omega + i0) \in (-\pi, 0)$. Since $\theta_+$ is real, also $\sin \theta_+$ is a real number. Thus (4.4) implies

$$\sum_{x,y \in \mathbb{Z}} \cos(\theta_+(x - y))h(y)\overline{h}(x) = 0,$$
and therefore
\[
\left| \sum_{x \in \mathbb{Z}} \cos(\theta_+ x) h(x) \right|^2 + \left| \sum_{x \in \mathbb{Z}} \sin(\theta_+ x) h(x) \right|^2 = 0.
\]

In summary, if \( \omega \in (0, 4) \) and \( h \) is such that (4.2) holds, then \( \hat{h}(\theta_+) = 0 \) for \( \theta_+ = \theta(\omega + i0) \). Moreover, equality \( \theta_- = \theta(\omega - i0) = -\theta_+ \) implies that \( \hat{h}(\theta_-) = 0 \). Hence the function \( \hat{\psi}(\theta) = \frac{\hat{h}(\theta)}{\phi(\theta) - \omega} \) is an entire function of \( \theta \in \mathbb{C} \). It is easy to check that the trigonometric polynomial \( \phi(\theta) - \omega \) has simple roots for \( \omega \in (0, 4) \), and therefore \( \hat{\psi}(\theta) \) is also a trigonometric polynomial. This implies that \( \psi(x) \) has a finite support; see [10, Thm. 9] for a similar argument. Moreover, \( \psi \) is the unique solution of the equation
\[
(-\Delta - \omega) \psi = h.
\] (4.5)

Next we prove that also \( \varphi = R_0(\omega + i0) h \) is a solution to (4.5). Indeed, the function \( R_0(\eta) h \) satisfies (4.5) with \( \omega = \eta \notin (0, 4) \), and from Lemma 3.1 it follows that one can pass to the limit in the equation as \( \eta \to \omega + i0 \). Thus the uniqueness for (4.5) yields that \( \psi = \varphi = R_0(\omega + i0) h \). Consequently,
\[
(-\Delta - \omega + V) \psi = 0,
\] (4.6)

since \( (-\Delta - \omega + V) \psi = h + V \psi = h + VR_0(\omega + i0) h = 0 \) by (4.2). But the only solution of (4.6) with a finite support is \( \psi \equiv 0 \), which implies \( h \equiv 0 \).

**Step ii** Fix \( \omega \in (0, 4) \) and \( \sigma > 1/2 \). Then Lemma 3.1 yields
\[
1 + VR_0(\omega \pm i\varepsilon) \xrightarrow{B(\sigma, \sigma)} 1 + VR_0(\omega \pm i0), \quad \varepsilon \to 0+;
\]

For this, recall that the potential \( V \) is assumed to be compactly supported in \( \mathbb{Z} \). Therefore the convergence \( R_0(\omega \pm i\varepsilon) \to R_0(\omega \pm i0) \) in \( B(\sigma, -\sigma) \) is improved to convergence in \( B(\sigma, \sigma) \) through multiplication by \( V \). By Step i), the operator \( 1 + VR_0(\omega \pm i0) \) has only a trivial kernel. Hence, being Fredholm if index zero, \( 1 + VR_0(\omega \pm i0) \) is invertible, and moreover
\[
(1 + VR_0(\omega \pm i\varepsilon))^{-1} \xrightarrow{B(\sigma, \sigma)} (1 + VR_0(\omega \pm i0))^{-1}, \quad \varepsilon \to 0+.
\]

Then the representation \( R = R_0(1 + VR_0)^{-1} \) implies (4.1).

**Remark 4.1.** Equation (3.2) implies
\[
R(\omega \pm i0, x, y) = \overline{R(\omega + i0, x, y)}, \quad \omega \in (0, 4).
\]
Fredholm alternative argument

In this section we are going to obtain an asymptotic expansion for the perturbed resolvent $R(\omega)$. In particular, we will show that no term of order $\omega^{-1/2}$ appears in the series for $R(\omega)$ in the case of a generic potential $V \in \mathcal{S}$, regardless of the singularity of $R_0(\omega)$.

Definition 5.1. i) A set $V \subset \mathcal{S}$ is called generic, if for each $V \in \mathcal{S}$ we have $\alpha V \in V$, with the possible exception of a discrete set of $\alpha \in \mathbb{C}$.

ii) We say that a property holds for a “generic” $V$, if it holds for all $V$ from a generic subset of $\mathcal{S}$.

