A METHOD FOR DETERMINING THE MOD-3^k BEHAVIOUR OF RECURSIVE SEQUENCES (WITH EXPANDED APPENDIX)

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Abstract. We present a method for obtaining congruences modulo powers of 3 for sequences given by recurrences of finite depth with polynomial coefficients. We apply this method to Catalan numbers, Motzkin numbers, Riordan numbers, Schröder numbers, Eulerian numbers, trinomial coefficients, Delannoy numbers, and to functions counting free subgroups of finite index in the inhomogeneous modular group and its lifts. This leads to numerous new results, including many extensions of known results to higher powers of 3.

1. Introduction

In [3], Deutsch and Sagan present a rather systematic study of the behaviour of Catalan and Motzkin numbers and related sequences (mostly) modulo 2 and 3. In order to arrive at their results, they use clever ad hoc arguments, combining combinatorial and computational techniques. The results in [3] which are of particular interest to us here are those concerning the modulus 3. One finds numerous such theorems there, determining the behaviour of Motzkin numbers, Motzkin prefix numbers, Riordan numbers, central trinomial coefficients, central binomial coefficients, Catalan numbers, central Delannoy numbers, Schröder numbers, hex tree numbers, central Eulerian numbers, and Apéry numbers modulo 3. The purpose of the present paper is to demonstrate that, with the exception of the Apéry numbers, all these — seemingly unrelated — results are in fact direct consequences of a *single* "master theorem," namely Theorem 17 in Section 6. Moreover, this same theorem also allows us to refine all these results to congruences modulo *arbitrary powers* of 3.

A second motivation for our paper comes from [11]. In Theorem 3(i) of that paper, the number of free subgroups of given index in the inhomogeneous modular group $PSL_2(\mathbb{Z})$ is considered modulo 3. Our original goal was to extend that theorem to higher powers of 3. The corresponding results are described in Section 16. In the course of our work on this problem, we became aware of the paper [3] of Deutsch and Sagan, which pointed the way to many further applications of the method used, and, in particular, to the unifying "master theorem," of which the numbers of free subgroups of given index in $PSL_2(\mathbb{Z})$ constitute one instance.

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This master theorem embeds itself into a general method which applies to sequences given by a recurrence of finite depth with polynomial coefficients, and gives rise to congruences modulo arbitrarily large powers of 3. More precisely, let

$$\Psi(z)=\sum_{k\ge0}\sum_{n_1>\dots>n_k\ge0} z^{3^{n_1}+3^{n_2}+\dots+3^{n_k}}=\prod_{j=0}^{\infty}(1+z^{3^j}). \tag{1.1}$$

Suppose we are given an integer sequence $(f_n)_{n\ge0}$, whose generating function $F(z)=\sum_{n\ge0} f_n\,z^n$ satisfies a (formal) differential equation of the form

$$\mathcal{P}(z;F(z),F'(z),F''(z),\dots,F^{(s)}(z))=0, \tag{1.2}$$

where $\mathcal{P}$ is a polynomial with integer coefficients, which has a unique power series solution $F(z)$ with integer coefficients. (In the literature, series obeying a polynomial relation of the form (1.2) are known as *differentially algebraic* series; see, for instance, [1].) In its simplest form, what our method affords is an algorithmic procedure to find a polynomial in the *basic series* $\Psi(z)$ given by (1.1) with coefficients which are Laurent polynomials in z and $1+z$, such that this polynomial agrees with $F(z)$ when coefficients are reduced modulo some given power of 3. It turns out that, to make the method more flexible, in applications one will also have to use variations of the basic series $\Psi(z)$ obtained by substitutions of the variable z. See Section 4 for the description of the method and these variations.

We point out that this method is in the spirit of the one developed in [7], where a generating function method was described to determine the behaviour of recursive sequences modulo powers of 2. The most significant difference lies in the choice of the basic series. In [7], one works with the series $\Phi(z)=\sum_{n\ge0} z^{2^n}$ and variations thereof, whereas here we put the series $\Psi(z)$ in the focus of our attention. A second notable difference is that, because of the particular behaviour of powers of $\Psi(z)$ modulo 3, we have to allow powers of $1+z$ as denominators in the coefficients of the polynomials of $\Psi(z)$ that we are considering here. (In [7], coefficients of the polynomials in $\Phi(z)$ were Laurent polynomials in z only.)

The special feature of the aforementioned master theorem, Theorem 17, is that it treats, in a uniform fashion, functional equations of the form (1.2), which, after reduction modulo 3, become *quadratic* in $F(z)$, apart from satisfying a few more technical conditions. The ubiquity of applications of Theorem 17, which we describe in Sections 7–17, ultimately stems from the fact that numerous combinatorial sequences have generating functions satisfying a quadratic functional equation.

In the remainder of this introduction, we briefly describe the contents of the paper. The next two sections discuss our main character, the formal power series $\Psi(z)$ defined in (1.1). The announced method for finding congruences for recursive sequences requires knowledge of certain theoretical properties of $\Psi(z)$. First of all, one needs to know how to extract the explicit value of a concrete coefficient in a polynomial expression in $\Psi(z)$ modulo a given 3-power. This is discussed in Section 2. The appendix provides the resulting congruences modulo 27 for the coefficients of $\Psi^3(z)$ and $\Psi^5(z)$. Section 2 also contains several theoretical results pertaining to dependence/independence of powers of $\Psi(z)$, which are used in Section 3. Second, it is of great interest to know polynomial identities for $\Psi(z)$ modulo a given 3-power which are of *minimal degree*. These *"minimal polynomials"* are the subject of Section 3. While it is indeed easy to see that $\psi(z)$ is algebraic modulo powers of 3, the finding of polynomial relations of *minimal* degree is quite challenging. We provide a lower bound on the minimal degree, which in particular shows that the series $\Psi(z)$ is transcendental over $\mathbb{Z}[z]$ (see Theorem 10). In Section 4, we describe our method of expressing the generating function of a recursive sequence, when reduced modulo a given 3-power, as a polynomial in $\Psi(z)$ with coefficients that are Laurent polynomials in z and $1+z$.

We then proceed with a first illustration of our method, by applying it to "almost central" binomial coefficients, see Section 5.

The "centrepiece" of our paper is Section 6, which presents the aforementioned master theorem. This theorem describes a general family of functional-differential equations, where our method from Section 4 works in a completely automatic fashion. We demonstrate the power of this theorem in the subsequent sections by applying it — in this order — to Motzkin numbers, Motzkin prefix numbers, Riordan numbers, central trinomial coefficients, central binomial coefficients and their sums, Catalan numbers, central Delannoy numbers, Schröder numbers, hex tree numbers, free subgroup numbers for lifts of the inhomogeneous modular group, and central Eulerian numbers. All congruence results modulo 3 from [3] are derived completely automatically, and moreover we display the generating function results modulo 9 and 27 which are obtained by applying our method. The case of central Eulerian numbers is in fact quite challenging, since it requires several preparations until one sees that our method is actually applicable, see Section 17. It would cover too much space to display the explicit description of the congruence classes modulo 9 or 27 (or higher 3-powers) for all the above sequences, although this only amounts to routine calculations. We therefore confine ourselves to just one illustration, concerning the free subgroup numbers of $PSL_2(\mathbb{Z})$, our original motivation for this paper; see Section 16.

We close our paper with a short discussion of congruences modulo powers of 3 for Apéry numbers. They provide examples of number sequences where our method does not work. Nevertheless, the pattern which they satisfy modulo 9 seems to be very interesting; see Conjectures 65 and 66.

Note. This paper is accompanied by a *Mathematica* file and a *Mathematica* notebook so that an interested reader is able to redo (most of) the computations that are presented in this article. File and notebook are available at the article's website `http://www.mat.univie.ac.at/~kratt/artikel/3psl2z.html`.

2. Coefficient extraction from powers of $\Psi(z)$

In Section 4 we are going to describe a method for expressing differentially algebraic formal power series, after the coefficients of the series have been reduced modulo 3^γ, as polynomials in the Cantor-like series[1] $\Psi(z)$ (see (1.1) for the definition), the coefficients being Laurent polynomials in z and $1+z$. Such a method would be without value if we could not, at the same time, provide a procedure for extracting coefficients from the series $(1+z)^{-L}\Psi^K(z)$, where K and L are non-negative integers. The description of such a procedure is the topic of this section. Roughly speaking, this procedure consists of expanding powers of $\Psi(z)$ in terms of the auxiliary series in (2.3) (see (2.6) and (2.7)),

[1] We call it "Cantor-like" since the exponents of the monomials appearing in $\Psi(z)$ all have a ternary expansion consisting of only 1's and 0's. This is reminiscent of the ternary expansion of the numbers in the classical Cantor set all of which have a ternary expansion consisting of 2's and 0's only.

which can be further expanded using only a subclass of these series (see Lemma 9), and then reading the coefficients from these auxiliary series, which is (more or less) straightforward (see the appendix).

We begin with an expansion of $\Psi^2(z)$, on which all subsequent arguments in this section will be based.

Lemma 1. *We have*

$$\Psi^2(z) = \frac{1}{1+z} \sum_{s\ge0} \sum_{k_1>\dots>k_s\ge0} 3^s \prod_{j=1}^{s} \frac{z^{3^{k_j}}(1+z^{3^{k_j}})}{1+z^{3^{k_j+1}}}. \tag{2.1}$$

In the sum, the term for $s=0$ has to be interpreted as 1.

Proof. We have

$$\begin{aligned}\Psi^2(z) &= \prod_{j=0}^{\infty} (1+z^{3^j})^2\\ &= \frac{1}{1+z} \prod_{j=0}^{\infty} \frac{(1+z^{3^j})^3}{1+z^{3^{j+1}}}\\ &= \frac{1}{1+z} \prod_{j=0}^{\infty} \left(1+3\frac{z^{3^j}(1+z^{3^j})}{1+z^{3^{j+1}}}\right).\end{aligned}$$

Expansion of the product directly results into (2.1). □

Remark. The significance of Lemma 1 lies largely in the fact that, if one considers (2.1) modulo a 3-power, then the sum over s on the right-hand side gets truncated. We shall exploit this frequently. The simplest example is the reduction of (2.1) modulo 3. Using the notation

$$f(z) = g(z) \text{ modulo } 3^\gamma$$

to mean that the coefficients of z^i in $f(z)$ and $g(z)$ agree modulo 3^γ for all i, this reduction of (2.1) explicitly reads

$$\Psi^2(z) = \frac{1}{1+z} \quad \text{modulo } 3. \tag{2.2}$$

We shall make use of this congruence later at several points.

For convenience, we write

$$\widetilde{H}_{a_1,a_2,\dots,a_s}(z) := \sum_{k_1>\dots>k_s\ge0} \prod_{j=1}^{s} \left(\frac{z^{3^{k_j}}(1+z^{3^{k_j}})}{1+z^{3^{k_j+1}}}\right)^{a_j}. \tag{2.3}$$

We point out that $\widetilde{H}_{a_1,a_2,\dots,a_s}(z)$ is the monomial quasi-symmetric function

$$M_{a_1,a_2,\dots,a_s}(R(0),R(1),\dots),$$

where

$$R(k) = \frac{z^{3^k}(1+z^{3^k})}{1+z^{3^{k+1}}} \tag{2.4}$$

(cf. [18, p. 357]). Using this notation, Lemma 1 can be rephrased as

$$\Psi^2(z) = \frac{1}{1+z} \sum_{s\ge0} 3^s \widetilde{H}_{\underbrace{1,1,\dots,1}_{s \text{ times}}}(z). \tag{2.5}$$

It is not difficult to see that powers of $\Psi(z)$ can be expressed in the form

$$\Psi^{2K}(z) = \frac{1}{(1+z)^K} \sum_{r=1}^{K} \sum_{a_1,\dots,a_r\ge1} c_K(a_1,a_2,\dots,a_r)\widetilde{H}_{a_1,a_2,\dots,a_r}(z), \tag{2.6}$$

respectively as

$$\Psi^{2K+1}(z) = \frac{1}{(1+z)^K}\Psi(z) \sum_{r=1}^{K} \sum_{a_1,\dots,a_r\ge1} c_K(a_1,a_2,\dots,a_r)\widetilde{H}_{a_1,a_2,\dots,a_r}(z), \tag{2.7}$$

where the coefficients $c_K(a_1,a_2,\dots,a_r)$ can be determined explicitly.

Consequently, the coefficient extraction problem will be solved if we are able to say how to extract coefficients from the series

$$(1+z)^{-L}\widetilde{H}_{a_1,a_2,\dots,a_r}(z) \quad \text{and} \quad (1+z)^{-L}\Psi(z)\widetilde{H}_{a_1,a_2,\dots,a_r}(z).$$

This is in fact rather straightforward, if one has the patience to distinguish enough subcases. Roughly speaking, one analyses coefficient extraction from the subexpressions

$$\Psi(z)\frac{1}{(1+z)^L} \prod_{j=1}^{s} \left(\frac{z^{3^{k_j}}(1+z^{3^{k_j}})}{1+z^{3^{k_j+1}}}\right)^{a_j}$$

and

$$\frac{1}{(1+z)^L} \prod_{j=1}^{s} \left(\frac{z^{3^{k_j}}(1+z^{3^{k_j}})}{1+z^{3^{k_j+1}}}\right)^{a_j}.$$

This is made more precise in the appendix. How this is used is exemplified for the case of free subgroup numbers of the inhomogeneous modular group in Section 16.

It is also important to have information on dependence or independence of the series $\widetilde{H}_{a_1,a_2,\dots,a_r}(z)$. When $a_1,a_2,\dots,a_r$ vary over all possible choices, the series $\widetilde{H}_{a_1,a_2,\dots,a_r}(z)$ are *not* linearly independent over the ring $(\mathbb{Z}/3\mathbb{Z})[z]$, or over $(\mathbb{Z}/3^\gamma\mathbb{Z})[z]$ for a positive integer γ.

However, we shall show below (see Corollary 3) that, if we restrict to a_i's which are not divisible by 3, then the corresponding series $\widetilde{H}_{a_1,a_2,\dots,a_r}(z)$, together with the (trivial) series 1, *are* linearly independent over $(\mathbb{Z}/3\mathbb{Z})[z]$, and hence as well over $(\mathbb{Z}/3^\gamma\mathbb{Z})[z]$ for arbitrary positive integers γ (and also over $\mathbb{Z}[z]$). This fact underpins the arguments in the next section.

All other series $\widetilde{H}_{b_1,b_2,\dots,b_s}(z)$, where we do *not* make any restriction on the b_i's, can be expressed as a linear combination over $(\mathbb{Z}/3^\gamma\mathbb{Z})[z,(1+z)^{-1}]$ of 1 and the former series, and this can be carried out in an algorithmic fashion; see Lemma 9.

The problem of linear dependence/independence will be disposed of by transferring it to the problem of dependence/independence of simpler series, namely the series

$$H_{a_1,a_2,\dots,a_r}(z) := \sum_{n_1>n_2>\dots>n_r\ge0} z^{a_1 3^{n_1}+a_2 3^{n_2}+\dots+a_r 3^{n_r}}.$$

These series are in fact the analogues for the prime 3 of series which were considered in [7, Sec. 3] for the prime 2, and which were instrumental there in a theory for solving differential and functional equations modulo powers of 2. To see their relevance in the present context, one should observe that

$$\begin{aligned}\widetilde{H}_{a_1,a_2,\dots,a_r}(z) &= \sum_{k_1>\dots>k_r\ge 0}\prod_{j=1}^{s}\left(\frac{z^{3^{k_j}}(1+z^{3^{k_j}})}{1+z^{3^{k_j+1}}}\right)^{a_j}\\ &= \sum_{k_1>\dots>k_r\ge 0}\prod_{j=1}^{s}\left(\frac{z^{3^{k_j}}(1+z)^{3^{k_j}}}{(1+z)^{3^{k_j+1}}}\right)^{a_j} \quad \text{modulo } 3\\ &= H_{a_1,a_2,\dots,a_r}\left(\frac{z}{(1+z)^2}\right) \quad \text{modulo } 3. \end{aligned} \tag{2.8}$$

The idea which we have in mind is that, as a result of the last congruence, independence of (certain) series $H_{a_1,a_2,\dots,a_r}(z)$ implies *a fortiori* independence of the corresponding series $\widetilde{H}_{a_1,a_2,\dots,a_r}(z)$.

The auxiliary results which follow now are analogues of Lemma 6, Corollary 7, and Lemma 9 in [7]. The first of these pertains to the uniqueness of representations of integers as sums of powers of 3 with multiplicities, tailor-made for application to the series $H_{a_1,a_2,\dots,a_r}(z)$.

Lemma 2. *Let d,r,s be positive integers with $r \ge s$, c an integer with $|c| \le d$, and let $a_1,a_2,\dots,a_r$ respectively $b_1,b_2,\dots,b_s$ be two sequences of integers, none of them divisible by 3, with $1 \le a_i \le d$ for $1 \le i \le r$, and $1 \le b_i \le d$ for $1 \le i \le s$. If*

$$a_1 3^{2rd} + a_2 3^{2(r-1)d} + \dots + a_r 3^{2d} = b_1 3^{n_1} + b_2 3^{n_2} + \dots + b_s 3^{n_s} + c \tag{2.9}$$

for integers $n_1,n_2,\dots,n_s$ with $n_1 > n_2 > \dots > n_s \ge 0$, then $r=s$, $c=0$, $a_i=b_i$, and $n_i = 2d(r+1-i)$ for $i=1,2,\dots,r$.

Proof. We use induction on r.

First, let $r=1$. Then $s=1$ as well, and (2.9) becomes

$$a_1 3^{2d} = b_1 3^{n_1} + c. \tag{2.10}$$

If $n_1 > 2d$, then the above equation, together with the assumption that a_1 is not divisible by 3, implies

$$a_1 3^{2d} \equiv c \quad \text{modulo } 3^{2d+1}.$$

However, by assumption, we have $|c| \le d < a_1 3^{2d}$, which is absurd.

If $d < n_1 < 2d$, then it follows from (2.10) that c must be divisible by 3^{n_1}. Again by assumption, we have $|c| \le d < 3^d < 3^{n_1}$, so that $c=0$. But then (2.10) cannot be satisfied since b_1 is assumed not to be divisible by 3.

If $0 \le n_1 \le d$, then we estimate

$$b_1 3^{n_1} + c \le d\left(3^d+1\right) \le (3^d-1)(3^d+1) < 3^{2d},$$

which is again a contradiction to (2.10).

The only remaining possibility is $n_1 = 2d$. If this is substituted in (2.10) and the resulting equation is combined with $|c| \le d < 3^{2d}$, then the conclusion is that the equation can only be satisfied if $c=0$ and $a_1=b_1$, in accordance with the assertion of the lemma.

We now perform the induction step. We assume that the assertion of the lemma is established for all $r < R$, and we want to show that this implies its validity for $r = R$. Let t be maximal such that $n_t \ge 2d$. Then reduction of (2.9) modulo 3^{2d} yields

$$b_{t+1}3^{n_{t+1}} + b_{t+2}3^{n_{t+2}} + \cdots + b_s 3^{n_s} + c \equiv 0 \quad \text{modulo } 3^{2d}. \tag{2.11}$$

Let us write $b \cdot 3^{2d}$ for the left-hand side in (2.11). Then, by dividing (2.9) (with R instead of r) by 3^{2d}, we obtain

$$a_1 3^{2(R-1)d} + a_2 3^{2(R-2)d} + \cdots + a_{R-1}3^{2d} = b_1 3^{n_1-2d} + b_2 3^{n_2-2d} + \cdots + b_t 3^{n_t-2d} + b - a_R. \tag{2.12}$$

We have

$$\begin{aligned} 0 \le b &\le 3^{-2d} d \left(3^{2d-1} + 3^{2d-2} + \cdots + 3^{2d-s+t} + 1\right) \\ &\le 3^{-2d} d \left(\tfrac{1}{2}(3^{2d} - 3^{2d-s+t}) + 1\right) \le d. \end{aligned}$$

Consequently, we also have $|b - a_R| \le d$. This means that we are in a position to apply the induction hypothesis to (2.12). The conclusion is that $t = R-1$, $b - a_R = 0$, $a_i = b_i$, and $n_i = 2d(R+1-i)$ for $i = 1, 2, \dots, R-1$. If this is used in (2.9) with $r = R$, then we obtain

$$a_R 3^{2d} = c$$

or

$$a_R 3^{2d} = b_R 3^{n_R} + c,$$

depending on whether $s = R-1$ or $s = R$. The first case is absurd since $c \le d < 3^{2d} \le a_R 3^{2d}$. On the other hand, the second case has already been considered in (2.10), and we have seen there that it follows that $c = 0$, $a_R = b_R$, and $n_R = 2d$.

