ON ATTRACTOR OF A SINGULAR NONLINEAR $U(1)$-INVARIANT KLEIN-GORDON EQUATION

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The long-time asymptotics is analyzed for all finite energy solutions to the 1D Klein-Gordon equation coupled to a nonlinear oscillator. The coupled system is invariant with respect to the rotation group $U(1)$. Each finite energy solution converges to the "eigenfunctions" $\psi_\pm(x)e^{i\omega_\pm t}$ as $t \to \pm\infty$. The problem is inspired by Schrödinger's identification of the quantum stationary states to the eigenfunctions in the quantum electrodynamics which is invariant with respect to the (global) gauge group $U(1)$.

1 Introduction

The eigenvalue problem is a core of quantum mechanics since Schrödinger had identified the solutions $\psi(x)e^{i\omega t}$ to "quantum stationary states". At first glance, this identification seems not compatible with the first Bohr postulate on the transitions between the stationary states, $|\omega_-\rangle \mapsto |\omega_+\rangle$. Namely, the transitions may be treated dynamically as a long-time asymptotics of the type

$$\psi(x,t) \sim \psi_\pm(x)e^{-i\omega_\pm t}, \quad t \to \pm\infty, \qquad (B)$$

which means that the set of the quantum stationary states is an attractor of corresponding dynamical system. However, such asymptotics generally is impossible for the linear autonomous Schrödinger, Klein-Gordon or Dirac equations because of the principle of superposition. The equations are autonomous in the presence of a static Maxwell field. For instance, a static Coulombic field gives the frequencies ω of the quantum stationary states of Hydrogen atom, with a high precision.

On the other hand, the transitory regime is followed by an electromagnetic radiation according to the second Bohr postulate. Hence, to get a consistent dynamical description of the transitions, it is necessary to take into account the coupling to the Maxwell field ("polarization of vacuum") which is the subject of Quantum Electrodynamics. The coupled *semiclassical* Klein-Gordon-Maxwell Equations have been introduced and discussed for the first time in [26]. The coupled equations are i) nonlinear and ii) invariant with respect to the (global gauge) group $U(1)$. This situation suggests that the features i), ii) may be responsible for the transitions (B) in the coupled semiclassical equations or their second-quantized version. The high precision of the linearized theory might be explained by a smallness of the nonlinear coupling.

The nonlinear coupling generates the perturbation series ("Feynman diagrams") in the calculation of the quantum stationary states $\psi(x)e^{-i\omega t}$ in Quantum Electrodynamics. The series give the perfect description of many quantum phenomena where the series converge numerically: for instance, the Lamb shift for Hydrogen and anomalous magnetic momentum of the electron. The perturbation series lead to the problem of renormalization, and their convergence has not been proved up to now. Moreover, the series diverge numerically in the theory of strong or electroweak interaction. Therefore, it would be instructive to identify the perturbation procedure as a tool, and not as a formulation of the problem. Namely, this situation suggests that the perturbation series is a tool to calculate the "nonlinear eigenfunctions" $\psi(x)e^{-i\omega t}$ to the coupled nonlinear equations, and the special role of the eigenfunctions might be explained by the long-time asymptotics (B) as an inherent mathematical property of the coupled system. Note that the existence of corresponding nontrivial solutions $\psi(x)e^{-i\omega t}$ is established recently in [4] for the coupled Dirac-Maxwell Equations without an external Maxwell field. In [1] similar result is established for a general class of nonlinear $U(1)$-invariant Klein-Gordon Eqns.

In a more general mathematical context, it is then natural to expect that the asymptotics (B) is an inherent mathematical property of a generic class of hyperbolic partial differential equations with the features i), ii). In the present paper, we check this for a model nonlinear $U(1)$-invariant Klein-Gordon Eqn. We prove that the transitions (B) are provided by radiation maintained by an energy flow over the spectrum, from low to higher modes. The spectral flow is provided by a polynomial character of the nonlinear term. The radiation cannot go forever as the total energy is finite. Therefore, the solution converges to the set of radiationless trajectories. A crucial point is the determination of this set: it consists only of the nonlinear eigenfunctions $\psi(x)e^{-i\omega t}$. Possible extensions must include more general 1D and 3D problems and higher symmetry groups as, for instance, $SU(2)$ or $SU(3)$ for the Yang-Mills Eqn [5,7,31].

We consider the long-time asymptotics and attractor of all finite energy solutions to a model nonlinear Klein-Gordon equations of the following type,

$$\ddot\psi(x,t) = \psi''(x,t) - m_0^2\psi(x,t) + \delta(x)F(\psi(0,t)), \quad (x,t)\in\mathbb{R}^2. \qquad (KG)$$

Here $m_0 > 0$, $\psi(x,t)$ is a continuous complex-valued "wave" function and F is a continuous function, the dots stand for the derivatives in t and the primes in x. All derivatives and the equation are understood in the distribution sense.