We consider the asymptotic behavior of $R(\omega)$ at the singular points $\omega = 0$ and $\omega = 4$. For instance, we focus on $\omega = 0$ and construct the resolvent $R(\omega)$ for small $|\omega|$ in the case of a generic potential $V$. This will be achieved by means of the relation

$$R(\omega) = (1 + R_0(\omega)V)^{-1}R_0(\omega).$$

According to Section 3, it remains to construct $(1 + R_0(\omega)V)^{-1}$. First we note that

$$T(\omega) = 1 + R_0(\omega)V = \text{Op}[\delta(x - y) + R_0(\omega, x, y)V(y)].$$

(5.1)

Taking into account (3.3) we decompose (5.1) as

$$T(\omega) = T_r(\omega) + T_s(\omega),$$

(5.2)

with

$$T_r(\omega) = \text{Op}[\delta(x - y) + \left(R_0(\omega, x, y) - \frac{i}{2} \omega^{-1/2}\right)V(y)]$$

(5.3)

and

$$T_s(\omega) = \text{Op}\left[\frac{i}{2} \omega^{-1/2}V(y)\right]$$

(5.4)

which isolates the singular term in the expansion of $T(\omega)$. This operator acts as

$$(T_s(\omega)u)(x) = \frac{i}{2} \omega^{-1/2}\langle V, u \rangle := \frac{i}{2} \omega^{-1/2}\sum_{y \in \mathbb{Z}} V(y)u(y),$$

(5.5)

and hence its range is the one-dimensional subspace of constant functions.

To determine

$$u(\omega) := R(\omega)\psi = (1 + R_0(\omega)V)^{-1}R_0(\omega)\psi$$
for a given function $\psi$, put $f(\omega) = R_0(\omega)\psi$. Thus we are looking for solutions $u(\omega) \in l^2_\sigma$, $\sigma > 3/2$ of the equation $T(\omega)u(\omega) = f(\omega)$. Accordingly, we decompose the space $l^2_{-\sigma}$ as the sum of orthogonal subspaces as $l^2_{-\sigma} = V^\perp + V^\parallel$, where the orthogonality refers to the $l^2$ inner product $\langle \cdot, \cdot \rangle$, and $V^\parallel$ is the one-dimensional subspace spanned by $V$. Therefore we can write
\[ u(\omega) = u^+(\omega) + c(\omega)v, \quad v := V/\|V\|, \] with suitable $u^+(\omega) \in V^\perp$ and $c(\omega) \in \mathbb{C}$; here $\|V\| = \|V\|_{\ell^2}$. By (5.5) we have $V^\perp \subset \ker T_\sigma(\omega)$. Thus $T_\sigma(\omega)u^+(\omega) = 0$, and consequently $T(\omega)u(\omega) = f(\omega)$ is equivalent to
\[ T_\sigma(\omega)u^+(\omega) + c(\omega)T(\omega)v = f(\omega). \] (5.7)

**Lemma 5.1.** Let $\sigma > 3/2$. Then for a generic potential $V \in S$ the operator $T_\sigma(\omega) : l^2_{-\sigma} \to l^2_{-\sigma}$ is invertible, provided that $|\omega|$ is sufficiently small.

**Proof.** First we show that for a generic potential $V \in S$ the operator $T_\sigma(0) : l^2_{-\sigma} \to l^2_{-\sigma}$ is invertible. Since
\[ T_\sigma(0) = \text{Op}\left[\delta(x - y) - \frac{1}{2}|x - y|V(y)\right], \]
it suffices to prove that the operator
\[ \text{Op}\left[(1 + x^2)^{-\sigma/2}\left(\delta(x - y) - \frac{1}{2}|x - y|V(y)\right)(1 + y^2)^{\sigma/2}\right] \]
is an invertible operator in $l^2$. And this holds generically. Indeed, for a given potential $V \in S$ we introduce
\[ \mathcal{A}(\alpha) = \text{Op}\left[(1 + x^2)^{-\sigma/2}\left(\delta(x - y) - \frac{\alpha}{2}|x - y|V(y)\right)(1 + y^2)^{\sigma/2}\right] = 1 + \alpha\mathcal{K}, \quad \alpha \in \mathbb{C}. \]
Due to $\sigma > 3/2$, the function
\[ K(x, y) = -\frac{1}{2}(1 + x^2)^{-\sigma/2}|x - y|V(y)(1 + y^2)^{\sigma/2} \in l^2(\mathbb{Z} \times \mathbb{Z}). \]
Hence $K(x, y)$ is a Hilbert-Schmidt kernel, and accordingly the operator $\mathcal{K} = \text{Op}(K(x, y)) : l^2 \to l^2$ is compact. Further, $\mathcal{A}(\alpha)$ is analytic in $\alpha \in \mathbb{C}$ and $\mathcal{A}(0)$ is invertible. It follows that $\mathcal{A}(\alpha)$ is invertible for all $\alpha \in \mathbb{C}$ outside a discrete set; see [2]. Thus we could replace the original potential $V$ by $\alpha V$ with $\alpha$ arbitrarily close to 1, if necessary, to have $T_\sigma(0)$ invertible. Since $T_\sigma(\omega) - T_\sigma(0) \to 0$ as $\omega \to 0$, also $T_\sigma(\omega)$ is invertible for sufficiently small $|\omega|$. \qed
Put 
\[ w(\omega) = (T_\omega^{-1}(\omega))^* v, \]
where \( T_\omega^{-1}(\omega) \) exists by Lemma 5.1. Since \( v \in l^2_\sigma \) for any \( \sigma \in \mathbb{R} \), we also get 
\[ w(\omega) \in \bigcap_{\sigma > 3/2} l^2_\sigma. \]
Furthermore, for \( v^\perp \in V^\perp \) one obtains 
\[ \langle w(\omega), T_\omega(z)v^\perp \rangle = \langle (T_\omega^{-1}(\omega))^* v, T_\omega(\omega)v^\perp \rangle = \langle v, v^\perp \rangle = 0, \]
so that 
\[ w(\omega) \perp T_\omega(\omega)V^\perp. \]