This completes the proof of the lemma. □

The independence of the series $H_{a_1,a_2,\dots,a_r}(z)$ with all a_i's not divisible by 3 is now an easy consequence.

Corollary 3. *The series $H_{a_1,a_2,\dots,a_r}(z)$, with all a_i's not divisible by 3, together with the series 1 are linearly independent over $(\mathbb{Z}/3\mathbb{Z})[z]$, and consequently as well over $(\mathbb{Z}/3^\gamma\mathbb{Z})[z]$ for an arbitrary positive integer γ, and over $\mathbb{Z}[z]$.*

Proof. Let us suppose that

$$p_0(z) + \sum_{i=1}^{N} p_i(z) H_{a_1^{(i)},a_2^{(i)},\dots,a_{r_i}^{(i)}}(z) = 0, \tag{2.13}$$

where the $p_i(z)$'s are non-zero polynomials in z over $\mathbb{Z}/3\mathbb{Z}$ (respectively over $\mathbb{Z}/3^\gamma\mathbb{Z}$ or over $\mathbb{Z}$), the r_i's are positive integers, and $a_j^{(i)}$, $j = 1, 2, \dots, r_i$, $i = 1, 2, \dots, N$, are integers not divisible by 3. We may also assume that the tuples $(a_1^{(i)}, a_2^{(i)}, \dots, a_{r_i}^{(i)})$, $i = 1, 2, \dots, N$, are pairwise distinct. Choose i_0 such that r_{i_0} is maximal among the r_i's. Without loss of generality, we may assume that the coefficient of z^0 in $p_{i_0}(z)$ is non-zero (otherwise we could multiply both sides of (2.13) by an appropriate power of z). Let d be the maximum of all $a_j^{(i)}$'s and the absolute values of exponents of z appearing in monomials with non-zero coefficient in the polynomials $p_i(z)$, $i = 0, 1, \dots, N$. Then, according to Lemma 2 with $r = r_{i_0}$, $a_j = a_j^{(i_0)}$, $j = 1, 2, \dots, r_{i_0}$, the coefficient of

$$z^{a_1^{(i_0)} 3^{2rd} + a_2^{(i_0)} 3^{2(r-1)d} + \cdots + a_r^{(i_0)} 3^{2d}}$$

is 1 in $H_{a_1^{(i_0)},a_2^{(i_0)},\dots,a_{r_{i_0}}^{(i_0)}}(z)$, while it is zero in series $z^e H_{a_1^{(i_0)},a_2^{(i_0)},\dots,a_{r_{i_0}}^{(i_0)}}(z)$, where e is a non-zero integer with $|e| \le d$, and in all other series $z^e H_{a_1^{(i)},a_2^{(i)},\dots,a_{r_i}^{(i)}}(z)$, $i = 1,\dots,i_0 - 1, i_0 + 1,\dots,N$, where e is a (not necessarily non-zero) integer with $|e| \le d$. This contradiction to (2.13) establishes the assertion of the corollary. □

In view of (2.8), the same result holds for the series $\widetilde{H}_{a_1,a_2,\dots,a_r}(z)$.

Corollary 4. *The series $\widetilde{H}_{a_1,a_2,\dots,a_r}(z)$, with all a_i's not divisible by 3, together with the series 1 are linearly independent over $(\mathbb{Z}/3\mathbb{Z})[z]$, and consequently as well over $(\mathbb{Z}/3^\gamma\mathbb{Z})[z]$ for an arbitrary positive integer γ, and over $\mathbb{Z}[z]$.*

The reader should recall from (2.6) and (2.7) that even powers of $\Psi(z)$ can be expanded in the series $\widetilde{H}_{a_1,a_2,\dots,a_s}(z)$, while odd powers can be expanded in the series $\Psi(z)\widetilde{H}_{a_1,a_2,\dots,a_s}(z)$. The next lemma shows that these two families of series are linearly independent from each other.

Lemma 5. *Let γ be a positive integer. If*

$$\sum_{j=0}^{N} a_{2j}(z)\Psi^{2j}(z) = \sum_{j=1}^{N} a_{2j-1}(z)\Psi^{2j-1}(z) \tag{2.14}$$

for polynomials $a_i(z)$ in z over $\mathbb{Z}/3^\gamma\mathbb{Z}$, then both sides in (2.14) *must be zero.*

Proof. We use the expansions (2.6) and (2.7). This leads to a relation of the form

$$\sum c_{a_1,a_2,\dots,a_s}(z)\widetilde{H}_{a_1,a_2,\dots,a_s}(z) = \Psi(z)\sum d_{a_1,a_2,\dots,a_s}(z)\widetilde{H}_{a_1,a_2,\dots,a_s}(z), \tag{2.15}$$

for certain polynomials $c_{a_1,a_2,\dots,a_s}(z)$ and $d_{a_1,a_2,h\dots,a_s}(z)$ in z. We consider this relation modulo 3. By (2.8), we then have

$$\sum c_{a_1,a_2,\dots,a_s}(z)H_{a_1,a_2,\dots,a_s}\left(\frac{z}{(1+z)^2}\right) \\ = \Psi(z)\sum d_{a_1,a_2,\dots,a_s}(z)H_{a_1,a_2,\dots,a_s}\left(\frac{z}{(1+z)^2}\right) \quad \text{modulo } 3.$$

Next we apply the square root to both sides of (2.2), to obtain

$$\Psi(z) = (1+z)^{-1/2} \quad \text{modulo } 3.$$

Substitution of this congruence in the congruence above then yields

$$\sum c_{a_1,a_2,\dots,a_s}(z)H_{a_1,a_2,\dots,a_s}\left(\frac{z}{(1+z)^2}\right) \\ = (1+z)^{-1/2}\sum d_{a_1,a_2,\dots,a_s}(z)H_{a_1,a_2,\dots,a_s}\left(\frac{z}{(1+z)^2}\right) \quad \text{modulo } 3. \tag{2.16}$$

Now we expand both sides as Laurent series in $1+z$, allowing only a finite number of positive powers of $1+z$. In order to do so, we observe that

$$\frac{z^k}{(1+z)^{2k}} = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell}(1+z)^{-k-\ell}.$$

This is used in each term of the expansions corresponding to $H_{a_1,a_2,\dots,a_s}(z/(1+z)^2)$. As a consequence, we see that the left-hand side of (2.16) is a (formal) Laurent series in $1+z$ with only finitely many positive powers of $1+z$, while the right-hand side is a Laurent series in $1+z$ of the same type, but multiplied by $(1+z)^{-1/2}$. This contradiction completes the proof of the lemma. □

We may now combine Corollary 4 and Lemma 5.

Theorem 6. *The series $\widetilde{H}_{a_1,a_2,\dots,a_r}(z)$ and $\Psi(z)\widetilde{H}_{a_1,a_2,\dots,a_r}(z)$, with all a_i's not divisible by 3, together with the series 1 are linearly independent over $(\mathbb{Z}/3\mathbb{Z})[z]$, and consequently as well over $(\mathbb{Z}/3^\gamma\mathbb{Z})[z]$ for an arbitrary positive integer γ, and over $\mathbb{Z}[z]$.*

In order to show that each $\widetilde{H}_{b_1,b_2,\dots,b_s}(z)$ can be expressed as a linear combination over $(\mathbb{Z}/3^\gamma\mathbb{Z})[z,(1+z)^{-1}]$ of the series 1 and series of the form $\tilde{H}_{a_1,a_2,\dots,a_r}(z)$, where none of the a_i's is divisible by 3 (see Lemma 9 below), we need two auxiliary identities, which are the subject of the following two lemmas.

Lemma 7. *Let n and k be positive integers. Then*

$$\left(\frac{z^{3^k}(1+z^{3^k})}{1+z^{3^{k+1}}}\right)^{3n}=\sum_{b\ge0}3^b\binom{-n}{b}\left(\frac{z^{3^{k+1}}(1+z^{3^{k+1}})}{1+z^{3^{k+2}}}\right)^{n+b}$$
$$+\sum_{\substack{a\ge1\\b\ge0}}\sum_{s_1,s_2,\dots,t_1,t_2,\dots\ge0}3^{a+b+\|s\|+\|t\|}\binom{n}{a}\binom{-n}{b}\left(\prod_{j\ge1}\binom{s_{j-1}}{s_j}\binom{-s_{j-1}}{t_j}\right)$$
$$\cdot\left(\frac{z^{3^k}(1+z^{3^k})}{1+z^{3^{k+1}}}\right)^{3(\|s\|+n+b)+\|t\|+a}, \qquad (2.17)$$

where, by convention, $s_0=n+b$, and we write $\|s\|$ for $s_1+s_2+\cdots$, with an analogous meaning for $\|t\|$.

Proof. First we argue that the sum in (2.17) is well-defined as a formal power series. Namely, given a positive integer $n+b$, non-zero summands in (2.17) arise only for $n+b=s_0\ge s_1\ge s_2\ge\cdots$. Hence, if we concentrate on vectors s and t for which $\|s\|+\|t\|=N$, with N a fixed positive integer, then there is only a finite number of vectors s for which non-zero summands exist. In particular, we have $s_i=0$ for $i>N$. In its turn, this entails $t_i=0$ for $i>N$ if one wants to have a non-vanishing summand. Therefore there are as well only finitely many possibilities for vectors t with $\|t\|\le N$.

Now, in order to prove the identity in (2.17), we start with the expansion

$$\left(\frac{z^{3^k}(1+z^{3^k})}{1+z^{3^{k+1}}}\right)^{3n}=z^{3^{k+1}n}\frac{(1+z^{3^{k+1}})^n}{(1+z^{3^{k+2}})^n}\frac{\left(1+3z^{3^k}\frac{1+z^{3^k}}{1+z^{3^{k+1}}}\right)^n}{\left(1+3z^{3^{k+1}}\frac{1+z^{3^{k+1}}}{1+z^{3^{k+2}}}\right)^n}$$
$$=\sum_{a,b\ge0}3^{a+b}\binom{n}{a}\binom{-n}{b}\left(\frac{z^{3^k}(1+z^{3^k})}{1+z^{3^{k+1}}}\right)^a\left(\frac{z^{3^{k+1}}(1+z^{3^{k+1}})}{1+z^{3^{k+2}}}\right)^{n+b}. \qquad (2.18)$$

The part of the double sum consisting of the terms with $a = 0$ directly yields the first sum on the right-hand side of (2.17). If $a \geq 1$, we want "to get rid of" the power with exponent $n+b$. In order to do so, we observe that, for an arbitrary positive integer m, we have

$$\left(\frac{z^{3^{k+1}}(1+z^{3^{k+1}})}{1+z^{3^{k+2}}}\right)^m = z^{3^{k+1}m}\frac{(1+z^{3^k})^{3m}}{(1+z^{3^{k+1}})^{3m}}\frac{\left(1+3z^{3^{k+1}}\frac{1+z^{3^{k+1}}}{1+z^{3^{k+2}}}\right)^m}{\left(1+3z^{3^k}\frac{1+z^{3^k}}{1+z^{3^{k+1}}}\right)^m}$$
$$= \sum_{s_1,t_1\geq 0} 3^{s_1+t_1}\binom{m}{s_1}\binom{-m}{t_1}\left(\frac{z^{3^k}(1+z^{3^k})}{1+z^{3^{k+1}}}\right)^{3m+t_1}\left(\frac{z^{3^{k+1}}(1+z^{3^{k+1}})}{1+z^{3^{k+2}}}\right)^{s_1}.$$

This relation is now iterated, to get

$$\left(\frac{z^{3^{k+1}}(1+z^{3^{k+1}})}{1+z^{3^{k+2}}}\right)^m$$
$$= \sum_{s_1,s_2,\dots,t_1,t_2,\dots\geq 0} 3^{\|s\|+\|t\|}\left(\prod_{j\geq 1}\binom{s_{j-1}}{s_j}\binom{-s_{j-1}}{t_j}\right)\left(\frac{z^{3^k}(1+z^{3^k})}{1+z^{3^{k+1}}}\right)^{3m+3\|s\|+\|t\|}, \quad (2.19)$$

where, by convention, $s_0 = m$. Finally, if this identity with $m = n+b$ is substituted in (2.18) in the terms with $a \geq 1$, then we obtain the right-hand side of (2.17). $\square$

Remark. What Lemma 7 affords is an expansion of a power of

$$z^{3^{k+1}}(1+z^{3^{k+1}})/(1+z^{3^{k+2}}) \quad (2.20)$$

with exponent divisible by 3 in terms which either have *coefficients* with higher divisibility by 3 or are a power of (2.20) with *exponent* of smaller 3-adic valuation than the original exponent (see the term for $b = 0$ in the first sum on the right-hand side of (2.17)).

Lemma 8. *For all non-negative integers j and positive integers α and β, we have*

$$\frac{1}{(1+z^{3^j})^\alpha} = \mathrm{Pol}_{j,\alpha,\beta}\left(z,(1+z)^{-1}\right) \quad \text{modulo } 3^\beta,$$

where $\mathrm{Pol}_{j,\alpha,\beta}\left(z,(1+z)^{-1}\right)$ is a polynomial in z and $(1+z)^{-1}$ with integer coefficients.

Proof. We perform an induction with respect to $j+\beta$. Clearly, for $\beta = 1$ and $j = 0$, there is nothing to prove.

For the induction step, we write

$$\begin{aligned}\frac{1}{(1+z^{3^j})^\alpha} &= \frac{1}{(1+z^{3^{j-1}})^{3\alpha}} + \frac{1}{(1+z^{3^j})^\alpha} - \frac{1}{(1+z^{3^{j-1}})^{3\alpha}}\\
&= \frac{1}{(1+z^{3^{j-1}})^{3\alpha}} + \frac{1}{(1+z^{3^{j-1}})^{3\alpha}(1+z^{3^j})^\alpha}\Big((1+z^{3^{j-1}})^{3\alpha} - (1+z^{3^j})^\alpha\Big)\\
&= \frac{1}{(1+z^{3^{j-1}})^{3\alpha}} + \frac{1}{(1+z^{3^{j-1}})^{3\alpha}(1+z^{3^j})^\alpha}\\
&\qquad\qquad \times\Big((1+3z^{3^{j-1}}+3z^{2\cdot 3^{j-1}}+z^{3^j})^\alpha - (1+z^{3^j})^\alpha\Big)\\
&= \frac{1}{(1+z^{3^{j-1}})^{3\alpha}} + \frac{1}{(1+z^{3^{j-1}})^{3\alpha}(1+z^{3^j})^\alpha}\\
&\qquad\qquad \times\sum_{\ell=1}^{\alpha}\binom{\alpha}{\ell}3^\ell\left(z^{3^{j-1}}+z^{2\cdot 3^{j-1}}\right)^\ell\left(1+z^{3^j}\right)^{\alpha-\ell}\\
&= \frac{1}{(1+z^{3^{j-1}})^{3\alpha}} + \sum_{\ell=1}^{\alpha}\binom{\alpha}{\ell}\frac{3^\ell z^{\ell\cdot 3^{j-1}}}{(1+z^{3^{j-1}})^{3\alpha-\ell}(1+z^{3^j})^\ell}.\end{aligned}$$

Now the induction hypothesis can be applied, and yields

$$\begin{aligned}\frac{1}{(1+z^{3^j})^\alpha} &= \operatorname{Pol}_{j-1,3\alpha,\beta}\left(z,(1+z)^{-1}\right)\\
&\quad+\sum_{\ell=1}^{\alpha}\binom{\alpha}{\ell}3^\ell z^{\ell\cdot 3^{j-1}}\operatorname{Pol}^{3\alpha-\ell}_{j-1,3\alpha-\ell,\beta-\ell}\left(z,(1+z)^{-1}\right)\operatorname{Pol}^{\ell}_{j,\ell,\beta-\ell}\left(z,(1+z)^{-1}\right)\\
&\qquad\text{modulo } 3^\beta.\end{aligned}$$

This completes the induction. □

We are now ready for the proof of the announced expansion result.

Lemma 9. *For positive integers $b_1, b_2, \dots, b_s$ and γ, the series $\widetilde{H}_{b_1,b_2,\dots,b_s}(z)$, can be expressed as a linear combination over $(\mathbb{Z}/3^\gamma\mathbb{Z})[z,(1+z)^{-1}]$ of the series 1 and series of the form $\widetilde{H}_{a_1,a_2,\dots,a_r}(z)$, where none of the a_i's is divisible by 3.*

Remark. What the proof below actually does is to find an expansion of $\widetilde{H}_{b_1,b_2,\dots,b_s}(z)$ over $\mathbb{Z}_3[z,(1+z)^{-1}]$, where $\mathbb{Z}_3$ denotes the ring of 3-adic integers. As this point of view will not play an explicit role in the sequel, we shall not pursue this aspect any further.

Proof of Lemma 9. With each term

$$c_{b_1,b_2,\dots,b_s}\widetilde{H}_{b_1,b_2,\dots,b_s}(z),$$

$c_{b_1,b_2,\dots,b_s}$ being some integer, we associate the quadruple (s,v,i,t), where

$$v = v_3(c_{b_1,b_2,\dots,b_s}),$$

i is the maximal index such that b_i is divisible by 3, and $t = v_3(b_i)$. We describe an algorithmic procedure for expressing $\widetilde{H}_{b_1,b_2,\dots,b_s}(z)$ in terms of series $\widetilde{H}_{a_1,a_2,\dots,a_r}(z)$ with associated quadruple less than the quadruple associated with $\widetilde{H}_{b_1,b_2,\dots,b_s}(z)$, according to the following total order of quadruples: we define $(s_1,v_1,i_1,t_1) \prec (s_2,v_2,i_2,t_2)$ if, and only if,

$$\begin{aligned} &s_1 < s_2,\\ \text{or } &s_1 = s_2 \text{ and } v_1 > v_2,\\ \text{or } &s_1 = s_2,\ v_1 = v_2,\ \text{and } i_1 < i_2,\\ \text{or } &s_1 = s_2,\ v_1 = v_2,\ i_1 = i_2,\ \text{and } t_1 < t_2. \end{aligned}$$

Our algorithmic procedure consists of four recurrence relations, (2.22)–(2.25) below. These are based on Hou's reduction idea in [6].

