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On Sommerfeld representation and uniqueness in scattering by wedges

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SUMMARY

We consider a non-stationary scattering of plane waves by a wedge. We prove the Sommerfeld-type representation and uniqueness of solution to the Cauchy problem in appropriate functional spaces developing the general method of complex characteristics (*Math. USSR Sb.* 1973; **21**(1):91–135, *Moscow Univ. Math. Bull.* 1974; **29**(2):140–145, *Oper. Theory Adv. Appl.* 1992; **57**:171–183, *Am. Math. Soc. Transl.* (2) 2002; **206**:125–159). Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: scattering; wedge; complex characteristics

1. INTRODUCTION

In this paper we start a mathematical justification of the time-dependent theory of scattering by wedges. The complete justification would include the well posedness in the sense of Hadamard, the limiting amplitude principle and the limiting absorption principle. In the present paper we make the first step: we justify the Sommerfeld-type representation for solutions to the Cauchy problem with an incident plane wave. It implies, in particular, the uniqueness of the solution.

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By the Sommerfeld-type representation we mean the inverse Fourier–Laplace transformation of the Sommerfeld integral representing the solution of the stationary problem with a complex parameter.

The remaining part of the programme will follow in a subsequent paper by an investigation of the representation. Our main issue is the general method of complex characteristics [1–4]. The method gives explicitly all the solutions of the problems in angles.

The Sommerfeld integral appeared first in his paper [5] and has played a key role in the theory of scattering by wedges: see References [6–10] and the survey [11]. However, its rigorous justification, for all solutions from a functional class, has been never done. We introduce an appropriate functional class and give a justification for the first time. More precisely, we prove that a solution in our class is unique and admits the Sommerfeld integral representation. First, we reduce the problem to the stationary one, for the Helmholtz equation, by Fourier transform in time. Next we solve the stationary problem by the method of complex characteristics. The method reduces the stationary problem to the Riemann–Hilbert problem on a Riemann surface of complex characteristics of the elliptic Helmholtz equation. We solve this problem explicitly: the solution is given by the Sommerfeld integral which is equal to the Cauchy integral over a contour on the Riemann surface.

The paper concerns two-dimensional scattering of plane waves by a wedge $W := \{y = (y_1, y_2) \in \mathbb{R}^2\colon\ y_1 = \rho\cos\theta,\ y_2 = \rho\sin\theta,\ \rho > 0,\ 0 < \theta < \phi\}$ of a magnitude $\phi \in (0, \pi)$. We consider an incident plane wave $u_{\rm in}(t, y)$ of the form,

$$u_{\rm in}(t, y) = e^{i(k_0\cdot y - \omega_0 t)} f(t - n_0 \cdot y) \quad \text{for } t \in \mathbb{R} \text{ and } y \in Q := \mathbb{R}^2 \backslash W \tag{1}$$

Here the frequency $\omega_0 > 0$ and the wave vector $k_0 \in \mathbb{R}^2$, $\omega_0 = |k_0|$ and $n_0 = k_0/\omega_0$, $a \cdot b$ stands for the scalar product in $\mathbb{R}^2$. The profile $f \in C^\infty(\mathbb{R})$, and for some s_1, $s_1 > 0$,

$$f(s) = \begin{cases} 0, & s \leqslant 0 \\ 1, & s \geqslant s_1 \end{cases} \tag{2}$$

Denote $n_0 = (\cos\alpha, \sin\alpha)$ and consider the case,

$$0 < \alpha < \phi < \pi/2 \tag{3}$$

for example (see Figure 1). Physically, in this case the incident wave $u_{\rm in}$ is reflected by both sides of the wedge. Other cases can be considered similarly.

Remark 1.1
$0 < \alpha < \pi/2$.

We consider the following wave problem in Q with the Dirichlet boundary conditions:

$$\left.\begin{cases} \Box u(t, y) = 0, & y \in Q \\ u(t, y) = 0, & y \in \partial Q \end{cases}\right| \quad t \in \mathbb{R} \tag{4}$$

where $\Box = \partial_t^2 - \Delta$. We will state the result also for the case of the Neumann boundary conditions. We include the ingoing wave $u_{\rm in}$ in the statement of the problem through the initial condition,

$$u(t, y) = u_{\rm in}(t, y), \quad y \in Q,\ t < 0 \tag{5}$$

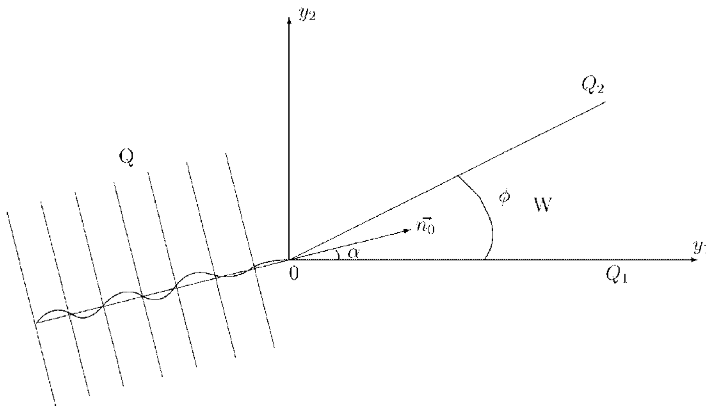

Figure 1. Incident wave.

It is possible since $u_{\text{in}}(t,y)$ is a solution to problem (4) for $t<0$: the boundary conditions in (4) hold for $t<0$ since u_{in} is then identically zero in a neighbourhood of ∂Q. Equivalently, $u(t,y)$ is the solution of the Cauchy problem for system (4) with the initial conditions,

$$\left\{\begin{array}{l} u(0,y)=u_{\text{in}}(0,y) \\ \dot{u}(0,y)=\dot{u}_{\text{in}}(0,y) \end{array}\right| \quad y\in Q \tag{6}$$

The initial functions $u_{\text{in}}(0,y)$ and $\dot{u}_{\text{in}}(0,y)$ can be determined from (1),

$$u_{\text{in}}(0,y)=\mathrm{e}^{\mathrm{i}k_0\cdot y}f(-n_0\cdot y), \quad y\in Q \tag{7}$$

$$\dot{u}_{\text{in}}(0,y)=\omega_0\mathrm{e}^{\mathrm{i}k_0\cdot y}(-\mathrm{i}f(-n_0\cdot y)+f'(-n_0\cdot y)), \quad y\in Q \tag{8}$$

A complete theory of scattering for problem (4), (5) would include the following steps: (i) The proof of the well-posedness of the problem in the sense of Hadamard, in appropriate functional spaces. (ii) The proof of the limiting amplitude principle, i.e.

$$u(t,y)\sim \mathrm{e}^{-\mathrm{i}\omega_0 t}u_\infty(y), \quad t\to\infty \tag{9}$$

where $u_\infty(y)$ is the *limiting amplitude*. (iii) The proof of the Sommerfeld formula for the limiting amplitude.

In the present paper we derive for the first time the Sommerfeld-type representation for the solution $u(t,y)$ of the non-stationary diffraction problem and accomplish the uniqueness statement from the first step of the programme.

The Sommerfeld representation plays a key role in the scattering by wedges, since it gives a representation of the solution as a superposition of plane waves. Our progress in the justification of the Sommerfeld representation is based on the general method of complex characteristics developed in References [1–4]. It has been used previously for (i) the proof of the completeness of the Ursell's trapped modes on a sloping beach [12,13] and (ii) the proof of the uniqueness of the Neumann problem in angles [14].

The method uses the complex Fourier transform and the Malyshev's method of automorphic functions [15]. We reduce the stationary problem to an algebraic *connection equation* with two unknown functions on a Riemann surface of complex characteristics of the wave equation.

The equation relates the Fourier transforms of the surface layer densities of the integral representation of the solution by the surface potentials.

The Malyshev's method allows us to reduce the equation to the Riemann–Hilbert problem which we solve explicitly.

The method provides an explicit representation for all solutions from the class of tempered distributions. It gives the solution as a Fourier integral which is a superposition of plane waves that is instructive to get the Sommerfeld representation. The present application of the method demonstrates that it provides a suitable technique for dealing with a diffraction by wedges.

Let us note that the solution could be constructed also by a separation of variables in polar co-ordinates, as in Reference [16]. Similarly, the method [17] gives the solutions for elliptic problems in angles. However, the methods do not give a representation of the solution as a superposition of plane waves. Moreover, the method [17] is applicable only to equations with real coefficients that is non-sufficient for the diffraction problems.

In 1992, Eskin [18] solved general boundary value problems for the wave equation in an angle developing a variant of our method. However, the scattering problems have not been considered in that paper.

Let us comment on previous works in the directions of the programme. Let us note that the Sommerfeld representation for non-stationary solutions, the uniqueness of the solutions in a functional class, and the limiting amplitude principle (9) for general solutions from a functional class, have been never discussed for the diffraction by wedges.

First, Sommerfeld [5] has obtained the formulas for the scattering amplitudes for the Dirichlet and Neumann boundary value conditions. He found the amplitudes as the solutions to the corresponding stationary Helmholtz equation with a radiation condition at infinity. The scattered wave formally satisfies also a limiting absorption principle. Also the papers of Malujinetz [10], Meister *et al.* [19,20], Oberhettinger [11] deal with the stationary diffraction problems on wedges.

Second, non-stationary problems were considered by Soboleff [21] and Sobolev [22]. Developing the method of Sommerfeld, he constructed a particular solution to a non-stationary problem with an incident wave of a special form. Keller and Blank [23] have also constructed a pulse solution to a non-stationary problem of diffraction by a wedge. An exponential incident plane wave is also considered. Borovikov [6] has proved the existence and uniqueness of a Green's function for problem (4), (6). The Green function is a particular solution to the scattering problem with a spherical incident wave. The corresponding solution with the incident plane wave is then constructed as the limit of the Green function when the source is going to infinity. The use of a particular Green function constructed in Reference [6] poses the following question: which class of solutions produces the Green function? The situation makes necessary a realization of the Hadamard's programme.

Further, in the paper of Petrashen' *et al.* [16], asymptotics (9) are proved for a particular solution to problem (4), (5). The method [16] is based on separation of variables. The authors also claim the Sommerfeld representation for the limiting amplitude but do not give a complete proof.

Rottbrand [24] (see, also Reference [25]) considered an aperiodic time-dependent plane wave field which falls on a half-plane. A particular solution is constructed.

Also, Rottbrand and others [25,26] constructed a particular solution of the non-stationary scattering problem by a wedge with an aperiodic time-dependent incident plane wave field. The authors develop the Wiener–Hopf technique. The uniqueness of the solution and well posedness are not considered.

Finally, the monographs [27–29] concern qualitative properties of solutions to the problems in angles. They develop the well known method of Kondrat'ev [30] based on Mellin's transform. The method gives a complete information on the smoothness and asymptotic expansion of the solutions at the wedge of the angle. An efficient application of the method has been done in the paper [31], to an exact determination of the von Neumann index of the stationary problem corresponding to a superconducting wave-guide.

The plan of our paper is the following. In Section 2, we introduce the functional classes and formulate the main result. In Section 3, we reduce the non-stationary problem to a stationary one. In Section 4, we extend the problem to the plane. In Section 5, we apply the Fourier transform. In Sections 6 and 7, we derive the key functional difference equation on the Riemann surface of complex characteristics. In Sections 8–10, we derive the Sommerfeld-type representation for the scattered wave.

2. DEFINITIONS AND MAIN RESULT

1. Let us consider a function $u(t)\in C(\mathbb{R})$, and assume that $u(t)=0$ for $t\leqslant -T$ and $|u(t)|\leqslant C(1+|t|)^N$ for some $C,N\in\mathbb{R}$. We denote its Fourier transform in time as

$$\hat{u}(\omega):=F_{t\to\omega}[u](\omega):=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega t}u(t)\,\mathrm{d}t=\int_{-T}^{\infty}\mathrm{e}^{\mathrm{i}\omega t}u(t)\,\mathrm{d}t,\quad \operatorname{Im}\omega>0 \tag{10}$$

Let us denote $\mathbb{C}^+:=\{\omega\in\mathbb{C}\colon \operatorname{Im}\omega>0\}$. Obviously, $\hat{u}(\omega)$ is an analytic function in $\omega\in\mathbb{C}^+$.

2. We will also use the real and complex Fourier transforms in the space variables. Let us consider $u(x)\in C_0^\infty(\mathbb{R}^n)$, $n=1,2$. We denote

$$\tilde{u}(\xi):=F_{x\to\xi}[u](\xi):=\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\xi x}u(x)\,\mathrm{d}x,\quad \xi\in\mathbb{R}^n \tag{11}$$

We will use similar notations for tempered distributions $u\in S'(\mathbb{R}^n)$. By the Paley–Wiener theorem [32, Theorem I.5.2], the distribution $\tilde{u}(\xi)$ has an analytic extension to the set $\{z\in\mathbb{C}^n\colon \operatorname{Im} z\in K_+^n\}$, if $\operatorname{supp} u\subset K_+^n:=\{x\in\mathbb{R}^n\colon x_i>0,\ i=1,\dots,n\}$.

3. We denote by $\mathscr{C}$ the Sommerfeld contour in the following (turned) form:

$$\mathscr{C}=\mathscr{C}_1\cup\mathscr{C}_2 \tag{12}$$

where $\mathscr{C}_1=\{w_1-\mathrm{i}\pi/2\mid w_1\geqslant 1\}\cup\{1+\mathrm{i}w_2\mid -5/2\pi\leqslant w_2\leqslant -\pi/2\}\cup\{w_1-5/2\mathrm{i}\pi\mid w_1\geqslant 1\}$. The contour $\mathscr{C}_2$ is a reflection of $\mathscr{C}_1$ with respect to the point $-3\pi/2$. We choose the orientation of the contour $\mathscr{C}$ clock-wise.

4. We denote by $\dot{Q}\equiv\bar{Q}\setminus\{0\}$, $\{y\}:=|y|/(1+|y|)$, $y\in\mathbb{R}^2$ or $y\in\mathbb{R}$.

Let us consider some $\varepsilon\geqslant 0$ and $N\geqslant 0$.

Definition 2.1

(i) E_ε is the space of functions $u(y) \in C(\bar{Q}) \cap C^1(\dot{Q})$ with the finite norm,

$$|u|_\varepsilon = \sup_{y \in \bar{Q}} |u(y)| + \sup_{y \in \dot{Q}} \{y\}^\varepsilon |\nabla u(y)| < \infty \tag{13}$$

(ii) $\mathscr{E}_{\varepsilon,N}$ is the space of functions $u(t,y)$ of $t \geqslant 0$ and $y \in \bar{Q}$ such that for each fixed $y \in \bar{Q}$ the function $u(t,y)$ is a continuous function of $t \geqslant 0$ with the finite norm,

$$\|u\|_{\varepsilon,N} := \sup_{t \geqslant 0} \left[\sup_{y \in \bar{Q}} |u(t,y)| + \sup_{y \in \dot{Q}} (1+t)^{-N} \{y\}^\varepsilon |\nabla_y u(t,y)| \right] < \infty \tag{14}$$

Let us denote $\Phi := 2\pi - \phi$, $q := \pi/2\Phi$ and

$$H(w,\alpha,\Phi) = \coth(q(w + \pi\mathrm{i}/2 - \mathrm{i}\alpha)) - \coth(q(w - 3\pi\mathrm{i}/2 + \mathrm{i}\alpha)), \quad w \in \mathbb{C} \tag{15}$$

The main result of this paper is the following theorem.