We identify a complex number $\psi = \psi_1 + i\psi_2$ with the real two-dimensional vector (ψ_1, ψ_2). Physically, Eqn (KG) describes small crosswise oscillations of an infinite string in three-dimensional space (x, ψ_1, ψ_2) stretched along the axis Ox. The string is subject to an "elastic" force $-m_0^2\psi(x,t)$ and coupled to an oscillator attached at the point $x = 0$: F is a nonlinear "oscillator force". We assume that the oscillator force F admits a real-valued potential $U \in C^2(\mathbb{R}^2)$,

$$F(\psi) = -\nabla U(\psi), \quad \psi\in\mathbb{R}^2. \qquad (P)$$

Then Eqn (KG) formally is a Hamiltonian system with the Hamiltonian functional

$$\mathcal{H}(\psi,\dot\psi) = \frac{1}{2}\int_{\mathbb{R}}[|\dot\psi|^2 + |\psi'|^2 + m_0^2|\psi|^2]dx + U(\psi(0)). \qquad (H)$$

It is conserved for finite energy solutions. To have a good a priori estimates, we assume that the potential is confining, i.e.

$$U(\psi)\to+\infty, \quad |\psi|\to\infty. \qquad (U)$$

Our key assumption concerns the $U(1)$-invariance (or the rotation-invariance) of the oscillator (cf [1]): $F(e^{i\theta}\psi) = e^{i\theta}F(\psi), \quad \theta \in [0, 2\pi]$, or equivalently,

$$F(\psi) = a(|\psi|)\psi, \quad \psi \in \mathbb{C}, \tag{I}$$

where $a(|\psi|)$ is real by (P). For instance, $F(0) = 0$. Obviously, the symmetry holds true iff the potential is radial: $U(\psi) = u(|\psi|)$. The symmetry implies that $e^{i\theta}\psi(x,t)$ is a solution to Eqn (KG) if $\psi(x,t)$ is. The Eqn is $U(1)$-invariant in the sense of [8]. Here $U(1)$ stands for the rotation group $e^{i\theta}$, $\theta \in [0, 2\pi]$. Main subject of this paper is an analysis of a special role of the "stationary states" of Eqn (KG), or Solitary Waves (see [8]) which are finite energy solutions of type

$$\psi_\omega(x,t) = \psi_\omega(x)e^{-i\omega t}, \quad \omega \in \mathbb{R}. \tag{S}$$

The frequency ω and the amplitude $\psi(x)$ give a solution to the following "nonlinear eigenvalue problem": (KG) implies by (I),

$$-\omega^2\psi_\omega(x) = \psi_\omega''(x) - m_0^2\psi_\omega(x) + \delta(x)F(\psi_\omega(x)), \quad x \in \mathbb{R}. \tag{ω}$$

Note that $\omega \in \mathbb{R}$ due to (U) and energy conservation. $\psi_\omega(x) = 0$ is always the solitary wave as $F(0) = 0$, and for $|\omega| > m_0$ only the zero solitary wave exist.

Definition S denotes the set of all solitary waves, $S = \{\psi_\omega(x) \in H^1(\mathbb{R}) : \omega \in \mathbb{R}\}$.

Here $H^1(\mathbb{R})$ denotes the Sobolev space. We give a complete analysis of the set S of all solitary waves $\psi(x)$ by an explicit calculation. For a polynomial F, the set S $\bmod U(1)$ is isomorphic to a finite union of one-dimensional intervals.

Our main results are the following two long-time asymptotics. First, we prove an attraction to the set S of all solitary waves:

$$\psi(\cdot, t) \to S, \quad t \to \pm\infty, \tag{A}$$

where the convergence holds in local energy seminorms. We prove it for a polynomial nonlinear term under following "True Nonlinearity" condition: $U(\psi) = \sum\limits_{n \le N} u_n|\psi|^{2n};\ u_N > 0,\ N \ge 2$. Equivalently, for the function a from (I),

$$a(|\psi|) = -\sum_{n \le N} a_n|\psi|^{2n-1};\ a_N > 0,\ N \ge 1. \tag{N}$$

Furthermore, we prove an attraction of type (B) where the asymptotics also holds in local energy seminorms. We prove it under Condition (N) together with following "Energy Nondegeneracy" Condition: for each $E \in [0, \infty)$

$$\text{the set } \{\omega \in \mathbb{R} : \mathcal{H}(\psi, -i\omega\psi) = E, \quad \psi \in S\} \text{ is discrete,} \tag{E}$$

where ω is the frequency corresponding to ψ. We give a simple sufficient criterion providing generic examples where (E) holds true.

For the case $m_0 = 0$, the Eqn (KG) is equivalent to the Lamb system of a wave equation coupled to an oscillator. The system has been introduced in [22] for linear function F. For nonlinear F such system has been studied in [12]: the solutions (S) exist only with $\omega = 0$. Then the asymptotics (A) and (B) mean a *global* attraction to the *static* stationary states. The attraction is established under a condition of type (N) with $a = 0$. These results were extended to more general 1D wave equations in [13,15] (see the survey [16]), to 3D wave-particle systems in [17,18,20] and to Maxwell-Lorentz system in [19] (see the survey [14]). The proofs have used the following common strategy:

i We analyze the energy radiation to the infinity to prove an attraction of the solution to a compact finite-dimensional subset in the phase space as $t \to \infty$;

ii We analyze the limit set of the trajectory: each *omega*-limit point is a stationary state. This means that the set of stationary states is a point attractor.

iii The convergence to a limit point of the attractor follows if it is discrete.

Here we adopt this general strategy to the Klein-Gordon equation with $m_0 > 0$. However, the key arguments in the proofs are considerably different. This is related to more complicate character of energy propagation in the Klein-Gordon Eqn: the dispersive relation $\omega^2 = k^2 + m_0^2$ provides the group velocities $v = \nabla\omega(k)$ (see [25]). For the wave equation then $|v| = 1$ for any k. On the other hand, for the Klein-Gordon each velocity v with $|v| < 1$ is possible while $|v| = 1$ is impossible in total controversy to the wave equation. Respectively, our strategy requires the following modifications:

I We consider $t > 0$ and split the solution in two components: dispersive and bound. The dispersive component is the union of the harmonics with frequencies $|\omega| \geq m_0$. The bound component is the union of the harmonics with $\omega \in (-m_0, m_0)$ which implies a compactness. Stationary phase arguments lead to a local decay of the dispersive component as $t \to +\infty$ and reduce the long-time behavior of the solution to the one of the bound component, i.e. to a compact set.