Let h be an integer between 1 and s. Furthermore, we assume that b_h is divisible by 3 and that all b_i's with $i > h$ are not divisible by 3. We write $b_h = 3b'_h$. Then, from the definition of the series $\widetilde H_{b_1,b_2,\dots,b_s}(z)$ and from Lemma 7, we have

$$\begin{aligned} \widetilde H_{b_1,b_2,\dots,b_s}(z) &= \sum_{k_1>\dots>k_{s-1}>k_s\ge0}\left(\prod_{\substack{j=1\\j\ne h}}^{s}\left(\frac{z^{3^{k_j}}(1+z^{3^{k_j}})}{1+z^{3^{k_j+1}}}\right)^{b_j}\right)\left(\frac{z^{3^{k_h}}(1+z^{3^{k_h}})}{1+z^{3^{k_h+1}}}\right)^{3b'_h}\\ &=\sum_{b\ge0}3^b\binom{-b'_h}{b}\sum_{k_1>\dots>k_{s-1}>k_s\ge0}\left(\prod_{\substack{j=1\\j\ne h}}^{s}\left(\frac{z^{3^{k_j}}(1+z^{3^{k_j}})}{1+z^{3^{k_j+1}}}\right)^{b_j}\right)\left(\frac{z^{3^{k_h+1}}(1+z^{3^{k_h+1}})}{1+z^{3^{k_h+2}}}\right)^{b'_h+b}\\ &\quad+\sum_{\substack{a\ge1\\b\ge0}}\sum_{s_1,s_2,\dots,t_1,t_2,\dots\ge0}3^{a+b+\|s\|+\|t\|}\binom{b'_h}{a}\binom{-b'_h}{b}\left(\prod_{j\ge1}\binom{s_{j-1}}{s_j}\binom{-s_{j-1}}{t_j}\right)\\ &\quad\cdot\sum_{k_1>\dots>k_{s-1}>k_s\ge0}\left(\prod_{\substack{j=1\\j\ne h}}^{s}\left(\frac{z^{3^{k_j}}(1+z^{3^{k_j}})}{1+z^{3^{k_j+1}}}\right)^{b_j}\right)\left(\frac{z^{3^{k_h}}(1+z^{3^{k_h}})}{1+z^{3^{k_h+1}}}\right)^{3(\|s\|+b'_h+b)+\|t\|+a}. \end{aligned} \tag{2.21}$$

Now we distinguish four cases. First let $1 < h < s$. In the first of the above sums on the right-hand side, let $k'_h = k_h + 1$ be a new summation index. We observe that we have

$$\begin{aligned} &\{(k_1,\dots,k_{h-1},k_h+1,k_{h+1},\dots,k_s) : k_1>\dots>k_{h-1}>k_h>k_{h+1}>\dots>k_s\ge0\}\\ &=\Big(\{(k_1,\dots,k_{h-1},k'_h,k_{h+1},\dots,k_s) : k_1>\dots>k_{h-1}>k'_h>k_{h+1}>\dots>k_s\ge0\}\\ &\qquad\dot\cup\{(k_1,\dots,k_{h-1},k_{h-1},k_{h+1},\dots,k_s) : k_1>\dots>k_{h-1}>k_{h+1}>\dots>k_s\ge0\}\Big)\\ &\qquad\Big\backslash\{(k_1,\dots,k_{h-1},k_{h+1}+1,k_{h+1},\dots,k_s) : k_1>\dots>k_{h-1}>k_{h+1}>\dots>k_s\ge0\}, \end{aligned}$$

where the union is a disjoint union, and the third set on the right-hand side is entirely contained in the union of the first two. If we apply this observation to the (new) index set of the first sum, then, from (2.21), we obtain

$$
\begin{aligned}
\widetilde{H}_{b_1,b_2,\dots,b_s}(z) = \sum_{b\ge0} 3^b \binom{-b'_h}{b} \Bigg(& \widetilde{H}_{b_1,b_2,\dots,b_{h-1},b'_h+b,b_{h+1},\dots,b_s}(z) \\
& \qquad\qquad + \widetilde{H}_{b_1,b_2,\dots,b_{h-1}+b'_h+b,b_{h+1},\dots,b_s}(z) \\
- \sum_{k_1>\dots>k_{s-1}>k_s\ge0} & \left(\prod_{\substack{j=1\\ j\ne h}}^{s} \left(\frac{z^{3^{k_j}}(1+z^{3^{k_j}})}{1+z^{3^{k_j+1}}}\right)^{b_j}\right) \left(\frac{z^{3^{k_{h+1}+1}}(1+z^{3^{k_{h+1}+1}})}{1+z^{3^{k_{h+1}+2}}}\right)^{b'_h+b}\Bigg) \\
+ \sum_{\substack{a\ge1\\ b\ge0}} \sum_{s_1,s_2,\dots,t_1,t_2,\dots\ge0} & 3^{a+b+\|s\|+\|t\|} \binom{b'_h}{a}\binom{-b'_h}{b} \left(\prod_{j\ge1}\binom{s_{j-1}}{s_j}\binom{-s_{j-1}}{t_j}\right) \\
& \cdot \widetilde{H}_{b_1,b_2,\dots,b_{h-1},3(\|s\|+b'_h+b)+\|t\|+a,b_{h+1},\dots,b_s}(z).
\end{aligned}
$$

To the term with exponent b'_h+b we apply the identity (2.19). Then the above relation becomes

$$
\begin{aligned}
\widetilde{H}_{b_1,b_2,\dots,b_s}(z) = \sum_{b\ge0} 3^b \binom{-b'_h}{b} \Bigg(& \widetilde{H}_{b_1,b_2,\dots,b_{h-1},b'_h+b,b_{h+1},\dots,b_s}(z) \\
& \qquad\qquad + \widetilde{H}_{b_1,b_2,\dots,b_{h-1}+b'_h+b,b_{h+1},\dots,b_s}(z) \\
- \sum_{u_1,u_2,\dots,v_1,v_2,\dots\ge0} & 3^{\|u\|+\|v\|} \left(\prod_{j\ge1}\binom{u_{j-1}}{u_j}\binom{-u_{j-1}}{v_j}\right) \\
& \cdot \widetilde{H}_{b_1,b_2,\dots,b_{h-1},3(b'_h+b+\|u\|)+\|v\|+b_{h+1},\dots,b_s}(z)\Bigg) \\
+ \sum_{\substack{a\ge1\\ b\ge0}} \sum_{s_1,s_2,\dots,t_1,t_2,\dots\ge0} & 3^{a+b+\|s\|+\|t\|} \binom{b'_h}{a}\binom{-b'_h}{b} \left(\prod_{j\ge1}\binom{s_{j-1}}{s_j}\binom{-s_{j-1}}{t_j}\right) \\
& \cdot \widetilde{H}_{b_1,b_2,\dots,b_{h-1},3(\|s\|+b'_h+b)+\|t\|+a,b_{h+1},\dots,b_s}(z).
\end{aligned}
\tag{2.22}
$$

It should be observed that the quadruples associated with the terms on the right-hand side of (2.22) are indeed less than the quadruple associated with $\widetilde{H}_{b_1,b_2,\dots,b_s}(z)$ in the order $\prec$.

Next we consider the case where $1 < h = s$. Reasoning in a way similar to the one which in the end led to (2.22), we obtain the recurrence relation

$$\widetilde{H}_{b_1,b_2,\dots,b_s}(z) = \sum_{b\ge0} 3^b \binom{-b'_s}{b} \Bigg(\widetilde{H}_{b_1,b_2,\dots,b_{s-1},b'_s+b}(z) \\ + \widetilde{H}_{b_1,b_2,\dots,b_{s-1}+b'_s+b}(z) - \left(\frac{z(1+z)}{1+z^3}\right)^{b'_s+b} \widetilde{H}_{b_1,b_2,\dots,b_{s-1}}(z) \Bigg) \\ + \sum_{\substack{a\ge1\\ b\ge0}} \sum_{s_1,s_2,\dots,t_1,t_2,\dots\ge0} 3^{a+b+\|s\|+\|t\|} \binom{b'_s}{a} \binom{-b'_s}{b} \left(\prod_{j\ge1} \binom{s_{j-1}}{s_j} \binom{-s_{j-1}}{t_j} \right) \\ \cdot \widetilde{H}_{b_1,b_2,\dots,b_{s-1},3(\|s\|+b'_s+b)+\|t\|+a}(z). \tag{2.23}$$

Again, the quadruples associated with the terms on the right-hand side of (2.23) are indeed less than the quadruple associated with $\widetilde{H}_{b_1,b_2,\dots,b_s}(z)$ in the order $\prec$.

Now let $1 = h < s$. In this case, we obtain the recurrence relation

$$\widetilde{H}_{b_1,b_2,\dots,b_s}(z) = \sum_{b\ge0} 3^b \binom{-b'_1}{b} \Bigg(\widetilde{H}_{b'_1+b,b_2,\dots,b_s}(z) \\ - \sum_{u_1,u_2,\dots,v_1,v_2,\dots\ge0} 3^{\|u\|+\|v\|} \left(\prod_{j\ge1} \binom{u_{j-1}}{u_j} \binom{-u_{j-1}}{v_j} \right) \widetilde{H}_{3(b'_1+b+\|u\|)+\|v\|+b_2,\dots,b_s}(z) \Bigg) \\ + \sum_{\substack{a\ge1\\ b\ge0}} \sum_{s_1,s_2,\dots,t_1,t_2,\dots\ge0} 3^{a+b+\|s\|+\|t\|} \binom{b'_1}{a} \binom{-b'_1}{b} \left(\prod_{j\ge1} \binom{s_{j-1}}{s_j} \binom{-s_{j-1}}{t_j} \right) \\ \cdot \widetilde{H}_{3(\|s\|+b'_1+b)+\|t\|+a,b_2,\dots,b_s}(z). \tag{2.24}$$

Finally, in the degenerate case $1 = h = s$, summation over k on both sides of (2.17) with $n = b'_1$ directly yields

$$\widetilde{H}_{b_1}(z) = \sum_{b\ge0} 3^b \binom{-b'_1}{b} \left(\widetilde{H}_{b'_1+b}(z) - \left(\frac{z(1+z)}{1+z^3}\right)^{b'_1+b} \right) \\ + \sum_{\substack{a\ge1\\ b\ge0}} \sum_{s_1,s_2,\dots,t_1,t_2,\dots\ge0} 3^{a+b+\|s\|+\|t\|} \binom{b'_1}{a} \binom{-b'_1}{b} \left(\prod_{j\ge1} \binom{s_{j-1}}{s_j} \binom{-s_{j-1}}{t_j} \right) \\ \cdot \widetilde{H}_{3(\|s\|+b'_1+b)+\|t\|+a}(z). \tag{2.25}$$

It is clear that, if we recursively apply (2.22)–(2.25) to a given series $\widetilde{H}_{b_1,b_2,\dots,b_s}(z)$, and use $\widetilde{H}_{\emptyset}(z) = 1$ as an initial condition, we will eventually arrive at a linear combination of 1, powers of $z(1+z)/(1+z^3)$, and series $\widetilde{H}_{a_1,a_2,\dots,a_r}(z)$ with all a_i's being not divisible by 3, where the coefficients are polynomials in z and $(1+z^3)^{-1}$. In view of Lemma 8, this completes the proof of the lemma. $\square$

How the results of this section are implemented to extract coefficients of concrete powers of z from powers of $\Psi(z)$ modulo a given 3-power is explained in more detail in the appendix. As an illustration, the coefficients of $\Psi^3(z)$ and of $\Psi^5(z)$ are worked out explicitly modulo 9 and modulo 27, see Propositions 68 and 71.

3. Minimal polynomials for $\Psi(z)$

Here we consider polynomial relations of the Cantor-like formal power series $\Psi(z)$ defined in (1.1), when coefficients are reduced modulo powers of 3. The series is transcendental over $\mathbb{Z}[z]$ (see Theorem 10 below). However, if the coefficients of $\Psi(z)$ are considered modulo a 3-power 3^γ, then $\Psi(z)$ obeys a polynomial relation with coefficients that are Laurent polynomials in z and $1+z$. The focus of this section is on what can be said concerning such polynomial relations, and, in particular, about those of minimal length.

Here and in the sequel, given integral power series (or Laurent series) $f(z)$ and $g(z)$, we write

$$f(z) = g(z) \text{ modulo } 3^\gamma$$

to mean that the coefficients of z^i in $f(z)$ and $g(z)$ agree modulo 3^γ for all i.

We say that a polynomial $A(z,t)$ in z and t is *minimal for the modulus* 3^γ, if it is monic (as a polynomial in t), has coefficients which are Laurent polynomials in z and $1+z$, satisfies $A(z,\Psi(z)) = 0$ modulo 3^γ, and there is no monic polynomial $B(z,t)$ whose coefficients are Laurent polynomials in z and $1+z$, whose t-degree is less than that of $A(z,t)$, and which satisfies $B(z,\Psi(z)) = 0$ modulo 3^γ. Minimal polynomials are not unique; see Remark 15 below.

Our results concerning minimal polynomials are analogous to the corresponding ones in [7, Sec. 2], albeit more difficult to prove. We start by providing a lower bound on degrees of minimal polynomials. We strongly believe that this lower bound is actually sharp; see Conjecture 12 below. In order to formulate the result, we let $v_3(\alpha)$ denote the 3-adic valuation of the integer α, that is, the maximal exponent e such that 3^e divides α.

Theorem 10. *The degree of a minimal polynomial for the modulus 3^γ, $\gamma \ge 1$, is $2d$, where d satisfies $d + v_3(d!) \ge \gamma$. In particular, the series $\Psi(z)$ is transcendental over $\mathbb{Z}[z]$.*

For the proof of this theorem, we need the following auxiliary result.

Lemma 11. *Let $a, b, c_1, c_2, \dots, c_d$ be positive integers with $a > b$. Then*

$$\sum_{k=0}^{b} (-1)^{b-k} \binom{b}{k} \prod_{j=1}^{a} 3^{c_j} \binom{k}{c_j} \tag{3.1}$$

is divisible by $3^a b!$. It is even divisible by $3^{a+1} b!$ if at least one of the c_j's is at least 3.

Proof. We need to recall some facts from classical difference operator calculus (cf. [14]). We denote by E the shift operator defined by $Ef(y) = f(y+1)$, and by $\triangle$ the forward difference operator defined by $\triangle f(y) = f(y+1) - f(y)$. The reader should observe that $\triangle = E - I$, where I stands for the identity operator. In terms of these operators, we may rewrite the expression in (3.1) as

$$\begin{aligned}\sum_{k=0}^{b}(-1)^{b-k}\binom{b}{k}E^k\prod_{j=1}^{a}3^{c_j}\binom{y}{c_j}\bigg|_{y=0}&=(E-I)^b\prod_{j=1}^{a}3^{c_j}\binom{y}{c_j}\bigg|_{y=0}\\&=\left(\prod_{j=1}^{a}\frac{(-3)^{c_j}}{c_j!}\right)\triangle^b\prod_{j=1}^{a}(-y)_{c_j}\bigg|_{y=0},\end{aligned}\tag{3.2}$$

where the Pochhammer symbol is defined by $(\alpha)_m=\alpha(\alpha+1)\cdots(\alpha+m-1)$ for $m\ge 1$, and $(\alpha)_0=1$, The second product over j is a polynomial in y with integer coefficients. It is well-known from difference calculus that the constant term of the b-fold application of the operator $\triangle$ on such a polynomial is divisible by $b!$. For the sake of self-containedness, we provide the simple argument here: let $p(y)$ be a polynomial in y with integer coefficients. Then we may expand $p(y)$ in Pochhammer symbols,

$$p(y)=\sum_{m\ge 0}\alpha_m(-y)_m,$$

where the coefficients α_m are all integers. Since $E(-y)_m=-m\cdot(-y)_{m-1}$, we infer

$$E^bp(y)\Big|_{y=0}=(-1)^b\sum_{m\ge 0}\alpha_m\frac{m!}{(m-b)!}(-y)_{m-b}\Big|_{y=0}=(-1)^b\alpha_b b!,$$

which is visibly divisible by $b!$.

If we use what we know in (3.2), together with the easily verified fact that

$$v_3(3^\alpha)\ge\begin{cases}v_3(\alpha!)+1,&\text{if }\alpha\ge 1,\\ v_3(\alpha!)+2,&\text{if }\alpha\ge 3,\end{cases}$$

then the assertion of the lemma follows immediately. □

Proof of Theorem 10. The first observation is that Lemma 5 implies that minimal polynomials must have even degree, and, moreover, that they can be chosen as even polynomials in t, that is, as polynomials in t^2.

So, for a contradiction, let us suppose that $A(z,t)$ is the minimal polynomial for the modulus 3^γ, that it is even, and that it has degree $2d$, where $d+v_3(d!)<\gamma$.

Since $A(z,t)$ is even in t, it does not matter whether we consider $A(z,t)$ as a polynomial in t^2 (and z and $(1+z)^{-1}$) or as a polynomial in $\left(t^2-\frac{1}{1+z}\right)$ (and z and $(1+z)^{-1}$). We take the latter point of view.

We now expand a power $\big(\Psi^2(z)-\frac{1}{1+z}\big)^b$, $1\le b\le d$, in terms of the series $\widetilde H_{a_1,a_2,\dots,a_s}(z)$ (see (2.3) for the definition) using (2.5) and the short notation (2.4):

$$\begin{aligned}\left(\Psi^2(z)-\frac{1}{1+z}\right)^b&=\frac{1}{(1+z)^b}\left(\sum_{s\ge1}3^s\widetilde H_{\underbrace{1,1,\dots,1}_{s\text{ times}}}(z)\right)^b\\
&=\frac{1}{(1+z)^b}\left(\sum_{s\ge1}3^s\sum_{k_1>\dots>k_s\ge0}R(k_1)R(k_2)\cdots R(k_s)\right)^b\\
&=\frac{1}{(1+z)^b}\left(\left(\prod_{j\ge0}(1+3R(j))\right)-1\right)^b\\
&=\frac{1}{(1+z)^b}\sum_{k=0}^b(-1)^{b-k}\binom bk\prod_{j\ge0}(1+3R(j))^k\\
&=\frac{1}{(1+z)^b}\sum_{s\ge1}\sum_{a_1,\dots,a_s\ge1}c(b;a_1,a_2,\dots,a_s)\,\widetilde H_{a_1,a_2,\dots,a_s}(z),\end{aligned}\tag{3.3}$$

where the $c(b;a_1,a_2,\dots,a_s)$'s are certain combinatorial (and, hence, integer) coefficients. Not all of the $\widetilde H$-series which occur with non-zero coefficient $c(b;a_1,a_2,\dots,a_s)$ will have the property that all indices a_j are not divisible by 3. So, in a second step, one will have to apply the reduction procedure described in Lemma 9 to arrive at an expansion of the form

$$\left(\Psi^2(z)-\frac{1}{1+z}\right)^b=\frac{1}{(1+z)^b}\sum_{s\ge1}\sum_{\substack{a_1,\dots,a_s\ge1\\3\nmid a_1,\dots,3\nmid a_s}}\widetilde c(b;a_1,a_2,\dots,a_s)\widetilde H_{a_1,a_2,\dots,a_s}(z),\tag{3.4}$$

where the $\widetilde c(b;a_1,a_2,\dots,a_s)$'s are certain modified coefficients. It should be observed (cf. Corollary 4) that the $\widetilde H$-series on the right-hand side of (3.4) are linearly independent over $(\mathbb Z/3^\gamma\mathbb Z)[z,(1+z)^{-1}]$.

We are interested in the coefficient $\widetilde c(b;1,1,\dots,1)$ with a occurrences of 1. We claim that

$$\widetilde c(b;\underbrace{1,1,\dots,1}_{a\text{ times}})=\begin{cases}3^{a+1}\,b!\,N_1,&\text{if }a<b,\\3^b\,b!+3^{b+v_3(b!)+1}N_2,&\text{if }a=b,\\3^a\,b!\,N_3,&\text{if }a>b,\end{cases}\tag{3.5}$$

where N_1,N_2,N_3 are integers (depending on a and b).

In order to see this, we must examine how terms $\widetilde H_{1,\dots,1}(z)$ with a non-zero coefficient $\widetilde c(b;1,1,\dots,1)$ in the expansion (3.4) arise from (3.3). From the proof of Lemma 9, we see that such terms come from terms

$$c(b;3^{d_1},3^{d_2},\dots,3^{d_a})\,\widetilde H_{3^{d_1},3^{d_2},\dots,3^{d_a}}(z),$$

in (3.3) with non-zero coefficient $c(\dots)$. (To be correct, there are also contributions to $\widetilde c(b;1,1,\dots,1)$ from other terms, but these will have a higher 3-divisibility than those discussed here. We shall not mention them any further. Suffice it to point out that they are subsumed in the terms $N_4(b;d_1,d_2,\dots,d_a)$ below.) If these observations are put together, one obtains

$$\widetilde{c}(b;\underbrace{1,1,\dots,1}_{a\text{ times}})=\sum_{d_1,\dots,d_a\ge0}c(b;3^{d_1},3^{d_2},\dots,3^{d_a})\left(1+3N_4(b;d_1,d_2,\dots,d_a)\right),\qquad(3.6)$$

where the $N_4(b;d_1,d_2,\dots,d_a)$'s are integers.

We must now compute the coefficient $c(b;3^{d_1},3^{d_2},\dots,3^{d_a})$. As we already said, this is a combinatorial coefficient arising in the expansion (3.3). Recalling the definition (2.3) of the $\widetilde{H}$-series, one sees that this coefficient equals the coefficient of

$$R(0)^{3^{d_1}}R(1)^{3^{d_2}}\cdots R(a-1)^{3^{d_a}}$$

in the line just above (3.3). An easy extraction yields

$$c(b;3^{d_1},3^{d_2},\dots,3^{d_a})=\sum_{k=0}^{b}(-1)^{b-k}\binom{b}{k}\prod_{j=1}^{a}3^{3^{d_j}}\binom{k}{3^{d_j}}.$$

By Lemma 11 we know that this expression is divisible by $3^a\,b!$, and, if one of the d_j's is positive, even by $3^{a+1}\,b!$. If this is used in (3.6), then our claim (3.5) follows without much effort, the term $3^b\,b!$ (occurring in (3.5) in the case where $a=b$) being contributed by

$$c(b;\underbrace{1,1,\dots,1}_{b\text{ times}})=3^b\,b!.$$

We are now in the position to complete the argument. Let us return to the minimal polynomial $A(z,t)$ for the modulus 3^γ. We expand it in powers of $\left(t^2-\frac{1}{1+z}\right)$,

$$A(z,t)=\sum_{b=0}^{d}A_b\left(t^2-\frac{1}{1+z}\right)^b,\qquad(3.7)$$

where $A_d=1$. (The reader should recall that minimal polynomials are monic by definition.) We substitute $\Psi(z)$ for t and claim that $A(z,\Psi(z))$ cannot vanish modulo 3^γ. In order to see this, we perform an induction on the index b in (3.7). The claim that we are going to prove by this induction is that the coefficient A_a is divisible by $3^{b+v_3(b!)-a-v_3(a!)}$ for all a with $1\le a\le b$. This is trivially true for $b=1$, since in that case the only possible a is $a=1$, in which case the condition is void.