Now, taking the inner product of (5.7) with \( w(\omega) \) we find 
\[ c(\omega) = \frac{\langle f(\omega), w(\omega) \rangle}{\langle T(\omega)v, w(\omega) \rangle}, \quad (5.8) \]
provided that 
\[ \langle T(\omega)v, w(\omega) \rangle \neq 0. \]

**Lemma 5.2.** For a generic potential \( V \in \mathcal{S} \) with \( \sum_{x \in \mathbb{Z}} V(x) \neq 0 \), the relation 
\[ \langle T(\omega)v, w(\omega) \rangle \neq 0 \]
holds for sufficiently small \( |\omega| \neq 0 \).

**Proof.** Denote 
\[ T_\omega(0, \alpha) = \text{Op} \left[ \delta(x - y) - \frac{\alpha}{2} |x - y| V(y) \right], \quad \alpha \in \mathbb{C}. \]
Then \( T_\omega(0, 1) = T_\omega(0), \ T_\omega(0, 0) = \text{Op} [\delta(x - y)], \) and \( \langle T_\omega(0, 0)^{-1}1, V \rangle = \langle 1, V \rangle \neq 0 \). Hence, the meromorphic function \( \alpha \mapsto \langle T_\omega(0, \alpha)^{-1}1, V \rangle \) does not vanish identically, and thus we have \( \langle T_\omega(0, \alpha)^{-1}1, V \rangle \neq 0 \) for all \( \alpha \in \mathbb{C} \) outside a discrete set. Therefore we could replace the original potential \( V \) by \( \alpha V \) with \( \alpha \) arbitrarily close to 1, if necessary, to ensure that 
\[ \langle T_\omega^{-1}(0)1, V \rangle \neq 0 \quad (5.9) \]
Then for a generic potential \( V \in \mathcal{S} \) with \( \langle 1, V \rangle = \sum_{x \in \mathbb{Z}} V(x) \neq 0 \), we have 
\[ \langle T(\omega)v, w(\omega) \rangle = \langle T_\omega(\omega)v, w(\omega) \rangle + \langle T_\omega^{-1}(\omega)^* v, \omega \rangle \]
\[ = \langle T_\omega(\omega)v, (T_\omega^{-1}(\omega))^* v \rangle + \frac{i}{2} \omega^{-1/2} \langle V, v \rangle \langle 1, w(\omega) \rangle \]
\[ = 1 + \frac{i}{2} \omega^{-1/2} \| V \| \langle T_\omega^{-1}(\omega)1, v \rangle \]
\[ = \frac{i}{2} \omega^{-1/2} \langle T_\omega^{-1}(0)1, V \rangle + o(\omega^{-1/2}) \neq 0 \quad (5.10) \]
for sufficiently small \( |\omega| \neq 0. \) \qed
By Lemma 5.1, (5.7) yields
\[ u^\perp(\omega) = T_r^{-1}(\omega) \left( f(\omega) - c(\omega) T(\omega)v \right). \]

Thus (5.6) implies that
\[ u(\omega) = T_r^{-1}(\omega) \left( f(\omega) - c(\omega) T(\omega)v \right) + c(\omega)v. \]

Hence we can summarize the foregoing arguments as follows:

**Theorem 5.1.** Let \( \sigma > 3/2 \). Then for a generic potential \( V \in S \) with \( \sum_{x \in \mathbb{Z}} V(x) \neq 0 \), the resolvent \( R(\omega) = (H - \omega)^{-1} \) can be expressed as
\[ R(\omega) \psi = T_r^{-1}(\omega) \left( f(\omega) - c(\omega) T(\omega)v \right) + c(\omega)v, \quad (5.11) \]
where \( T_r(\omega) \) is from (5.3) and invertible by Lemma 5.1, \( f(\omega) = R_0(\omega) \psi \), \( c(\omega) \) is given by (5.8), and \( T(\omega) = 1 + R_0(\omega)V \).

### 6 Puiseux expansion

**Theorem 6.1.** Let \( \sigma > 7/2 \). Then for a generic potential \( V \in S \) with \( \sum_{x \in \mathbb{Z}} V(x) \neq 0 \), the resolvent \( R(\omega) \) has the expansion
\[ R(\omega) = R^0 + \mathcal{O}(|\omega|^{1/2}), \quad \omega \to 0, \quad (6.1) \]
where the asymptotics hold in the norm of \( B(\sigma, -\sigma) \). See (6.6) below for the explicit form of \( R^0 \).