II We analyze the trajectory starting from any *omega*-limit point, as $t \to +\infty$, of the bound component: first, the time spectrum of the trajectory is embedded in $[-m_0, m_0]$. Secondly, the trajectory is radiationless as the energy is bounded. These two facts allow us reduce the time-spectrum of the trajectory to a unique harmonic with a frequency $\omega_+ \in (-m_0, m_0)$: otherwise, Eqn (KG) and (N) imply that the time-spectrum of the component is not embedded in $[-m_0, m_0]$. Then (A) follows.

III We prove that the energy has a limit as $t \to +\infty$. Then (E) implies that ω_+ is the same for each *omega*-limit point. This means the asymptotics of type (B).

Note that the compact attracting set in **I** is infinite-dimensional in contrast to **i**. The argument **II** physically means the following radiative mechanism: the low-frequency perturbation of the stationary state does not radiate the energy until it generates a spectral line (t.e. a point of the spectrum) embedded in the continuous spectrum outside $[-m_0, m_0]$. First, this mechanism has been discovered numerically in the experiments with the relativistic Ginzburg-Landau equation. Here we deduce this mechanism from a "spectral condition" which follows from Eqn (KG). The deduction is based on the Titchmarsh Theorem concerning the support of the convolution of the distributions. In our case the convolution arises for the time-spectrum of the solution.

The plan of the paper is the following. In Section 2 we state main results. In

Section 3 we give a complete description of the set of all solitary waves. Section 4 concerns a splitting of the solution in a dispersive and bound components. In Sections 5 and 6 we derive the complete description of the limiting radiationless trajectories. Namely, in Section 5 we obtain the spectral condition for all limiting trajectories, and in Section 6 we apply the Titchmarsh Theorem to the spectral condition.

The asymptotics of type (A), (B) were discovered first with $\psi_\pm = 0$ in the scattering theory for nD nonlinear wave, Klein-Gordon, Schrödinger and Yang-Mills equations for the case when the attractor S is a point zero: see [6,7,9,11,23,29]. Then the asymptotics mean well known *local energy decay*.

The asymptotics with $\psi_\pm \neq 0$ and $\omega_\pm = 0$, were established in [12−20] for a list of 1D and 3D nonlinear wave problems corresponding to $m_0 = 0$, and without a symmetry group. There the attractor S is the set of all stationary states. It can be an infinite set and even contain some continuous components.

First results on the asymptotics of type (A), (B) with $\psi_\pm \neq 0$ and $\omega_\pm \neq 0$ were obtained for nonlinear $U(1)$-invariant Schrödinger Eqn: in [27,28] for a multidimensional Eqn with a potential (see also [24]), and in [3] for translation-invariant 1D Eqn. In these papers the asymptotics are established for the solutions close to a solitary wave, which means the attraction to a *local* attractor.

In the present paper, we establish the *global* attraction to the solitary waves (S) with $\psi_\pm \neq 0$ and $\omega_\pm \neq 0$ for all finite energy solutions to the model $U(1)$-invariant 1D nonlinear Klein-Gordon equation (KG). The global attractor is isomorphic-mod$\,U(1)$ to a finite union of one-dimensional intervals. Our results demonstrate that the long-time asymptotics (A) and (B) are the properties of *generic* equations from a class of nonlinear $U(1)$-invariant equations. For instance, the asymptotics (A) holds true for $U(1)$-invariant equations (KG) under Condition (N). This Condition defines an open and dense everywhere subset in the class of confining polynomial $U(1)$-invariant potentials.

Our results suggest possible extension to a *generic* class of nonlinear hyperbolic equations with a Lie symmetry group G: corresponding attractor probably consists of solitary waves $e^{\Omega t}\psi(x)$ introduced in [8]. Here Ω is an element of corresponding Lie algebra $\mathcal{G}$ and $e^{\Omega t}$ is the one-parametric subgroup of $\mathcal{G}$, then $\Omega, \psi(x)$ is a solution to a "nonlinear eigenmatrix problem". However, this extension is an open question.

Note that the results [12−21] on convergence to static stationary states (i.e. with $\Omega = 0$), concern the equations with the trivial symmetry group $G = \{e\}$.

Remark If we have two Lie groups $G_1 \subset G_2$, then each G_2-invariant equation also is G_1-invariant. At first glance, this contradicts our conjecture as it means that the larger symmetry group G_2 leads to more sophisticate long-time asymptotics. However, the conjecture only concerns *generic* G-invariant equations, while G_2-invariant equations combine an *exceptional* subset of the equations among G_1-invariant ones.