We proceed to the induction step. In the following, we shall use the short notation $\widetilde{H}_{1^r}(z)$ for $\widetilde{H}_{1,1,\dots,1}(z)$ with r occurrences of 1. We consider the coefficient of $\widetilde{H}_{1^b}(z)$ in $\left(t^2-\frac{1}{1+z}\right)^b$. Let us denote it by C. According to (3.5), it equals $3^b\,b!+3^{b+v_3(b!)+1}N_2$, where N_2 is some integer. In particular, it is divisible by $3^{b+v_3(b!)}$, but not by a higher power of 3. By using (3.5) again (this time with a and b interchanged), in combination with the induction hypothesis, we see that the coefficient of $\widetilde{H}_{1^b}(z)$ in the term $A_a\left(\Psi^2(z)-\frac{1}{1+z}\right)^a$, $a<b$, is divisible by $3^{b+v_3(a!)}3^{b+v_3(b!)-a-v_3(a!)}=3^{2b-a+v_3(b!)}\ge 3^{b+v_3(b!)+1}$. This is a higher divisibility by 3 than that of the coefficient C. Moreover, by another application of (3.5), the coefficient of $\widetilde{H}_{1^b}(z)$ in the term $A_a\left(\Psi^2(z)-\frac{1}{1+z}\right)^a$, $a>b$, is divisible by

$$3^{b+1+v_3(a!)}\ge3^{b+v_3((b+1)!)+1}\ge3^{b+v_3(b!)+1},\qquad(3.8)$$

which is again a higher 3-divisibility than that of the coefficient C. So none of these terms can cancel C, unless A_b is divisible by 3. Assuming the latter, one considers the coefficient of $\widetilde{H}_{1^{b-1}}$ in $A_{b-1}\left(\Psi^2(z)-\frac{1}{1+z}\right)^{b-1}$ and compares its 3-divisibility to the 3-divisibility of the corresponding coefficients in the other terms. The conclusion is that also A_{b-1} must be divisible by another 3, that is (the reader should recall the induction hypothesis), it must be divisible by $3^{b+v_3(b!)-(b-1)-v_3((b-1)!)+1}$. One repeats this same argument with the coefficient of $\widetilde{H}_{1^a}$ in $A_a\left(\Psi^2(z)-\frac{1}{1+z}\right)^a$, $a=b-2,\dots,2,1$, to conclude that A_a must be divisible by $3^{b+v_3(b!)-a-v_3(a!)+1}$, $a=1,2,\dots,b$.

Now one returns to the term $A_b\left(\Psi^2(z)-\frac{1}{1+z}\right)^b$. So far, we know that A_b must be divisible by 3, and that the coefficient of $\widetilde{H}_{1^b}$ in $\left(\Psi^2(z)-\frac{1}{1+z}\right)^b$ is divisible by $3^{b+v_3(b!)}$, but not by a higher power of 3. If we repeat the arguments of the previous paragraph, but take into account that they led to an "improvement" of the induction hypothesis, then one is led to the conclusion that A_b must have an even higher 3-divisibility, entailing analogous higher 3-divisibility of the coefficients A_a with $a<b$, until one reaches the conclusion that A_b must be divisible by $3^{b+1+v_3((b+1)!)-b-v_3(b!)}$ (cf. the middle term in (3.8)), and consequently A_a by $3^{b+1+v_3((b+1)!)-a-v_3(a!)}$, $a=1,2,\dots,b-1$. This is exactly the induction hypothesis with b replaced by $b+1$.

When we use what we have just proven for $b=d$, then this produces a contradiction. Namely, if A_a is divisible by $3^{d+v_3(d!)-a-v_3(a!)}$, $a=1,2,\dots,d$, then the above arguments would imply that A_d must be divisible by (at least) 3. This, however, is impossible since $A_d=1$.

We have thus shown that, indeed, a minimal polynomial for the modulus 3^γ, where $d+v_3(d!)<\gamma$ must have degree larger than $2d$. This completes the proof of the theorem. □

Based on the observations in Proposition 14 later in this section and Theorem 10, we propose the following conjecture.

Conjecture 12. *The degree of a minimal polynomial for the modulus 3^γ, $\gamma\ge 1$, is $2d$, where d is the least positive integer such that $d+v_3(d!)\ge\gamma$.*

Remark 13. (1) Given the ternary expansion of d, say

$$d=d_0+d_1\cdot 3+d_2\cdot 9+\cdots+d_r\cdot 3^r,\quad 0\le\gamma_i\le 1,$$

we have, by the well-known formula of Legendre [9, p. 10],

$$\begin{aligned}v_3(d!)&=\sum_{\ell=1}^{\infty}\left\lfloor\frac{d}{3^\ell}\right\rfloor=\sum_{\ell=1}^{\infty}\left\lfloor\sum_{i=0}^{r}d_i3^{i-\ell}\right\rfloor=\sum_{\ell=1}^{\infty}\sum_{i=\ell}^{r}d_i3^{i-\ell}\\&=\sum_{i=1}^{r}\sum_{\ell=1}^{i}d_i3^{i-\ell}=\sum_{i=1}^{r}\frac{d_i}{2}\left(3^i-1\right)=\frac{1}{2}(d-s(d)),\end{aligned}\tag{3.9}$$

where $s(d)$ denotes the sum of digits of d in its ternary expansion. Consequently, an equivalent way of phrasing Theorem 10 is to say that the degree of a minimal polynomial for the modulus 3^γ is $2d$, where d is the least positive integer with $\frac{1}{2}(3d-s(d))\ge\gamma$.

(2) We claim that, in order to establish Conjecture 12, it suffices to prove the conjecture for $\gamma=\frac{1}{2}(3^{\delta+1}-1)$, $\delta=0,1,\dots$. If we take into account the remark in (1), this

means that it is sufficient to prove that, for each $\delta \ge 0$, there is a polynomial $A_\delta(z,t)$ of degree $2\cdot 3^\delta$ such that

$$A_\delta(z,\Psi(z)) = 0 \quad \text{modulo } 3^{(3^{\delta+1}-1)/2}. \tag{3.10}$$

For, arguing by induction, let us suppose that we have already constructed $A_0(z,t)$, $A_1(z,t),\dots,A_m(z,t)$ satisfying (3.10). Let

$$\alpha = \alpha_0 + \alpha_1\cdot 3 + \alpha_2\cdot 9 + \dots + \alpha_m\cdot 3^m, \quad 0\le \alpha_i \le 2,$$

be the ternary expansion of the positive integer α. In this situation, we have

$$\prod_{\delta=0}^{m} A_\delta^{\alpha_\delta}(z,\Psi(z)) = 0 \quad \text{modulo } \prod_{\delta=0}^{m} 3^{\alpha_\delta(3^{\delta+1}-1)/2} = 3^{(3\alpha-s(\alpha))/2}. \tag{3.11}$$

On the other hand, the degree of the left-hand side of (3.11) as a polynomial in $\Psi(z)$ is $\sum_{\delta=0}^m \alpha_\delta\cdot 2\cdot 3^\delta = 2\alpha$.

Let us put these observations together. In view of (3.9), our lower bound assumption says that the degree of a minimal polynomial for the modulus 3^γ cannot be smaller than the least even integer, $2d^{(\gamma)}$ say, for which $\frac{1}{2}(3d^{(\gamma)} - s(d^{(\gamma)})) \ge \gamma$. If we take into account that the quantity $\frac{1}{2}(3\alpha - s(\alpha))$, as a function in α, is weakly monotone increasing in α, then (3.11) tells us that, as long as $d^{(\gamma)} \le 2 + 2\cdot 3 + 2\cdot 9 + \dots + 2\cdot 3^m = 3^{m+1}-1$, we have found a monic polynomial of degree $2d^{(\gamma)}$, $B_\gamma(z,t)$ say, for which $B_\gamma(z,\Psi(z)) = 0$ modulo 3^γ, namely the left-hand side of (3.11) with α replaced by $d^{(\gamma)}$, to wit

$$B_\gamma(z,t) = \prod_{\delta=1}^{m} A_\delta^{d_\delta^{(\gamma)}}(z,t),$$

where $d^{(\gamma)} = d_0^{(\gamma)} + d_1^{(\gamma)}\cdot 3 + d_2^{(\gamma)}\cdot 9 + \dots + d_m^{(\gamma)}\cdot 3^m$ is the ternary expansion of $d^{(\gamma)}$. Hence, it must necessarily be a minimal polynomial for the modulus 3^γ.

Since $\frac{1}{2}\big(3(3^{m+1}-1) - s(3^{m+1}-1)\big) = \frac{1}{2}(3^{m+2}-3-2(m+1))$, we have thus found minimal polynomials for all moduli 3^γ with $\gamma \le \frac{1}{2}(3^{m+2}-2m-5)$. Now we should note that the quantity $\frac{1}{2}(3\alpha - s(\alpha))$ makes a jump from $\frac{1}{2}(3^{m+2}-2m-5)$ to $\frac{1}{2}(3^{m+2}-1)$ when we move from $\alpha = 3^{m+1}-1$ to $\alpha = 3^{m+1}$. If we take $A_m^3(z,t)$, which has degree $2\cdot 3^{m+1}$, then, by (3.10), we also have a minimal polynomial for the modulus $\left(3^{(3^{m+1}-1)/2}\right)^3 = 3^{(3^{m+2}-3)/2}$ and, in view of the preceding remark, as well for all moduli 3^γ with γ between $\frac{1}{2}(3^{m+2}-2m-5)$ and $\frac{1}{2}(3^{m+2}-3)$.

So, indeed, the first modulus for which we do not have a minimal polynomial is the modulus $3^{(3^{m+2}-1)/2}$. This is the role which $A_{m+1}(z,t)$ (see (3.10) with $m+1$ in place of δ) would have to play.

(3) The arguments in Item (2) show at the same time that, supposing that we have already constructed $A_0(z,t), A_1(z,t),\dots,A_m(z,t)$, the polynomial $A_m^3(z,t)$ is a very close "approximation" to the polynomial $A_{m+1}(z,t)$ that we are actually looking for next, which is only "off" by a factor of 3. This can be used to recursively compute polynomials $A_\delta(z,t)$ satisfying (3.10), by expanding $A_m(z,\Psi(z))$ in terms of the series $\widetilde{H}_{a_1,a_2,\dots,a_s}(z)$ with all a_i's not divisible by 3, as discussed in the previous section (see in particular (2.5) and Lemma 9), dividing the result through by $3^{(3^{m+1}-1)/2}$, and

considering the quotient, say

$$3^{-(3^{m+1}-1)/2}A_m(z,\Psi(z))=\sum_{s\ge1}\sum_{a_1,\dots,a_s\ge1}c(a_1,a_2,\dots,a_s;z)\widetilde H_{a_1,a_2,\dots,a_s}(z),$$

where the coefficients $c(a_1,a_2,\dots,a_s;z)$ are necessarily in $\mathbb{Z}[z,(1+z)^{-1}]$, modulo 3. Then one computes

$$\left(3^{-(3^{m+1}-1)/2}A_m(z,\Psi(z))\right)^3 \quad = \quad \sum_{s\ge1}\sum_{a_1,\dots,a_s\ge1}c^3_{a_1,a_2,\dots,a_s}(z)\widetilde H^3_{a_1,a_2,\dots,a_s}(z) \quad \text{modulo } 3, \tag{3.12}$$

uses Lemma 9 again to perform the expansion

$$\left(\widetilde H_{a_1,a_2,\dots,a_s}(z)\right)^3=\widetilde H_{a_1,a_2,\dots,a_s}(z)+\text{other terms} \quad \text{modulo } 3,$$

substitutes this back into (3.12), and expresses whatever remains in terms of powers of $A_i(z,\Psi(z))$'s, with $i=0,1,\dots,m$. We are not able to actually show that this always works, but it does for small m. The corresponding results are listed in Proposition 14 below.

Proposition 14. *Minimal polynomials for the moduli* $3,9,27,81,243,729,2187,\dots,3^{13}$ *are*

$$\begin{aligned}
&A_0(z,t):=t^2-\tfrac{1}{1+z} && \text{modulo } 3,\\
&A_0^2(z,t) && \text{modulo } 9,\\
&A_0^3(z,t) && \text{modulo } 27,\\
&A_1(z,t):=\left(t^2-\tfrac{1}{1+z}\right)^3-\tfrac{9}{(1+z)^2}\left(t^2-\tfrac{1}{1+z}\right)+\tfrac{27z}{(1+z)^5} && \text{modulo } 81,\\
&A_0(z,t)A_1(z,t) && \text{modulo } 243,\\
&A_0^2(z,t)A_1(z,t) && \text{modulo } 729,\\
&A_1^2(z,t) && \text{modulo } 2189,\\
&A_1^2(z,t) && \text{modulo } 3^8,\\
&A_0(z,t)A_1^2(z,t) && \text{modulo } 3^9,\\
&A_0^2(z,t)A_1^2(z,t) && \text{modulo } 3^{10},\\
&A_1^3(z,t) && \text{modulo } 3^{11},\\
&A_1^3(z,t) && \text{modulo } 3^{12},\\
&A_2(z,t):=A_1^3(z,t)-\tfrac{3^8}{(1+z)^6}A_1(z,t)\\
&\qquad\qquad+\tfrac{3^{10}z}{(1+z)^9}A_0^2(z,t)-\tfrac{3^{11}z(1+z^2)}{(1+z)^{12}}A_0(z,t)+\tfrac{3^{12}z^4}{(1+z)^{17}} && \text{modulo } 3^{13}.
\end{aligned}$$

Proof. To begin with, the reader should recall from Theorem 10 that minimal polynomials always have even degree.

Then, considering (2.5), we see that $A_0(z,t)$ is indeed a minimal polynomial for the modulus 3.

Continuing, we follow the procedure outlined in Item (3) of Remark 13. By (2.5), we have

$$3^{-1}A_0(z,\Psi(z)) = \frac{1}{1+z}\widetilde{H}_1(z) \quad \text{modulo } 3, \tag{3.13}$$

whence

$$\begin{aligned} 3^{-3}A_0^3(z,\Psi(z)) &= \frac{1}{(1+z)^3}\widetilde{H}_1^3(z) \quad \text{modulo } 3\\ &= \frac{1}{(1+z)^3}\left(\widetilde{H}_1(z) - \frac{z(1+z)}{1+z^3}\right) \quad \text{modulo } 3. \end{aligned}$$

Putting this together with (3.13) and the simple observation that

$$\frac{z(1+z)}{1+z^3} = \frac{z}{(1+z)^2} \quad \text{modulo } 3,$$

it follows immediately that $A_1(z,\Psi(z))$ vanishes indeed modulo 3^4. The fact that $A_1(z,t)$ is *minimal* for the modulus 3^4 follows from the degree bound in Theorem 10. This bound also implies the assertions about minimal polynomials for the moduli $9, 27, 3^5, 3^6, \dots, 3^{12}$.

The reasoning for $A_2(z,t)$ is analogous, albeit more laborious. We leave the details to the reader. □

Remark 15. Minimal polynomials are highly non-unique: for example, the polynomial

$$\left(t^2 - \frac{1}{1+z}\right)^2 + 3\left(t^2 - \frac{1}{1+z}\right)$$

is obviously also a minimal polynomial for the modulus 9.

In the remainder of this section, we show that the derivative of our basic series $\Psi(z)$, when reduced modulo a given power of 3, can be expressed as a multiple of $\Psi(z)$, where the multiplicative factor is a polynomial in z and $(1+z)^{-1}$. This is one of the crucial facts which make the method described in Section 4 work.

By straightforward differentiation, we obtain

$$\Psi'(z) = \Psi(z)\sum_{j=0}^{\infty}\frac{3^j z^{3^j-1}}{1+z^{3^j}}. \tag{3.14}$$

Modulo a given 3-power 3^β, the sum on the right-hand side of (3.14) is finite. Hence, by Lemma 8, our claim follows.

4. The method

We consider a (formal) differential equation of the form (1.2), satisfied by the integral power series $F(z)$, In this situation, we propose the following algorithmic approach to determining $F(z)$ modulo a 3-power 3^{3^α}, for some positive integer α. Let us fix $\varepsilon \in \{1,-1\}$ and a positive integer γ. We make the Ansatz

$$F(z) = \sum_{i=0}^{2\cdot 3^\alpha - 1} a_i(z)\Psi^i(\varepsilon z^\gamma) \quad \text{modulo } 3^{3^\alpha}, \tag{4.1}$$

where $\Psi(z)$ is given by (1.1), and where the $a_i(z)$'s are (at this point) undetermined elements of $\mathbb{Z}[z,z^{-1},(1+\varepsilon z^\gamma)^{-1}]$. Now we substitute (4.1) into (1.2), and we shall gradually determine approximations $a_{i,\beta}(z)$ to $a_i(z)$ such that (1.2) holds modulo 3^β, for $\beta = 1, 2, \dots, 3^\alpha$. To start the procedure, we consider the differential equation (1.2) modulo 3, with

$$F(z) = \sum_{i=0}^{2\cdot 3^\alpha - 1} a_{i,1}(z)\Psi^i(\varepsilon z^\gamma) \quad \text{modulo } 3. \tag{4.2}$$

We have

$$\frac{d}{dz}\left(\Psi(\varepsilon z^\gamma)\right) = \frac{\varepsilon\gamma z^{\gamma-1}}{1+\varepsilon z^\gamma}\Psi(\varepsilon z^\gamma) \quad \text{modulo } 3.$$

Consequently, we see that the left-hand side of (1.2) is a polynomial in $\Psi(\varepsilon z^\gamma)$ with coefficients in $\mathbb{Z}[z, z^{-1}, (1+\varepsilon z^\gamma)^{-1}]$. We reduce powers $\Psi^k(\varepsilon z^\gamma)$ with $k \ge 2\cdot 3^\alpha$ using the relation

$$\left(\Psi^2(\varepsilon z^\gamma) - \frac{1}{1+\varepsilon z^\gamma}\right)^{3^\alpha} = 0 \quad \text{modulo } 3^{3^\alpha}, \tag{4.3}$$

which is implied by the minimal polynomial for the modulus 3 given in Proposition 14.[2] Since, at this point, we are only interested in finding a solution to (1.2) modulo 3, Relation (4.3) simplifies to

$$\Psi^{2\cdot 3^\alpha}(\varepsilon z^\gamma) - \frac{1}{(1+\varepsilon z^\gamma)^{3^\alpha}} = 0 \quad \text{modulo } 3. \tag{4.4}$$

Now we compare coefficients of powers $\Psi^k(\varepsilon z^\gamma)$, $k = 0, 1, \dots, 2\cdot 3^\alpha - 1$. This yields a system of $2\cdot 3^\alpha$ (differential) equations (modulo 3) for unknown functions $a_{i,1}(z)$ in $\mathbb{Z}[z, z^{-1}, (1+\varepsilon z^\gamma)^{-1}]$, $i = 0, 1, \dots, 2\cdot 3^\alpha - 1$, which may or may not have a solution.