**Proof.** Step i). Fix \( \sigma > 7/2 \). Equations (3.3) and (5.3) imply that for small \( |\omega| \),
\[ T_r(\omega) = T_0 + \omega^{1/2} T_1 + \mathcal{O}(|\omega|) \]
in \( B(-\sigma, -\sigma) \), where
\[
T_0 = T_r(0) = \text{Op} \left[ \delta(x - y) - \frac{1}{2} |x - y| V(y) \right],
\]
\[
T_1 = \text{Op} \left[ \sum_{k=0}^{2} c_k |x - y|^k V(y) \right] = \frac{i}{4} \text{Op} \left[ \left( \frac{1}{4} - |x - y|^2 \right) V(y) \right].
\]
Note that again the compact support of $V$ is used here. Next we write down the Neumann series for $T_r^{-1}(\omega)$ about the invertible $T_0 = T_r(0)$ to obtain
\[ T_r^{-1}(\omega) = S_0 + \omega^{1/2}S_1 + O(|\omega|), \quad \omega \to 0, \]
(6.2)
in $B(-\sigma, -\sigma)$, where
\[ S_0 = T_0^{-1} = T_r(0)^{-1}, \quad S_1 = -T_0^{-1}T_1T_0^{-1}. \]

**Step ii.** Now let us calculate $c(\omega)$. From (6.2) we deduce
\[ (T_r^{-1}(\omega))^* = S_0^* + \omega^{1/2}S_1^* + O(|\omega|) \]
in $B(\sigma, \sigma)$ for $\sigma > 7/2$. Thus
\[ w(\omega) = (T_r^{-1}(\omega))^* v = w_0 + \omega^{1/2}w_1 + O(|\omega|) \]
(6.3)
in $l^2_\sigma$ for $\sigma > 7/2$, where
\[ w_0 = S_0^* v, \quad w_1 = S_1^* v. \]

By (3.3),
\[ R_0(\omega) = \frac{i}{2} \omega^{-1/2} \text{Op}(1) + R_0^0 + \omega^{1/2}R_1^0 + O(|\omega|) \]
(6.4)
in $B(\sigma, -\sigma)$ for $\sigma > 7/2$. Hence the numerator of (5.8) admits the asymptotic expansion
\[ \langle f(\omega), w(\omega) \rangle = \langle R_0(\omega)\psi, w(\omega) \rangle = \left( \frac{i}{2} \omega^{-1/2} \text{Op}(1)\psi + R_0^0\psi + \omega^{1/2}R_1^0\psi + O(|\omega|), \right. \]
\[ w_0 + \omega^{1/2}w_1 + O(|\omega|) \]
\[ = \frac{i}{2} \omega^{-1/2} \langle 1, \psi \rangle \langle 1, w_0 \rangle + \frac{i}{2} \langle 1, \psi \rangle \langle 1, w_1 \rangle + \langle R_0^0\psi, w_0 \rangle \]
\[ + O(|\omega|^{1/2}). \]

Next we have to expand the denominator of (5.8). By (5.10) and (6.3),
\[ \langle T(\omega)v, w(\omega) \rangle = 1 + \frac{i}{2} \omega^{-1/2}\|V\|\langle 1, (T_r^{-1}(\omega))^* v \rangle \]
\[ = 1 + \frac{i}{2} \omega^{-1/2}\|V\|\langle 1, w_0 + \omega^{1/2}w_1 + O(|\omega|) \rangle \]
\[ = \frac{i}{2} \omega^{-1/2}\|V\|\langle 1, w_0 \rangle + 1 + \frac{i}{2}\|V\|\langle 1, w_1 \rangle + O(|\omega|^{1/2}). \]
We already noticed that for a generic potential

\[ \langle 1, w_0 \rangle = \langle 1, S_0^* v \rangle = \langle 1, (T_r^{-1}(0))^* v \rangle = \langle T_r^{-1}(0) 1, v \rangle \neq 0, \]

recall (5.9). Hence (5.8) implies

\[
c(\omega) = \frac{\langle f(\omega), w(\omega) \rangle}{\langle T(\omega)v, w(\omega) \rangle}
= \frac{i}{2} \omega^{-1/2} \langle 1, \psi \rangle \langle 1, w_0 \rangle + \frac{i}{2} \langle 1, \psi \rangle \langle 1, w_1 \rangle + \langle R_0^0 \psi, w_0 \rangle + \mathcal{O}(\omega^{1/2})
= \frac{i}{2} \omega^{-1/2} ||V|| \langle 1, w_0 \rangle + 1 + \frac{i}{2} ||V|| \langle 1, w_1 \rangle + \mathcal{O}(\omega^{1/2})
= c_0 + \omega^{1/2} c_1 + \mathcal{O}(\omega), \tag{6.5}
\]

where \( c_0 = ||V||^{-1} \langle 1, \psi \rangle \) and \( c_1 \in \mathbb{C} \) is appropriate.