Theorem 2.2
Let $u(t,y)$ be a solution to the scattering problem (4), (5) and $u(t,y) \in \mathscr{E}_{\varepsilon,N}$ with an $\varepsilon \in [0,1)$ and an $N \geqslant 0$. Then the solution $u(t,y)$

(i) is unique
(ii) is given by the inverse Fourier transform,

$$u(t,\rho,\theta) = F^{-1}_{\omega \to t}[\hat{u}(\omega,\rho,\theta)], \quad t \geqslant 0, \ (\rho,\theta) \in \bar{Q} \tag{16}$$

where $\hat{u}(\omega,\rho,\theta)$ is the Sommerfeld-type integral

$$\hat{u}(\omega,\rho,\theta) = \frac{\mathrm{i}\hat{f}(\omega-\omega_0)}{4\Phi} \int_{\mathscr{C}} \mathrm{e}^{-\rho\omega \sinh(w - \mathrm{i}\theta)} H(w,\alpha,\Phi)\,\mathrm{d}w, \quad \rho \geqslant 0, \ \phi \leqslant \theta \leqslant 2\pi, \ \omega \in \mathbb{C}^+ \tag{17}$$

Remark 2.3
(i) We reduce the non-stationary problem to corresponding stationary problem by Fourier transform in time. We show that the solution to the stationary problem is unique in E_ε and is expressed by the Sommerfeld integral.
(ii) Similar results hold for the problem of type (4) with the Neumann boundary value conditions. The proofs for this alternative problem can be done by the same methods. In this case expression (17) is replaced by

$$\hat{u}(\omega,\rho,\theta) = \frac{\mathrm{i}\hat{f}(\omega-\omega_0)}{4\Phi} \int_{\mathscr{C}} \mathrm{e}^{-\rho\omega \sinh(w - \mathrm{i}\theta)} H_N(w,\alpha,\Phi)\,\mathrm{d}w, \quad \rho > 0, \ \phi < \theta < 2\pi, \ \omega \in \mathbb{C}^+$$

where

$$H_N(w,\alpha,\Phi) = \coth(q(w + \pi\mathrm{i}/2 - \mathrm{i}\alpha)) + \coth(q(w - 3\pi\mathrm{i}/2 + \mathrm{i}\alpha)), \quad w \in \mathbb{C}$$

Here, in contrast to (17), the integral converges absolutely only for $\phi < \theta < 2\pi$.
(iii) The existence of the solution $u(t,y)$ will be proved in a forthcoming paper by an analysis of function (16): it is the solution to the scattering problem (4), (5) and belongs to the class $\mathscr{E}_{\varepsilon,1}$ with $\varepsilon = 1 - \pi/\Phi$.

3. FOURIER TRANSFORM IN TIME

Let us consider problem (4), (5). We are going to apply the Fourier transform in time, (10), to Equation (4) to get the Helmholtz stationary equation with a parameter. First, we reduce the problem to the zero initial conditions. Namely, define the *scattered wave*,

$$u_s(t,y) \equiv u(t,y) - u_{\text{in}}(t,y), \quad t \in \mathbb{R}, \; y \in Q \tag{18}$$

where $u(t,y)$ is a solution to problem (4), (5). Then (5) implies that

$$u_s(t,y) \equiv 0, \quad t \leqslant 0, \; y \in Q \tag{19}$$

Furthermore, $u_s(t,y)$ is a solution to the problem,

$$\left.\begin{cases} \Box u_s = 0, & y \in Q \\ u_s|_{\partial Q} = -(u_{\text{in}})|_{\partial Q} \end{cases}\right|, \; t>0, \quad \{u_s(0,y)=0, \dot{u}_s(0,y)=0|, \; y \in Q \tag{20}$$

Let us note that $u_{\text{in}}(t,y) \in \mathscr{E}_{0,0}$. Therefore, the condition $u(t,y) \in \mathscr{E}_{\varepsilon,N}$ is equivalent to $u_s(t,y) \in \mathscr{E}_{\varepsilon,N}$. Hence, we get obviously from (19).

Lemma 3.1
The Fourier transform $\hat{u}_s(\omega,y)$ is an analytic function in $\omega \in \mathbb{C}^+$ with the values in E_ε.

In particular, $\hat{u}_s(\omega,y)$ is a continuous function of $(\omega,y) \in \mathbb{C}^+ \times \bar{Q}$. Let us apply the Fourier transform in time to problem (20). Split $\partial Q = Q_1 \cup Q_2$ where $Q_1 := \{y=(y_1,y_2) \in \partial Q : y_2 = 0\}$ and $Q_2 := \{y=(y_1,y_2) \in \partial Q : y_2 = y_1 \tan\phi\}$. Calculate the Dirichlet data of $\hat{u}_{\text{in}}(\omega,\cdot)$ on the sides Q_1 and Q_2 of the angle Q:

$$\hat{u}_{\text{in}}(\omega,y) = g(\omega)\mathrm{e}^{\mathrm{i}\omega y_1\cos\alpha}, \; y \in Q_1; \quad \hat{u}_{\text{in}}(\omega,y) = g(\omega)\mathrm{e}^{-\mathrm{i}\omega y_2 \cos(\alpha+\Phi)/\sin\Phi}, \; y \in Q_2 \tag{21}$$

where

$$g(\omega) := \hat{f}(\omega - \omega_0) \tag{22}$$

Therefore, the scattering problem (20) is reduced to the following stationary problem.

Lemma 3.2
Let $u_s(t,y) \in \mathscr{E}_{\varepsilon,N}$ be a solution to problem (20); then

(i) The function $\hat{u}_s(\omega,y)$ is a solution to the following boundary value problem with a parameter $\omega \in \mathbb{C}^+$:

$$\begin{cases} (-\Delta - \omega^2)\hat{u}_s(\omega,y) = 0, & y \in Q \\ \hat{u}_s(\omega,y) = -g(\omega)\mathrm{e}^{\mathrm{i}\omega y_1\cos\alpha}, & y \in Q_1 \\ \hat{u}_s(\omega,y) = -g(\omega)\mathrm{e}^{-\mathrm{i}\omega y_2[\cos(\alpha+\Phi)/\sin\Phi]}, & y \in Q_2 \end{cases} \tag{23}$$

(ii) $\hat{u}_s(\omega,\cdot) \in E_\varepsilon$ for $\omega \in \mathbb{C}^+$.

It remains to solve the stationary problem (23) because it obviously yields a solution of problem (20). We will solve it by the method [1,3,4,12]. Let us fix an $\omega \in \mathbb{C}^+$ throughout the paper, and denote

$$v_1^0(y) := -g(\omega)e^{i\omega y_1 \cos\alpha}, \quad y \in Q_1, \quad v_2^0(y) := -g(\omega)e^{i\omega y_2 \cos(\alpha+\Phi)/\sin\Phi}, \quad y \in Q_2 \tag{24}$$

We call these functions 'Dirichlet data of the solution $\hat{u}_s$'.

4. REDUCTION TO A PROBLEM IN THE PLANE

In this section we reduce problem (23) in the angle, to a problem in the plane. Let us change the variables $(x_1, x_2) = \mathscr{L}(y)$, where transformation corresponding to the matrix $\mathscr{L}$ maps the angle Q onto $K := \{(x_1, x_2): x_1 < 0 \text{ or } x_2 < 0\}$:

$$x_1 = y_1 + y_2 \cot\Phi, \quad x_2 = -y_2/\sin\Phi \tag{25}$$

Then system (23) for the function,

$$v(x_1, x_2) = \hat{u}_s(\omega, \mathscr{L}^{-1}(x_1, x_2)) \tag{26}$$

takes the form,

$$\begin{cases} \mathscr{H}(D)v(x) := \left(-\dfrac{1}{\sin^2\Phi}\left[\Delta - 2\cos\Phi \dfrac{\partial^2}{\partial x_1 \partial x_2}\right] - \omega^2\right) v(x) = 0, \quad x \in K \\ v(x_1, 0) = v_1^0(x_1), \ x_1 > 0, \quad v(0, x_2) = v_2^0(x_2), \ x_2 > 0 \end{cases} \tag{27}$$

where

$$v_1^0(x_1) = -g(\omega)e^{i\omega x_1 \cos\alpha}, \ x_1 > 0, \quad v_2^0(x_2) = -g(\omega)e^{i\omega x_2 \cos(\alpha+\Phi)}, \ x_2 > 0 \tag{28}$$

Let us denote $\dot{K} := \bar{K} \backslash \{0\}$. Then norm (13) in the co-ordinates (x_1, x_2) reads

$$|v|_\varepsilon = \sup_{x \in \bar{K}} |v(x)| + \sup_{x \in \dot{K}} \{x\}^\varepsilon |\nabla_x v| < \infty \tag{29}$$

Since $v \in E_\varepsilon$, the function v possesses the following Neumann data on $\partial\dot{K}$:

$$v_1^1(x_1) := \partial_{x_2} v(x_1, 0), \ x_1 > 0, \quad v_2^1(x_2) := \partial_{x_1} v(0, x_2), \ x_2 > 0 \tag{30}$$

Let us extend $v_l^\beta(x_l)$ by zero for $x_l < 0$. Then, by (29),

$$\left. \begin{aligned} &|v_l^1(x_l)| \leqslant C\{x_l\}^{-\varepsilon}, \quad x_l \in \mathbb{R} \setminus 0 \\ &|v_l^0(x)| \leqslant C_0, \quad x_l \in \mathbb{R} \end{aligned} \right| \quad l = 1, 2 \tag{31}$$

Therefore, $v_l^\beta(x_l) \in S'(\mathbb{R})$, $l = 1, 2$, $\beta = 0, 1$. We extend $v(x)$ by zero outside K and denote

$$v_0(x) = \begin{cases} v(x), & x \in \bar{K} \\ 0, & x \notin \bar{K} \end{cases} \tag{32}$$

The following lemma expresses the distribution $\mathscr{H} v_0$ in terms of the Cauchy data of v.

Lemma 4.1
Let $v(x) \in E_\varepsilon$ be a solution to (27). Then, in the sense of distributions,

$$\mathscr{H}(D)v_0(x) = d_0(x), \quad x \in \mathbb{R}^2 \tag{33}$$

where $d_0(x)$ is the distribution of the form,

$$\begin{aligned} d_0(x) = \frac{1}{\sin^2 \Phi} [&\delta(x_2)v_1^1(x_1) + \delta'(x_2)v_1^0(x_1) + \delta(x_1)v_2^1(x_2) + \delta'(x_1)v_2^0(x_2) \\ &- 2\cos\Phi\delta(x_2)\partial_{x_1}v_1^0(x_1) - 2\cos\Phi\delta(x_1)\partial_{x_2}v_2^0(x_2)] \end{aligned} \tag{34}$$

Proof
For $\rho > 0$ let us denote $K_\rho := \{(x_1, x_2) \in K\colon\ x_1 < -\rho \text{ or } x_2 < -\rho\}$; $v_\rho := v|_{K_\rho}$. The function $v \in C^\infty(K)$, since it is a solution of the elliptic transformed Helmholtz equation in K. Denote

$$v_\rho(x) = \begin{cases} v(x), & x \in \bar{K}_\rho \\ 0, & x \notin \bar{K}_\rho \end{cases} \tag{35}$$

Then $v_\rho \to v_0$ in $L^1_{\text{loc}}(\mathbb{R}^2)$ as $\rho \to 0+$ since $v \in E_\varepsilon$ with $\varepsilon < 1$. Therefore,

$$v_\rho(x) \to v_0(x) \text{ in } S'(\mathbb{R}^2), \quad \rho \to 0+ \tag{36}$$

Now we apply the operator $\mathscr{H}(D)$ from (27) to the distribution v_ρ. Since v_ρ is a discontinuous function and v_0 is a solution of the homogeneous transformed Helmholtz equation in K, we have,

$$\mathscr{H}(\omega, D)v_\rho(x) = d_\rho(x), \quad x \in \mathbb{R}^2 \tag{37}$$

where d_ρ is the distribution with the support in ∂K_ρ, given by the expression,

$$\left.\begin{aligned} d_\rho(x) = \frac{1}{\sin^2 \Phi}[&\delta(x_2+\rho)\Theta(x_1+\rho)\partial_{x_2}v(x_1,-\rho) + \delta'(x_2+\rho)\Theta(x_1+\rho)v(x_1,-\rho) \\ &+ \delta(x_1+\rho)\Theta(x_2+\rho)\partial_{x_1}v(-\rho,x_2) + \delta'(x_1+\rho)\Theta(x_2+\rho)v(-\rho,x_2) \\ &- 2\cos\Phi\delta(x_2+\rho)\Theta(x_1+\rho)\partial_{x_1}v(x_1,-\rho) \\ &- 2\cos\Phi\delta(x_1+\rho)\Theta(x_2+\rho)\partial_{x_2}v(-\rho,x_2)] \end{aligned}\right| \tag{38}$$

where Θ is the Heaviside function. Equation (36) implies that

$$d_\rho(x) \to \mathscr{H}v_0(x) \text{ in } S'(\mathbb{R}^2), \quad \rho \to 0+ \tag{39}$$

It remains to check that

$$d_\rho \to d_0, \quad \rho \to 0+ \tag{40}$$

Step 1: The continuity of $v(x)$ in $\bar{K}$ implies that

$$v(x_1, -\rho) \to v_1^0(x_1), \quad \rho \to 0+,\ x_1 > 0 \tag{41}$$

By (29) we have

$$|v(x_1, -\rho)| \leqslant C, \quad \rho \geqslant 0, \ \ x_1 > 0 \tag{42}$$

Therefore, (41) implies that,

$$\Theta(x_1 + \rho)v(x_1, -\rho) \to \Theta(x_1)v_1^0(x_1) \ \text{ in } S'(\mathbb{R}), \quad \rho \to 0+$$

Hence we get,

$$\delta'(x_2 + \rho)\Theta(x_1 + \rho)v(x_1, -\rho) \to \delta'(x_2)\Theta(x_1)v_1^0(x_1) \ \text{ in } S'(\mathbb{R}^2), \quad \rho \to 0+ \tag{43}$$

Similarly we have,

$$\delta'(x_1 + \rho)\Theta(x_2 + \rho)v_\rho^0(-\rho, x_2) \to \delta'(x_1)\Theta(x_2)v_2^0(x_2) \ \text{ in } S'(\mathbb{R}^2), \quad \rho \to 0+ \tag{44}$$

Step 2: The continuity of $\nabla v_0(x)$ in $\dot{K}$ implies that,

$$\partial_{x_2} v(x_1, -\rho) \to \partial_{x_2} v(x_1) \quad \forall x_1 > 0, \ \ \rho \to 0+ \tag{45}$$

By (29) we have,

$$|\partial_{x_2} v(x_1, -\rho)| \leqslant C\{(x_1, \rho)\}^{-\varepsilon} \leqslant C_1\{x_1\}^{-\varepsilon}, \quad x_1 > -\rho \tag{46}$$

Since $0 \leqslant \varepsilon < 1$, (45) and (46) imply by the Lebesgue theorem that,

$$\Theta(x_1 + \rho)\partial_{x_2} v(x_1, -\rho) \to \Theta(x_1)\partial_{x_2} v(x_1, 0) \ \text{ in } S'(\mathbb{R}^2), \quad \rho \to 0+$$

Hence we get,

$$\delta'(x_2 + \rho)\Theta(x_1 + \rho)\partial_{x_2} v(x_1, -\rho) \to \delta'(x_2)v_1^1(x_1) \ \text{ in } S'(\mathbb{R}^2), \quad \rho \to 0+ \tag{47}$$

Similarly we have,

$$\left.\begin{array}{r} \delta'(x_1 + \rho)\Theta(x_2 + \rho)\partial_{x_1} v(-\rho, x_2) \to \delta'(x_1)v_2^1(x_2) \\ -2\cos\Phi\delta(x_2 + \rho)\Theta(x_1 + \rho)\partial_{x_1} v(x_1, -\rho) \to -2\cos\Phi\delta(x_2)\partial_{x_1} v_1^0(x_1) \\ -2\cos\Phi\delta(x_1 + \rho)\Theta(x_2 + \rho)\partial_{x_2} v(x_2, -\rho) \to -2\cos\Phi\delta(x_1)\partial_{x_2} v_2^0(x_2) \end{array}\right| \quad \rho \to 0+ \tag{48}$$

in $S'(\mathbb{R}^2)$. Finally, (43), (44), (47) and (48) imply (40). □

Remark 4.2
Generally, one could expect the presence of additional terms $c_\alpha \delta^{(\alpha)}(x)$ in the RHS of (34) (cf. Reference [4]). Our proof demonstrates that all $c_\alpha = 0$ for the solution $v \in E_\varepsilon$ with $\varepsilon < 1$. Our result [14] shows that the same is true for all finite energy solutions.