Provided we have already found functions $a_{i,\beta}(z)$ in $\mathbb{Z}[z, z^{-1}, (1+\varepsilon z^\gamma)^{-1}]$, $i = 0, 1, \dots, 2\cdot 3^\alpha - 1$, for some β with $1 \le \beta \le 3^\alpha - 1$, such that

$$F(z) = \sum_{i=0}^{2\cdot 3^\alpha - 1} a_{i,\beta}(z)\Psi^i(\varepsilon z^\gamma) \tag{4.5}$$

solves (1.2) modulo 3^β, we put

$$a_{i,\beta+1}(z) := a_{i,\beta}(z) + 3^\beta b_{i,\beta+1}(z), \quad i = 0, 1, \dots, 2\cdot 3^\alpha - 1, \tag{4.6}$$

where the $b_{i,\beta+1}(z)$'s are (at this point) undetermined elements in $\mathbb{Z}[z, z^{-1}, (1+\varepsilon z^\gamma)^{-1}]$. Next we substitute

$$F(z) = \sum_{i=0}^{2\cdot 3^\alpha - 1} a_{i,\beta+1}(z)\Psi^i(\varepsilon z^\gamma)$$

[2]Actually, if we would like to obtain an optimal result, we should use the relation implied by the minimal polynomial for the modulus 3^{3^α} in the sense of Section 3. But since we have no general formula available for such a minimal polynomial (cf. Items (2) and (3) of Remark 13 in that section), and since we wish to prove results for arbitrary moduli, to choose instead powers of the minimal polynomial for the modulus 3 is the best compromise. In principle, it may happen that there exists a polynomial in $\Psi(\varepsilon z^\gamma)$ with coefficients in $\mathbb{Z}[z, z^{-1}, (1+\varepsilon z^\gamma)^{-1}]$, which is identical with $F(z)$ after reduction of its coefficients modulo 3^{3^α}, but the Ansatz (4.1) combined with the reduction (4.3) fails because it is too restrictive. However, in all the cases that we treat, this obstruction does not occur. Moreover, once we are successful using this (potentially problematic) Ansatz, then the result can be easily converted into an optimal one by reducing the obtained polynomial further using the relation implied by the actual minimal polynomial for the modulus 3^{3^α}.

in (1.2). Using (3.14) and Lemma 8, we see that derivatives of $\Psi(\varepsilon z^\gamma)$ can be expressed as multiples of $\Psi(z)$, where multiplicative factors are polynomials in z and $(1+\varepsilon z^\gamma)^{-1}$. Consequently, we may expand the left-hand side of (1.2) as a polynomial in $\Psi(\varepsilon z^\gamma)$ with coefficients in $\mathbb{Z}[z,z^{-1},(1+\varepsilon z^\gamma)^{-1}]$. Subsequently, we apply again the reduction using relation (4.3). By comparing coefficients of powers $\Psi^k(\varepsilon z^\gamma)$, $k=0,1,\dots,2\cdot 3^\alpha-1$, we obtain a system of $2\cdot 3^\alpha$ (differential) equations (modulo $3^{\beta+1}$) for the unknown functions $b_{i,\beta+1}(z)$, $i=0,1,\dots,2\cdot 3^\alpha-1$, which may or may not have a solution. If we manage to push this procedure through until $\beta=3^\alpha-1$, then, setting $a_i(z)=a_{i,3^\alpha}(z)$, $i=0,1,\dots,2\cdot 3^\alpha-1$, the series $F(z)$ as given in (4.1) is a solution to (1.2) modulo 3^{3^α}, as required.

We should point out that, in order for the method to be meaningful, we need to assume that the differential equation (1.2), when considered modulo an arbitrary 3-power 3^e, determines the solution $F(z)$ *uniquely* modulo 3^e. Otherwise, our method could produce several different "solutions," and it might be difficult to decide which of them actually corresponds to the series $F(z)$ we are interested in. Concerning this point, see the discussion in Section 18.

5. A sample application

In this section, we demonstrate how to apply the method from the previous section, by considering the sequence of "almost central" binomial coefficients $\left(\binom{2n+2}{n}\right)_{n\ge0}$. To get started, we need a functional equation for the generating function $A(z)=\sum_{n\ge0}\binom{2n+2}{n}z^n$. It is not difficult to see that $A(z)$ satisfies the equation

$$z^2A^2(z)+A(z)-\frac{1}{1-4z}=0. \tag{5.1}$$

We fix α, choose $\varepsilon=-1$ and $\gamma=1$ in the method from Section 4, and make the Ansatz

$$A(z)=\sum_{i=0}^{2\cdot 3^\alpha-1}a_i(z)\Psi^i(-z)\quad\text{modulo }3^{3^\alpha},$$

where $\Psi(z)$ is given by (1.1), and where the $a_i(z)$'s are (at this point) undetermined Laurent polynomials in z and $1-z$.

The reader should recall that the idea is to gradually determine approximations $a_{i,\beta}(z)$ to $a_i(z)$ such that

$$A_\beta(z):=\sum_{i=0}^{2\cdot 3^\alpha-1}a_{i,\beta}(z)\Psi^i(-z) \tag{5.2}$$

solves (5.1) modulo 3^β, for $\beta=1,2,\dots,3^\alpha$.

For the "base step" (that is, for $\beta=1$), we claim that

$$A_1(z)=\frac{1}{z^2}-\frac{1}{z^2}(1+z)(1-z)^{\frac12(3^\alpha-1)}\Psi^{3^\alpha}(-z)$$

solves (5.1) modulo 3. Indeed, substitution of $A_1(z)$ for $A(z)$ on the left-hand side of (5.1) yields the expression

$$\frac{1}{z^2}\Big(1-2(1+z)(1-z)^{\frac{1}{2}(3^\alpha-1)}\Psi^{3^\alpha}(-z)+(1+z)^2(1-z)^{3^\alpha-1}\Psi^{2\cdot 3^\alpha}(-z)\Big)\\ +\frac{1}{z^2}-\frac{1}{z^2}(1+z)(1-z)^{\frac{1}{2}(3^\alpha-1)}\Psi^{3^\alpha}(-z)-\frac{1}{1-z}\quad\text{modulo } 3.$$

Now one uses the relation (4.4) (with $\varepsilon=-1$ and $\gamma=1$) to "get rid of" $\Psi^{2\cdot 3^\alpha}(-z)$. This leads to the expression

$$\frac{1}{z^2}\Big(1-2(1+z)(1-z)^{\frac{1}{2}(3^\alpha-1)}\Psi^{3^\alpha}(-z)+(1+z)^2(1-z)^{-1}\Big)\\ +\frac{1}{z^2}-\frac{1}{z^2}(1+z)(1-z)^{\frac{1}{2}(3^\alpha-1)}\Psi^{3^\alpha}(-z)-\frac{1}{1-z}\\ =\frac{3}{z^2(1-z)}-\frac{3}{z^2}(1+z)(1-z)^{\frac{1}{2}(3^\alpha-1)}\Psi^{3^\alpha}(-z)\quad\text{modulo } 3,$$

which is indeed 0 modulo 3.

We proceed to the "iteration step." Supposing that $A_\beta(z)$ as given in (5.2) solves (5.1) modulo 3^β, we put

$$a_{i,\beta+1}(z):=a_{i,\beta}(z)+3^\beta b_{i,\beta+1}(z),\quad i=0,1,\dots,2\cdot 3^\alpha-1,$$

as in (4.6), where the $b_{i,\beta+1}(z)$'s are (at this point) undetermined Laurent polynomials in z and $1-z$. Next we substitute

$$\sum_{i=0}^{2\cdot 3^\alpha-1}a_{i,\beta+1}(z)\Psi^i(-z)$$

in (5.1) and consider the result modulo $3^{\beta+1}$. This leads to the congruence

$$z^2A_\beta^2(z)+A_\beta(z)-\frac{1}{1-4z}+3^\beta\sum_{i=0}^{2\cdot 3^\alpha-1}b_{i,\beta+1}(z)\Psi^i(-z)\\ +2\cdot 3^\beta z^2\sum_{i=0}^{2\cdot 3^\alpha-1}\sum_{j=0}^{2\cdot 3^\alpha-1}a_{i,\beta}(z)b_{j,\beta+1}(z)\Psi^{i+j}(-z)=0\quad\text{modulo } 3^{\beta+1}.$$

By our assumption on $A_\beta(z)$, we may divide by 3^β, so that we obtain the congruence

$$\sum_{i=0}^{2\cdot 3^\alpha-1}\operatorname{Rat}_i(z)\Psi^i(-z)+\sum_{i=0}^{2\cdot 3^\alpha-1}b_{i,\beta+1}(z)\Psi^i(-z)\\ -z^2\sum_{i=0}^{2\cdot 3^\alpha-1}\sum_{j=0}^{2\cdot 3^\alpha-1}a_{i,\beta}(z)b_{j,\beta+1}(z)\Psi^{i+j}(-z)=0\quad\text{modulo } 3,\tag{5.3}$$

where $\operatorname{Rat}_i(z)$, $i=0,1,\dots,2\cdot 3^\alpha-1$, are certain Laurent polynomials in z and $1-z$ with integer coefficients. Again, we may reduce powers $\Psi^{i+j}(-z)$ with $i+j\ge 2\cdot 3^\alpha$ by means of (4.4). Furthermore, by construction, we know that $a_{0,\beta}(z)=1/z^2$ modulo 3 and that

$$a_{3^\alpha,\beta}(z)=\frac{1}{z^2}(1+z)(1-z)^{\frac{1}{2}(3^\alpha-1)}\quad\text{modulo } 3.$$

If we use all this in (5.3), then we obtain

$$\sum_{i=0}^{2\cdot 3^\alpha-1} \operatorname{Rat}_i(z)\Psi^i(-z)-(1+z)(1-z)^{\frac{1}{2}(3^\alpha-1)}\sum_{j=0}^{3^\alpha-1} b_{j,\beta+1}(z)\Psi^{j+3^\alpha}(-z)\\ -\frac{1+z}{(1-z)^{\frac{1}{2}(3^\alpha+1)}}\sum_{j=3^\alpha}^{2\cdot 3^\alpha-1} b_{j,\beta+1}(z)\Psi^{j-3^\alpha}(-z)=0 \quad \text{modulo } 3.$$

Now we compare coefficients of powers of $\Psi(-z)$. This produces the system of congruences

$$(1+z)(1-z)^{g_i}b_{i,\beta+1}(z)=\operatorname{Rat}_{i+3^\alpha}(z) \quad \text{modulo } 3, \qquad i=0,1,\dots,2\cdot 3^\alpha-1,$$

where $g_i=\frac{1}{2}(3^\alpha-1)$ if $0\le i<3^\alpha$, $g_i=-\frac{1}{2}(3^\alpha+1)$ if $3^\alpha\le i<2\cdot 3^\alpha$, and the index $i+3^\alpha$ in $\operatorname{Rat}_{i+3^\alpha}(z)$ has to be taken modulo $2\cdot 3^\alpha$. Clearly, this system is trivially uniquely solvable. Moreover, in all cases which we looked at, the numerator of the rational function $\operatorname{Rat}_i(z)$ is divisible by $1+z$, $i=0,1,\dots,2\cdot 3^\alpha-1$. We have no explanation for this, yet. However, under the assumption that this is indeed the case, we see that the algorithm will go through for $\beta=1,2,\dots,3^\alpha$, and, consequently, for all non-negative integers α the series $A(z)$, when coefficients are reduced modulo 3^{3^α}, can be represented as a polynomial in $\Psi(-z)$ with coefficients that are Laurent polynomials in z and $1-z$.

We have implemented the algorithm explained above. For $\alpha=1$, it produces the following result.

Theorem 16. *Let $\Psi(z)$ be given by* (1.1). *Then, we have*

$$\sum_{n\ge 0}\binom{2n+2}{n}z^n=\frac{13}{z^2}+3\left(-\frac{z+1}{z^2}+3\frac{1-z-z^2}{z^2(1-z)}\right)\Psi(-z)\\ +\left(4+\frac{6}{z}-\frac{4}{z^2}\right)\Psi^3(-z)+3\left(4z+2-\frac{1}{z}-\frac{5}{z^2}\right)\Psi^5(-z)\\ \text{modulo } 27. \quad (5.4)$$

The congruence results for $\Psi^3(z)$ and $\Psi^5(z)$, which one can obtain by following the recipe provided in the appendix, and which are presented explicitly in Propositions 68 and 71, could now be used to find explicit criteria when $\binom{2n+2}{n}$ is congruent to $0,1,2,\dots,26$ modulo 27.[3] Since a display of such results is rather lengthy, we omit them here, also in most of the subsequent sections. As we already said in the Introduction, we confine ourselves to providing only one illustration of coefficient extraction, namely for the case of free subgroup numbers for $PSL_2(\mathbb{Z})$; see Section 16.

6. A general theorem on functional–differential equations of quadratic type

The purpose of this section is to show that the method from Section 4 always applies to a certain class of functional-differential equations. The characteristic feature

[3]In principle, such results could also be obtained via the extensions of Lucas' theorem given in [2] or [5, Theorem 1], which provide a means for determining the congruence class of a binomial coefficient modulo a given prime power.

of the equations in this class is that, when reduced modulo 3, they become quadratic equations. Since there are numerous combinatorial objects whose generating functions satisfy quadratic equations, the main theorem of this section has a wide range of applications. These will be discussed in Sections 7–17.

Theorem 17. *Let $\Psi(z)$ be given by* (1.1)*, and let α be some positive integer. Furthermore, suppose that the formal power series $F(z)$ with integer coefficients satisfies the functional-differential equation*

$$c_2(z)F^2(z) + c_1(z)F(z) + c_0(z) + 3\mathcal{Q}(z; F(z), F'(z), F''(z), \dots, F^{(s)}(z)) = 0, \qquad (6.1)$$

where

(1) $c_2(z) = z^{e_1}(1 + \varepsilon z^\gamma)^{e_2}$ modulo 3*, with non-negative integers e_1, e_2 and $\varepsilon \in \{1, -1\}$;*
(2) $c_1^2(z) - c_0(z)c_2(z) = z^{2f_1}(1 + \varepsilon z^\gamma)^{2f_2+1}$ modulo 3*, with non-negative integers f_1, f_2;*
(3) *$\mathcal{Q}$ is a polynomial with integer coefficients;*
(4) *the equation* (6.1) *has a uniquely determined formal power solution, namely $F(z)$.*

Then $F(z)$, when coefficients are reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(\varepsilon z^\gamma)$ of the form

$$F(z) = a_0(z) + \sum_{i=0}^{2\cdot 3^\alpha - 1} a_i(z)\Psi^i(\varepsilon z^\gamma) \quad \text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i = 0, 1, \dots, 2\cdot 3^\alpha - 1$, are Laurent polynomials in z and $1 + \varepsilon z^\gamma$.

Before we are able to embark on the proof of Theorem 17, we need to establish an auxiliary result. There, the notion of *formal Laurent series* is used, by which we mean a formal series $\sum_{n\ge s} f_n z^n$, where s is some integer.

Lemma 18. *Let β be a positive integer. Under the assumptions of Theorem* 17*, the equation* (6.1) *has exactly two different solutions in formal Laurent series modulo 3^β.*

Proof. We proceed by induction on β. For $\beta = 1$, the equation (6.1) reduces to

$$c_2(z)F^2(z) + c_1(z)F(z) + c_0(z) = 0 \qquad \text{modulo } 3.$$

This is a quadratic equation over a field (namely the field of formal Laurent series over $\mathbb{Z}/3\mathbb{Z}$). Hence, the two solutions are

$$\frac{-c_1(z) \pm \sqrt{c_1^2(z) - 4c_0(z)c_2(z)}}{2c_2(z)} = \frac{-c_1(z) \pm z^{f_1}(1+\varepsilon z^\gamma)^{f_2+\frac{1}{2}}}{2c_2(z)} \qquad \text{modulo } 3. \qquad (6.2)$$

Indeed, the square root $(1+\varepsilon z^\gamma)^{1/2}$ can be computed without any difficulties modulo 3 by means of the binomial theorem. Furthermore, it does not vanish so that (6.2) yields indeed two *different* formal Laurent series.

Now we suppose that we already know that there are exactly two formal Laurent series, say $F_1(z)$ and $F_2(z)$, which satisfy (6.1) modulo 3^β. In order to find solutions modulo $3^{\beta+1}$, we make the Ansatz $F_i + 3^\beta G_i(z)$, $i = 1, 2$, where the $G_i(z)$'s are two unknown formal Laurent series to be determined, we substitute this in (6.1) and reduce modulo $3^{\beta+1}$:

$$c_2(z)(F_i^2(z) + 2\cdot 3^\beta F_i(z)G_i(z)) + c_1(z)(F_i(z) + 3^\beta G_i(z)) + c_0(z) \\ + 3\mathcal{Q}(z; F_i(z), F_i'(z), F_i''(z), \dots, F_i^{(s)}(z)) = 0 \qquad \text{modulo } 3^{\beta+1}.$$

Rearrangement of terms converts this congruence into

$$3^\beta G_i(z)\big(c_1(z) + 2c_2(z)F_i(z)\big) = -c_2(z)F_i^2(z) - c_1(z)F_i(z) - c_0(z) \\ + 3\mathcal{Q}(z; F_i(z), F_i'(z), F_i''(z), \dots, F_i^{(s)}(z)) = 0 \qquad \text{modulo } 3^{\beta+1}.$$

By induction hypothesis, we know that the right-hand side is divisible by 3^β. We divide both sides by 3^β and obtain

$$G_i(z)\big(c_1(z) + 2c_2(z)F_i(z)\big) = S(z) \qquad \text{modulo } 3,$$

for some explicit formal Laurent series $S(z)$. On the left-hand side, the factor $c_1(z) + 2c_2(z)F_i(z)$ is non-zero since $c_0(z)$, $c_1(z)$, and $c_2(z)$ are polynomials, while $F_i(z)$ contains a square root (cf. (6.2)). Hence, for $i = 1, 2$, we obtain uniquely determined $G_i(z)$ modulo 3 such that $F_i(z) + 3^\beta G_i(z)$ is a solution to (6.1) modulo $3^{\beta+1}$. This completes the induction step. $\square$

Proof of Theorem 17. We apply the method from Section 4. As we discussed there and in the sample application in the previous section, it consists of two steps, a *base step*, which determines a solution to the functional-differential equation (6.1) modulo 3, and an *iteration step*, which affords solutions modulo higher and higher powers of 3.

In the entire proof, we always assume without loss of generality that $c_0(z)$, $c_1(z)$, and $c_2(z)$ are already reduced modulo 3 (otherwise we could put terms which are divisible by 3 into the polynomial $\mathcal{Q}(\,.\,)$).

Base step. We start by substituting the Ansatz (4.2) in (6.1) and reducing the result modulo 3. In this way, we obtain

$$c_2(z) \sum_{i=0}^{2\cdot 3^\alpha - 1} \sum_{j=0}^{2\cdot 3^\alpha - 1} a_{i,1}(z)a_{j,1}(z)\Psi^{i+j}(\varepsilon z^\gamma) \\ + c_1(z) \sum_{i=0}^{2\cdot 3^\alpha - 1} a_{i,1}(z)\Psi^i(\varepsilon z^\gamma) + c_0(z) = 0 \quad \text{modulo } 3.$$

When $i+j\ge 2\cdot 3^\alpha$, we may reduce $\Psi^{i+j}(\varepsilon z^\gamma)$ further using relation (4.4) (with z replaced by εz^γ). This leads to

$$c_2(z)\sum_{\substack{0\le i,j\le 2\cdot 3^\alpha-1\\ i+j<2\cdot 3^\alpha}} a_{i,1}(z)a_{j,1}(z)\Psi^{i+j}(\varepsilon z^\gamma)\\ + c_2(z)\sum_{\substack{0\le i,j\le 2\cdot 3^\alpha-1\\ i+j\ge 2\cdot 3^\alpha}} a_{i,1}(z)a_{j,1}(z)\frac{1}{(1+\varepsilon z^\gamma)^{3^\alpha}}\Psi^{i+j-2\cdot 3^\alpha}(\varepsilon z^\gamma)\\ + c_1(z)\sum_{i=0}^{2\cdot 3^\alpha-1} a_{i,1}(z)\Psi^i(\varepsilon z^\gamma)+c_0(z)=0 \quad \text{modulo } 3. \quad (6.3)$$

Now we compare coefficients of $\Psi^i(\varepsilon z^\gamma)$, for $i=0,1,\dots,2\cdot 3^\alpha-1$. We claim that the choice of

$$a_{0,1}(z)=\frac{c_1(z)}{c_2(z)}=z^{-e_1}(1+\varepsilon z^\gamma)^{-e_2}c_1(z), \quad (6.4)$$

$$a_{3^\alpha,1}(z)=\pm\frac{(c_1^2(z)-c_0(z)c_2(z))^{1/2}}{c_2(z)}(1+\varepsilon z^\gamma)^{3^\alpha/2}=\pm z^{f_1-e_1}(1+\varepsilon z^\gamma)^{f_2-e_2+\frac{1}{2}(3^\alpha+1)}, \quad (6.5)$$

with all other $a_{i,1}(z)$'s vanishing, provides two solutions (depending on the sign in front of the expression for $a_{3^\alpha,1}(z)$) modulo 3 to the system of congruences resulting from (6.3) in rational functions $a_{i,1}(z)$. Namely, if we substitute this choice in the left-hand side of (6.3), then we obtain

$$c_2(z)\frac{c_1^2(z)}{c_2^2(z)}+2c_2(z)\frac{c_1(z)}{c_2(z)}a_{3^\alpha,1}(z)\Psi^{3^\alpha}(\varepsilon z^\gamma)+c_2(z)\frac{c_1^2(z)-c_0(z)c_2(z)}{c_2^2(z)}\\ +c_1(z)\frac{c_1(z)}{c_2(z)}+c_1(z)a_{3^\alpha,1}(z)\Psi^{3^\alpha}(\varepsilon z^\gamma)+c_0(z),$$

which indeed vanishes modulo 3.