2 Main results

Consider the Cauchy problem for Eqn (KG), with the initial conditions $\psi|_{t=0} = \psi_0(x)$, $\dot\psi|_{t=0} = \pi_0(x)$. Write the Cauchy problem as

$$\dot Y(t) = \mathcal{V}(Y(t)), \quad t \in \mathbb{R}; \quad Y(0) = Y_0, \tag{2.1}$$

where $Y(t) = (\psi(\cdot,t), \dot\psi(\cdot,t))$ and $Y_0 = (\psi_0, \pi_0)$. We introduce the phase space $\mathcal{E}$ of finite energy states for Eqn (KG). Denote by L^2 the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$ with the norm $|\cdot|$, and denote by $|\cdot|_R$ the norm in $L^2(-R, R; \mathbb{C})$ for $R > 0$. Denote by H^1 the Sobolev space $\{\psi(x) \in L^2 : \ \psi'(x) \in L^2\}$

Definition 2.1 *i) $\mathcal{E} = H^1 \oplus L^2$ is the Hilbert space of the pairs $(\psi(x), \pi(x))$, with the norm*

$$\|(\psi, \pi)\|_{\mathcal{E}} = |\psi'| + |\psi| + |\pi|. \tag{2.2}$$

ii) $\mathcal{E}_F$ is the space $\mathcal{E}$ endowed with the Fréchet topology defined by the seminorms

$$\|(\psi, \pi)\|_R \equiv |\psi'|_R + |\psi|_R + |\pi|_R, \quad R > 0. \tag{2.3}$$

Note that both spaces $\mathcal{E}$ and $\mathcal{E}_F$ are metrisable and $\mathcal{E}_F$ is not a complete space. With the assumptions (P), (U), Eqn (KG) is formally a Hamiltonian system with the phase space $\mathcal{E}$ and the Hamiltonian functional

$$\mathcal{H}(\psi, \pi) = \frac{1}{2} \int_{\mathbb{R}} [|\pi(x)|^2 + |\psi'(x)|^2 + m_0^2 |\psi(x)|^2] dx + U(\psi(0)), \quad (\psi, \pi) \in \mathcal{E}, \tag{$\mathcal{H}$}$$

which is continuous in $\mathcal{E}$.

Proposition 2.2 *Let Conditions (P) and (U) be fulfilled. Then*
i) for every $Y_0 \in \mathcal{E}$ the Cauchy problem (2.1) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.
ii) The map $W_t : \ Y_0 \mapsto Y(t)$ is continuous in $\mathcal{E}$ and $\mathcal{E}_F$ for each $t \in \mathbb{R}$.
iii) The energy is conserved, $\mathcal{H}(Y(t)) = \mathcal{H}(Y_0) \ \ t \in \mathbb{R}$.
iv) The a priory bounds hold, $\sup\limits_{t \in \mathbb{R}} \Big(\|\dot\psi(\cdot,t)\| + \|\psi'(\cdot,t)\| + \|\psi(\cdot,t)\| + \|\psi(0,t)\| \Big) < \infty.$

Definition 2.3 *i) The solitary waves of Eqn (2.1) are the solutions $Y(t) = (\psi_\omega(x), -i\omega\psi_\omega(x))e^{-i\omega t}$ with $\psi_\omega \in S$.*
ii) $\mathcal{S}$ is the set of the initial dates $(\psi_\omega(x), -i\omega\psi_\omega(x)) \in \mathcal{E}$ of all solitary waves.

The set $\mathcal{S}$ obviously is invariant under a multiplication by $e^{i\theta}$, $\theta \in [0, 2\pi]$. Let us give a simple criterion providing (E). Consider the following algebraic equation:

$$F(r) = 2m_0 r, \quad r \geq 0. \tag{2.4}$$

Let $r_- < r_+$ be any neighboring roots of the equation: we also include $r = +\infty$ as a "root". Then the criterion is the following:

$$\text{either } \min_{[r_-, \ r_+]} F(r) \leq 0, \text{ or } \min_{[r_-, \ r_+]} (F(r) - 2m_0 r) \geq 0. \tag{C}$$

Example Consider the "Ginzburg-Landau" potential $U(\psi) = g(|\psi|^2 - 1)^2/2$. Then $F(r) = -2g(r^3 - r)$ and $F'(0) = 2g$. For $g \leq m_0$ we have two "roots" $r = 0, +\infty$, and the first condition holds true. For $g > m_0$ we have three "roots" $r = 0, r_1, +\infty$: for the neighboring roots $r_1, +\infty$, the first condition holds true, while for $r = 0, r_1$, the second condition holds true.

Let us call a subset $D \subset L^2$ "finite $\mathrm{mod}\, U(1)$" if the set $D \ \mathrm{mod}\, U(1)$ is finite.

Proposition 2.4 *Let Conditions* (P), (U), (I) *and* (N) *be fulfilled. Then*
i) for any fixed $\omega \in \mathbb{R}$,

$$\text{the set of solutions to } (\omega) \text{ is finite} \bmod U(1). \qquad (\Omega)$$

ii) Let additionally, Criterion (C) *be fulfilled. Then Condition* (E) *holds true and moreover, the set in* (E) *is finite.*

Our main result is the following theorem.

Theorem A *Let Conditions* (P), (U), (I) *and* (N) *be fulfilled. Then*
i) For any solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *to Eqn (2.1),*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}, \quad t \to \pm\infty. \qquad (\mathcal{A})$$

ii) Let additionally, (E) *and* (Ω) *hold. Then there exist solitary waves* $(\psi_\pm(x), -i\omega_\pm\psi_\pm(x))e^{-i\omega_\pm t}$ *such that for some* $\theta_\pm(t) \in [0, 2\pi)$ *we have*

$$Y(t) \overset{\mathcal{E}_F}{\sim} (\psi_\pm(x), -i\omega_\pm\psi_\pm(x))e^{-i(\omega_\pm t + \theta_\pm(t))}, \quad t \to \pm\infty. \qquad (\mathcal{B})$$

3 Solitary waves

Here we prove Proposition 2.4.