5. FOURIER TRANSFORM IN SPACE

Let us apply the Fourier transform to Equation (33). We obtain,

$$\mathscr{H}(\omega, \xi)\tilde{v}_0(\xi) \equiv \left[\frac{1}{\sin^2\Phi}(\xi_1^2 + \xi_2^2 - 2\cos\Phi\xi_1\xi_2) - \omega^2\right]\tilde{v}_0(\xi) = \tilde{d}_0(\omega, \xi), \quad \xi \in \mathbb{R}^2 \tag{49}$$

where $\tilde{v}_0(\xi)$ resp. $\tilde{d}_0(\xi)$ denotes the Fourier transform of the tempered distribution v_0 resp. d_0. Identity (49) is also understood in the sense of distributions. Formula (34) implies that,

$$\tilde{d}_0(\omega,\xi) = \frac{1}{\sin^2\Phi}[\tilde{v}_1^1(\xi_1) - \tilde{v}_1^0(\xi_1)(\mathrm{i}\xi_2 - 2\mathrm{i}\xi_1\cos\Phi) + \tilde{v}_2^1(\xi_2) - \tilde{v}_2^0(\xi_2)(\mathrm{i}\xi_1 - 2\mathrm{i}\xi_2\cos\Phi)] \quad (50)$$

Identity (49) allows us to define the solution,

$$\tilde{v}_0(\xi) = \frac{\tilde{d}_0(\omega,\xi)}{\mathscr{H}(\omega,\xi)}, \quad \xi\in\mathbb{R}^2 \quad (51)$$

since $\mathscr{H}(\omega,\xi)\neq 0$ for $\xi\in\mathbb{R}^2$ and $\omega\in\mathbb{C}^+$. It remains to determine the unknown functions $\tilde{v}_l^1(\xi)$ (the Neumann data) entering into $\tilde{d}_0(\xi)$ (the Dirichlet data $\tilde{v}_l^0$ are known from (28)). We prove that the functions $\tilde{v}_l^1(\xi)$ satisfy a connection equation which is an algebraic relation on the Riemann surface of the complex characteristics of the Helmholtz operator $\mathscr{H}$. We will find a particular solution to this connection equation, reducing it to a difference equation. We will prove that this particular solution satisfies a certain growth estimate on the Riemann surface. Any solution from E_ε satisfies these growth estimates. This allows us to identify the particular solution with the unique solution from the space E_ε. This identification leads to the uniqueness and the Sommerfeld-type representation.

Let us calculate the complex Fourier transform $\tilde{v}_l^\beta(z_l)$ of the Cauchy data $v_l^\beta(x_l)$. For the Dirichlet data $v_l^0(x_l)$ we have by (28):

$$\begin{aligned}\tilde{v}_1^0(z_1) &= \frac{g(\omega)}{\mathrm{i}(z_1+\omega\cos\alpha)}, \quad \operatorname{Im} z_1>0\\ \tilde{v}_2^0(z_2) &= \frac{g(\omega)}{\mathrm{i}(z_2+\omega\cos(\alpha+\Phi))}, \quad \operatorname{Im} z_2>0\end{aligned} \quad (52)$$

Next we analyse the growth of the functions $\tilde{v}_l^1(z_l)$ in $\mathbb{C}^+$, $l=1,2$.

Lemma 5.1
For $l=1,2$ we have

$$|\tilde{v}_l^1(z_l)| \leqslant C\,\frac{(1+\operatorname{Im} z_l)^\varepsilon}{\operatorname{Im} z_l}, \quad \operatorname{Im} z_l>0 \quad (53)$$

Proof
Estimates (31) imply that,

$$\int_0^\infty |\mathrm{e}^{\mathrm{i}z_lx_l}v_l^1(x_l)|\,\mathrm{d}x_l \leqslant C_1\int_0^\infty \mathrm{e}^{-\operatorname{Im} z_lx_l}\{x_l\}^{-\varepsilon}\,\mathrm{d}x_l$$

Obviously the integral over $[1,\infty)$ satisfies the estimate of type (53). Making the change of variable $x_1\operatorname{Im} z_1=\eta$ in the integral over $[0,1]$, we also obtain the estimate of type (53) for this integral. Summing up these two estimates, we obtain (53).

6. RIEMANN SURFACE AND CONNECTION EQUATION

6.1. Riemann surface

To state the connection equation, we recall some notations from References [3,4]. Let us denote by $V = V(\omega)$ the Riemann surface,

$$V = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^2 + z_2^2 - 2\cos\Phi z_1 z_2 - \omega^2 \sin^2\Phi = 0\} \tag{54}$$

The equation is equivalent to

$$(z_1 \sin\Phi)^2 + (z_2 - z_1\cos\Phi)^2 = \omega^2 \sin^2\Phi$$

Therefore the formulas,

$$\left.\begin{aligned} z_1 &:= \omega \sin\varphi \\ z_2 - z_1\cos\Phi &:= \omega\sin\Phi\cos\varphi \end{aligned}\right| \quad \varphi \in \mathbb{C}$$

give a parametrization of V. It is convenient to change the parameter $w := \mathrm{i}\varphi$. The surface V has a universal covering surface $\check{V} \cong \mathbb{C}$ with the projection $p : \check{V} \to V$ defined by

$$p : w \mapsto (z_1, z_2), \quad \begin{cases} z_1 = z_1(w) := -\mathrm{i}\omega\sinh w \\ z_2 = z_2(w) := -\mathrm{i}\omega\sinh(w + \mathrm{i}\Phi) \end{cases} \tag{55}$$

Let us define $\check{V}_l^+$ for $l=1$ resp. $l=2$ as the connected component of the set $\{w \in \mathbb{C}\colon \operatorname{Im} z_l(w) > 0\}$ which contains the point $w = \mathrm{i}\pi/2$ resp. $w = \mathrm{i}((\pi/2) - \Phi)$. Then $\partial\check{V}_l^+ = \check{\Gamma}_l^+ \cup \check{\Gamma}_l^-$, where

$$\left.\begin{aligned} \check{\Gamma}_1^- &= \{w \in \mathbb{C}\colon \operatorname{Im} z_1(w) = 0,\ 0 \in \check{\Gamma}_1^-\} \\ \check{\Gamma}_1^+ &= \{w \in \mathbb{C}\colon \operatorname{Im} z_1(w) = 0,\ \mathrm{i}\pi \in \check{\Gamma}_1^+\} \\ \check{\Gamma}_2^- &= \{w \in \mathbb{C}\colon \operatorname{Im} z_2(w) = 0,\ \mathrm{i}(\pi - \Phi) \in \check{\Gamma}_2^-\} \\ \check{\Gamma}_2^+ &= \{w \in \mathbb{C}\colon \operatorname{Im} z_2(w) = 0,\ -\mathrm{i}\Phi \in \check{\Gamma}_2^+\} \end{aligned}\right| \tag{56}$$

It is easy to check that

$$\check{\Gamma}_1^- = \left\{ w = (w_1 + \mathrm{i}w_2)\colon\ w_{1,2} \in \mathbb{R},\ w_2 = \arctan\left(\frac{\omega_1}{\omega_2}\tanh w_1\right)\right\} \tag{57}$$

with the gauge $\arctan 0 = 0$. The same representation holds for $\check{\Gamma}_1^+$ with the gauge $\arctan 0 = \pi$. Therefore, the contour $\check{\Gamma}_1^+$ is the translation of $\check{\Gamma}_1^-$ by the vector $\pi\mathrm{i}$: $\check{\Gamma}_1^+ = \check{\Gamma}_1^- + \pi\mathrm{i}$. Similarly, the contour $\check{\Gamma}_2^-$ is the translation of $\check{\Gamma}_2^+$ by $\pi\mathrm{i}$, and $\check{\Gamma}_2^+$ is the translation of $\check{\Gamma}_1^-$ by $-\mathrm{i}\Phi$. Thus, all contours (56) are identical up to translations. For $v \in \mathbb{R}$, let us define the contour

$$\gamma(v) \equiv \check{\Gamma}_1^- + \mathrm{i}v \tag{58}$$

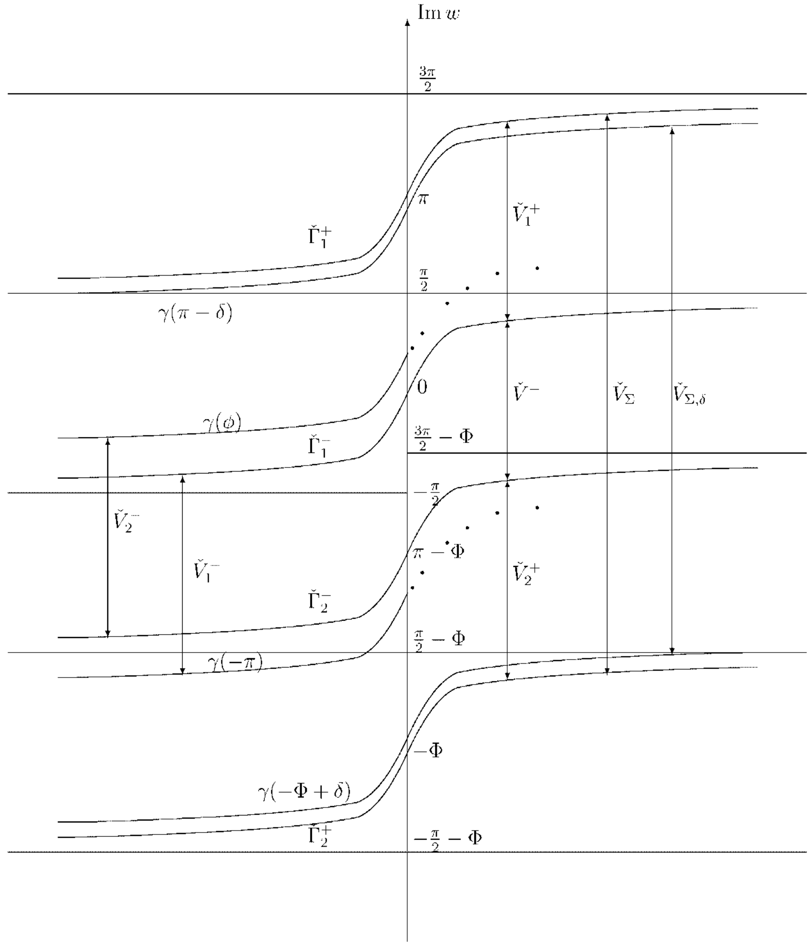

Figure 2. Universal covering.

Then contours (56) can be represented in the following form:

$$\left.\begin{array}{l} \check{\Gamma}_1^- = \gamma(0), \quad \check{\Gamma}_1^+ = \gamma(\pi) \\ \check{\Gamma}_2^- = \gamma(\pi-\Phi), \quad \check{\Gamma}_2^+ = \gamma(-\Phi) \end{array}\right| \tag{59}$$

Let us define the region $\check{V}_l^-$ for $l=1,2$ as the connected component of the set $\{w\in\mathbb{C}\colon \operatorname{Im} z_l(w) <0\}$ which contains the point $w=-\mathrm{i}\pi/2$. Set $\check{V}^- := \check{V}_1^- \cap \check{V}_2^-$ and

$$\check{V}_\Sigma := \check{V}_1^+ \cup \overline{\check{V}^-} \cup \check{V}_2^+ \tag{60}$$

(see Figure 2, which corresponds to the case $\operatorname{Re}\omega>0$).

Using the definitions of $\check{V}_l^{\pm}$, $\check{V}^{-}$, $\check{V}_{\Sigma}$ we can represent the regions by the contours $\gamma(v)$:

$$\begin{aligned}\check{V}_1^{+}&=\{w:\gamma(0)<w<\gamma(\pi)\},\quad \check{V}_2^{+}=\{w:\gamma(-2\pi+\phi)<w<\gamma(-\pi+\phi)\}\\ \check{V}_1^{-}&=\{w:\gamma(-\pi)<w<\gamma_0\},\quad \check{V}_2^{-}=\{w:\gamma(-\pi+\phi)<w<\gamma(\phi)\}\\ \check{V}^{-}&=\{w:\gamma(-\pi+\phi)<w<\gamma(0)\},\quad \check{V}_{\Sigma}=\{w:\gamma(-\Phi)<w<\gamma(\pi)\}\end{aligned} \tag{61}$$

(see Figure 2). Here the symbol '$<$' means that the point w lies between corresponding curves. Also, we will consider the following subregion $\check{V}_{\Sigma,\delta}$ with a $\delta>0$:

$$\check{V}_{\Sigma,\delta}=\{w:\gamma(-\Phi+\delta)<w<\gamma(\pi-\delta)\} \tag{62}$$

(see Figure 2).

6.2. *Lifting onto universal covering*

Now we 'lift' the functions $\tilde{v}_l^{\beta}$ onto $\check{V}_l^{+}$ with covering (55). Namely, we denote by $\check{v}_l^{\beta}(w)$ the composition of $\tilde{v}_l^{\beta}(z_l)$ and $z_l(w)$:

$$\check{v}_l^{\beta}(w)=\tilde{v}_l^{\beta}(z_l(w)),\quad z_l\in\mathbb{C}^{+},\ \ l=1,2,\ \ \beta=0,1 \tag{63}$$

The analyticity of the functions $\tilde{v}_l^{\beta}$ in $\mathbb{C}^{+}$ implies the analyticity of $\check{v}_l^{\beta}$ in $\check{V}_l^{+}$, $l=1,2$. We calculate these lifting for the known Dirichlet data of solution. Namely, (52) and (63) give

$$\check{v}_1^{0}(w)=\frac{g(\omega)}{\omega(\sinh w+\mathrm{i}\cos\alpha)},\quad \check{v}_2^{0}(w)=\frac{g(\omega)}{\omega(\sinh(w+\mathrm{i}\Phi)+\mathrm{i}\cos(\alpha+\Phi))} \tag{64}$$

Hence,

$$|\check{v}_l^{0}(w)|\leqslant C\mathrm{e}^{-|w|},\quad |\mathrm{Re}\,w|\geqslant 1 \tag{65}$$

Similarly, estimate (53) imply that,

$$|\check{v}_1^{1}(w)|\leqslant C\,\frac{(1+\mathrm{Im}(-\mathrm{i}\omega\sinh w))^{\varepsilon}}{\mathrm{Im}(-\mathrm{i}\omega\sinh w)},\quad w\in\check{V}_1^{+} \tag{66}$$

$$|\check{v}_2^{1}(w)|\leqslant C\,\frac{(1+\mathrm{Im}(-\mathrm{i}\omega\sinh(w+\mathrm{i}\Phi)))^{\varepsilon}}{\mathrm{Im}(-\mathrm{i}\omega\sinh(w+\mathrm{i}\Phi))},\quad w\in\check{V}_2^{+} \tag{67}$$

6.3. *Connection equation*

Now we can formulate our basic connection equation. Let us recall that the function $\check{v}_l^{\beta}(w)$, defined by (63), is analytic in the region $\check{V}_l^{+}$. By $H(V)$ we denote the set of analytic functions in an open set $V\subset\mathbb{C}$. By $[\check{v}(w)]_l$, $l=1,2$ we denote the analytic continuation of a function $\check{v}(w)\in H(V_l^{+})$ to the complex region $\check{V}_{\Sigma}$ (see Figure 2) if the continuation exists. Let us denote

$$\check{v}_1(w):=\check{v}_1^{1}(w)+\omega\sinh(w-\mathrm{i}\Phi)\check{v}_1^{0}(w),\quad \check{v}_2(w):=\check{v}_2^{1}(w)+\omega\sinh(w+2\mathrm{i}\Phi)\check{v}_2^{0}(w) \tag{68}$$

The following connection equation has been proved in Reference [3].