The two solutions given by (6.4) and (6.5) are different modulo 3 since $a_{3^\alpha,1}(z)\ne 0$. They must be the two different solutions modulo 3 guaranteed by Lemma 18 with $\beta=1$. With both these solutions, we enter the iteration step.

Iteration step. We consider the Ansatz (4.5)–(4.6), where the coefficients $a_{i,\beta}(z)$ are supposed to provide a solution $F_\beta(z)=\sum_{i=0}^{2\cdot 3^\alpha-1}a_{i,\beta}(z)\Psi^i(\varepsilon z^\gamma)$ to (6.1) modulo 3^β. This Ansatz, substituted in (6.1), produces the congruence

$$c_2(z)F_\beta^2(z)+c_1(z)F_\beta(z)+c_0(z)+3\mathcal{Q}(z;F_\beta(z),F'_\beta(z),F''(z),\dots,F_\beta^{(s)}(z))\\ +3^\beta c_1(z)\sum_{i=0}^{2\cdot 3^\alpha-1} b_{i,\beta+1}(z)\Psi^i(\varepsilon z^\gamma)\\ +2\cdot 3^\beta c_2(z)\sum_{i=0}^{2\cdot 3^\alpha-1}\sum_{j=0}^{2\cdot 3^\alpha-1} a_{i,\beta}(z)b_{j,\beta+1}(z)\Psi^{i+j}(\varepsilon z^\gamma)=0 \quad \text{modulo } 3^{\beta+1}.$$

By our assumption on $F_\beta(z)$, we may divide by 3^β. Furthermore, by construction, we know that $a_{0,\beta}(z)=c_1(z)/c_2(z)$ modulo 3 and that

$$a_{3^\alpha,\beta}(z)=\pm z^{f_1-e_1}(1+\varepsilon z^\gamma)^{f_2-e_2+\frac{1}{2}(3^\alpha+1)}\quad\text{modulo } 3$$

(the sign $\pm$ depending on the sign in (6.5)), while all other $a_{i,\beta}(z)$'s vanish modulo 3. If we use these facts and also perform the reduction (4.4), then we arrive at the congruence

$$\sum_{i=0}^{2\cdot 3^\alpha-1}\mathrm{Rat}_i(z)\Psi^i(\varepsilon z^\gamma)+c_1(z)\sum_{i=0}^{2\cdot 3^\alpha-1}b_{i,\beta+1}(z)\Psi^i(\varepsilon z^\gamma)\\
+2\cdot c_2(z)\Bigg(\frac{c_1(z)}{c_2(z)}\sum_{j=0}^{2\cdot 3^\alpha-1}b_{j,\beta+1}(z)\Psi^j(\varepsilon z^\gamma)\\
\pm z^{f_1-e_1}(1+\varepsilon z^\gamma)^{f_2-e_2+\frac{1}{2}(3^\alpha+1)}\sum_{j=0}^{3^\alpha-1}b_{j,\beta+1}(z)\Psi^{j+3^\alpha}(\varepsilon z^\gamma)\\
\pm z^{f_1-e_1}(1+\varepsilon z^\gamma)^{f_2-e_2-\frac{1}{2}(3^\alpha-1)}\sum_{j=3^\alpha}^{2\cdot 3^\alpha-1}b_{j,\beta+1}(z)\Psi^{j-3^\alpha}(\varepsilon z^\gamma)\Bigg)=0\quad\text{modulo } 3,$$

where $\mathrm{Rat}_i(z)$, $i=0,1,\dots,2\cdot 3^\alpha-1$, are certain Laurent polynomials in z and $1+\varepsilon z^\gamma$ with integer coefficients. Reducing further modulo 3, this simplifies to

$$\sum_{i=0}^{2\cdot 3^\alpha-1}\mathrm{Rat}_i(z)\Psi^i(\varepsilon z^\gamma)\mp z^{f_1}(1+\varepsilon z^\gamma)^{f_2+\frac{1}{2}(3^\alpha+1)}\sum_{j=0}^{3^\alpha-1}b_{j,\beta+1}(z)\Psi^{j+3^\alpha}(\varepsilon z^\gamma)\\
\mp z^{f_1}(1+\varepsilon z^\gamma)^{f_2-\frac{1}{2}(3^\alpha-1)}\sum_{j=3^\alpha}^{2\cdot 3^\alpha-1}b_{j,\beta+1}(z)\Psi^{j-3^\alpha}(\varepsilon z^\gamma)=0\quad\text{modulo } 3.$$

Comparison of powers of $\Psi(\varepsilon z^\gamma)$ then yields the system of congruences

$$\pm z^{f_1}(1+\varepsilon z^\gamma)^{g_i}b_{i,\beta+1}(z)=\mathrm{Rat}_{i+3^\alpha}(z)\quad\text{modulo } 3,\qquad i=0,1,\dots,2\cdot 3^\alpha-1,$$

where $g_i=f_2+\frac{1}{2}(3^\alpha+1)$ if $0\le i<3^\alpha$, $g_i=f_2-\frac{1}{2}(3^\alpha-1)$ if $3^\alpha\le i<2\cdot 3^\alpha$, and the index $i+3^\alpha$ in $\mathrm{Rat}_{i+3^\alpha}(z)$ has to be taken modulo $2\cdot 3^\alpha$. This system is trivially uniquely solvable. Since we had two different solutions modulo 3 to start with, we now obtain two different solutions $F_\beta(z)$ to (6.1) modulo 3^β. These must be the two different solutions modulo 3^β guaranteed by Lemma 18. When we arrive at $\beta=3^\alpha$, the uniqueness condition (4) guarantees that only one of the two obtained solutions corresponds to $F(z)$ modulo 3^{3^α}. Thus, we have proven that the algorithm of Section 4 will produce a solution $F_{3^\alpha}(z)$ to (6.1) modulo 3^{3^α} which is a polynomial in $\Psi(\varepsilon z^\gamma)$ with coefficients that are Laurent polynomials in z and $1+\varepsilon z^\gamma$, and which equals any formal power series solution $F(z)$ to (6.1) when coefficients are reduced modulo 3^{3^α}. □

7. Motzkin numbers

Let M_n be the n-th *Motzkin number*, that is, the number of lattice paths from $(0,0)$ to $(n,0)$ consisting of steps taken from the set $\{(1,0),(1,1),(1,-1)\}$ never running below the x-axis. It is well-known (cf. [18, Ex. 6.37]) that the generating function $M(z) = \sum_{n\ge0} M_n\, z^n$ is given by

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}, \tag{7.1}$$

and hence satisfies the functional equation

$$z^2 M^2(z) + (z-1)M(z) + 1 = 0. \tag{7.2}$$

Theorem 19. *Let $\Psi(z)$ be given by* (1.1)*, and let α be some positive integer. Then the generating function $M(z) = \sum_{n\ge0} M_n\, z^n$ for the Motzkin numbers, reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(z^3)$ of the form*

$$M(z) = a_0(z) + \sum_{i=0}^{2\cdot 3^\alpha - 1} a_i(z)\Psi^i(z^3) \quad \text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i = 0, 1, \dots, 2\cdot 3^\alpha - 1$, are Laurent polynomials in z and $1 + z$.

Proof. Choosing $\gamma = 1$, $c_2(z) = z^2$, $c_1(z) = z - 1$, $c_0(z) = 1$, we see that the functional equation (7.2) fits into the framework of Theorem 17, since

$$c_1^2(z) - c_0(z)c_2(z) = 1 + z \quad \text{modulo } 3.$$

The constants e_1, e_2, f_1, f_2 are given by $e_1 = 2$, $e_2 = 0$, $f_1 = 0$, $f_2 = 0$, $\varepsilon = 1$, and we have $\mathcal{Q}(\dots) = 0$. Consequently, the generating function $M(z)$, when reduced modulo 3^{3^α}, can be written as a polynomial in the basic series $\Psi(z) = (1 + z)\Psi(z^3)$. In the formulation of the theorem, we have chosen to express everything in terms of $\Psi(z^3)$, since this leads to the more elegant expressions for the moduli 9 and 27 given below. ☐

We have implemented the algorithm explained in the above proof. For $\alpha = 1$, it produces the following result.

Theorem 20. *Let $\Psi(z)$ be given by* (1.1)*. Then, we have*

$$\begin{aligned}\sum_{n\ge0} M_n\, z^n &= 13z^{-1} + 14z^{-2} + \left(9z + 12 + 24z^{-1} + 21z^{-2}\right)\Psi(z^3)\\ &\quad + \left(9z^5 + 12z^4 + 10z^3 + 23z^2 + 25z + 19 + 14z^{-1} + 4z^{-2}\right)\Psi^3(z^3)\\ &\quad - \left(9z^7 + 3z^6 + 24z^5 + 30z^4 + 6z^3 + 21z^2 + 6z + 3 + 24z^{-1} + 12z^{-2}\right)\Psi^5(z^3)\\ &\qquad \text{modulo } 27.\end{aligned} \tag{7.3}$$

In the case of modulus 9, this theorem reduces to the following result.

Corollary 21. *Let $\Psi(z)$ be given by* (1.1)*. Then, we have*

$$\begin{aligned}\sum_{n\ge0} M_n\, z^n &= 4z^{-1} + 5z^{-2} - \left(3 + 6z^{-1} + 3z^{-2}\right)\Psi(z^3)\\ &\quad + \left(3z^4 + 4z^3 + 2z^2 + 1z + 4 + 2z^{-1} + 7z^{-2}\right)\Psi^3(z^3) \quad \text{modulo } 9.\end{aligned} \tag{7.4}$$

These results generalise Corollary 4.10 in [3], which is an easy consequence.

8. Motzkin prefix numbers

Let MP_n be the n-th *Motzkin prefix number*, that is, the number of lattice paths from $(0,0)$ consisting of n steps taken from the set $\{(1,0),(1,1),(1,-1)\}$ never running below the x-axis. It is easy to see that the generating function $MP(z)=\sum_{n\ge0} MP_n\,z^n$ satisfies the functional equation

$$z(1-3z)MP^2(z)+(1-3z)MP(z)-1=0. \tag{8.1}$$

Namely, let again $M(z)$ denote the generating function for Motzkin numbers (see Section 7). Then, by an elementary combinatorial decomposition (a Motzkin prefix can be empty, it can start with a horizontal step followed by a Motzkin prefix, it can start with a step $(1,1)$ and never return to the x-axis, or start with a step $(1,1)$, return to the x-axis for the first time by a step $(1,-1)$, and then be followed by a Motzkin prefix), we have

$$MP(z)=1+2z\,MP(z)+z^2M(z)MP(z).$$

Combined with (7.1), this yields an explicit expression for $MP(z)$, from which the functional equation (8.1) can be derived.

Theorem 22. *Let $\Psi(z)$ be given by* (1.1)*, and let α be some positive integer. Then the generating function $MP(z)=\sum_{n\ge0} MP_n\,z^n$ for the Motzkin prefix numbers, reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(z)$ of the form*

$$MP(z)=a_0(z)+\sum_{i=0}^{2\cdot3^\alpha-1} a_i(z)\Psi^i(z)\quad\text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i=0,1,\dots,2\cdot3^\alpha-1$, are Laurent polynomials in z and $1+z$.

Proof. Choosing $\gamma=1$, $c_2(z)=z(1-3z)$, $c_1(z)=1-3z$, $c_0(z)=-1$, we see that the functional equation (8.1) fits into the framework of Theorem 17, since

$$c_1^2(z)-c_0(z)c_2(z)=1+z\quad\text{modulo }3.$$

The constants e_1,e_2,f_1,f_2 are given by $e_1=1$, $e_2=0$, $f_1=0$, $f_2=0$, $\varepsilon=1$, and we have $\mathcal{Q}(\dots)=0$. Consequently, the generating function $MP(z)$, when reduced modulo 3^{3^α}, can be written as a polynomial in the basic series $\Psi(z)$, as claimed. □

We have implemented the algorithm explained in the above proof. For $\alpha=1$, it produces the following result.

Theorem 23. *Let $\Psi(z)$ be given by* (1.1)*. Then, we have*

$$\sum_{n\ge0} MP_n\,z^n=13z^{-1}+\left(9z+6z^{-1}+15\right)\Psi(z)-\left(6z^2+16z+4z^{-1}+14\right)\Psi^3(z)\\ -\left(9z^3+15z^2+18z+15z^{-1}\right)\Psi^5(z)\quad\text{modulo }27. \tag{8.2}$$

In the case of modulus 9, this theorem reduces to the following result.

Corollary 24. *Let $\Psi(z)$ be given by* (1.1)*. Then, we have*

$$\sum_{n\ge0} MP_n\,z^n=4z^{-1}+3\left(1+z^{-1}\right)\Psi(z)+\left(3z^2-z+7+2z^{-1}\right)\Psi^3(z)\quad\text{modulo }9. \tag{8.3}$$

These results generalise Corollary 4.11 in [3], which is an easy consequence.

9. Riordan numbers

Let R_n be the n-th *Riordan number*, that is, the number of lattice paths from $(0,0)$ to $(n,0)$ consisting of steps taken from the set $\{(1,0),(1,1),(1,-1)\}$ never running below the x-axis, and where steps $(1,0)$ are not allowed on the x-axis. It is easy to see that the generating function $R(z)=\sum_{n\ge0} R_n\,z^n$ satisfies the functional equation

$$z(1+z)R^2(z)-(z+1)R(z)+1=0. \tag{9.1}$$

Namely, let again $M(z)$ denote the generating function for Motzkin numbers (see Section 7). Then, by an elementary combinatorial decomposition (a path as above can be empty, or it may start with a step $(1,1)$, return to the x-axis for the first time by a step $(1,-1)$, and then be followed by a path of the above type), we have

$$R(z)=1+z^2M(z)R(z).$$

Combined with (7.1), this yields an explicit expression for $R(z)$, from which the functional equation (9.1) can be derived.

Theorem 25. *Let $\Psi(z)$ be given by* (1.1)*, and let α be some positive integer. Then the generating function $R(z)=\sum_{n\ge0} R_n\,z^n$ for the Riordan numbers, reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(z^3)$ of the form*

$$R(z)=a_0(z)+\sum_{i=0}^{2\cdot3^\alpha-1} a_i(z)\Psi^i(z^3)\quad\text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i=0,1,\dots,2\cdot3^\alpha-1$, are Laurent polynomials in z and $1+z$.

Proof. Choosing $\gamma=1$, $c_2(z)=z(1+z)$, $c_1(z)=-z-1$, $c_0(z)=1$, we see that the functional equation (9.1) fits into the framework of Theorem 17, since

$$c_1^2(z)-c_0(z)c_2(z)=1+z\quad\text{modulo } 3.$$

The constants e_1,e_2,f_1,f_2 are given by $e_1=1$, $e_2=1$, $f_1=0$, $f_2=0$, $\varepsilon=1$, and we have $\mathcal{Q}(\dots)=0$. Consequently, the generating function $R(z)$, when reduced modulo 3^{3^α}, can be written as a polynomial in the basic series $\Psi(z)=(1+z)\Psi(z^3)$. In the formulation of the theorem, we have chosen to express everything in terms of $\Psi(z^3)$, since this leads to the more elegant expressions for the moduli 9 and 27 given below. ☐

We have implemented the algorithm explained in the above proof. For $\alpha=1$, it produces the following result.

Theorem 26. *Let $\Psi(z)$ be given by* (1.1)*. Then, we have*

$$\sum_{n\ge0} R_n\,z^n = 14z^{-1}+\left(9+21z^{-1}\right)\Psi(z)+\left(9z^2+30z+25+4z^{-1}\right)\Psi^3(z)\\ +\left(18z^2+24z+21+15z^{-1}\right)\Psi^5(z)\quad\text{modulo } 27. \tag{9.2}$$

In the case of modulus 9, this theorem reduces to the following result.

Corollary 27. *Let $\Psi(z)$ be given by* (1.1)*. Then, we have*

$$\sum_{n\ge0} R_n\,z^n = -4z^{-1}-3z^{-1}\Psi(z)+\left(3z+10+7z^{-1}\right)\Psi^3(z)\quad\text{modulo } 9. \tag{9.3}$$

These results generalise Corollary 4.12 in [3], which is an easy consequence.

10. Central trinomial coefficients

Let T_n be the n-th *central trinomial coefficient*, that is, the coefficient of t^n in $(1+t+t^2)^n$. Already Euler knew that the generating function $T(z) = \sum_{n\ge0} T_n\, z^n$ equals $1/\sqrt{1-2z-3z^2}$ (cf. [18, solution to Exercise 6.42]). Consequently, $T(z)$ satisfies the functional equation

$$(1-2z-3z^2)T^2(z) - 1 = 0. \tag{10.1}$$

Theorem 28. *Let $\Psi(z)$ be given by* (1.1)*, and let α be some positive integer. Then the generating function $T(z) = \sum_{n\ge0} T_n\, z^n$ for the trinomial coefficients, reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(z^3)$ of the form*

$$T(z) = a_0(z) + \sum_{i=0}^{2\cdot3^\alpha-1} a_i(z)\Psi^i(z^3) \quad \text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i = 0,1,\dots,2\cdot3^\alpha-1$, are Laurent polynomials in z and $1+z$.

Proof. Choosing $\gamma = 1$, $c_2(z) = 1-2z-3z^2$, $c_1(z) = 0$, $c_0(z) = -1$, we see that the functional equation (10.1) fits into the framework of Theorem 17, since

$$c_1^2(z) - c_0(z)c_2(z) = 1+z \quad \text{modulo } 3.$$

The constants e_1, e_2, f_1, f_2 are given by $e_1 = 0$, $e_2 = 1$, $f_1 = 0$, $f_2 = 0$, $\varepsilon = 1$, and we have $\mathcal{Q}(\dots) = 0$. Consequently, the generating function $T(z)$, when reduced modulo 3^{3^α}, can be written as a polynomial in the basic series $\Psi(z) = (1+z)\Psi(z^3)$. In the formulation of the theorem, we have chosen to express everything in terms of $\Psi(z^3)$, since this leads to the more elegant expressions for the moduli 9 and 27 given below. □

We have implemented the algorithm explained in the above proof. For $\alpha = 1$, it produces the following result.

Theorem 29. *Let $\Psi(z)$ be given by* (1.1)*. Then, we have*

$$\sum_{n\ge0} T_n\, z^n = -\left(9z^2+24z+15\right)\Psi(z^3) + \left(15z^5+25z^4+4z^3+12z^2+10z+19\right)\Psi^3(z^3) \\ + \left(9z^8+6z^7+6z^6+9z^5+21z^4+3z^3+15z+24\right)\Psi^5(z^3) \quad \text{modulo } 27. \tag{10.2}$$

In the case of modulus 9, this theorem reduces to the following result.

Corollary 30. *Let $\Psi(z)$ be given by* (1.1)*. Then, we have*

$$\sum_{n\ge0} T_n\, z^n = -3(z+1)\Psi(z^3) + \left(-3z^5+z^4+7z^3+3z^2+4z+4\right)\Psi^3(z^3) \quad \text{modulo } 9. \tag{10.3}$$

These results generalise Corollary 4.9 in [3], which is an easy consequence.

11. Central binomial coefficients and their sums

In this section we consider the sequence $\left(\binom{2n}{n}\right)_{n\ge0}$ of *central binomial numbers.* It follows from the binomial theorem that the corresponding generating function $CB(z)$ is given by

$$CB(z) := \sum_{n\ge0} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}},$$

and hence satisfies the functional equation

$$(1-4z)CB^2(z) - 1 = 0. \tag{11.1}$$

Theorem 31. *Let $\Psi(z)$ be given by* (1.1)*, and let α be some positive integer. Then the generating function $CB(z) = \sum_{n\ge0}\binom{2n}{n} z^n$ for the central binomial numbers, reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(-z)$ of the form*

$$CB(z) = a_0(z) + \sum_{i=0}^{2\cdot3^\alpha-1} a_i(z)\Psi^i(-z) \quad \text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i = 0,1,\dots,2\cdot3^\alpha-1$, are Laurent polynomials in z and $1-z$.

Proof. Choosing $\gamma = 1$, $c_2(z) = 1-4z$, $c_1(z) = 0$, $c_0(z) = -1$, we see that the functional equation (11.1) fits into the framework of Theorem 17, since

$$c_1^2(z) - c_0(z)c_2(z) = 1 - z \quad \text{modulo } 3.$$

The constants e_1, e_2, f_1, f_2 are given by $e_1 = 0$, $e_2 = 1$, $f_1 = 0$, $f_2 = 0$, $\varepsilon = -1$, and we have $\mathcal{Q}(\dots) = 0$. Consequently, the generating function $CB(z)$, when reduced modulo 3^{3^α}, can be written as a polynomial in the basic series $\Psi(-z)$, as claimed. $\square$

We have implemented the algorithm explained in the above proof. For $\alpha = 1$, it produces the following result.