*Step iii). Substituting (5.2), (5.4), (6.2), (6.4), and (6.5) into (5.11), and noting the key relation

\[
\frac{i}{2} \omega^{-1/2} \text{Op}(1) \psi - c_0 \text{Op} \left[ \frac{i}{2} \omega^{-1/2} V(y) \right] v = \frac{i}{2} \omega^{-1/2} \left( \langle 1, \psi \rangle - c_0 \langle V, v \rangle \right) = 0,
\]

we obtain the following asymptotic expansion for \( R(\omega) \psi \).

\[
R(\omega) \psi = T_r^{-1}(\omega) \left( R_0(\omega) \psi - c(\omega) [T_r(\omega) + T_s(\omega)] v \right) + c(\omega) v
= T_r^{-1}(\omega) \left( \frac{i}{2} \omega^{-1/2} \text{Op}(1) \psi + R_0^0 \psi + \mathcal{O}(\omega^{1/2}) - (c_0 + \omega^{1/2} c_1 + \mathcal{O}(\omega)) \text{Op} \left[ \frac{i}{2} \omega^{-1/2} V(y) \right] v \right)
= T_r^{-1}(\omega) \left( R_0^0 \psi + \mathcal{O}(\omega^{1/2}) - \frac{i}{2} (c_1 + \mathcal{O}(\omega^{1/2})) ||V|| \right)
= \left( S_0 + \mathcal{O}(\omega^{1/2}) \right) \left( R_0^0 \psi - \frac{i}{2} c_1 ||V|| + \mathcal{O}(\omega^{1/2}) \right)
= S_0 \left( R_0^0 \psi - \frac{i}{2} c_1 ||V|| \right) + \mathcal{O}(\omega^{1/2}).
\]

This expansion does not contain singular terms in \( \omega^{-1/2} \), since they have cancelled. Therefore defining \( R^0 \psi = S_0 (R_0^0 \psi - \frac{i}{2} c_1 ||V||) \), the proof of Theorem 6.1 is complete; the explicit form of the operator \( R^0 \) can be obtained by calculating \( c_1 = c_1(\psi) \in \mathbb{C} \) from (6.5). More precisely, it is found that

\[
c_1 = \frac{||V|| \langle R_0^0 \psi, w_0 \rangle - \langle 1, \psi \rangle}{\frac{i}{2} ||V||^2 \langle 1, w_0 \rangle},
\]
so that
\[
R^0 \psi = \left( S_0 R^0_0 \psi - \frac{\langle S_0 R^0_0 \psi, V \rangle}{\langle S_0(1), V \rangle} S_0(1) \right) + \frac{\langle \psi, 1 \rangle}{\langle S_0(1), V \rangle} S_0(1)
\] (6.6)
is obtained. Here the first operator makes the projection of \( S_0 R^0_0 \psi \) onto the space \( V^\perp \) along the vector \( S_0(1) \) and the second operator is of range 1.

**Corollary 6.1.** Let \( \sigma > 7/2 \). Then for a generic potential \( V \in S \) with \( \sum_{x \in \mathbb{Z}} V(x) \neq 0 \), the resolvent expansion of \( R(\omega) \) from (6.1) may be differentiated in \( \omega \) three times, and for \( r = 1, 2, 3 \),
\[
(d/d\omega)^r R(\omega) = \mathcal{O}(|\omega|^{1/2-r}), \quad \omega \to 0,
\]
in \( B(\sigma, -\sigma) \).

**Proof.** Note that
\[
R(\omega) = (1 + R_0(\omega)V)^{-1} R_0(\omega),
\]
and \( R_0(\omega) \) has a differentiable asymptotic series by Lemma 3.2. Hence it suffices to argue that the asymptotic series for \( (1+R_0(\omega)V)^{-1} \) is differentiable. For the latter, it may be shown that
\[
(d/d\omega)(1 + R_0V)^{-1} = -(1 + R_0V)^{-1} R'_0 V (1 + R_0V)^{-1},
\]
and after some lengthy but straightforward calculation also (6.7) is found.

**Remark 6.1.** A similar expansion of \( R(\omega) \) is valid as \( \omega \to 4 \).

### 7 Long-time asymptotics

**Theorem 7.1.** Let \( \sigma > 7/2 \). Then for a generic potential \( V \in S \) with \( \sum_{x \in \mathbb{Z}} V(x) \neq 0 \), the asymptotics (1.5) hold, i.e.,
\[
\left\| e^{-itH} - \sum_{j=1}^{n} e^{-it\omega_j} P_j \right\|_{B(\sigma, -\sigma)} = \mathcal{O}(t^{-3/2}), \quad t \to \infty.
\]
Here \( P_j \) denote the projections on the eigenspaces corresponding to the eigenvalues \( \omega_j \in \mathbb{R} \setminus [0, 4] \), \( j = 1, \ldots, n \).
Proof. The estimate for $e^{-itH}$ is based on the formula

$$e^{-itH} = -\frac{1}{2\pi i} \oint_{|\omega|=C} e^{-it\omega} R(\omega) d\omega, \quad C > \max\{4; |\omega_j|, \ j=1,\ldots,n\}. \quad (7.1)$$