Step 1 Let us calculate all solitary waves. Denote $\kappa^2 = m_0^2 - \omega^2$. Then (ω) implies $\psi''(x) = \kappa^2\psi(x)$, $x \neq 0$, hence $\psi(x) = C_\pm e^{\kappa x}$, $\pm x > 0$. Since $\psi'(x) \in L^2$, the function $\psi(x)$ is continuous, hence $C_- = C_+ = C$ and $\psi(x) = Ce^{\kappa x}$, $x \in \mathbb{R}$. Furthermore, $\psi(x) \in L^2$, therefore κ is real and negative if $C \neq 0$:

$$\psi(x) = Ce^{-\kappa|x|}, \quad \kappa = \sqrt{m_0^2 - \omega^2} > 0, \quad \omega \in (-m_0,\ m_0). \qquad (3.1)$$

At last, we get an algebraic equation for the constant C equating the coefficients of $\delta(x)$ in both sides of (ω):

$$0 = \psi'(0+) - \psi'(0-) + F(\psi(0)). \qquad (3.2)$$

This implies $0 = -2\kappa C + F(C)$, or equivalently,

$$\kappa = \kappa_C := \frac{F(C)}{2C}, \quad C \in \mathbb{C}, \quad \kappa \in (0, m_0]. \qquad (3.3)$$

Now we can prove (Ω): for a fixed ω, we also have a fixed κ, hence the set of the roots of Eqn (3.3) is finite $\bmod U(1)$ as κ_C is a polynomial of a degree ≥ 2 by (N). It remains to prove (E).

Step 2 Eqn (3.3) demonstrates that the set $\mathcal{S}$ of all solitary waves allow a simple parameterization with the complex constant C. First, $C = 0$ gives the zero solitary wave $\psi_0(x) = 0$. Further consider all complex $C \neq 0$, then $\kappa = \kappa_C \in (0, m_0]$ for the finite energy solitary waves. For a complex constant $C \neq 0$ with $\kappa_C \in [0, m_0]$, denote by $\psi_C(x)$ the function (3.1). If $\kappa_C \in (0, m_0]$, the solitary waves $\psi_C(x)$ has a finite energy, while $\kappa_C = 0$ with $C \neq 0$ correspond to the infinite energy of $\psi_C(x)$. Now we can evaluate the set in (E):

$$\{\omega \in \mathbb{R} : \mathcal{H}(\psi, -i\omega\psi) = E, \quad \psi \in \mathcal{S}\} = \{\omega_C : \mathcal{H}(\psi_C, -i\omega_C\psi_C) = E, \quad C \in [0, \infty)\}, \tag{3.4}$$

where $\omega_C^2 = m_0^2 - \kappa_C^2$. We restrict here $C \in \mathbb{C}$ to $C \in [0, \infty)$ after division by $e^{i\theta}$ as it does not change ω_C and $\mathcal{H}(\psi_C, -i\omega_C\psi_C)$ that follows from (I) and (H).

Step 3 It suffices to prove that the set of possible values of $C \in [0, \infty)$ in RHS of (3.4) is finite.

Consider the set of the roots $C \in [0, \infty)$ of the equation (3.3) with the maximal value $\kappa_C = m_0$ and add $r = +\infty$ as a "root". The set is finite as κ_C is a polynomial of a degree ≥ 2.

Let us consider any neighboring roots $r_- < r_+$ and prove that the interval $[r_-, r_+]$ contains at most a finite number of the C with $\mathcal{H}(\psi_C, -i\omega_C\psi_C) = E$.

First, consider the case when $\min\limits_{[r_-,\ r_+]} (F(r) - 2m_0 r) \geq 0$. Then $\kappa_C \geq m_0$, $C \in [r,\ r_+]$, and only $\kappa_C = m_0$ correspond to a finite energy. Hence, the set of such C is finite as κ_C is a polynomial of a degree ≥ 2.

Second, consider the case when $\min\limits_{[r_-,\ r_+]} F(r) \leq 0$. Then the "finite energy"-set $\{C \in (r_-, r_+) : \kappa_C \in (0, m_0)\}$ is open, hence it is a union of a finite set of the intervals (α, β). Obviously, $\kappa_C = 0$ at least in one end of each interval. Let us consider arbitrary of these intervals, (α, β). The function $\mathcal{H}(\psi_C, -i\omega_C\psi_C)$ is finite and real analytic in (α, β) as all the functions $\kappa_C, \omega_C = \sqrt{m_0^2 - \kappa^2(C)}$ and $U(\psi_C(0))$ are real-analytic functions of C until $\kappa_C \in (0, m_0)$. If $\mathcal{H}(\psi_C, -i\omega_C\psi_C) = E$ for an infinite number of the points in (α, β), then $\mathcal{H}(\psi_C, -i\omega_C\psi_C) = E$ everywhere in (α, β). However, $\mathcal{H}(\psi_C, -i\omega_C\psi_C) = \infty$ at the end where $\kappa_C = 0$. $\square$

4 Dispersive and bound components

We split the solution in two components: a dispersive and a bound. The dispersive component describes a radiation to infinity, while the bound is responsible for the long-time asymptotics in the finite space.