Proposition 6.1

(i) The function $\check{v}_1(w)$ admits the analytic continuation from $\check{V}_1^+$ to $\check{V}_\Sigma$, and the function $\check{v}_2(w)$ admits the analytic continuation from $\check{V}_2^+$ to $\check{V}_\Sigma$.
(ii) For the analytic continuations the following connection equation holds:

$$[\check{v}_1(w)]_1 + [\check{v}_2(w)]_2 = 0, \quad w \in \check{V}_\Sigma \tag{69}$$

(iii) The following estimate holds for the analytic continuation:

$$|[\check{v}_l(w)]| \leqslant C_\delta(1 + \mathrm{e}^{|w|})^q, \quad w \in V_{\Sigma,\delta}, \ \ l = 1, 2 \tag{70}$$

for any $\delta \in (0, \Phi/2 + \pi/2)$, with a $q \in \mathbb{R}$ depending on the parameter ε from (66), (67).

7. AUTOMORPHISMS AND DIFFERENCE EQUATION

In this section, we reduce the connection equation (69) to a difference equation. In the next section, we will construct a meromorphic and then an analytic solution to the difference equation.

The functions $\check{v}_l^0(w)$ are meromorphic in $\check{V} \cong \mathbb{C}$ by (64). Therefore, Proposition 6.1 implies that the functions $\check{v}_l^1(w)$ are meromorphic in $\check{V}_\Sigma$ for $l = 1, 2$. Hence, (69) implies that,

$$\check{v}_1^1(w) + \check{v}_2^1(w) = G(w), \quad w \in \check{V}_\Sigma \tag{71}$$

where the function $G(w)$ is given by

$$G(w) = g(\omega)\left(-\frac{\sinh(w - \mathrm{i}\Phi)}{\sinh w + \mathrm{i}\cos\alpha} - \frac{\sinh(w + 2\mathrm{i}\Phi)}{\sinh(w + \mathrm{i}\Phi) + \mathrm{i}\cos(\alpha + \Phi)}\right), \quad w \in \mathbb{C} \tag{72}$$

according to (64).

Definition 7.1
An automorphism $\check{h}_l : \check{V} \to \check{V}$ for $l = 1$ resp. $l = 2$ is the reflection in the point $\mathrm{i}\pi/2$ resp. $\mathrm{i}\pi/2 - \mathrm{i}\Phi$:

$$\left.\begin{aligned} \check{h}_1 w &= -w + \mathrm{i}\pi \\ \check{h}_2 w &= -w + \mathrm{i}\pi - 2\mathrm{i}\Phi \end{aligned}\right| \quad w \in \mathbb{C} \tag{73}$$

Let us note that for $l = 1, 2$ the automorphism $\check{h}_l$ does not change the projection $w \mapsto z_l(w)$:

$$z_l(\check{h}_l w) = z_l(w), \quad w \in \mathbb{C} \tag{74}$$

Therefore, $\check{h}_l \check{V}_l^+ = \check{V}_l^+$, and the functions $\check{v}_l^\beta$ are invariant with respect to the automorphism $\check{h}_l$:

$$\check{v}_l^\beta(\check{h}_l w) = \check{v}_l^\beta(w), \quad w \in \check{V}_l^+, \ \ \beta = 0, 1 \tag{75}$$

We are going to extend the results [3] (cf. also References [1–4,13,32]) from the case $\Phi = 3\pi/2$ to an arbitrary $\Phi > \Pi$. Let us denote

$$G_2(w) = G(w) - G(\hat{h}_2 w), \quad w \in \mathbb{C} \tag{76}$$

Theorem 7.2
Let $\check{v}_0(x)$ be a solution to (33), where $d_0(\omega, x)$ is defined by (34). Then $\check{v}_1^1(w) \in H(\check{V}_1^+)$, and

(i) $\check{v}_1^1$ admits a meromorphic continuation from $\check{V}_1^+$ onto the entire complex plane $\mathbb{C}$,
(ii) $\check{v}_1^1$ satisfies the following difference and invariance equations:

$$\check{v}_1^1(w) - \check{v}_1^1(w + 2\mathrm{i}\Phi) = G_2(w), \quad w \in \mathbb{C} \tag{77}$$

$$\check{v}_1^1(-w + \mathrm{i}\pi) = \check{v}_1^1(w), \quad w \in \mathbb{C} \tag{78}$$

(iii) The following estimate holds:

$$|\check{v}_1^1(w)| \leqslant \mathrm{e}^{p|\mathrm{Re}\, w|}, \quad |w| \geqslant 1 \tag{79}$$

with a $p \in \mathbb{R}$.

Proof

Step 1: *Analytic continuation*. Proposition 6.1 implies that the function $\check{v}_1(w) := \check{v}_1^1(w) + \omega \sinh(w - \mathrm{i}\Phi)\check{v}_1^0(w)$ is analytic in $\check{V}_\Sigma$. On the other hand, the function $\check{v}_1^0$ is meromorphic in $\mathbb{C}$ by (64), hence $\check{v}_1^1$ is also meromorphic in $\check{V}_\Sigma$. Let us consider the region $\check{V}_\Sigma \cup \check{V}_\Sigma^{\check{h}_1}$, where $\check{V}_\Sigma^{\check{h}_1}$ is the image of the region $\check{V}_\Sigma$ under the transform $\check{h}_1$ (see Figure 3).

From (60) and (73) it follows that $\check{V}_\Sigma \cup \check{V}_\Sigma^{\check{h}_1}$ is a curvilinear strip bounded by $\check{\Gamma}_2^+$ and $\check{h}_1 \check{\Gamma}_2^+ = \check{\Gamma}_2^+ + \mathrm{i}\pi + 2\mathrm{i}\Phi$.

The invariance of $\check{v}_1^1$ under $\check{h}_1$ in $\check{V}_1^+$ implies that the function $\check{v}_1^1(w)$ admits the meromorphic continuation to the region $\check{V}_\Sigma \cup \check{V}_\Sigma^{\check{h}_1}$. Now identity (71) implies that the function $\check{v}_2^1$ also admits the meromorphic continuation to $\check{V}_\Sigma \cup \check{V}_\Sigma^{\check{h}_1}$ since G is meromorphic everywhere by (72).

We can proceed by induction: the function $\check{v}_2^1$ admits the meromorphic continuation to the region $V_\Sigma \cup V_\Sigma^{h_1} \cup (V_\Sigma \cup V_\Sigma^{h_1})^{h_2}$, and the function $\check{v}_1^1$ admits the meromorphic continuation to the same region. The induction implies that the functions $\check{v}_l^1$ are meromorphic in $\mathbb{C}$. Therefore, the connection equation (71) holds for the meromorphic continuations everywhere in the complex plane since the RHS is meromorphic everywhere by (72):

$$\check{v}_1^1(w) + \check{v}_2^1(w) = G(w), \quad w \in \mathbb{C} \tag{80}$$

Furthermore, for $l = 1, 2$ the invariance conditions (75) also hold everywhere:

$$\check{v}_l^1(\check{h}_l(w)) = \check{v}_l^1(w), \quad w \in \mathbb{C} \tag{81}$$

Hence, (78) follows by (73).

Step 2: *Difference equation*. Applying the automorphism $\check{h}_2$ to identity (80), we get,

$$\check{v}_1^1(\check{h}_2 w) + \check{v}_2^1(\check{h}_2 w) = G(\omega, \check{h}_2 w), \quad w \in \mathbb{C} \tag{82}$$

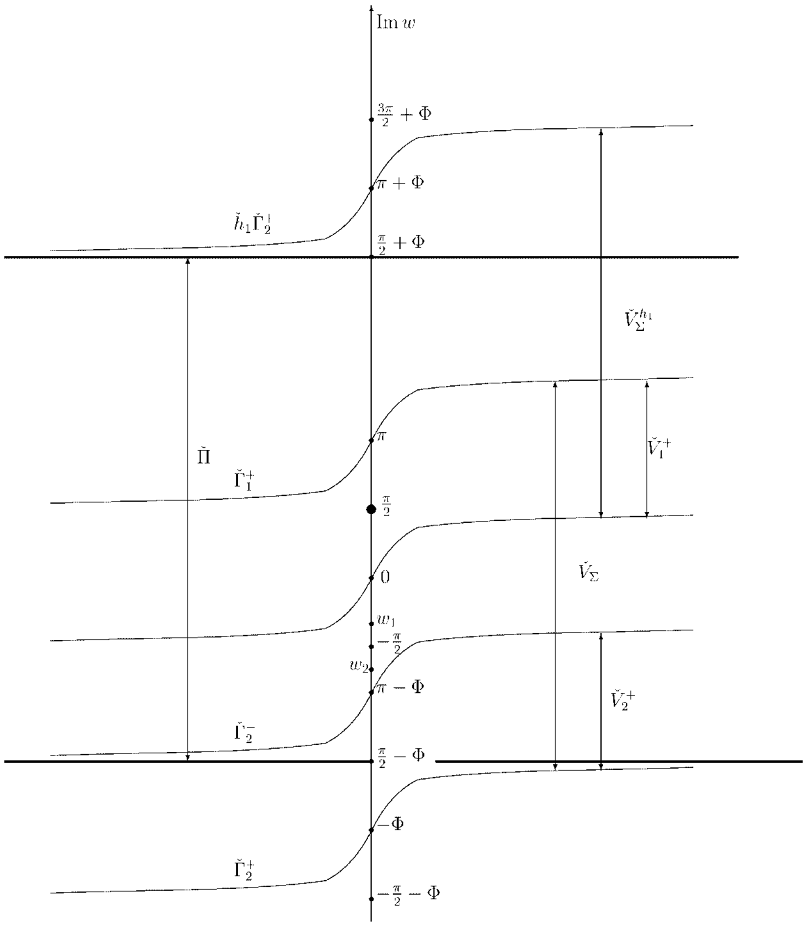

Figure 3. Analytic continuation.

However, $\check{v}_2^1(\check{h}_2 w)=\check{v}_2^1(w)$ by (81) with $l=2$. Hence, subtracting identity (82) from (80), we obtain the difference equation,

$$\check{v}_1^1(w)-\check{v}_1^1(\check{h}_2 w)=G_2(w), \quad w\in\mathbb{C} \tag{83}$$

Finally, $v_1^1(\check{h}_2 w)=v_1^1(\check{h}_1(\check{h}_2 w))$ by (81) with $l=1$. Hence, (77) follows, since $\check{h}_1(\check{h}_2 w)=-\check{h}_2 w+\pi\mathrm{i}=w+2\mathrm{i}\Phi$ by (73). Let us note that (72) and (76) imply the following representation for the function G_2:

$$G_2(w)=g(\omega)\left[-\frac{\sinh(w-\mathrm{i}\Phi)}{\sinh w+\mathrm{i}\cos\alpha}+\frac{\sinh(w+3\mathrm{i}\Phi)}{\sinh(w+2\mathrm{i}\Phi)+\mathrm{i}\cos\alpha}\right.$$
$$\left.-\frac{\sinh(w+2\mathrm{i}\Phi)-\sinh w}{\sinh(w+\mathrm{i}\Phi)+\mathrm{i}\cos(\alpha+\Phi)}\right] \tag{84}$$

Step 3: *Exponential estimates.* Equation (65) implies that the function $\check{v}_1^0(w)$ satisfies estimate (79). Let us take a $\delta<\pi/2$. Then (70) and (68) for $\check{v}_1$ imply that $\check{v}_1^1$ satisfies estimate (79) in $\check{V}_{\Sigma,\delta}\cap\{\mathrm{Re}\, w\geqslant 1\}$. Invariance (78) of the function $\check{v}_1^1(w)$ implies that the same estimate holds in the region $(\check{V}_{\Sigma,\delta}\cup\check{V}_{\Sigma,\delta})\cap\{\mathrm{Re}\, w\geqslant 1\}$. This region lies between $\gamma(-\Phi+\delta)$ and $\gamma(\pi+\Phi-\delta)$ and has a width exceeding 2Φ since $\delta<\pi/2$. Now the difference equation (77) and estimate (79) for G_2 from (84) imply the same estimate for $\check{v}_1^1(w)$ for all w with $|\mathrm{Re}\, w|\geqslant 1$. □

8. SOLUTION OF THE DIFFERENCE EQUATION

8.1. Meromorphic solution

First, let us construct a meromorphic in $\mathbb{C}$ solution of (77), (78) decreasing as $\mathrm{e}^{-|\mathrm{Re}\, w|}$ for $\mathrm{Re}\, w\to\infty$. Denote

$$w_{1,n}=-\mathrm{i}\pi/2+\mathrm{i}\alpha+2\mathrm{i}\pi n,\quad w_{2,n}=-\mathrm{i}\pi/2-\mathrm{i}\alpha+2\mathrm{i}\pi n,\ \ n\in\mathbb{Z} \tag{85}$$

Let us define the function

$$T_1(w)=\mathrm{i}g(\omega)\,\frac{\cos(\alpha+\Phi)}{2\sin\alpha}\left[\coth\left(\frac{w-w_{1,0}}{2}\right)-\coth\left(\frac{w-w_{2,0}}{2}\right)\right],\quad w\in\mathbb{C} \tag{86}$$

Obviously, $T_1(w)$ is a meromorphic function in $\mathbb{C}$ with poles at $w_{1,n}$ and $w_{2,n}$ and residues

$$\mathrm{res}_{w_{1,n}}T_1=\mathrm{i}g(\omega)\,\frac{\cos(\alpha+\Phi)}{\sin\alpha},\quad \mathrm{res}_{w_{2,n}}T_1=-\mathrm{i}g(\omega)\,\frac{\cos(\alpha+\Phi)}{\sin\alpha} \tag{87}$$

Lemma 8.1

(i) The function T_1 is analytic in $\hat{V}_1^+$.
(ii) It is a solution to (77) and (78).
(iii) The estimate holds

$$|T_1(w)|\leqslant C|g(\omega)|\mathrm{e}^{-|\mathrm{Re}\, w|},\quad w\in\hat{V}_1^+ \tag{88}$$

Proof
The poles $w_{1,n}$ and $w_{2,n}$ do not belong to $\check{V}_1^+$, since $0<\alpha<\pi/2$ by Remark 1.1. Identity (78) follows from (86), (85) directly. Estimate (88) also follows from (86). □

It remains to prove the difference equation (77). It is easy to check that the function

$$T_1(w)-T_1(w+2\mathrm{i}\Phi)-G_2(w) \tag{89}$$

is analytic for all $w\in\mathbb{C}$, periodic with the period $2\pi\mathrm{i}$, and bounded by $C\mathrm{e}^{-|w|}$. Hence the function is identically zero.

8.2. Analytic solution

By Lemma 8.1, the function $\check{v}_1^1(w)=T_1(w)$ is a particular meromorphic solution to the inhomogeneous difference equation (77). However, the corresponding function $\check{v}_1(w)$ from (68) is not analytic in $\check{V}_\Sigma$ that does not correspond to Proposition 6.1. Hence we have to construct a general solution to (77), (78). The general solution admits the representation,

$$\check{v}_1^1(w) = T_1(w) + T_0(w) \tag{90}$$

where $T_0(w)$ is a solution to the corresponding homogeneous equations. Let us check that

$$T_0(w) = \frac{\mathrm{i}\pi \sin\Phi}{\Phi}\, g(\omega) H(w, \alpha, \Phi) \tag{91}$$

where

$$H(w, \alpha, \Phi) = \coth \frac{\pi}{2\Phi}(w - w_{1,0}) - \coth \frac{\pi}{2\Phi}(w - w_{2,0} - 2\mathrm{i}\pi) \tag{92}$$

First, T_0 is a solution to the homogeneous equations corresponding to (77), (78). In the following lemma we also check that the corresponding function $\check{v}_1(w)$ from (68) is analytic in V_Σ. The poles of the function T_0 are

$$\tau_{1,k} = w_{1,0} + 2\mathrm{i}\Phi k, \quad \tau_{2,k} = w_{2,0} + 2\mathrm{i}\pi + 2\mathrm{i}\Phi k, \;\; k \in \mathbb{Z} \tag{93}$$

and the corresponding residues are

$$\mathrm{res}_{\tau_{1,k}} T_0(\omega, w) = 2\mathrm{i}g(w)\sin\Phi, \quad \mathrm{res}_{\tau_{2,k}} T_0(\omega, w) = -2\mathrm{i}g(w)\sin\Phi \tag{94}$$

Obviously, function (91) satisfies the estimate

$$|T_0(w)| \leqslant C\mathrm{e}^{-(\pi/\Phi)|\mathrm{Re}\, w|}, \quad |\mathrm{Re}\, w| \geqslant 1 \tag{95}$$

Lemma 8.2

(i) $\check{v}_1^1$ is a solution to (77), (78),
(ii) $\check{v}_1^1$ is meromorphic in $\mathbb{C}$ and analytic in $\check{V}_1^+$,
(iii) $\check{v}_1^1$ satisfies the estimate,

$$|\check{v}_1^1(w)| \leqslant C\mathrm{e}^{-(\pi/\Phi)|\mathrm{Re}\, w|}, \quad |\mathrm{Re}\, w| \geqslant 1 \tag{96}$$

(iv) The function $\check{v}_1(w)$ from (68) is analytic in a neighbourhood of $\overline{\check{V}_\Sigma}$.