Encircling the spectrum $[0,4] \cup \{\omega_j : j=1,\ldots,n\}$ of $H$ by smaller and smaller pathes, it follows from

$$P_j = -\frac{1}{2\pi i} \oint_{|\omega-\omega_j|=\varepsilon} R(\omega) d\omega$$

for $\varepsilon > 0$ sufficiently small and Remark 4.1 that

$$e^{-itH} - \sum_{j=1}^n e^{-it\omega_j} P_j = \frac{1}{2\pi i} \int_{[0,4]} e^{-it\omega} (R(\omega + i0) - R(\omega - i0)) d\omega = \frac{1}{\pi} \int_{[0,4]} e^{-it\omega} \text{Im} R(\omega + i0) d\omega = \int_{[0,4]} e^{-it\omega} P(\omega) d\omega,$$

where $P(\omega) = \frac{1}{\pi} \text{Im} R(\omega + i0)$. The asymptotic expansion for $P(\omega)$ at the singular points $\mu = 0$ and $\mu = 4$ can be deduced from (6.1). Thus, restricting to $\omega \in \mathbb{R}$, we have

$$P(\mu + \omega) = O(|\omega|^{1/2}), \ \omega \to 0. \quad (7.2)$$

To prove the desired decay for large $t$, it is convenient to represent the function $P(\omega)$ for $\omega \in [0,4]$ as

$$P(\omega) = \phi_1(\omega) P(\omega) + \phi_2(\omega) P(\omega), \quad (7.3)$$

where $\phi_j(\omega) \in C_0^\infty(\mathbb{R})$ for $j=1,2$, $\phi_1(\omega) + \phi_2(\omega) = 1$ for $\omega \in [0,4]$, supp $\phi_1 \subset (-1,3)$, and supp $\phi_2 \subset (1,5)$. Due to (7.2) and Corollary 6.1, we can apply Lemma 7.1 below with $F = \phi_1 P$, $a = 3$, $B = B(\sigma,-\sigma)$ where $\sigma > 7/2$, and $\theta = 1/2$ to get

$$\int_{[0,4]} e^{-it\omega} \phi_1(\omega) P(\omega) d\omega = O(t^{-3/2}), \quad t \to \infty,$$

in $B(\sigma,-\sigma)$. Since the same argument can be used for $F = \phi_2 P$, (7.3) shows that the proof is complete. \hfill \Box

The following result is a special case of [6, Lemma 10.2].
Lemma 7.1. Assume \( \mathcal{B} \) is a Banach space, \( a > 0 \), and \( F \in C((0,a;\mathcal{B}) \) satisfies \( F(0) = F(a) = 0 \), \( F' \in L^1(0,a;\mathcal{B}) \), as well as \( F''(\omega) = O(\omega^{\theta-2}) \) as \( \omega \downarrow 0 \) for some \( \theta \in (0,1) \). Then
\[
\int_0^a e^{-it\omega} F(\omega) d\omega = O(t^{-1-\theta}), \quad t \to \infty.
\]

8 The Klein-Gordon equation

Now we extend the results of Sections 5-7 to the case of the Klein-Gordon equation (1.6)-(1.7). The operator \( \mathcal{H} \) is not selfadjoint in \( l^2 \oplus l^2 \). First we prove the existence and uniqueness of the global solution \( \Psi := e^{-it H} \Psi_0 \).

Lemma 8.1. For any initial data \( \Psi_0(x) \in l^2 \oplus l^2 \) there exists a unique solution \( \Psi(x,t) \in C(\mathbb{R}, l^2 \oplus l^2) \) of (1.7).

Proof. The existence of a local solution for sufficiently small \( |t| \) is shown by the contraction mapping method. That this local solution can be extended to a global solution follows from the energy a priori estimate. In fact, multiplying (1.6) by \( \dot{\psi}(x,t) \) and taking the sum over \( x \in \mathbb{Z} \), we have
\[
\frac{d}{dt} \left( \| \dot{\psi}(t) \|_{l^2}^2 + \| \nabla \psi(t) \|_{l^2}^2 + m^2 \| \psi(t) \|_{l^2}^2 \right) + 2 \sum_{x \in \mathbb{Z}} V(x) \psi(x,t) \dot{\psi}(x,t) = 0,
\]
where \( (\nabla \psi)(x) = \psi(x+1) - \psi(x) \) for \( x \in \mathbb{Z} \). Put \( \alpha = -\min_{x \in \mathbb{Z}} V(x) \geq 0 \). Since \( \| \nabla \psi \|_{l^2} \leq 2 \| \psi \|_{l^2} \), we get
\[
\| \dot{\psi}(t) \|_{l^2}^2 + \| \nabla \psi(t) \|_{l^2}^2 + m^2 \| \psi(t) \|_{l^2}^2 \leq (4 + m^2) \| \Psi_0 \|_{l^2 \oplus l^2}^2 + \alpha \int_0^t \| \Psi(s) \|_{l^2 \oplus l^2}^2 \, ds
\]
and therefore
\[
\| \Psi(t) \|_{l^2 \oplus l^2}^2 \leq C \| \Psi_0 \|_{l^2 \oplus l^2}^2 + \alpha_1 \int_0^t \| \Psi(s) \|_{l^2 \oplus l^2}^2 \, ds.
\]
for suitable constants \( C > 0 \) and \( \alpha_1 > 0 \). The Gronwall inequality implies that
\[
\| \Psi(t) \|_{l^2 \oplus l^2}^2 \leq Ce^{\alpha_1 t} \| \Psi_0 \|_{l^2 \oplus l^2}^2, \quad t > 0.
\]
which gives the desired bound. \( \square \)
Now we can apply the Fourier-Laplace transform
\[
\tilde{\Psi}(x, \omega) = \int_0^\infty e^{i\omega t} \Psi(x, t) \, dt, \quad \text{Im} \omega > \alpha_1 > 0,
\]
and get the stationary equation
\[
(H - \omega)\tilde{\Psi}(\omega) = -i\Psi_0, \quad \text{Im} \omega > \alpha_1.
\]

Let us first consider the resolvent \( R(\omega) = (H - \omega)^{-1} \) of the operator \( H \).