Step 1 First, we split the solution $\psi(x, t)$ in two components $\phi_0(x, t), \phi_1(x, t)$: for $x \in \mathbb{R}$, $t > 0$,

$$\begin{cases} \ddot{\phi}_0(x,t) = \phi_0''(x,t) - m_0^2\phi_0(x,t), \\ \quad \phi_0|_{t=0} = \psi_0, \quad \dot{\phi}_0|_{t=0} = \psi_1, \end{cases} \qquad \begin{cases} \ddot{\phi}_1(x,t) = \phi_1''(x,t) - m_0^2\phi_1(x,t) + \delta(x)f(t), \\ \quad \phi_1|_{t=0} = 0, \quad \dot{\phi}_1|_{t=0} = 0, \end{cases}$$

where $f(t) := F(\psi(0, t))$. We will study the properties of each component ϕ_i, $i = 0, 1$.

Lemma 4.1 *Let Conditions (P) and (U) hold. Then the component ϕ_0 decays as $t \to \infty$: $\forall R, T > 0$,*

$$\sup_{|x|<R} |\phi_0(x,t)| \to 0, , \quad \sup_{|x|<R} \int_t^{t+T} \Big(|\dot{\phi}_0(x,s)|^2 + |\phi_0'(x,s)|^2\Big) ds \to 0. \tag{4.1}$$

Step 2 Next we split $\phi_1(x,t)$ further in two components: $\phi_1(x,t) = \phi_2(x,t) + \phi_3(x,t),\ t > 0$. Here $\phi_2(x,t)$ is a dispersive component which decays as $t \to \infty$ similar to $\phi_0(x,t)$, while $\phi_3(x,t)$ is a "bound" component with the time spectrum in $[-m_0, m_0]$. To split $\phi_1(x,t)$, we calculate in the Fourier-Laplace transform

$$\tilde{\phi}_1(x,\omega) = \mathcal{F}^+\phi_1 := \int_0^\infty e^{i\omega t}\phi_1(x,t)dt, \quad x \in \mathbb{R}, \quad \omega \in \mathbb{C}^+ := \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}.$$

The integral converges and is an analytic function in $\mathbb{C}^+$ for each $x \in \mathbb{R}$ due to Proposition 2.2 iv). Eqn for ϕ_1 implies (cf (ω))

$$-\omega^2\tilde{\phi}_1(x,\omega) = \tilde{\phi}_1''(x,\omega) - m_0^2\tilde{\phi}_1(x,\omega) + \delta(x)\tilde{f}(\omega), \quad \omega \in \mathbb{C}^+.$$

The solution $\phi_1(x,\omega)$ is a linear combination of the fundamental solutions $E_\pm(x,\omega) = \dfrac{e^{\pm ik|x|}}{\pm 2ik}$ where k stands for an analytic function $k(\omega) = \sqrt{\omega^2 - m_0^2}$, $\omega \in \mathbb{C}^+$ with $\operatorname{Im} k(\omega) > 0$. Further we use the standard "limit absorption principle" for the selection of $E_\pm$: only E_+ is appropriate as $\phi_1(\cdot,\omega) \in H^1$ for $\omega \in \mathbb{C}^+$. Thus, $\tilde{\phi}_1(x,\omega) = -\tilde{f}(\omega)E_+(x,\omega), \quad x \in \mathbb{R}, \quad \omega \in \mathbb{C}^+$. Extend $k(\omega)$ to $\omega \in \overline{\mathbb{C}^+}$ by continuity. Then $k(\omega)$ is real for $\omega \in \mathbb{R}$ with $|\omega| \ge m_0$ and imaginary with $|\omega| < m_0$. Denote $\kappa = \kappa(\omega) := -ik(\omega) = \sqrt{m_0^2 - \omega^2} > 0$ for $\omega \in (-m_0, m_0)$. Then

$$\phi_1(x,t) = -\frac{1}{2\pi}\int\limits_{|\omega|\ge m_0} e^{-i\omega t}\tilde{f}(\omega)\frac{e^{ik|x|}}{2ik}d\omega + \frac{1}{2\pi}\int\limits_{|\omega|<m_0} e^{-i\omega t}\tilde{f}(\omega)\frac{e^{-\kappa|x|}}{2\kappa}d\omega, \quad (4.2)$$

or $\phi_1(x,t) = \phi_2(x,t) + \phi_3(x,t),\ t \in \mathbb{R}$. We call $d(x,t) := \phi_0(x,t) + \phi_2(x,t)$ a dispersive component and $b(x,t) := \phi_3(x,t)$ a bound component of the solution $\psi(x,t)$.

Lemma 4.2 *Let Conditions (P) and (U) hold. Then for $d(x,t)$ the decay (4.1) holds true.*

The sketch of proof The functions $e^{ik|x|}$ have "an infinite energy", while ϕ_1 has a finite energy, and $e^{-\kappa|x|}$ as well. This is possible only if the density $\tilde{f}(\omega)$ is absolutely continuous. Then the oscillatory integral form of the function ϕ_2 implies its decay of type (4.1).