Proof

(i) $\check{v}_1^1(w)$ satisfies (77), (78) by (90) and Lemma 8.1.
(ii) $\check{v}_1^1$ is meromorphic in $\mathbb{C}$ by (90), since T_1 and T_0 are meromorphic. The definition of $\hat{V}_1^+$, the conditions

$$\Phi > 3/2\pi \quad \text{and} \quad 0 < \alpha < \pi/2 \tag{97}$$

and (93), imply that the poles $\tau_{1,k}$ and $\tau_{2,k}$ of $T_0(\omega, w)$ do not belong to $\check{V}_1^+$ for all $k \in \mathbb{Z}$, since $\omega \in \mathbb{C}^+$. Hence, the analyticity of $\check{v}_1^1$ in $\check{V}_1^+$ follows from (90) and Lemma 8.1.
(iii) Estimate (96) follows from (90) by estimates (88) for T_1 and (95) for T_0.
(iv) Equation (64) implies that $w_{1,n}$ and $w_{2,n}$ are the poles of $\check{v}_1^0(w)$, and only $w_{1,0}, w_{2,0}$ belong to $\overline{\check{V}_\Sigma}$.

Moreover, $w_{1,0}, w_{2,0}$ belong to $\check{V}_\Sigma$ (see Figure 3), and

$$\begin{aligned}\operatorname{res}_{w_{1,0}} \omega \sinh(w - \mathrm{i}\Phi)\check{v}_1^0(w) &= -\mathrm{i}g(\omega)\,\frac{\cos(\alpha-\Phi)}{\sin\alpha}\\ \operatorname{res}_{w_{2,0}} \omega \sinh(w - \mathrm{i}\Phi)\check{v}_1^0(w) &= \mathrm{i}g(\omega)\,\frac{\cos(\alpha+\Phi)}{\sin\alpha}\end{aligned} \tag{98}$$

Similarly, the function T_1 from (86) has only the poles $w_{1,0}, w_{2,0}$ in $\check{V}_\Sigma$ with the residues (87). At last, (93) implies that the set of poles of T_0, in $\check{V}_\Sigma$ consists of only the point $\tau_{1,0}$ defined by (93), with residue (94). Therefore, the analyticity of $\check{v}_1(w)$ in $\overline{\check{V}_\Sigma}$ follows from (90), (98) and (68). $\square$

8.3. Uniqueness

We prove Theorem 2.2(i).

Proposition 8.3
Let us consider a function $\check{v}_1^1(w)$ which satisfies all statements (i), (ii), (iv) of Lemma 8.2, estimates (66) and an estimate of type (96),

$$|\check{v}_1^1(w)| \leqslant C\mathrm{e}^{p|\operatorname{Re} w|}, \quad |\operatorname{Re} w| \geqslant 1 \tag{99}$$

with a $p \in \mathbb{R}$. Then the function is unique and given by (90).

Proof
Let $\check{v}^*(w)$ be another function with the same characteristics. Consider the difference

$$D(w) := \check{v}_1^1(w) - \check{v}^*(w), \quad w \in \mathbb{C} \tag{100}$$

It is meromorphic in $\mathbb{C}$ by the statement (ii) and periodic with period $2\mathrm{i}\Phi$ by the difference equation (77) for $\check{v}_1^1(w)$ and $\check{v}^*(w)$. Furthermore, $D(w)$ is analytic in $\check{V}_\Sigma$ by (68) and the statement (iv) holds. Hence, $D(w)$ is also analytic in $V_\Sigma \cup V_\Sigma^{h_1}$ by invariance (78) for $\check{v}_1^1(w)$ and $\check{v}^*(w)$. On the other hand, the strip $\check{\Pi} := \{w \in \mathbb{C}\colon \pi/2 - \Phi \leqslant \operatorname{Im} w \leqslant \pi/2 + \Phi\}$ lies in $\check{V}_\Sigma \cup \check{V}_\Sigma^{\check{h}_1}$, since $\omega \in \mathbb{C}^+$ (see Figure 3). Hence, the function $D(w)$ is analytic in $\check{\Pi}$ and everywhere in $\mathbb{C}$ by the periodicity.

Moreover, (78) implies that $D(-w + \pi\mathrm{i}) = D(w)$, $w \in \mathbb{C}$. Thus, the analytic function $D(w)$ in $\mathbb{C}$ is invariant with respect to the group G generated by translations in the period $2\mathrm{i}\Phi$ and the symmetry in the point $\pi\mathrm{i}/2$. In other words, D is an analytic function in the factorspace $F := \mathbb{C}/G$. Let us construct an analytic transformation of the factorspace F to the extended complex plane $\mathbb{C}^*$. An elementary calculation gives (cf. References [4,12,13]),

$$S(w) \equiv \coth^2 \frac{\pi}{2\Phi}(w - \pi\mathrm{i}/2) \tag{101}$$

The transformation is single-valued in the factorspace F, since it is even and periodic with the period $2\mathrm{i}\Phi$. The rays $\{w : w = \pi\mathrm{i}/2 \pm \Phi + w_1,\ \ w_1 \geqslant 0\}$ are transformed to the interval $\{s : 0 \leqslant s < 1\}$. The intervals $\{w : w = \mathrm{i}w_2,\ \pi/2 - \Phi \leqslant w_2 \leqslant \pi/2\}$ and $\{w : w = \mathrm{i}w_2,\ \pi/2 \leqslant w_2 \leqslant \pi/2 + \mathrm{i}\Phi\}$ are transformed to $\{s : -\infty \leqslant s \leqslant 0\}$. Therefore, the map $w \mapsto S(w)$ is an analytic isomorphism $S : F \to \mathbb{C}^* \setminus \{1\}$ and there exists an inverse analytic isomorphism $W : \mathbb{C}^* \setminus \{1\} \to F$.

Then the complex function $D(W(s))$ is analytic in $\mathbb{C}^* \setminus \{1\}$ and possibly has only one singular point at $s=1$.

Let us check that this singular point is a pole. Namely, the function $\check{v}^*$ satisfies an estimate (99), and $\check{v}_1^1$ satisfies the similar estimate (96) corresponding to $p=-\pi/\Phi$. Hence, $D(w)$ also satisfies the similar estimate. On the other hand, (101) implies that

$$S(w) - 1 = \frac{1}{\sinh^2 \alpha(w)}, \quad \alpha(w) = \frac{\pi}{2\Phi}(w - \pi i/2) \tag{102}$$

Hence, $\operatorname{Re} w \geqslant 1$ for $|S(w)-1| \leqslant r$ for a small $r>0$. Therefore, the estimate of type (99) for $D(w)$ implies the following power estimate for $D(W(s))$ in a certain neighbourhood of the point $s=1$:

$$|D(W(s))| \leqslant C|s-1|^{p_1}, \quad |s-1| \leqslant r \tag{103}$$

Therefore the point s = 1 is the pole of the function $D(W(s))$. Let us check that this point is a regular one, and $D(W(1))=0$. For this purpose we use estimate (66). Namely, the points $w=\pi i/2 + w_1$, with $w_1 \geqslant 0$, belong to $\check{V}_1^+$ (see Figure 2). The function $D(w)$ satisfies the estimate of type (66) as well as both functions $\check{v}_1^1$ and $\check{v}^*$. Hence,

$$|D(\pi i/2 + w_1)| \leqslant C \frac{(1+\omega_2 \cosh w_1)^{\varepsilon}}{\omega_2 \cosh w_1}, \quad w_1 \geqslant 0$$

and therefore,

$$D(\pi i/2 + w_1) = \mathcal{O}(e^{-(1-\varepsilon)w_1}), \quad w_1 \to +\infty \tag{104}$$

At last, (102) implies that for $w_1 > 0$,

$$S(\pi i/2 + w_1) - 1 = \frac{1}{\sinh^2(\pi w_1/2\Phi)} \sim e^{-\pi w_1/\Phi}, \quad w_1 \to +\infty \tag{105}$$

Hence, $S(\pi i/2 + w_1) > 1$ and $S(\pi i/2 + w_1) \to 1$, $w_1 \to +\infty$. Therefore, (104) and (105) imply that $D(W(s)) = \mathcal{O}((1-s)^{(1-\varepsilon)\Phi/\pi}) \to 0$, $s \to 1+$. Finally, $D(W(s))$ is analytic at $s=1$ and $D(W(s)) \equiv 0$ by the Liouville Theorem. The proposition is proved. □

Corollary 8.4
A solution to problem (4), (5) in the class $\mathscr{E}_{\varepsilon,N}$ with a positive $\varepsilon<1$ is unique.

Proof
By Lemma 3.2, it suffices to prove the uniqueness of a solution $u_s \in E_\varepsilon$ to problem (23), or equivalently, the uniqueness of the corresponding solution $v(x)$ to problem (27). Proposition 8.3 demonstrates that it suffices to prove that estimate (99) holds for a certain p. Finally, (99) follows from estimate (79) in Theorem 7.2. □

This corollary proves the statement (i) of Theorem 2.2. In the following sections we prove the statement (ii).

9. THE SCATTERED WAVE

We state a representation of the Sommerfeld type for the solution to the stationary problem (23) and analyse a class of contours of integration. For $y=(y_1,y_2)\in\mathbb{R}^2$ we introduce the polar co-ordinates by $y=(\rho\cos\theta,\rho\sin\theta)$, $\phi\leqslant\theta\leqslant 2\pi$, $\rho>0$. Let us define the contours $\Gamma(\theta)$,

$$\Gamma(\theta)=\begin{cases}\gamma(\phi)\cup\gamma(\phi-2\pi), & \phi<\theta\leqslant\pi\\ \gamma(\pi)\cup\gamma(-\pi), & \pi<\theta<2\pi\end{cases} \tag{106}$$

We choose orientations of the contours such that the region $\check{V}_1^-$ remains on the left (see Figure 4).

Now we can state the Sommerfeld type representation for the scattered wave.

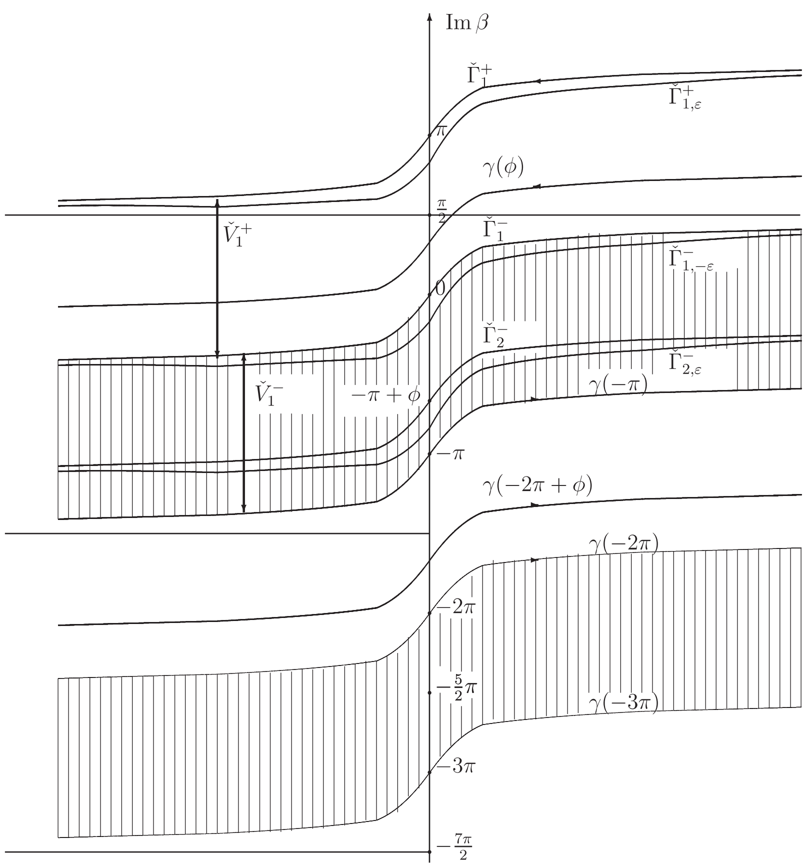

Figure 4. Contours of integration.

Theorem 9.1
If a solution to problem (23), with an $\omega \in \mathbb{C}^+$, exists in the space E_ε with $\varepsilon \in (0,1)$, then it is expressed by the following integral:

$$\hat{u}_s(\omega,\rho,\theta) = -\frac{1}{4\pi \sin \Phi} \int_{\Gamma(\theta)} \mathrm{e}^{-\rho\omega \sinh(w-\mathrm{i}\theta)} \check{v}_1(w)\,\mathrm{d}w, \quad \rho > 0, \;\; \phi < \theta < 2\pi \tag{107}$$

where the function $\check{v}_1$ is given by (68) with the function $\check{v}_1^0$ defined by (64) and $\check{v}_1^1$ defined by (90)–(92), (86).

The complete proof will be given in Sections 10–12. In this section we analyse some properties of the contours. Let us consider a parameter $v \in \mathbb{R}$ s.t.

$$2k\pi < v < \pi + 2k\pi \tag{108}$$

for some $k \in \mathbb{Z}$. Then $\check{\Gamma}_1^- - \mathrm{i}v \subset \check{V}_1^- - 2k\pi\mathrm{i}$ by (61) and (58) (see Figure 2). On the other hand, (58) implies that $\check{\Gamma}_1^- - \mathrm{i}v = \gamma(-v)$. Hence, (108) is equivalent to

$$\gamma(-v) \subset \check{V}_1^- - 2k\pi\mathrm{i} \tag{109}$$

Let us check that the exponent in integrand (107) decays 'superexponentially' on these contours.

Lemma 9.2
Let $\omega := \omega_1 + \mathrm{i}\omega_2 \in \mathbb{C}^+$, and let (108) (or (109)) be satisfied. Then the following estimate holds:

$$|\mathrm{e}^{-\rho\omega \sinh w}| \leqslant \mathrm{e}^{-C\rho \sin v\, \omega_2 \exp|w_1|}, \quad w := w_1 + \mathrm{i}w_2 \in \gamma(-v) \tag{110}$$

where $C > 0$ does not depend on w, ω, v.