**Lemma 8.2.** If \( \omega^2 - m^2 \in \mathbb{C} \setminus [0, 4] \), then the resolvent \( R(\omega) \) can be expressed in terms of the resolvent \( R_0(\omega) \) from (1.3) as
\[
R(\omega) = \begin{pmatrix}
\omega R(\omega^2 - m^2) & iR(\omega^2 - m^2) \\
-i(1 + \omega^2 R(\omega^2 - m^2)) & \omega R(\omega^2 - m^2)
\end{pmatrix}.
\]

**Proof.** The expression for the resolvent \( R_0(\omega) = (H_0 - \omega)^{-1} \) of the free equation with \( V = 0 \) in the case where \( \omega^2 - m^2 \in \mathbb{C} \setminus [0, 4] \) can be obtained by inverse Fourier transform
\[
F_{\theta \to x-y}^{-1}\left( \frac{1}{\omega(\phi(\theta) - (\omega^2 - m^2))} \right) = R_0(\omega^2 - m^2, x, y).
\]
Using that by (2.1)
\[
F_{\theta \to x-y}^{-1}\left( \frac{1}{\omega(\phi(\theta) - (\omega^2 - m^2))} \right) = R_0(\omega^2 - m^2, x, y),
\]
we get
\[
R_0(\omega) = \begin{pmatrix}
\omega R_0(\omega^2 - m^2) & iR_0(\omega^2 - m^2) \\
-i(1 + \omega^2 R_0(\omega^2 - m^2)) & \omega R_0(\omega^2 - m^2)
\end{pmatrix}.
\]
Put
\[
V = \begin{pmatrix}
0 & 0 \\
\overline{V} & 0
\end{pmatrix}.
\]
Then the formula
\[
R(\omega) = (I - iR_0(\omega)V)^{-1}R_0(\omega)
\]
for the full resolvent yields (8.1). \( \square \)
The representation (8.1) implies the following properties of the operator $H$.

1) By Lemma 4.1 we have that

$$\text{Spec}_{\text{ess}} H = [-\sqrt{m^2 + 4}, -m] \cup [m, \sqrt{m^2 + 4}].$$

The discrete spectrum of $H$ is $\tilde{\omega}^\pm_j = \pm \sqrt{m^2 + \omega_j}$, where $\omega_j$ are the eigenvalues of the operator $H$. Note that either $\tilde{\omega}^+_j \in \mathbb{R}$ or $\tilde{\omega}^-_j \in i\mathbb{R}$.

2) Let $\sigma > 1/2$. By Lemma 4.2, the following limits exist as $\varepsilon \to 0+$.

$$R(\omega \pm i\varepsilon) \xrightarrow{B(\sigma, -\sigma)} R(\omega \pm i0),$$

and moreover

$$R(\omega - i0, x, y) = \overline{R}(\omega + i0, x, y).$$

Both relations hold for $\omega \in (-\sqrt{m^2 + 4}, -m) \cup (m, \sqrt{m^2 + 4})$.

3) Let $\sigma > 7/2$. By Theorem 6.1, we have for a generic potential $V \in S$ with $\sum_{x \in \mathbb{Z}} V(x) \neq 0$ the following asymptotic expansion of the resolvent $R$ in $B(\sigma, -\sigma)$:

$$R(\mu + \omega) = R_0'' + O(|\omega|^{1/2}), \quad \omega \to 0,$$

where $\mu = \pm m$ or $\mu = \pm \sqrt{m^2 + 4}$.

4) Let $\sigma > 7/2$. By Theorem 7.1, for a generic potential $V \in S$ with $\sum_{x \in \mathbb{Z}} V(x) \neq 0$, the following asymptotics hold:

$$\left\| e^{-itH} - \sum_{j=1}^n e^{-it\tilde{\omega}^+_j} P^+_j \right\|_{B(\sigma, -\sigma)} = O(t^{-3/2}), \quad t \to \infty.$$

Here $P^+_j$ are the projections onto the eigenspaces corresponding to the eigenvalues $\tilde{\omega}^+_j$, $j = 1, \ldots, n$.

**ACKNOWLEDGMENTS** The authors thank very much B. Vainberg for fruitful discussions. A.I.K. was supported partly by Faculty of Mathematics of Vienna University, Max-Planck Institute for Mathematics in the Sciences (Leipzig) and Wolfgang Pauli Institute. E.A.K. was supported partly by research grant of RFBR-NNIO (no.01-01-04002) and the FWF project “Asymptotics and Attractors of Hyperbolic Equations” (FWF P-16105-N05).
A Appendix

Let the number of the points in the support of the potential $V$ equal 1 or 2. We will show that for such a potential the operator $H = -\Delta + V$ always has a real eigenvalue outside the interval $[0, 4]$.