Theorem 32. *Let $\Psi(z)$ be given by* (1.1)*. Then, we have*

$$\sum_{n\ge0}\binom{2n}{n} z^n = \left(9\frac{1+z}{1-z} + 3\right)\Psi(-z) - (4z+8)\Psi^3(-z) \\ - \left(12z^2 + 12z + 3\right)\Psi^5(-z) \quad \text{modulo } 27. \tag{11.2}$$

In the case of modulus 9, this theorem reduces to the following result.

Corollary 33. *Let $\Psi(z)$ be given by* (1.1)*. Then, we have*

$$\sum_{n\ge0}\binom{2n}{n} z^n = -3\Psi(-z) + (2z+4)\Psi^3(-z) \quad \text{modulo } 9. \tag{11.3}$$

These results generalise Theorem 4.3 in [3], which is an easy consequence.

We next turn to sums of central binomial coefficients. From (11.1) it follows directly that

$$(1-4z)(1-z)^2 SCB^2(z) - 1 = 0, \tag{11.4}$$

where

$$SCB(z) := \sum_{n\ge0} z^n \sum_{k=0}^{n} \binom{2k}{k}.$$

Theorem 34. *Let $\Psi(z)$ be given by* (1.1), *and let α be some positive integer. Then the generating function $SCB(z) = \sum_{n\ge0} z^n \sum_{k=0}^{n} \binom{2k}{k}$ for sums of central binomial numbers, reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(-z)$ of the form*

$$SCB(z) = a_0(z) + \sum_{i=0}^{2\cdot3^\alpha-1} a_i(z)\Psi^i(-z) \quad \text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i = 0, 1, \dots, 2\cdot3^\alpha - 1$, are Laurent polynomials in z and $1 - z$.

Proof. Choosing $\gamma = 1$, $c_2(z) = (1-4z)(1-z)^2$, $c_1(z) = 0$, $c_0(z) = -1$, we see that the functional equation (11.4) fits into the framework of Theorem 17, since

$$c_1^2(z) - c_0(z)c_2(z) = (1-z)^3 \quad \text{modulo } 3.$$

The constants e_1, e_2, f_1, f_2 are given by $e_1 = 0$, $e_2 = 3$, $f_1 = 0$, $f_2 = 1$, $\varepsilon = -1$, and we have $\mathcal{Q}(\dots) = 0$. Consequently, the generating function $SCB(z)$, when reduced modulo 3^{3^α}, can be written as a polynomial in the basic series $\Psi(-z) = (1-z)\Psi(-z^3)$. In the formulation of the theorem, we have chosen to express everything in terms of $\Psi(-z^3)$, since this leads to the more elegant expressions for the moduli 9 and 27 given below. □

We have implemented the algorithm explained in the above proof. For $\alpha = 1$, it produces the following result.

Theorem 35. *Let $\Psi(z)$ be given by* (1.1). *Then, we have*

$$\sum_{n\ge0} z^n \sum_{k=0}^{n} \binom{2k}{k} = +\left(3 + 9\frac{z+1}{1-z}\right)\Psi(-z^3) + \left(-4z^3 + 12z + 19\right)\Psi^3(-z^3) \\ + \left(15z^6 + 9z^5 + 15z^3 + 18z^2 + 24\right)\Psi^5(-z^3) \quad \text{modulo } 27. \tag{11.5}$$

In the case of modulus 9, this theorem reduces to the following result.

Corollary 36. *Let $\Psi(z)$ be given by* (1.1). *Then, we have*

$$\sum_{n\ge0} z^n \sum_{k=0}^{n} \binom{2k}{k} = -3\Psi(-z^3) + \left(4 + 3z + 2z^3\right)\Psi^3(-z^3) \quad \text{modulo } 9. \tag{11.6}$$

These results generalise Theorem 5.7 in [3], which is an easy consequence.

12. Catalan numbers

Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the n-th *Catalan number*. It is well-known (cf. [18, equation below (6.18) with $k = 2$]) that the generating function $C(z) = \sum_{n\ge0} C_n z^n$ satisfies the functional equation

$$zC^2(z) - C(z) + 1 = 0. \tag{12.1}$$

Theorem 37. *Let $\Psi(z)$ be given by* (1.1)*, and let α be some positive integer. Then the generating function $C(z) = \sum_{n\ge0} C_n z^n$ for the Catalan numbers, reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(-z)$ of the form*

$$C(z) = a_0(z) + \sum_{i=0}^{2\cdot3^\alpha-1} a_i(z)\Psi^i(-z) \quad \text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i = 0, 1, \dots, 2\cdot3^\alpha - 1$, are Laurent polynomials in z and $1 - z$.

Proof. Choosing $\gamma = 1$, $c_2(z) = z$, $c_1(z) = -1$, $c_0(z) = 1$, we see that the functional equation (12.1) fits into the framework of Theorem 17, since

$$c_1^2(z) - c_0(z)c_2(z) = 1 - z \quad \text{modulo } 3.$$

The constants e_1, e_2, f_1, f_2 are given by $e_1 = 1$, $e_2 = 0$, $f_1 = 0$, $f_2 = 0$, $\varepsilon = -1$, and we have $\mathcal{Q}(\dots) = 0$. Consequently, the generating function $C(z)$, when reduced modulo 3^{3^α}, can be written as a polynomial in the basic series $\Psi(-z)$, as claimed. $\square$

We have implemented the algorithm explained in the above proof. For $\alpha = 1$, it produces the following result.

Theorem 38. *Let $\Psi(z)$ be given by* (1.1)*. Then, we have*

$$\sum_{n\ge0} C_n\, z^n = -13z^{-1} - 3\left(4 + 2z^{-1}\right)\Psi(-z) + \left(-8z - 14 + 4z^{-1}\right)\Psi^3(-z) \\ + 3\left(z^2 - 6z + 9 - 4z^{-1}\right)\Psi^5(-z) \quad \text{modulo } 27. \tag{12.2}$$

In the case of modulus 9, this theorem reduces to the following result.

Corollary 39. *Let $\Psi(z)$ be given by* (1.1)*. Then, we have*

$$\sum_{n\ge0} C_n\, z^n = -4z^{-1} + 3\left(1 - z^{-1}\right)\Psi(-z) + \left(4z - 2 - 2z^{-1}\right)\Psi^3(-z) \quad \text{modulo } 9. \tag{12.3}$$

These results generalise Theorem 5.2 in [3], which is an easy consequence.

13. Central Delannoy numbers

Let D_n be the n-th *central Delannoy number*, that is, the the number of lattice paths from $(0,0)$ to (n,n) consisting of steps taken from the set $\{(1,0),(0,1),(1,1)\}$. It is well-known (see [18, Ex. 6.3.8]) that the generating function $D(z) = \sum_{n\ge0} D_n\, z^n$ equals $1/\sqrt{1 - 6z + z^2}$. Consequently, $D(z)$ satisfies the functional equation

$$(1 - 6z + z^2)D^2(z) - 1 = 0. \tag{13.1}$$

Theorem 40. *Let $\Psi(z)$ be given by* (1.1)*, and let α be some positive integer. Then the generating function $D(z) = \sum_{n\ge0} D_n\, z^n$ for the central Delannoy numbers, reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(z^2)$ of the form*

$$D(z) = a_0(z) + \sum_{i=0}^{2\cdot3^\alpha-1} a_i(z)\Psi^i(z^2) \quad \text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i = 0, 1, \dots, 2\cdot3^\alpha - 1$, are Laurent polynomials in z and $1 + z^2$.

Proof. Choosing $\gamma = 2$, $c_2(z) = 1 - 6z + z^2$, $c_1(z) = 0$, $c_0(z) = -1$, we see that the functional equation (13.1) fits into the framework of Theorem 17, since

$$c_1^2(z) - c_0(z)c_2(z) = 1 + z^2 \quad \text{modulo } 3.$$

The constants e_1, e_2, f_1, f_2 are given by $e_1 = 0$, $e_2 = 1$, $f_1 = 0$, $f_2 = 0$, $\varepsilon = 1$, and we have $\mathcal{Q}(\dots) = 0$. Consequently, the generating function $D(z)$, when reduced modulo 3^{3^α}, can be written as a polynomial in the basic series $\Psi(z^2)$, as claimed. $\square$

We have implemented the algorithm explained in the above proof. For $\alpha = 1$, it produces the following result.

Theorem 41. *Let $\Psi(z)$ be given by* (1.1). *Then, we have*

$$\sum_{n\ge0} D_n\, z^n = \left(3 + 9\frac{(z+2)^2}{1+z^2}\right)\Psi(z^2) + \left(19z^2 + 3z + 19\right)\Psi^3(z^2) \\ - \left(3z^4 + 9z^3 + 6z^2 + 9z + 3\right)\Psi^5(z^2) \quad \text{modulo } 27. \tag{13.2}$$

In the case of modulus 9, this theorem reduces to the following result.

Corollary 42. *Let $\Psi(z)$ be given by* (1.1). *Then, we have*

$$\sum_{n\ge0} D_n\, z^n = -3\Psi(z^2) + \left(4z^2 + 3z + 4\right)\Psi^3(z^2) \quad \text{modulo } 9. \tag{13.3}$$

These results generalise the part on central Delannoy numbers in Theorem 5.15 of [3], which is an easy consequence.

14. Schröder numbers

Let S_n be the n-th (large) *Schröder number*, that is, the number of lattice paths from $(0,0)$ to $(2n,0)$ consisting of steps taken from the set $\{(2,0),(1,1),(1,-1)\}$ never running below the x-axis. It is well-known (cf. [18, Second Problem on page 178]) that the generating function $S(z) = \sum_{n\ge0} S_n\, z^n$ is given by

$$S(z) = \frac{1 - z - \sqrt{1 - 6z + z^2}}{2z},$$

and hence satisfies the functional equation

$$zS^2(z) + (z-1)S(z) + 1 = 0. \tag{14.1}$$

Theorem 43. *Let $\Psi(z)$ be given by* (1.1), *and let α be some positive integer. Then the generating function $S(z) = \sum_{n\ge0} S_n z^n$ for the Schröder numbers, reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(z^2)$ of the form*

$$S(z) = a_0(z) + \sum_{i=0}^{2\cdot3^\alpha-1} a_i(z)\Psi^i(z^2) \quad \text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i = 0, 1, \dots, 2\cdot3^\alpha - 1$, are Laurent polynomials in z and $1 + z^2$.

Proof. Choosing $\gamma = 2$, $c_2(z) = z$, $c_1(z) = z - 1$, $c_0(z) = 1$, we see that the functional equation (14.1) fits into the framework of Theorem 17, since

$$c_1^2(z) - c_0(z)c_2(z) = 1 + z^2 \quad \text{modulo } 3.$$

The constants e_1, e_2, f_1, f_2 are given by $e_1 = 1$, $e_2 = 0$, $f_1 = 0$, $f_2 = 0$, $\varepsilon = 1$, and we have $\mathcal{Q}(\dots) = 0$. Consequently, the generating function $S(z)$, when reduced modulo 3^{3^α}, can be written as a polynomial in the basic series $\Psi(z^2)$, as claimed. $\square$

We have implemented the algorithm explained in the above proof. For $\alpha = 1$, it produces the following result.

Theorem 44. *Let $\Psi(z)$ be given by (1.1). Then, we have*

$$\sum_{n\ge0} S_n\, z^n = 13 + 14z^{-1} + \left(21z + 18 + 21z^{-1}\right)\Psi(z^2) \\ + \left(4z^3 + 15z^2 + 17z + 15 + 4z^{-1}\right)\Psi^3(z^2) \\ + \left(15z^5 + 9z^4 + 18z^3 + 18z^2 + 18z + 9 + 15z^{-1}\right)\Psi^5(z^2) \quad \text{modulo } 27. \tag{14.2}$$

In the case of modulus 9, this theorem reduces to the following result.

Corollary 45. *Let $\Psi(z)$ be given by (1.1). Then, we have*

$$\sum_{n\ge0} S_n\, z^n = 4 + 5z^{-1} - 3\left(z + z^{-1}\right)\Psi(z^2) - \left(2z^3 + 3z^2 + 4z + 3 + 2z^{-1}\right)\Psi^3(z^2) \\ \text{modulo } 9. \tag{14.3}$$

Schröder numbers have not been considered in [3].

15. Hex tree numbers

Let H_n be the n-th *hex tree number*, that is, the number of planar rooted trees where each vertex may have a left, a middle, or a right descendant, but never a left *and* middle descendant, and never a middle *and* right descendant. By an elementary combinatorial decomposition, it is easy to see that the generating function $H(z) = \sum_{n\ge0} H_n\, z^n$ satisfies the functional equation

$$z^2H^2(z) + (3z - 1)H(z) + 1 = 0. \tag{15.1}$$

Theorem 46. *Let $\Psi(z)$ be given by (1.1), and let α be some positive integer. Then the generating function $H(z) = \sum_{n\ge0} H_n\, z^n$ for the hex tree numbers, reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(-z^2)$ of the form*

$$H(z) = a_0(z) + \sum_{i=0}^{2\cdot3^\alpha-1} a_i(z)\Psi^i(-z^2) \quad \text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i = 0, 1, \dots, 2\cdot3^\alpha - 1$, are Laurent polynomials in z and $1 - z^2$.

Proof. Choosing $\gamma = 2$, $c_2(z) = z^2$, $c_1(z) = 3z - 1$, $c_0(z) = 1$, we see that the functional equation (15.1) fits into the framework of Theorem 17, since

$$c_1^2(z) - c_0(z)c_2(z) = 1 - z^2 \quad \text{modulo } 3.$$

The constants e_1, e_2, f_1, f_2 are given by $e_1 = 2$, $e_2 = 0$, $f_1 = 0$, $f_2 = 0$, $\varepsilon = -1$, and we have $\mathcal{Q}(\dots) = 0$. Consequently, the generating function $H(z)$, when reduced modulo 3^{3^α}, can be written as a polynomial in the basic series $\Psi(z^2)$, as claimed. □

We have implemented the algorithm explained in the above proof. For $\alpha = 1$, it produces the following result.

Theorem 47. *Let $\Psi(z)$ be given by (1.1). Then, we have*

$$\sum_{n\ge0} H_n\,z^n = 12z^{-1} + 14z^{-2} - 3\left(4 + 3z^{-1} + 2z^{-2}\right)\Psi(-z^2) \\ + \left(z^2 + 21z + 13 + 15z^{-1} + 4z^{-2}\right)\Psi^3(-z^2) \\ + 3\left(z^4 + 3z^3 + 12z^2 + 21z + 3z^{-1} + 14z^{-2}\right)\Psi^5(-z^2) \quad \text{modulo } 27. \tag{15.2}$$

In the case of modulus 9, this theorem reduces to the following result.

Corollary 48. *Let $\Psi(z)$ be given by (1.1). Then, we have*

$$\sum_{n\ge0} H_n\,z^n = -4z^{-2} + 3\left(1 + 3z^{-1} + 2z^{-2}\right)\Psi(-z^2) \\ + \left(4z^2 + 7 + 7z^{-2}\right)\Psi^3(-z^2) \quad \text{modulo } 9. \tag{15.3}$$

These results generalise Theorem 5.3 in [3], which is an easy consequence.

16. Free subgroup numbers for lifts of the inhomogeneous modular group

Let f_λ be the number of free subgroups of $PSL_2(\mathbb{Z})$ of index 6λ. More generally, let $f_\lambda(m)$ denote the number of free subgroups of index $6m\lambda$ in the lift $\Gamma_m(3)$ of $PSL_2(\mathbb{Z})$ defined by

$$\Gamma_m(3) = C_{2m} \underset{C_m}{*} C_{3m} = \left\langle x, y \,\middle|\, x^{2m} = y^{3m} = 1,\ x^2 = y^3 \right\rangle, \quad m \ge 1.$$

Clearly, we have $f_\lambda = f_\lambda(1)$.

The generating function $F_m(z) = 1 + \sum_{\lambda\ge1} f_\lambda(m) z^\lambda$ satisfies the Riccati differential equation (see [10, Eq. (18)] with $\mathfrak{G} = \Gamma_m(q)$, or [7, Eq. (8.1)])

$$zF_m^2(z) - (1 - (6m-2)z)F_m(z) + 6mz^2F_m'(z) + 1 + (1 - 6m + 5m^2)z = 0. \tag{16.1}$$

It is not hard to see that, if m is divisible by 3, then $v_3(f_\lambda(m))$ will increase with λ, that is, the sequence $f_\lambda(m)$ is ultimately zero modulo any given 3-power.

Theorem 49. *Let $\Psi(z)$ be given by (1.1), let m be a positive integer not divisible by 3, and let α be some positive integer. Then the generating function $F_m(z)$ for the free subgroup numbers $f_\lambda(m)$, when reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(z^6)$ of the form*

$$F_m(z) = a_0(z) + \sum_{i=0}^{2\cdot3^\alpha-1} a_i(z)\Psi^i(z^6) \quad \text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i = 0, 1, \dots, 2\cdot3^\alpha - 1$, are Laurent polynomials in z and $1 + z^2$.

Proof. Choosing $\gamma = 2$, $c_2(z) = -z$, $c_1(z) = 1-(6m-2)z$, $c_0(z) = 1-(1-6m+5m^2)z$, we see that the functional equation (16.1) fits into the framework of Theorem 17, since

$$c_1^2(z) - c_0(z)c_2(z) = 1 + z^2 \quad \text{modulo } 3.$$

The constants e_1, e_2, f_1, f_2 are given by $e_1 = 1$, $e_2 = 0$, $f_1 = 0$, $f_2 = 0$, $\varepsilon = 1$, and we have $\mathcal{Q}(\cdots) = 6mz^2F_m'(z)$. Consequently, the generating function $F_m(z)$, when reduced modulo 3^{3^α}, can be written as a polynomial in the basic series $\Psi(z^2) = (1+z^2)\Psi(z^6)$. In the formulation of the theorem, we have chosen to express everything in terms of $\Psi(z^6)$, since this leads to the more elegant expressions for the moduli 9 and 27 given below. ☐

We have implemented the algorithm explained in the above proof. For $\alpha = 1$, it produces the following result.

Theorem 50. *Let $\Psi(z)$ be given by* (1.1)*. Then, for all positive integers $m \equiv 1 \pmod 3$, we have*

$$\begin{aligned} 1 + \sum_{\lambda\ge1} f_\lambda(m)\, z^\lambda &= 1 - z^{-1} - 3\frac{(z+2)\,(z^2+2z+2)}{z\,(z^2+1)} - \frac{9+3(m-1)z^4}{(z^2+1)^2} \\ &\quad + \left(3mz^3 + (3m+21)z + 21z^{-1}\right)\Psi(z^6) \\ &\quad + \big((-2m+18)z^9 + (-m^2+6m+18)z^7 + (-3m+25)z^5 \\ &\qquad + (m+15)z^3 + (2m^2-3m+6)z + 4z^{-1}\big)\,\Psi^3(z^6) \\ &\quad + \big((-3m+9)z^{15} + (-3m+6)z^{13} + 15z^{11} + (3m+9)z^9 + (3m+3)z^7 \\ &\qquad + 3z^5 + (-3m+9)z^3 + (-3m+6)z + 15z^{-1}\big)\,\Psi^5(z^6) \quad \text{modulo } 27. \end{aligned} \tag{16.2}$$

Furthermore, for all positive integers $m \equiv 2 \pmod 3$, we have

$$\begin{aligned} 1 + \sum_{\lambda\ge1} f_\lambda(m)\, z^\lambda &= 1 - z^{-1} + 3\frac{(z+1)\,(z^2+z+2)}{z\,(z^2+1)} \\ &\quad + \frac{9+18z+18z^2+9z^3+9z^4-(3m-24)z^5}{(z^2+1)^2} \\ &\quad + \left(-3mz^3 + (-3m+21)z + 21z^{-1}\right)\Psi(z^6) \\ &\quad + \big((15m^2-4m+24)z^9 + (-m^2+3m)z^7 + (3m-2)z^5 \\ &\qquad + (-m+15)z^3 + (14m^2+18)z + 4z^{-1}\big)\,\Psi^3(z^6) \\ &\quad + \big((3m+9)z^{15} + (3m+6)z^{13} + 15z^{11} + (-3m+9)z^9 + (-3m+3)z^7 \\ &\qquad + 3z^5 + (3m+9)z^3 + (3m+6)z + 15z^{-1}\big)\,\Psi^5(z^6) \quad \text{modulo } 27. \end{aligned} \tag{16.3}$$

In the case of modulus 9, this theorem reduces to the following result.