Proof
Equation (110) is equivalent to

$$|\mathrm{e}^{-\rho\omega \sinh(w-\mathrm{i}v)}| \leqslant \mathrm{e}^{-C\rho \sin v\, \omega_2 \exp|w_1|}, \quad w := w_1 + \mathrm{i}w_2 \in \check{\Gamma}_1^- \tag{111}$$

On the other hand, (57) implies that the function $z_1 = -\mathrm{i}\omega \sinh w$ is a diffeomorphism of $\check{\Gamma}_1^-$ onto $\mathbb{R}$. Hence, for $w \in \check{\Gamma}_1^-$ we have

$$\begin{aligned}\mathrm{Re}\,(\omega \sinh(w - \mathrm{i}v)) &= \mathrm{Re}\,(\omega \sinh w \cos v - \mathrm{i}\omega \cosh w \sin v)\\ &= \mathrm{Re}\,(\mathrm{i}z_1 \cos v - \mathrm{i}\omega \cosh w \sin v) = \sin v\, \mathrm{Im}\,(\omega \cosh w)\end{aligned} \tag{112}$$

Furthermore, (57) implies that for $w = w_1 + \mathrm{i}w_2 \in \check{\Gamma}_1^-$ we have,

$$\mathrm{Im}\,(\omega \cosh w) = \omega_2 \cosh w_1 \left[\cos w_2 + \left(\frac{\omega_1}{\omega_2} \tanh w_1\right) \sin w_2\right] = \frac{\omega_2 \cosh w_1}{\cos w_2} > 0 \tag{113}$$

since $w_2 \in (-\pi/2, \pi/2)$ and $\omega_2 > 0$. Finally, again by (57), we have for $w \in \check{\Gamma}_1^-$:

$$\frac{\omega_2 \cosh w_1}{\cos w_2} = \cosh w_1 \sqrt{\omega_2^2 + \omega_1^2 \tanh^2 w_1} \tag{114}$$

It remains to note that $\sqrt{\omega_2^2 + \omega_1^2 \tanh^2 w_1} \geqslant \omega_2$. Therefore, $\mathrm{Im}(\omega \cosh w) \geqslant \omega_2 \cosh w_1$ for $w = w_1 + w_2 \in \check{\Gamma}_1^-$. □

Corollary 9.3
Let $\gamma(v_1) \subset \check{V}_1^- + 2k\pi\mathrm{i}$, $k \in \mathbb{Z}$. Then the exponent $\mathrm{e}^{-\rho\omega \sinh w}$ decays superexponentially on $\gamma(v_1)$.

Proof
This follows from (110) with $v = -v_1$, since condition (109) holds for this v. □

Let us introduce two forms of comparison of continuous contours $\Gamma_l := \{\sigma + \mathrm{i}\tau_l(\sigma): \sigma \in \mathbb{R}\}$, $l = 1, 2$, where $\tau_l(\sigma) \in C(\mathbb{R})$.

Definition 9.4

(i) $\Gamma_1 < \Gamma_2$ if $\tau_1(\sigma) < \tau_2(\sigma)$ for $\sigma \in \mathbb{R}$.
(ii) $\Gamma_1 \prec \Gamma_2$ if there exists a $C > 0$ s.t. $\tau_1(\sigma) < \tau_2(\sigma)$ for $|\sigma| \geqslant C$.

Obviously, $\Gamma_1 < \Gamma_2$ implies $\Gamma_1 \prec \Gamma_2$. Moreover, definition (58) implies that $\gamma(v) < \gamma(\mu)$ if $v < \mu$. Now we can strengthen Corollary 9.3.

Corollary 9.5
Let us assume that $\gamma(v_1) \prec \gamma \prec \gamma(v_2)$ and $\gamma(v_1), \gamma(v_1) \subset \check{V}_1^- + 2k\pi\mathrm{i}$, $k \in \mathbb{Z}$ for some $k \in \mathbb{Z}$. Then the function $\mathrm{e}^{-\rho\omega \sinh w}$ decays superexponentially on γ:

$$|\mathrm{e}^{-\rho\omega \sinh w}| \leqslant \mathrm{e}^{-C\rho \exp|w_1|}, \quad w \in \gamma \tag{115}$$

for some $C > 0$.

Proof
By Corollary 9.3, the exponent decays on the contours $\gamma(v)$ with $v_1 \leqslant v \leqslant v_2$. This decay is uniform that follows from the proof of the corollary (or Lemma 9.2). □

For $\varepsilon > 0$ let us denote $\check{\Gamma}_{1,-\varepsilon}^- := \{w \in \check{V}^-: \mathrm{Im}\, z_1(\omega, w) = -\varepsilon\}$, (cf. (56) and Figure 4).

Remark 9.6
We will construct in Appendix A an explicit parametrization for the curve $\check{\Gamma}_{1,-\varepsilon}^-$ with an $\varepsilon < \omega_2$. The parametrization demonstrates that the curve is homotopic to $\check{\Gamma}_1^-$ in the class of contours 'with the ends at infinity'. We will use the homotopy for the deformation of the contours when applying the Cauchy theorem.

Lemma 9.7
Let $\delta > 0$ and $0 < \varepsilon < \omega_2$. Then $\gamma(-\delta) \prec \check{\Gamma}_{1,-\varepsilon}^- < \check{\Gamma}_1^-$.

Corollary 9.8
For any $\theta \in (0, \pi)$, the function $\mathrm{e}^{-\rho\omega \sinh w}$ decays superexponentially on the contour $\check{\Gamma}_{1,-\varepsilon}^- - \mathrm{i}\theta$.

The lemma and the corollary are proved in Appendix A.

10. INVERSE FOURIER TRANSFORM IN SPACE

Let us start the proof of Theorem 9.1. First note that $u_s(y) = v(x)$ by (26). Further, $\mathscr{H}(\omega, z) \neq 0$ for all $z \in \mathbb{R}^2$, $\omega \in \mathbb{C}^+$. Therefore, (51) and (32) imply that

$$v(x) = F^{-1}_{z \mapsto x} \frac{\tilde{d}_0(\omega, z)}{\mathscr{H}(\omega, z)}, \quad x \in K \tag{116}$$

We are going to evaluate the inverse Fourier transform in the complex domain. For $\varepsilon \in \mathbb{R}$ let us denote by $\Gamma(\varepsilon, \varepsilon) = \{z \in \mathbb{C}^2 \colon \operatorname{Im} z_l = \varepsilon, \ l = 1, 2\}$. There exists a positive $\varepsilon_0(\omega)$ such that for $|\varepsilon| < \varepsilon_0(\omega)$ we have

$$|z_1^2 + z_2^2 - 2z_1 z_2 \cos\Phi - \sin^2\Phi\, \omega^2| \geqslant C(\varepsilon, \Phi)(1 + |z|)^2, \quad z \in \Gamma(\varepsilon, \varepsilon) \tag{117}$$

Therefore, (116) implies that for $0 < \varepsilon < \varepsilon_0(\omega)$ we have

$$v(x) = F^{-1}_{z \mapsto x} \left(\left. \frac{\tilde{d}_0(\omega, z)}{\mathscr{H}(\omega, z)} \right|_{\Gamma_{\varepsilon,\varepsilon}} \right), \quad x \in K \tag{118}$$

This follows by the Cauchy Theorem since the distribution $\tilde{d}_0(\omega, z)$ admits an analytic extension to the tube domain $\operatorname{Im} z_l > 0$, $l = 1, 2$ by the Paley–Wiener theorem [32, Theorem I.5.2]. The quotient in (118) is considered as an analytic functional in the sense of [32] with the local density on the plane $\Gamma_{\varepsilon,\varepsilon}$, and $F^{-1}_{z \to x}$ denotes the inverse Fourier transform of analytic functionals. Now (50) implies that $v(x)$ admits the splitting in two summands:

$$v(x) = I_1(x) + I_2(x), \quad x \in K \tag{119}$$

where we denote

$$\left.\begin{aligned} I_1(x) &= F^{-1}_{z \to x} \left(\left. \frac{\tilde{v}_2^1(z_2) - \tilde{v}_2^0(z_2)(\mathrm{i}z_1 - 2\mathrm{i}z_2 \cos\Phi)}{z_1^2 + z_2^2 - 2z_1 z_2 \cos\Phi - \omega^2 \sin^2\Phi} \right|_{\Gamma(\varepsilon,\varepsilon)} \right) \\ I_2(x) &= F^{-1}_{z \to x} \left(\left. \frac{\tilde{v}_1^1(z_1) - \tilde{v}_1^0(z_1)(\mathrm{i}z_2 - 2\mathrm{i}z_1 \cos\Phi)}{z_1^2 + z_2^2 - 2z_1 z_2 \cos\Phi - \omega^2 \sin^2\Phi} \right|_{\Gamma(\varepsilon,\varepsilon)} \right) \end{aligned}\right| \tag{120}$$

We can rewrite each function $I_l(x)$ as follows:

$$\begin{aligned} I_1(x) &= -\frac{(1 - \partial_{x_2}^2)(1 - \partial_{x_1})^2}{(2\pi)^2} \int_{\Gamma(\varepsilon,\varepsilon)} \frac{\mathrm{e}^{-\mathrm{i}z_1 x_1 - \mathrm{i}z_2 x_2} [\tilde{v}_2^1(z_2) - \tilde{v}_2^0(z_2)(\mathrm{i}z_1 - 2\mathrm{i}z_2 \cos\Phi)]\, \mathrm{d}z_1\, \mathrm{d}z_2}{(z_1^2 + z_2^2 - 2z_1 z_2 \cos\Phi - \omega^2 \sin^2\Phi)(1 + z_2^2)(z_1 - \mathrm{i})^2} \\ I_2(x) &= -\frac{(1 - \partial_{x_1}^2)(1 - \partial_{x_2})^2}{(2\pi)^2} \int_{\Gamma(\varepsilon,\varepsilon)} \frac{\mathrm{e}^{-\mathrm{i}z_1 x_1 - \mathrm{i}z_2 x_2} [\tilde{v}_1^1(z_1) - \tilde{v}_1^0(z_1)(\mathrm{i}z_2 - 2\mathrm{i}z_1 \cos\Phi)]\, \mathrm{d}z_1\, \mathrm{d}z_2}{(z_1^2 + z_2^2 - 2z_1 z_2 \cos\Phi - \omega^2 \sin^2\Phi)(1 + z_1^2)(z_2 - \mathrm{i})^2} \end{aligned} \tag{121}$$

Here the integrals converge absolutely by (52), (53) and (117).

Now we are going to reduce the integrals in (121) by the Cauchy theorem using the method [3]. Namely, let us factorize the denominators as follows:

$$\begin{aligned} z_1^2 + z_2^2 - 2z_1z_2\cos\Phi - \omega^2\sin^2\Phi &= (z_1 - z_1^+(z_2))(z_1 - z_1^-(z_2)) \\ &= (z_2 - z_2^+(z_1))(z_2 - z_2^-(z_1)) \end{aligned} \tag{122}$$

Here the roots $z_l^{\pm}$ are given by the following formulas:

$$z_1^{\pm}(z_2) = z_2\cos\Phi \pm \sin\Phi\sqrt{\omega^2 - z_2^2}, \quad z_2^{\pm}(z_1) = z_1\cos\Phi \pm \sin\Phi\sqrt{\omega^2 - z_1^2} \tag{123}$$

where $\operatorname{Im}\sqrt{\omega^2 - z_l^2} \leqslant 0$ for $\operatorname{Im} z_l \geqslant 0$. Since $\sin\Phi < 0$, we have,

$$\left.\begin{aligned} &\operatorname{Im} z_1^+(z_2) \geqslant 0 \quad \text{and} \quad \operatorname{Im} z_1^-(z_2) \leqslant 0 \quad \text{for } \operatorname{Im} z_2 \geqslant 0 \\ &\operatorname{Im} z_2^+(z_1) \geqslant 0 \quad \text{and} \quad \operatorname{Im} z_2^-(z_1) \leqslant 0 \quad \text{for } \operatorname{Im} z_1 \geqslant 0 \end{aligned}\right| \tag{124}$$

Substituting (122) to (121), we get,

$$\begin{aligned} I_1(x) = &-\frac{(1-\partial_{x_2}^2)(1-\partial_{x_1})^2}{(2\pi)^2}\int_{\operatorname{Im} z_2 = \varepsilon} \frac{\mathrm{e}^{-\mathrm{i}z_2x_2}}{1+z_2^2} \\ &\times \left(\int_{\operatorname{Im} z_1 = \varepsilon} \frac{\mathrm{e}^{-\mathrm{i}z_1x_1}[\tilde{v}_2^1(z_2) - \tilde{v}_2^0(z_2)(\mathrm{i}z_1 - 2\mathrm{i}z_2\cos\Phi)]\,\mathrm{d}z_1}{(z_1 - z_1^+(z_2))(z_1 - z_1^-(z_2))(z_1 - \mathrm{i})^2}\right)\mathrm{d}z_2 \end{aligned} \tag{125}$$

$$\begin{aligned} I_2(x) = &-\frac{(1-\partial_{x_1}^2)(1-\partial_{x_2})^2}{(2\pi)^2}\int_{\operatorname{Im} z_1 = \varepsilon} \frac{\mathrm{e}^{-\mathrm{i}z_1x_1}}{1+z_1^2} \\ &\times \left(\int_{\operatorname{Im} z_2 = \varepsilon} \frac{\mathrm{e}^{-\mathrm{i}z_2x_2}[\tilde{v}_1^1(z_1) - \tilde{v}_1^0(z_1)(\mathrm{i}z_2 - 2\mathrm{i}z_1\cos\Phi)]\,\mathrm{d}z_2}{(z_2 - z_2^+(z_1))(z_2 - z_2^-(z_1))(z_2 - \mathrm{i})^2}\right)\mathrm{d}z_1 \end{aligned}$$

11. REDUCTION TO THE RIEMANN SURFACE

Let us apply the Cauchy theorem to the inner integrals in (125). The iterated integrals converge absolutely by (117), (123) and estimates (52), (53). Moreover, for $0 < \varepsilon \leqslant \varepsilon_0(\omega)$, we can close the contour of integration over z_1 in I_1 to the lower complex half-plane for $x_1 \geqslant 0$ and to the upper complex half-plane for $x_1 \leqslant 0$. Similarly, we close the contour of integration over z_2 in the expression for I_2. For example, let us consider $x_1 \geqslant 0$, $x_2 \leqslant 0$. Then we close (i) the contour of integration over z_1 in I_1 to the lower complex half-plane and (ii) the contour of integration over z_2 in I_2 to the upper complex half-plane. Then the inner integrals are given by the residues at the poles $z_1^-(z_2)$ and $z_2^+(z_1)$, respectively. Therefore,

$$I_1(x) = \frac{\mathrm{i}}{2\pi}(1-\partial_{x_2}^2)(1-\partial_{x_1})^2 \times \int_{\mathrm{Im}\, z_2=\varepsilon} \frac{\mathrm{e}^{-\mathrm{i}z_2x_2-\mathrm{i}z_1^-(z_2)x_1}[\tilde{v}_2^1(z_2)-\tilde{v}_2^0(z_2)(\mathrm{i}z_1^-(z_2)-2\mathrm{i}z_2\cos\Phi)]}{(1+z_2^2)(z_1^-(z_2)-z_1^+(z_2))(z_1^-(z_2)-\mathrm{i})^2}\,\mathrm{d}z_2, \quad x_1 \geqslant 0 \tag{126}$$

$$I_2(x) = -\frac{\mathrm{i}}{2\pi}(1-\partial_{x_1}^2)(1-\partial_{x_2})^2 \times \int_{\mathrm{Im}\, z_1=\varepsilon} \frac{\mathrm{e}^{-\mathrm{i}z_1x_1-\mathrm{i}z_2^+(z_1)x_2}[\tilde{v}_1^1(z_1)-\tilde{v}_1^0(z_1)(\mathrm{i}z_2^+(z_1)-2\mathrm{i}z_1\cos\Phi)]}{(1+z_1^2)(z_2^+(z_1)-z_2^-(z_1))(z_2^+(z_1)-\mathrm{i})^2}\,\mathrm{d}z_1, \quad x_2 \leqslant 0$$

Now the key observation is the following: the integrands in (126) are restrictions of integrands (121) to the Riemann surface V. Thus, the integral I_1 resp. I_2 is equal to an integral over the following contours $\Gamma_{2,\varepsilon}^-$ resp. $\Gamma_{1,\varepsilon}^+$:

$$\Gamma_{2,\varepsilon}^- := \{z \in V\colon\ \mathrm{Im}\, z_2 = \varepsilon,\ z_1 = z_1^-(z_2)\}, \quad \Gamma_{1,\varepsilon}^+ := \{z \in V\colon\ \mathrm{Im}\, z_1 = \varepsilon,\ z_2 = z_2^+(z_1)\} \tag{127}$$

lying in $V_2^+ \cap V_1^-$ resp. $V_1^+ \cap V_2^+$. Roughly speaking, (126) are integrals (125) reduced to V.