**Example I.** Let $V(x) = V_1 \delta(x - x_1)$. We seek the solution of the equation

$$(-\Delta - \omega + V)\psi = 0$$  \hfill (A.1)

in the form

$$\psi = (-\Delta - \omega)^{-1} h.$$  

Then (A.1) becomes

$$h(x) + V(x)((-\Delta - \omega)^{-1} h)(x) = 0.$$  \hfill (A.2)

Substituting the explicit formula (2.2) for the resolvent in (A.2) we obtain

$$h(x) + V_1 \delta(x - x_1) \left[ -i \sum_{y \in \mathbb{Z}} \frac{e^{-i\theta(\omega)|x-y|}}{2 \sin \theta(\omega)} h(y) \right] = 0.$$  \hfill (A.3)

Thus $h(x) = 0$ for $x \neq x_1$, and (A.3) simplifies to

$$h(x_1) \left( 1 - \frac{iV_1}{2 \sin \theta(\omega)} \right) = 0.$$  \hfill (A.4)

Hence one has to solve the following equation for the eigenvalue $\omega$ of the operator $H$.

$$2 \sin \theta(\omega) = iV_1.$$  \hfill (A.5)

First we consider the case where $V_1 < 0$ and seek the solution to (A.5) in the form $\theta(\omega) = is$ for $s \in \mathbb{R}$. Then (A.5) implies $s = \text{arcsinh}(V_1/2) < 0$. Therefore $\theta(\omega) = is \in \Gamma_c$, and consequently $\omega \in (-\infty, 0)$ is a real eigenvalue of the operator $H$. Similarly, if $V_1 > 0$, then we get a real eigenvalue $\omega \in (4, \infty)$. It is easy to check that the corresponding eigenfunctions belong to $l^2$.

**Example II.** Let $V(x) = V_1 \delta(x - x_1) + V_2 \delta(x - x_2)$. Similarly to (A.4), we now get the system

$$\begin{cases} h(x_1) \left( \frac{iV_1}{2 \sin \theta(\omega)} - 1 \right) + h(x_2) \frac{iV_1}{2 \sin \theta(\omega)} e^{-i\theta(\omega)|x_2 - x_1|} = 0 \\ h(x_1) \frac{iV_2}{2 \sin \theta(\omega)} e^{-i\theta(\omega)|x_2 - x_1|} + h(x_2) \left( \frac{iV_2}{2 \sin \theta(\omega)} - 1 \right) = 0 \end{cases}.$$
The determinant of this system equals
\[ D(\omega) = (iV_1 - 2 \sin \theta(\omega)) \left( iV_2 - 2 \sin \theta(\omega) \right) + V_1 V_2 e^{-2i\theta(\omega)|x_2 - x_1|}. \]

We want to determine a real \( \omega \) which is a solution to the equation \( D(\omega) = 0 \). Denoting \( z = e^{-i\theta(\omega)} \), this reads as
\[ (V_1 + \frac{1}{z} - z)(V_2 + \frac{1}{z} - z) = V_1 V_2 z^{2|x_2 - x_1|}. \] (A.6)

Put \( N = |x_2 - x_1| \geq 1 \), \( a = 1/V_1 \), and \( b = 1/V_2 \). Then (A.6) becomes
\[ (az^2 - z - a)(bz^2 - z - b) = z^{2N+2}. \] (A.7)

Denote by \( L(z) \) and \( R(z) \) the left hand side and the right hand side of (A.7), respectively. It is easy to check that the graphs \( y = L(z) \) and \( y = R(z) \) intersect each other at the points \( z = \pm 1 \). Moreover, \( R(0) = 0 \) and \( R(z) > 0 \) for \( z \neq 0 \).

First we consider the case where \( a, b > 0 \). Then the polynomial \( L(z) \) has two roots in the interval \((-1, 0)\), and \( L(0) = ab > 0 \). Therefore these graphs also have an intersection at a point \( z = z_0 \), with \(-1 < z_0 < 0\). It is straightforward to prove that this point corresponds to a value \( \omega \in (4, \infty) \).

The case where \( a, b < 0 \) is handled similarly, and in this case we get a solution \( \omega \in (-\infty, 0) \) of the equation \( D(\omega) = 0 \).

Finally, if \( a \) and \( b \) have opposite signs, then \( L(0) < 0 \). Calculating the first derivatives of \( L(z) \) and \( R(z) \) at \( z = \pm 1 \), we obtain
\[ L'(-1) = -2a - 2b - 2, \quad L'(1) = -2a - 2b + 2, \]
\[ R'(-1) = -2N - 2, \quad R'(1) = 2N + 2. \]

If \( N > a + b \), then \( R'(-1) < L'(-1) \) and \( R(z) < L(z) \) for \( z > -1 \) and \( z + 1 \) small enough. On the other hand, \( L(0) < R(0) \). Thus the graphs of \( L(z) \) and \( R(z) \) have an intersection in \((-1, 0)\). Similarly, if \( N > -a - b \), then these graphs have an intersection in \((0, 1)\). Therefore we have at least one root of (A.7) in \((-1, 1) \setminus \{0\} \).

References