5 Radiationless solutions: spectral condition

Previous analysis demonstrates that long-time asymptotics of the solution $\psi(x,t)$, $t \to \infty$ depends only on the one of the bound component $b(x,t)$. Set $c(t) := b(0,t)$, then we have for $t \in \mathbb{R}$

$$b(x,t) = \frac{1}{2\pi}\int\limits_{|\omega|<m_0} e^{-i\omega t}\tilde{f}(\omega)\frac{e^{-\kappa|x|}}{2\kappa}d\omega,\ c(t) = \frac{1}{2\pi}\int\limits_{|\omega|<m_0} e^{-i\omega t}\tilde{f}(\omega)\frac{1}{2\kappa}d\omega. \quad (5.1)$$

Denote Spec $c(\cdot)$:= supp $\tilde{c}$. Then obviously,

$$\text{Spec } b(x,\cdot) = \text{Spec } c(\cdot), \ \ x \in \mathbb{R}. \qquad (\sigma_b)$$

We are going to reduce the time-spectrum of $b(x,t)$ to a unique frequency $\omega_+ \in [-m_0, m_0]$ taking into account Condition (N). More precisely, we prove it for each *omega - limiting* function

$$\beta(x,t) = \lim_{\tau_k \to \infty} b(x, \tau_k + t), \quad (x,t) \in \mathbb{R}^2. \tag{5.2}$$

This implies, first, the asymptotics (A). Then (B) would follow under Condition (D).

Let us describe our strategy.

1) First, we prove a compactness of the trajectory $b(\cdot, t)$, $t \in \mathbb{R}$ to establish a convergence of the type (5.2). Then we prove that the limiting trajectory $\beta(x,t)$

i) is a solution to the equation (KG) (though $b(x,t)$ does not!!):

$$\ddot{\beta}(x,t) = \beta''(x,t) - m_0^2 \beta(x,t) + \delta(x) F(\beta(0,t)), \quad (x,t) \in \mathbb{R}^2. \tag{KG_β}$$

ii) admits the representations of type (5.1):

$$\beta(x,t) = \frac{1}{2\pi} \int\limits_{|\omega| \le m_0} e^{-i\omega t} \tilde{g}(\omega) \frac{e^{-\kappa|x|}}{2\kappa} d\omega, \ \gamma(t) := \beta(0,t) = \frac{1}{2\pi} \int\limits_{|\omega| \le m_0} e^{-i\omega t} \tilde{g}(\omega) \frac{1}{2\kappa} d\omega. \tag{5.3}$$

Therefore, similarly to (σ_b),

$$\operatorname{Spec} \beta(x, \cdot) = \operatorname{Spec} \gamma(\cdot), \ \ x \in \mathbb{R}. \tag{σ_β}$$

2) Next we use the algebraic equation (cf (3.2)) $0 = \beta'(0+,t) - \beta'(0-,t) + F(\gamma(t)), t \in \mathbb{R}$, which follows from (KG_β) equating the coefficients of δ-function in both sides. It implies, $\operatorname{Spec} F(\gamma(\cdot)) \subset \operatorname{Spec} \beta'(0+, \cdot) \cup \operatorname{Spec} \beta'(0-, \cdot)$. Therefore, (σ_β) implies

$$\operatorname{Spec} F(\gamma(\cdot)) \subset \operatorname{Spec} \gamma(\cdot). \tag{σ_F}$$

Finally, we have from (5.3),

$$\operatorname{Spec} \gamma(\cdot)) \subset [-m_0, m_0]. \tag{σ_γ}$$

3) Now we use Conditions (I), (N): we deduce from (σ_F) and (σ_γ) that $\gamma(t)$ has a constant amplitude,

$$|\gamma(t)| = \text{const}, \ \ t \in \mathbb{R}. \tag{a}$$

4) This implies that Spec $\gamma(\cdot)$ is a point ω_+: $\gamma(t) = Ce^{-i\omega_+ t}$.

5)Finally, (I) implies that $f(t) := F(\gamma(t)) = F(C)e^{-i\omega_+ t}$, hence (5.3) implies that $\beta(x,t)$ is a solitary wave $\psi_+(x)e^{-i\omega_+ t}$.

Main point of the program is the implication 3): $(\sigma_F)\&(\sigma_\gamma) \Longrightarrow (a)$. We prove it by the classical Titchmarsh Theorem on the support of the convolution (see [10], p. 178). Note that 4) follows by the same argument. This leads to the proof of Theorem A.

We start with the following compactness argument. The set of distributions $D = \{e^{-i\tau\omega}\tilde{f}(\omega) : \tau \in \mathbb{R}\}$ is compact in the space of distributions with the support in $[-m_0, m_0]$. Hence, for any sequence $\tau_k \to \infty$ there exists a subsequence $\tau_{k'} \to \infty$

such that $e^{-i\tau_{k'}\omega}\tilde{f}(\omega) \to \tilde{g}(\omega),\ \ \omega \in \mathbb{R}$, where the convergence holds in the sense of distributions. Therefore, (5.1) implies the convergences of type (5.2),

$$b(x, t + \tau_{k'}) \to \beta(x,t) := \frac{1}{2\pi} \int\limits_{|\omega| \le m_0} e^{-i\omega t}\tilde{g}(\omega)\frac{e^{-\kappa|x|}}{2\kappa} d\omega,\ \ (x,t) \in \mathbb{R}^2. \qquad (5.4)$$

Furthermore, the convergence (5.4) holds uniformly in $(x,t) \in \mathbb{R}^n \times [-T,T]$ for any $T > 0$. Therefore, (KG) implies (KG_β). Hence, the algebraic equation (δ_β) and the spectral conditions $(\sigma_F), (\sigma_\gamma)$ hold true for each ω-limiting function $\beta(x,t)$.

Remark We call the limiting trajectories $\beta(x,t)$ "radiationless" as we suggest that the radiation of energy goes to zero as $t \to \infty$.

In next section, we will use the spectral condition $(\sigma_F), (\sigma_\gamma)$ to determine all the radiationless solutions. We will show that each such solution has the Schrödinger's form $\psi_+(x)e^{i\omega_+ t}$ with certain $\omega_+ \in (-m,m)$.