12. INTEGRATION OVER UNIVERSAL COVERING

Now we are going to rewrite I_l as the integrals in the parameter w on the universal covering of the Riemann surface.

Evaluation of $I_1(x)$ *for* $x_1 \geqslant 0$. Let us identify the contour of integration $\Gamma_{2,\varepsilon}^-$ with the covering contour $\check{\Gamma}_{2,\varepsilon}^-$ lying in $\check{V}_2^+ \cap \check{V}_1^-$. Definition (61) of the regions $\check{V}_1^+$ and $\check{V}_2^+$, implies that

$$\gamma(-\pi) < \check{\Gamma}_{2,\varepsilon} < \gamma(-\pi+\phi) \tag{128}$$

(see Figure 4). Formulas (123) give

$$z_1^-(z_2(w)) = -\mathrm{i}\omega \sinh w, \quad z_1^+(z_2(w)) = -\mathrm{i}\omega\sinh(w+2\mathrm{i}\Phi), \quad w \in \check{\Gamma}_{2,\varepsilon}^- \tag{129}$$

where $z_2(w)$ is defined by (55). It is convenient now to express x_1, x_2 in the polar co-ordinates $(y_1, y_2) = (\rho\cos\theta, \rho\sin\theta)$ (see (25)):

$$x_1 = \frac{\rho\sin(\Phi+\theta)}{\sin\Phi}, \quad x_2 = -\frac{\rho\sin\theta}{\sin\Phi} \tag{130}$$

Let us note that

$$\left.\begin{array}{l} x \in K \quad \text{and} \quad x_1 \geqslant 0 \quad \Leftrightarrow \quad \phi+\pi \leqslant \theta \leqslant 2\pi \\ x \in K \quad \text{and} \quad x_2 \leqslant 0 \quad \Leftrightarrow \quad \pi \leqslant \theta \leqslant 2\pi \end{array}\right| \tag{131}$$

Now (129) implies for $(x_1,x_2)\in K$, with $x_1\geqslant 0$, that

$$-\mathrm{i}z_2(w)x_2-\mathrm{i}z_1^-(z_2(w))x_1=-\omega\rho\sinh(w-\mathrm{i}\theta),\quad \phi+\pi\leqslant\theta\leqslant 2\pi,\ \ w\in\check{\Gamma}_{2,\varepsilon}^-(\omega) \tag{132}$$

From (129) and (55) we get similarly that,

$$\left.\begin{aligned}\mathrm{i}z_1^-(z_2)-2\mathrm{i}z_2\cos\Phi&=-\omega\sinh(w+2\mathrm{i}\Phi)\\ z_1^-(z_2)-z_1^+(z_2)&=-2\omega\sin\Phi\cosh(w+\mathrm{i}\Phi)\\ (z_1^-(z_2)-\mathrm{i})^2&=-(1+\omega\sinh w)^2\\ \mathrm{d}z_2&=-\mathrm{i}\omega\cosh(w+\mathrm{i}\Phi)\,\mathrm{d}w\end{aligned}\right|\quad w\in\check{\Gamma}_{2,\varepsilon}^-(\omega) \tag{133}$$

Substituting expressions (132)–(133) to the first integral (126), we get,

$$\begin{aligned}I_1(\rho,\theta)=&\frac{(1-\partial_{x_2}^2)(1-\partial_{x_1})^2}{4\pi\sin\Phi}\\ &\times\int_{\check{\Gamma}_{2,\varepsilon}^-}\frac{\mathrm{e}^{-\omega\rho\sinh(w-\mathrm{i}\theta)}[\check{v}_2^1(w)+\omega\sinh(w+2\mathrm{i}\Phi)\check{v}_2^0]\,\mathrm{d}w}{(1-\omega^2\sinh^2(w+\mathrm{i}\Phi))(1+\omega\sinh w)^2}\\ &\phi+\pi<\theta<2\pi\end{aligned} \tag{134}$$

A formal differentiation gives

$$\begin{aligned}I_1(\rho,\theta)=&\frac{1}{4\pi\sin\Phi}\int_{\check{\Gamma}_{2,\varepsilon}^-}\mathrm{e}^{-\omega\rho\sinh(w-\mathrm{i}\theta)}[\check{v}_2^1(w)+w\sinh(w+2\mathrm{i}\Phi)\check{v}_2^0(w)]\,\mathrm{d}w,\\ &\phi+\pi<\theta<2\pi\end{aligned} \tag{135}$$

since the denominator cancels obviously by representation (126). Let us justify the formal differentiation by uniform convergence of the integral. First, we rewrite (135) using the connection equation (69). Namely,

$$I_1(\rho,\theta)=-\frac{1}{4\pi\sin\Phi}\int_{\check{\Gamma}_{2,\varepsilon}^-}\mathrm{e}^{-\omega\rho\sinh(w-\mathrm{i}\theta)}\check{v}_1(w)\,\mathrm{d}w \tag{136}$$

since $\check{\Gamma}_{2,\varepsilon}\subset\check{V}_\Sigma$. Now we state the uniform convergence of the integral and extend it analytically to the region $S_0:=\{\rho,\theta)\colon\ \rho>0,\ \ \theta\in(\pi,2\pi)\}$.

Lemma 12.1

(i) Integral (136) converges uniformly for $\theta:\phi+\pi<\theta<2\pi$ and defines a real-analytic function (ρ,θ).
(ii) The function admits an analytic extension to the region S_0 by the following formula:

$$I_1(\rho,\theta)=\frac{1}{4\pi\sin\Phi}\int_{\gamma(-\pi)}\mathrm{e}^{-\omega\rho\sinh(w-\mathrm{i}\theta)}\check{v}_1(w)\,\mathrm{d}w,\quad \pi<\theta<2\pi \tag{137}$$

Proof

(i) Let us choose any $\theta_\pm > 0$ such that

$$\pi + \phi + \theta_- \leqslant \theta \leqslant 2\pi - \theta_+ \tag{138}$$

Then (128) implies that

$$\gamma(-3\pi + \theta_-) \leqslant \check{\Gamma}^-_{2,\varepsilon} - \mathrm{i}\theta \leqslant \gamma(-2\pi - \theta_+) \tag{139}$$

Therefore, the function $\mathrm{e}^{-\omega\rho \sinh(w-\mathrm{i}\theta)}$ decays superexponentially on the contour $\check{\Gamma}^-_{2,\varepsilon}(\omega)$ by Corollary 9.5. Hence, estimates (65), (67) imply the uniform convergence of the integral (135). Its derivatives in ρ, θ also converge uniformly.

(ii) Let us consider $\theta \in (\phi + \pi, 2\pi)$ and $v \in (-\pi, -\pi + \phi)$. Then condition (108) holds with $-v + \theta$ instead of v. Hence, $\mathrm{e}^{-\omega\rho \sinh w}$ decays superexponentially on $\gamma(v - \theta)$ by Lemma 9.2. Therefore, $\mathrm{e}^{-\omega\rho \sinh(w-\mathrm{i}\theta)}$ decays superexponentially on $\gamma(v)$.

Now, by the Cauchy Theorem and Remark 9.6, we can deform the contour $\check{\Gamma}^-_{2,\varepsilon}(\omega)$ to the contour $\gamma(-\pi)$ and obtain representation (137) for $\theta \in (\pi + \phi, 2\pi)$. Finally, we see (by the same lemma) that the integral converges absolutely for $\theta \in (\pi, 2\pi)$, since then $\gamma(-\pi - \theta) \subset \check{V}^-_1 - 2\mathrm{i}\pi$. It is clear that extension (137) is real-analytic in θ. □

Evaluation of $I_2(x)$ for $x_2 \leqslant 0$. Now we identify the contour of integration $\Gamma^+_{1,\varepsilon}$ with the covering contour $\check{\Gamma}^+_{1,\varepsilon}$ lying in $\check{V}^+_1 \cap \check{V}^+_2$. Then (123) and (55) imply

$$z^-_2(z_1) = -\mathrm{i}\omega \sinh(w - \mathrm{i}\Phi), \quad z^+_2(z_1) = -\mathrm{i}\omega \sinh(w + \mathrm{i}\Phi), \quad w \in \check{\Gamma}^+_{1,\varepsilon} \tag{140}$$

Now (131) implies for $(x_1, x_2) \in K$, with $x_2 \leqslant 0$,

$$-\mathrm{i}z_1 x_1 - \mathrm{i}z^+_2(z_1) x_2 = -\omega\rho \sin(w - \mathrm{i}\theta), \quad \pi \leqslant \theta < 2\pi, \quad w \in \check{\Gamma}^+_{1,\varepsilon} \tag{141}$$

From (131) and (55) we get,

$$\left.\begin{aligned} \mathrm{i}z^+_2(z_1) - 2\mathrm{i}\cos\Phi z_1 &= -\omega \sinh(w - \mathrm{i}\Phi) \\ z^+_2(z_1) - z^-_2(z_1) &= 2\omega \sin\Phi \cosh w \\ (z^+_2(z_1) - \mathrm{i})^2 &= -(1 + \omega \sinh(w + \mathrm{i}\Phi))^2 \\ \mathrm{d}z_1 &= -\mathrm{i}\omega \cosh w \, \mathrm{d}w \end{aligned}\right| \quad w \in \check{\Gamma}^+_{1,\varepsilon} \tag{142}$$

similarly to (133). Substituting expressions (141)–(142) into the second integral (126), we get:

$$I_2(\rho, \theta) = \frac{(1 - \partial^2_{x_1})(1 - \partial_{x_2})^2}{4\pi \sin\Phi} \times \int_{\check{\Gamma}^+_{1,\varepsilon}} \frac{\mathrm{e}^{-\omega\rho \sinh(w-\mathrm{i}\theta)}[\check{v}^1_1(w) + \omega \sinh(w - \mathrm{i}\Phi)\check{v}^0_1(w)]}{(1 - \omega^2 \sinh^2 w)(1 + \omega \sinh(w + \mathrm{i}\Phi))^2} \, \mathrm{d}w, \quad \pi < \theta < 2\pi \tag{143}$$

Further we get similar to (135),

$$I_2(\rho,\theta)=\frac{1}{4\pi\sin\Phi}\int_{\check{\Gamma}^+_{1,\varepsilon}} \mathrm{e}^{-\omega\rho\sinh(w-\mathrm{i}\theta)}\check{v}_1(w)\,\mathrm{d}w,\quad \pi<\theta<2\pi \tag{144}$$

From the definition of $\check{\Gamma}^+_{1,\varepsilon}(\omega)$ it follows that $\check{\Gamma}^+_{1,\varepsilon}(\omega)=\check{\Gamma}^-_{1,-\varepsilon}+\pi\mathrm{i}$. Then by Corollary 9.8, the function $\mathrm{e}^{-\omega\rho\sinh(w-\mathrm{i}\theta)}$ decays superexponentially on the contour $\check{\Gamma}^+_{1,\varepsilon}$ for $\theta\in(\pi,2\pi)$. Hence, integral (144) converges uniformly by (65), (66) and (68).

Finally, we can deform the contour $\check{\Gamma}^+_{1,\varepsilon}$ to the contour $\gamma(\pi)$ similar to the evaluation of I_1 above. Then we obtain the following final representation for I_2:

$$I_2(\rho,\theta)=\frac{1}{4\pi\sin\Phi}\int_{\gamma(\pi)} \mathrm{e}^{-\omega\rho\sinh(w-\mathrm{i}\theta)}\check{v}_1(w)\,\mathrm{d}w,\quad \pi<\theta<2\pi \tag{145}$$

Substituting (137) and (145) into (119), we obtain representation (107) for $\theta\in(\pi,2\pi)$. For $\theta\in(\phi,\pi]$, the proof is similar. Let us note that integral (107) converges absolutely for $\theta=\pi$. Indeed, the contours $\gamma(\phi)-\pi\mathrm{i}$ and $\gamma(-2\pi+\phi)-\pi\mathrm{i}$ lie inside $\check{V}^-_1$ and $\check{V}^-_1-2\pi\mathrm{i}$, respectively, and, therefore, the exponent in integral (145) decays superexponentially by Corollary 9.3. Theorem 9.1 is proved. □

Remark 12.2

Let us note that the proof of Theorem 9.1 does not depend on the boundary conditions from (23). Moreover, it is possible to extend the proof from the solutions $\hat{u}_s(\omega,\dot{})\in E_s$ to arbitrary tempered distributions $\hat{u}_s(\omega,\dot{})\in S'(Q)$. Therefore, the representation has a more general character than is indicated in Theorem 9.1: it expresses any solutions to the Helmholtz equation in the angle from the class of tempered distributions, in terms of the Cauchy data on one side of the angle. Of course, the representation is most useful in the cases, when we can determine the Cauchy data, as we have done in Section 8.2.

In the following section we choose the concrete solution (90) of the difference equation (77) and obtain the Sommerfeld representation.

12.1. Sommerfeld representation for scattered wave

In this section, we obtain the final representation of a solution to the stationary problem in the standard form of the Sommerfeld integral. This form is the direct consequence of the representation obtained in the previous section.

Theorem 12.3

If a solution to problem (23) with $\omega\in\mathbb{C}^+$ exists in the space E_ε with $\varepsilon\in(0,1)$, then it is expressed by the Sommerfeld-type integral,

$$\hat{u}_s(\omega,\rho,\theta)=\mathrm{i}\,\frac{g(\omega)}{4\Phi}\int_{\Gamma(\theta)} \mathrm{e}^{-\rho\omega\sinh(w-\mathrm{i}\theta)}H(w,\alpha,\Phi)\,\mathrm{d}w,\quad \phi\leqslant\theta\leqslant 2\pi \tag{146}$$

where $H(w,\alpha,\Phi)$ is given by (15).

Proof
By Theorem 9.1, a solution of problem (23) is expressed by the Sommerfeld-type integral (107), where $\check{v}_1$ is defined by (68). Now we use the explicit form (90), (86), (91) of the function $\check{v}_1^1(w)$ entering into expression (68) for the integrand $\check{v}_1(w)$.

Lemma 8.2(iv) implies that $\check{v}_1$ does not have poles on $\Gamma(\theta)$. Let us substitute the expression (90) for $\check{v}_1(w)$ and use the periodicity of the functions $T_1(w)$ and $\check{v}_1^0$ with the period $2\pi\mathrm{i}$ (see (64)). Then we obtain representation (146) for the solution to problem (23) for $\theta \in (\phi, 2\pi)$ since the contour $\Gamma(\theta)$ is a union of two contours $\gamma(v)$ and $\gamma(v-2\pi)$. It remains only to observe, that this integral is continuous at the ends, $\theta = 0, 2\pi$ since the integrand H decreases exponentially by (92). Theorem 12.3 is proved. □

13. INVERSE FOURIER TRANSFORM IN TIME

In this section we complete the proof of Theorem 2.2. Let us assume that there exists a solution $u(t, y)$ to problem (4), (5) and $u(t, y) \in \mathscr{E}_{\varepsilon,N}$ with an $\varepsilon \in [0, 1)$ and an $N \geqslant 0$. The corresponding scattered wave $u_s(t, y)$ is defined by (18). Its Fourier transform in time is expressed by (146) according to Lemma 3.2 and Theorem 12.3.

Let us apply the Fourier transform in time to (18). First, (1) implies that

$$\hat{u}_{\text{in}}(\omega, y) = g(\omega)\mathrm{e}^{\mathrm{i}\omega n_0 \cdot y}, \quad y \in \bar{Q}, \ \ \omega \in \mathbb{C}^+ \tag{147}$$

where $g(\omega)$ is defined from (22). Therefore, (18) implies that

$$\hat{u}(\omega, y) = \hat{u}_s(\omega, y) + g(\omega)\mathrm{e}^{\mathrm{i}\omega n_0 \cdot y}, \quad y \in \bar{Q}, \ \ \omega \in \mathbb{C}^+ \tag{148}$$

Remark
The function $u(t, y)$ is called the *total field* i.e. the sum of the incident and the scattered wave ([5]).