6 Titchmarsh Theorem

We have to reduce the spectrum of each limiting trajectory $\beta(x,t)$ to a unique point.

Step 1 Let us prove (a). We deduce it from the spectral conditions $(\sigma_F), (\sigma_\gamma)$. The inclusion (σ_F) is possible for a *linear* function $F(\psi) = a(|\psi|)\psi$ with $a(|\psi|)$ =const. For the nonlinear functions F satisfying Conditions (N), we will prove that $(\sigma_F), (\sigma_\gamma)$ imply $|\gamma(t)|$ =const.

Proposition 6.1 *Let a complex-valued continuous function $F(\psi) = a(|\psi|)\psi$, $\psi \in C$, satisfy Condition (N), and $\gamma(t)$ be a complex-valued bounded continuous function in $\mathbb{R}$ with a bounded spectrum, i.e. (σ_γ) holds with an $m_0 > 0$. Then (σ_F) implies (a).*

Proof $\gamma(t)$ and $\alpha(t) := a(|\gamma(t)|)$ are bounded continuous functions in $\mathbb{R}$. Hence, $\gamma(t)$ and $\alpha(t)$ are tempered distributions. Then the product $f(t) = \alpha(t)\gamma(t)$ becomes the convolution in the Fourier transform: $\tilde{f} = C\tilde{\alpha} * \tilde{\gamma}$. The convolution is defined and the identity holds true due to (σ_γ). Now Condition (σ_F) means that

$$\operatorname{supp} \tilde{\alpha} * \tilde{\gamma} \subset \operatorname{supp} \tilde{\gamma}, \qquad (\sigma_*)$$

while generally, $\operatorname{supp} \tilde{\alpha} * \tilde{\gamma} \subset \operatorname{supp} \tilde{\alpha} + \operatorname{supp} \tilde{\gamma}$ (see [25]). This situation suggests that the set $\operatorname{supp} \tilde{\alpha}$ must be a point zero that we prove below. First, (N) provides that $\alpha(t) := a(|\gamma(t)|)$ is a polynomial in $\gamma(t)$ and $\overline{\gamma}(t)$. Therefore, the set $\operatorname{supp} \tilde{\alpha}$ also is compact. Hence, we have $\operatorname{supp} \tilde{\alpha} \subset [s_-, s_+]$, $\quad \operatorname{supp} \tilde{\gamma} \subset [\sigma_-, \sigma_+]$ for some finite $s_\pm, \sigma_\pm \in \mathbb{R}$. It remains to prove that we can take here $s_\pm = 0$. Indeed, then $\tilde{\alpha}(\omega)$ is a finite linear combination of the derivatives of $\delta(\omega)$. This implies that $\alpha(t)$ is a polynomial, hence it is constant as $\alpha(t)$ is bounded by Proposition 2.2 iv).

Theorem B (Titchmarsh) *Let $\tilde{\alpha}, \tilde{\gamma}$ be two tempered distributions with compact supports as above, and $s_\pm \in \operatorname{supp} \tilde{\alpha}$, $\sigma_\pm \in \operatorname{supp} \tilde{\gamma}$. Then we have also*

$$s_\pm + \sigma_\pm \in \operatorname{supp} \tilde{\alpha} * \tilde{\gamma}. \qquad (6.1)$$

This theorem and (σ_*) imply $[s_- + \sigma_-, s_+ + \sigma_+] \subset [\sigma_-, \sigma_+]$, i.e. $s_\pm = 0$. $\square$

Step 2 It remains to prove that $\gamma(t) = Ce^{-i\omega_+ t}$, $t \in \mathbb{R}$. This follows from (a) by the same Theorem B. Namely, (a) implies $\gamma(t)\overline{\gamma}(t) \equiv C$, hence in the Fourier transform $\tilde{\gamma} * \tilde{\overline{\gamma}} = C_1\delta(\omega)$. Therefore, Theorem B implies $\operatorname{supp}\tilde{\gamma} + \operatorname{supp}\tilde{\overline{\gamma}} = \{0\}$. Hence, $\operatorname{supp}\tilde{\gamma} = \{\omega_+\}$ and $\operatorname{supp}\tilde{\overline{\gamma}} = \{-\omega_+\}$ with some $\omega_+ \in \mathbb{R}$.

Remarks i) We have used essentially that the spectrum of γ is bounded (see (σ_γ)). ii) We used the polynomial character of the nonlinear term to deduce the same for $F(\gamma(\cdot))$.

Let us comment on a special role of Spectral Condition (σ_F). Generally, one could expect that the spectrum of $F(\gamma(\cdot))$ contains new "spectral lines" which were not presented in the spectrum of $\gamma(\cdot)$. Generally, this "inflation of spectrum" holds for $F(\gamma) = \gamma^n$, $n \geq 2$, if $\operatorname{Spec}\gamma(\cdot)$ contains a point outside zero: it is obvious in the case of the discrete finite spectrum. Our results demonstrate that this inflation also holds for F satisfying Conditions (I) and (N) if $\gamma(\cdot)$ has a spectrum containing at least two distinct points. The iteration of F leads then to a generation of a spectral line outside $[-m_0, m_0]$ belonging to the dispersive component, i.e. to the radiation of energy to infinity. Thus, Condition (σ_F) means the absence of the energy flow from low to higher modes. This characterization of the limit "radiationless" trajectories serves as an equation for the determination of the attractor.

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