Let us prove the Sommerfeld-type representation (17) for $\hat{u}(\omega, y)$. It uses the 'two-loop' contour $\mathscr{C}$. We will deduce the representation applying the Cauchy Residue Theorem to integral (146). Let us introduce the following contours:

$$\tilde{\Gamma}(\theta) = \begin{cases} \Gamma'(\theta) \cup \Gamma_+ \cup \Gamma_-, & \pi < \theta < 2\pi \\ (\Gamma'(\theta) \cup \Gamma_+ \cup \Gamma_-) - \mathrm{i}(\pi - \phi), & \phi < \theta \leqslant \pi \end{cases} \tag{149}$$

where we denote

$$\Gamma'(\theta) = \{w \in \Gamma(\theta) \colon |\mathrm{Re}\, w| \geqslant 1\}, \ \Gamma_\pm = \{w \,|\, \mathrm{Re}\, w = \pm 1, \ \gamma(-\pi) \leqslant w \leqslant \check{\Gamma}_1^+\}$$

and the corresponding orientations are shown at Figure 5. (For a better visualization the vertical lines $\Gamma_\pm$ and $\Gamma_\pm - \mathrm{i}(\pi - \phi)$ are somewhat moved apart at this figure.)

Lemma 13.1
Function (148) can be represented in the form,

$$\hat{u}(\omega, \rho, \theta) = \frac{\mathrm{i}g(\omega)}{4\Phi} \int_{\mathscr{C}} \mathrm{e}^{-\omega\rho \sinh \beta} H(\beta + \mathrm{i}\theta, \alpha, \Phi)\, \mathrm{d}\beta, \ \ \rho \geqslant 0, \ \ \phi \leqslant \theta \leqslant 2\pi, \ \ \omega \in \mathbb{C}^+ \tag{150}$$

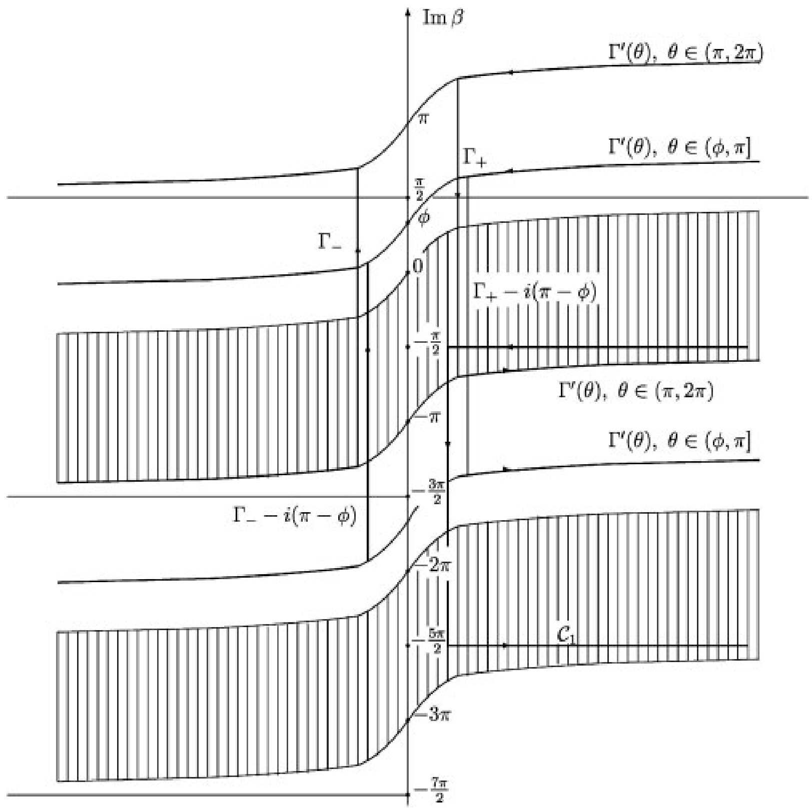

Figure 5. Sommerfeld type contours.

Proof
Formula (15) and (85), (93) imply that the function $H(\cdot, \alpha, \Phi)$ has a unique pole in $\check{V}_\Sigma$ (see (60)) at the point $w_{1,0} = -i\pi/2 + i\alpha$ (see (85)), and

$$\operatorname{res}_{w_{1,0}} H(w, \alpha, \Phi) = 2\Phi/\pi \tag{151}$$

Let us define the contours $\Gamma''(\theta) = \{w \in \Gamma(\theta)\colon |\operatorname{Re} w| \leqslant 1\}$ for $\theta \in (\pi, 2\pi)$, and represent $\Gamma(\theta)$ in the form,

$$\Gamma(\theta) = \begin{cases} \tilde{\Gamma}(\theta) + [\Gamma''(\theta) \cup (-\Gamma_+) \cup (-\Gamma_-)], & \pi < \theta < 2\pi \\ [\tilde{\Gamma}(\theta) + (\Gamma''(\theta) \cup (-\Gamma_+) \cup (-\Gamma_-)] - i(\Phi - \pi), & \phi < \theta \leqslant \pi \end{cases} \tag{152}$$

where sign '+' means an 'algebraic' summation of the contours.

Applying the Cauchy residues theorem to (146) and using (151) we obtain an intermediate representation for the function $\hat{u}(\omega,\rho,\theta)$:

$$\hat{u}(\omega,\rho,\theta)=\frac{\mathrm{i}g(\omega)}{4\Phi}\int_{\tilde{\Gamma}(\theta)}\mathrm{e}^{-\omega\rho\sinh(w-\mathrm{i}\theta)}H(w,\alpha,\Phi)\,\mathrm{d}w,\quad \rho\geqslant 0,\ \phi\leqslant\theta\leqslant 2\pi,\ w\in\mathbb{C}^+ \tag{153}$$

It remains to replace the contours $\tilde{\Gamma}(\theta)$, depending from θ, by the two loop-contour $\mathscr{C}$ which does not depend on θ. For this purpose, we change the variable in integral (153) by

$$\beta=w-\mathrm{i}\theta \tag{154}$$

Then the contour $\tilde{\Gamma}(\theta)$ transforms to the contour $\tilde{\Gamma}(\theta)-\mathrm{i}\theta$. Equation (92) implies that the function $H(\cdot+\mathrm{i}\theta,\alpha,\Phi)$ decays exponentially on the contour $\tilde{\Gamma}(\theta)-\mathrm{i}\theta$ for $\omega\in\overline{\mathbb{C}^+}$. Therefore, we can deform the contour $\tilde{\Gamma}(\theta)-\mathrm{i}\theta$, in integral (153), to the contour $\mathscr{C}$. Namely, the contour lies in the region, where the function $\mathrm{e}^{-\omega\rho\sinh\beta}$ is bounded with respect to β for all $\omega\in\overline{\mathbb{C}^+}$. Indeed, the lines $\{w_1-\mathrm{i}\pi/2\,|\,w_1\in\mathbb{R}\}$ and $\{w_1-5/2\mathrm{i}\pi\,|\,w_1\in\mathbb{R}\}$ lie in the region $\overline{\check{V}_1^-}\cup(\overline{\check{V}_1^-}-2\mathrm{i}\pi)$ for all $\omega\in\overline{\mathbb{C}^+}$ (see Figure 5). In this region, the function $\mathrm{e}^{-\rho\omega\sinh w}$ is bounded for all $\omega\in\overline{\mathbb{C}^+}$ by Corollary 9.5. Hence we can use the Cauchy theorem and prove representation (150) and Lemma 13.1. □

Now representation (17) for the solution $u(t,y)$ follows from (150). This completes the proof of Theorem 2.2. □

APPENDIX A

We prove Lemma 9.7. Identity (57) implies that, for $w_1+\mathrm{i}w_2\in\check{\Gamma}_1^-$, we have

$$\sin w_2=\frac{\omega_1\sinh w_1}{\sqrt{\omega_2^2\cosh^2 w_1+\omega_1^2\sinh^2 w_1}},\quad w_1\in\mathbb{R} \tag{A1}$$

Similarly, for $w_\varepsilon:=w_1+\mathrm{i}w_{2,\varepsilon}\in\check{\Gamma}_{1,-\varepsilon}^-$, with $0<\varepsilon<\omega_2$, we have by definition, $\operatorname{Im}z_1(w_\varepsilon)=\operatorname{Im}(-\mathrm{i}\omega\sinh w_\varepsilon)=-\varepsilon$, or

$$\sin w_{2,\varepsilon}=\frac{-\varepsilon\omega_2\cosh w_1}{\omega_2^2\cosh^2 w_1+\omega_1^2\sinh^2 w_1}+\frac{\omega_1\sinh w_1\sqrt{\omega_2^2\cosh^2 w_1+\omega_1^2\sinh^2 w_1-\varepsilon^2}}{\omega_2^2\cosh^2 w_1+\omega_1^2\sinh^2 w_1},\quad w_1\in\mathbb{R} \tag{A2}$$

By definition of $\check{\Gamma}_1^-$, $w_2\in(-\pi/2,\pi/2)$. The following lemma implies that Equation (A2) has a unique real solution $w_{2,\varepsilon}$ in the interval $(-\pi/2,\pi/2)$.

Lemma A.1
Let us consider $\omega = \omega_1 + i\omega_2 \in \mathbb{C}^+$ with $\omega_{1,2} \in \mathbb{R}$ and $\omega_2 > 0$. Then for any $\varepsilon \in [0, \omega_2)$

$$\left| \frac{-\varepsilon\omega_2 \cosh w_1}{\omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1} + \frac{\omega_1 \sinh w_1 \sqrt{\omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1 - \varepsilon^2}}{\omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1} \right| \leqslant 1, \quad w_1 \in \mathbb{R} \quad \text{(A3)}$$

Proof
Let $\omega_1 \geqslant 0$. The case $\omega_1 \leqslant 0$ is analysed similarly. It suffices to prove that for $w_1 \geqslant 0$

$$\frac{\varepsilon\omega_2 \cosh w_1}{\omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1} + \frac{\omega_1 \sinh w_1 \sqrt{\omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1 - \varepsilon^2}}{\omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1} \leqslant 1 \qquad \text{(A4)}$$

Step (i): First, (A4) holds obviously for $\varepsilon = 0$.
Step (ii): Let us prove that (A4) holds for $\varepsilon = \omega_2$, i.e.

$$\frac{\omega_2^2 \cosh w_1}{\omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1} + \frac{\omega_1 \sinh w_1 \sqrt{\omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1 - \omega_2^2}}{\omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1} \leqslant 1 \qquad \text{(A5)}$$

Multiplying inequality (A5) by the denominator and substituting $\sinh^2 w_1 = \cosh^2 w_1 - 1$ we obtain the equivalent inequality:

$$\omega_2^2(\cosh w_1 - 1) \leqslant (\cosh w_1^2 - 1) \left[(\omega_1^2 + \omega_2^2 - \omega_1 \sqrt{\omega_1^2 + \omega_2^2} \right] \qquad \text{(A6)}$$

Replacing $\cosh w_1 = t$, $\sqrt{\omega_1^2 + \omega_2^2} := a$, $\omega_1 := b$, we rewrite it as follows,

$$(a^2 - b^2)(t - 1) \leqslant a(a - b)(t^2 - 1) \qquad \text{(A7)}$$

It holds since $a \geqslant b$, $t \geqslant 1$.
Step (iii): Let us prove that the function

$$f(\varepsilon) := \varepsilon\omega_2 \cosh w_1 + \omega_1 \sinh w_1 \sqrt{\omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1 - \varepsilon^2} \qquad \text{(A8)}$$

is non-decreasing for $\varepsilon \in [0, \omega_2]$. Namely, differentiating (A8), we get,

$$f'(\varepsilon) = \omega_2 \cosh w_1 - \frac{\varepsilon\omega_1 \sinh w_1}{\sqrt{\omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1 - \varepsilon^2}} \qquad \text{(A9)}$$

The inequality $f'(\varepsilon) \geqslant 0$ is equivalent to

$$\varepsilon\omega_1 \sinh w_1 \leqslant \omega_2 \cosh w_1 \sqrt{\omega_2^2 \cosh w_1^2 + \omega_1^2 \sinh w_1^2 - \varepsilon^2} \qquad \text{(A10)}$$

To prove this, let us note first that

$$\varepsilon\omega_1 \sinh w_1 \leqslant \omega_1\omega_2 \sinh w_1 \qquad \text{(A11)}$$

since $\varepsilon \in [0, \omega_2]$. Second,

$$\omega_2 \cosh w_1 \sinh w_1 \sqrt{\omega_2^2 + \omega_1^2} \leqslant \omega_2 \cosh w_1 \sqrt{\omega_2^2 \cosh w_1^2 + \omega_1^2 \sinh w_1^2 - \varepsilon^2} \tag{A12}$$

that follows if we replace ε by ω_2. Finally (A11) and (A12) imply (A10). □

The lemma implies that Equation (A2) admits a unique solution in the interval $(-\pi/2, \pi/2)$. On the other hand, $w_{2,0} \in (-\pi/2, \pi/2)$, hence $w_{2,\varepsilon} \in (-\pi/2, \pi/2)$ for $w \in \check{\Gamma}_{1,-\varepsilon}^{-}$ by continuity. Finally, a comparison of (A1) and (A2) demonstrates that

$$\sin w_{2,\varepsilon} = \sin w_2 - \Delta(\omega, w_1), \quad w_1 \in \mathbb{R} \tag{A13}$$

where $\Delta(\omega, w_1) \to 0$ as $|w_1| \to \infty$. Let us prove that

$$\Delta(\omega, w_1) > 0 \tag{A14}$$

We have assumed that $\omega_1 \geqslant 0$. By (A1) and (A2), it suffices to prove (A14) for $w_1 \leqslant 0$. In this case (A14) is equivalent to the inequality

$$\frac{\varepsilon\omega_2 \cosh w_1}{a} + \frac{\omega_1 \sinh w_1 \sqrt{a - \varepsilon^2}}{a} - \frac{\omega_1 \sinh w_1}{\sqrt{a}} \geqslant 0, \quad w_1 \geqslant 0 \tag{A15}$$

where $a = \omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1$. Multiplying the last inequality by a, multiplying and dividing the sum of the two last terms by $\sqrt{a - \varepsilon^2} + \sqrt{a}$, we obtain an equivalent inequality

$$\frac{\varepsilon\omega_1 \sinh w_1}{\sqrt{\omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1 - \varepsilon^2} + \sqrt{\omega_2^2 \cosh^2 w_1 + \omega_1^2 \sinh^2 w_1}} \leqslant \omega_2 \cosh w_1 \tag{A16}$$

which holds by the hypothesis $\omega_2 < \varepsilon$. It proves (A14). Finally, (A13), (A14) imply that $\gamma(-\Delta) \prec \check{\Gamma}_{1,-\varepsilon}^{-}$ since $\gamma(-\Delta) = \check{\Gamma}_1^{-} - \mathrm{i}\Delta$. Lemma 9.7 is proved. □

Remark A.2 (*cf. Remark 9.6*)
Parametrization (A2) demonstrates that the curve $\check{\Gamma}_{1,-\varepsilon}^{-}$ with $\varepsilon < \omega_2$ is homotopic to $\check{\Gamma}_1^{-}$ in the class of contours 'with the ends at infinity' which we use in the deformation of the contours by the Cauchy Theorem.

Proof of Corollary 9.8
Let us consider $\theta_- < \theta < \pi - \theta_+$ with some positive θ_+ and θ_-. Then $\gamma(-\pi + \theta_+ - \delta) \prec \check{\Gamma}_{1,-\varepsilon} - \mathrm{i}\theta < \gamma(-\theta_-)$ for positive δ since $\gamma(-\delta) \prec \check{\Gamma}_{1,-\varepsilon}^{-} < \check{\Gamma}_1^{-}$ by Lemma 9.7. At last, let us choose $\delta < \theta_+$ (see Figure 4). Then Corollary 9.5 implies the superexponential decay. □

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