Corollary 51. *Let $\Psi(z)$ be given by* (1.1)*. Then, for all positive integers $m \equiv 1 \pmod 3$, we have*

$$\begin{aligned} 1 + \sum_{\lambda\ge1} f_\lambda(m)\, z^\lambda &= 1 - 4z^{-1} - 3\frac{z^2}{z^2+1} - 3z^{-1}\left(z^2+1\right)^2\Psi(z^6) \\ &\quad + \left(mz^9 + (m+1)z^7 + 7z^5 + mz^3 + (m+1)z + 7z^{-1}\right)\Psi^3(z^6) \quad \text{modulo } 9. \end{aligned} \tag{16.4}$$

Furthermore, for all positive integers $m \equiv 2 \pmod 3$, we have

$$1+\sum_{\lambda\ge1} f_\lambda(m)\,z^\lambda = 1+5z^{-1}+3\frac{z^2}{z^2+1}-3z^{-1}(1+z^2)^2\Psi(z^6)\\ -(mz^9+(m-1)z^7+2z^5+mz^3+(m-1)z+2z^{-1})\Psi^3(z^6) \quad \text{modulo } 9. \tag{16.5}$$

We have chosen this section to demonstrate that generating function results such as the ones in Theorem 50 and Corollary 51, in combination with the coefficient extraction procedure described in the beginning of the appendix, lead naturally to an explicit description of the congruence classes modulo any given 3-power of the elements of the sequence that we are considering. We illustrate this point with the sequence of free subgroup numbers $f_\lambda = f_\lambda(1)$ of the inhomogeneous modular group $PSL_2(\mathbb{Z})$. In the statement of the result below, we use notation from word calculus: by S^*, where S is a set of letters, we mean the set of all words with letters from S. By a slight misuse of notation, the symbol a^*, where a is a letter, will denote the set $\{\varepsilon, a, aa, aaa, \dots\}$, with ε standing for the empty word. Finally, if we write S_1S_2, where S_1 and S_2 are two sets of words, then we mean the set of all concatenations w_1w_2 with $w_1 \in S_1$ and $w_2 \in S_2$. Since our words represent ternary strings, we make the convention that 0's at the beginning of a word may be added or deleted at will. (The point of view which we shall adopt later in the statements of Theorem 54, Proposition 68, Corollary 69, Proposition 71, and Corollary 72 is to consider these strings as left-infinite strings, where one has added an infinite string of 0's on the left of a finite word if necessary.)

Theorem 52. *We have:*

(1) $f_\lambda \equiv 1 \pmod 9$, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\{0\}\cup\bigcup_{k\ge0}\left(00^*22^*\right)^{3k}0^*10^*00\cup\bigcup_{k\ge0}\left(00^*22^*\right)^{3k+2}0^*10\\ \cup\bigcup_{k\ge0}\left(22^*00^*\right)^{3k+2}2^*12\cup\bigcup_{k\ge0}\left(22^*00^*\right)^{3k}2^*12^*22;$$

(2) $f_\lambda \equiv 4 \pmod 9$, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\bigcup_{k\ge0}\left(00^*22^*\right)^{3k+2}0^*10^*00\cup\bigcup_{k\ge0}\left(00^*22^*\right)^{3k+1}0^*10\\ \cup\bigcup_{k\ge0}\left(22^*00^*\right)^{3k+1}2^*12\cup\bigcup_{k\ge0}\left(22^*00^*\right)^{3k+2}2^*12^*22;$$

(3) $f_\lambda \equiv 7 \pmod 9$, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\{10\}\cup\bigcup_{k\ge0}\left(00^*22^*\right)^{3k+1}0^*10^*00\cup\bigcup_{k\ge0}\left(00^*22^*\right)^{3k}0^*10\\ \cup\bigcup_{k\ge0}\left(22^*00^*\right)^{3k}2^*12\cup\bigcup_{k\ge0}\left(22^*00^*\right)^{3k+1}2^*12^*22;$$

(4) $f_\lambda \equiv 2 \pmod 9$, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\bigcup_{k\ge0}\left(00^*22^*\right)^{3k+2}0^*01\cup\bigcup_{k\ge0}\left(00^*22^*\right)^{3k+1}1;$$

(5) $f_\lambda \equiv 5 \pmod 9$, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\bigcup_{k\ge0} (00^*22^*)^{3k} 0^*01 \cup \bigcup_{k\ge0} (00^*22^*)^{3k+2} 1;$$

(6) $f_\lambda \equiv 8 \pmod 9$, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\bigcup_{k\ge0} (00^*22^*)^{3k+1} 0^*01 \cup \bigcup_{k\ge0} (00^*22^*)^{3k+3} 1;$$

(7) $f_\lambda \equiv 3 \pmod 9$, *if, and only if,* $\lambda \equiv 0 \pmod 4$ *or the* 3*-adic expansion of* n *is an element of*

$$\{0,2\}^*11\{0,2\}^*1,$$

(8) $f_\lambda \equiv 6 \pmod 9$, *if, and only if,* $\lambda \equiv 2 \pmod 4$ *or the* 3*-adic expansion of* n *is an element of*

$$\{0,2\}^*1022^* \cup \{0,2\}^*1200^* \cup \{0,2\}^*11\{0,2\}^*100^* \cup \{0,2\}^*11\{0,2\}^*122^*.$$

(9) *In all other cases, this coefficient is divisible by* 9*.*

Proof. One performs a (lengthy) case analysis using Corollary 51 with $m = 1$ in combination with the descriptions of the coefficients of $\Psi^3(z)$ modulo 9, which one obtains by carrying out the approach sketched in Section 2 (see in particular the paragraphs after (2.6) and (2.7)) and the appendix, the resulting congruences being explicitly displayed in Corollary 69. $\square$

The next corollary corrects Theorem 3(i) in [11].

Corollary 53. *We have:*

(1) $f_\lambda \equiv -1 \pmod 3$ *if, and only if, the* 3*-adic expansion of* λ *is an element of* $\{0,2\}^*1$*;*

(2) $f_\lambda \equiv 1 \pmod 3$ *if, and only if, the* 3*-adic expansion of* λ *is an element of*

$$\{0,2\}^*100^* \cup \{0,2\}^*122^*;$$

(3) *for all other* λ*, we have* $f_\lambda \equiv 0 \pmod 3$*.*

Remark. The conditions in Items (1) and (2) imply that the 3-adic expansion of λ always contains exactly one digit 1. In Item (1), this 1 is the first digit from the right. In Item (2), this 1 is not the right-most digit, and it has only 0's or only 2's to its right.

With some additional effort, by using Theorem 50 with $m = 1$ and the coefficient congruences modulo 27 for $\Psi^3(z)$ and $\Psi^5(z)$ worked out in Propositions 68 and 71, one is also able to refine the result in Theorem 52 to a description of the congruence classes of f_λ modulo 27. The outcome is lengthy. As an illustration, we provide the explicit characterisation of all λ for which $f_\lambda \equiv 1 \pmod{27}$. For the statement of the result, we need to introduce some statistics on strings consisting of 0's, 1's, and 2's. Given a string $s = \dots s_2 s_1 s_0$, we define:

$$\begin{aligned}
\#\mathrm{2str}(s) &= \#(\text{maximal strings of 2's in } s),\\
\#\mathrm{Iso}_3(s) &= \#(\text{isolated 0's and 2's in } s, \text{ excluding } s_0, s_1, s_2),\\
\#022(s) &= \#(\text{occurrences of the substring 022 in } s),
\end{aligned}$$

and similarly for $\#200(s)$, $\#201(s)$, $\#21(s)$, $\#021(s)$. Here, by an isolated letter, we mean a letter a whose neighbours are different from a. We use again the word notation explained before Theorem 52. In particular, we consider all strings as infinite strings with infinitely many 0's added to the left, so that #022(.) also counts an occurrence of 22 at the beginning of a(n originally finite) string.

Theorem 54. *We have* $f_\lambda \equiv 1 \pmod{27}$, *if, and only if, the* 3*-adic expansion of* n *is an element of* $\{0,2\}^*0100$ *and satisfies* $\#2\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)}{3} - \#\mathrm{Iso}_3(s) + \#022(s) + \#200(s) + \#201(s) \equiv 0 \pmod 3,$$

or of $\{0,2\}^*2100$ *and satisfies* $\#2\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)}{3} - \#\mathrm{Iso}_3(s) + \#022(s) + \#200(s) \equiv 1 \pmod 3,$$

or of $\{0,2\}^*10^*0000$ *and satisfies* $\#2\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)}{3} - \#\mathrm{Iso}_3(s) - \#21(s) + \#022(s) + \#200(s) + \#201(s) \equiv 2 \pmod 3,$$

or of $\{0,2\}^*1000$ *and satisfies* $\#2\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)}{3} - \#\mathrm{Iso}_3(s) - \#21(s) + \#022(s) + \#200(s) + \#201(s) \equiv 1 \pmod 3,$$

or of $\{0,2\}^*2210$ *and satisfies* $\#2\mathrm{str}(s) \equiv 2 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)-1}{3} - \#\mathrm{Iso}_3(s) + \#022(s) + \#200(s) \equiv 0 \pmod 3,$$

or of $\{0,2\}^*0210$ *and satisfies* $\#2\mathrm{str}(s) \equiv 2 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)-1}{3} - \#\mathrm{Iso}_3(s) + \#022(s) + \#200(s) \equiv 1 \pmod 3,$$

or of $\{0,2\}^*010$ *and satisfies* $\#2\mathrm{str}(s) \equiv 2 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)-1}{3} - \#\mathrm{Iso}_3(s) + \#022(s) + \#200(s) \equiv 2 \pmod 3,$$

or of $\{0,2\}^*0010$ *and satisfies* $\#2\mathrm{str}(s) \equiv 2 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)-1}{3} - \#\mathrm{Iso}_3(s) + \#022(s) + \#200(s) \equiv 0 \pmod 3,$$

or of $\{0,2\}^*0012$ *and satisfies* $\#2\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)-3}{3} - \#\mathrm{Iso}_3(s) + \#022(s) + \#200(s) \equiv 0 \pmod 3,$$

or of $\{0,2\}^*0212$ *and satisfies* $\#2\mathrm{str}(s) \equiv 1 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)-2}{3} - \#\mathrm{Iso}_3(s) + \#022(s) + \#200(s) \equiv 0 \pmod 3,$$

or of $\{0,2\}^*2012$ *and satisfies* $\#2\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)-3}{3} - \#\mathrm{Iso}_3(s) + \#022(s) + \#200(s) \equiv 1 \pmod 3,$$

or of $\{0,2\}^*2212$ *and satisfies* $\#1\mathrm{str}(s) \equiv 1 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-2}{3} - \#\mathrm{Iso}_3(s) + \#022(s) + \#200(s) \equiv 1 \pmod 3,$$

or of $\{0,2\}^*0122$ *and satisfies* $\#2\mathrm{str}(s) \equiv 1 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)-2}{3} - \#\mathrm{Iso}_3(s) + \#022(s) + \#200(s) \equiv 0 \pmod 3,$$

or of $\{0,2\}^*2122$ *and satisfies* $\#2\mathrm{str}(s) \equiv 2 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)-4}{3} - \#\mathrm{Iso}_3(s) + \#022(s) + \#200(s) - \#021(s) \equiv 2 \pmod 3,$$

or of $\{0,2\}^*01222$ *and satisfies* $\#2\mathrm{str}(s) \equiv 1 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)-2}{3} - \#\mathrm{Iso}_3(s) + \#022(s) + \#200(s) \equiv 2 \pmod 3,$$

or of $\{0,2\}^*21222$ *and satisfies* $\#2\mathrm{str}(s)\equiv 2 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)-1}{3}-\#\mathrm{Iso}_3(s)+\#022(s)+\#200(s)-\#021(s)\equiv 2 \pmod 3,$$

or of $\{0,2\}^*012^*2222$ *and satisfies* $\#2\mathrm{str}(s)\equiv 1 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)-2}{3}-\#\mathrm{Iso}_3(s)+\#022(s)+\#200(s)\equiv 2 \pmod 3,$$

or of $\{0,2\}^*212^*2222$ *and satisfies* $\#2\mathrm{str}(s)\equiv 2 \pmod 3$ *and*

$$\tfrac{2\,\#2\mathrm{str}(s)-1}{3}-\#\mathrm{Iso}_3(s)+\#022(s)+\#200(s)-\#021(s)\equiv 2 \pmod 3.$$

17. Central Eulerian numbers

The *Eulerian number* $A(n,k)$ is defined as the number of permutations of $\{1,2,\dots,n\}$ with exactly $k-1$ descents. It is well-known that

$$A(n,k)=\sum_{j=0}^{k}(-1)^{k-j}\binom{n+1}{k-j}j^n; \tag{17.1}$$

see [17, Eq. (1.37)] (one has to multiply both sides of that equation by $(1-x)^{d+1}$ and compare coefficients of powers of x). In this section, we analyse the behaviour of *central Eulerian numbers*, that is, the numbers $A(2n,n)=A(2n,n+1)$ and $A(2n-1,n)$, modulo powers of 3. Theorems 59 and 62 below say that, again, the generating functions for central Eulerian numbers, when coefficients are reduced modulo a given power of 3, can be written in terms of a polynomial in the basic series $\Psi(z)$. However, the arguments establishing this claim are much more elaborate. The principal reason for this is that it is (likely) not possible to find a *single* differential equation of the form (1.2), which we could use as starting point for the application of our method. Rather, we have to first reduce the central Eulerian numbers partially modulo the relevant 3-power, and *only* then do we succeed to show that the generating function for the reduced numbers satisfies a functional equation of the type (6.1) so that Theorem 17 can be applied. In order to accomplish this, we need a number of auxiliary results.

Lemma 55. *For any positive integer S, we have the (differential) operator equation*

$$\left((x+1)\frac{d}{dx}\right)^S=\sum_{m=1}^{S}\frac{1}{m!}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^S\right)(x+1)^m\frac{d^m}{dx^m}. \tag{17.2}$$

Proof. We proceed by induction on S. Clearly, Equation (17.2) holds for $S=1$. For the induction step, we compute

$$\begin{aligned}
\left((x+1)\frac{d}{dx}\right)^{S+1} &= (x+1)\frac{d}{dx}\sum_{m=1}^{S}\frac{1}{m!}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^S\right)(x+1)^m\frac{d^m}{dx^m}\\
&=\sum_{m=1}^{S}\frac{1}{m!}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^S\right)m(x+1)^m\frac{d^m}{dx^m}\\
&\qquad+\sum_{m=1}^{S}\frac{1}{m!}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^S\right)(x+1)^{m+1}\frac{d^{m+1}}{dx^{m+1}}\\
&=\sum_{m=1}^{S}\frac{1}{m!}\left(\sum_{k=1}^{m}(-1)^{m-k}m\left(\binom{m}{k}-\binom{m-1}{k}\right)k^S\right)(x+1)^m\frac{d^m}{dx^m}\\
&\qquad+\frac{1}{S!}\left(\sum_{k=1}^{S}(-1)^{S-k}\binom{S}{k}k^S\right)(x+1)^{S+1}\frac{d^{S+1}}{dx^{S+1}}\\
&=\sum_{m=1}^{S}\frac{1}{m!}\left(\sum_{k=1}^{m}(-1)^{m-k}m\binom{m-1}{k-1}k^S\right)(x+1)^m\frac{d^m}{dx^m}\\
&\qquad+\frac{1}{S!}\left(\sum_{k=1}^{S}(-1)^{S-k}\binom{S}{k}k^S\right)(x+1)^{S+1}\frac{d^{S+1}}{dx^{S+1}}\\
&=\sum_{m=1}^{S}\frac{1}{m!}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{S+1}\right)(x+1)^m\frac{d^m}{dx^m}\\
&\qquad+\frac{1}{S!}\left(\sum_{k=1}^{S}(-1)^{S-k}\binom{S}{k}k^S\right)(x+1)^{S+1}\frac{d^{S+1}}{dx^{S+1}}. \qquad (17.3)
\end{aligned}$$

At this point we use again the shift operator E and the forward difference operator $\triangle$ defined in the proof of Lemma 11. The symbol I stands for the identity operator. In terms of these operators, we may write

$$\begin{aligned}
\frac{1}{S!}\left(\sum_{k=1}^{S}(-1)^{S-k}\binom{S}{k}k^S\right) &= \frac{1}{S!}\left(\sum_{k=1}^{S}(-1)^{S-k}\binom{S}{k}E^k y^S\Big|_{y=0}\right)\\
&=\frac{1}{S!}(E-I)^S y^S\Big|_{y=0}=\frac{1}{S!}\triangle^S\, y^S\Big|_{y=0}=1,
\end{aligned}$$

the last equality being due to the fact that the difference operator $\triangle$ lowers the degree of the polynomial to which it is applied by 1, and it multiplies the leading coefficient by the degree of the corresponding monomial. But then

$$\frac{1}{S!}\left(\sum_{k=1}^{S}(-1)^{S-k}\binom{S}{k}k^S\right)=\frac{1}{(S+1)!}\left(\sum_{k=1}^{S+1}(-1)^{S+1-k}\binom{S+1}{k}k^{S+1}\right),$$

and if this is substituted back into the last line of (17.3), then we get the required result. $\square$

Lemma 56. *For any positive integer M, we have*

$$\left(-1+\sqrt{1+4z}\right)^M-\left(-1-\sqrt{1+4z}\right)^M=(-1)^{M+1}2^M\sqrt{1+4z}\sum_{\ell\ge0}\binom{M-\ell-1}{\ell}z^\ell \tag{17.4}$$

and

$$\left(-1+\sqrt{1+4z}\right)^M+\left(-1-\sqrt{1+4z}\right)^M=(-1)^M2^M\sum_{\ell\ge0}\frac{M}{M-\ell}\binom{M-\ell}{\ell}z^\ell. \tag{17.5}$$

Remark. The sums over ℓ on the right-hand sides of (17.4) and (17.5) are essentially Chebyshev polynomials of the second and first kind, respectively (cf. [13]). It is easy to see (and well-known) that they are polynomials in z with integer coefficients.

Proof of Lemma 56. We start with the proof of (17.4). According to the binomial theorem, the left-hand side can be written as

$$\begin{aligned}\left(-1+\sqrt{1+4z}\right)^M&-\left(-1-\sqrt{1+4z}\right)^M\\&=(-1)^M\sum_{j=0}^{M}\binom{M}{j}\left((-1)^j(1+4z)^{j/2}-(1+4z)^{j/2}\right)\\&=-2(-1)^M\sum_{j\ge0}\binom{M}{2j+1}(1+4z)^{j+\frac12}\\&=-2(-1)^M\sqrt{1+4z}\sum_{j\ge0}\sum_{\ell=0}^{j}\binom{M}{2j+1}\binom{j}{\ell}4^\ell z^\ell.\end{aligned}$$

Using the standard hypergeometric notation

$${}_pF_q\left[\begin{matrix}a_1,\dots,a_p\\b_1,\dots,b_q\end{matrix};z\right]=\sum_{m=0}^{\infty}\frac{(a_1)_m\cdots(a_p)_m}{m!\,(b_1)_m\cdots(b_q)_m}z^m,$$

where, as before, the Pochhammer symbol is defined by $(\alpha)_m=\alpha(\alpha+1)\cdots(\alpha+m-1)$ for $m\ge1$, and $(\alpha)_0=1$, we have

$$\sum_{j\ge0}\binom{M}{2j+1}\binom{j}{\ell}=\binom{M}{2\ell+1}{}_2F_1\left[\begin{matrix}\frac12+\ell-\frac M2,1+\ell-\frac M2\\\frac32+\ell\end{matrix};1\right].$$

This ${}_2F_1$-series can be summed by means of Gauß' summation formula (see [15, (1.7.6); Appendix (III.3)])

$${}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};1\right]=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)},$$

where $\Gamma(\,.\,)$ denotes the gamma function. After some simplification, this leads to

$$\sum_{j\ge0}\binom{M}{2j+1}\binom{j}{\ell}=2^{M-2\ell-1}\binom{M-\ell-1}{\ell}.$$

The proof of (17.5) is completely analogous, and is left to the reader. □

Proposition 57. *For any positive integer s, we have*

$$\sum_{n\ge0} z^n \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{2n+1}{n+1-j} j^{2s} = \frac{1}{2}\left(1+\sqrt{1+4z}\right)\left(1+p_s^{(1)}(z)\right), \tag{17.6}$$

where $p_s^{(1)}(z)$ is a polynomial in z with integer coefficients all of which are divisible by 3, and which satisfies $p_s^{(1)}(0)=0$. An explicit expression for $p_s^{(1)}(z)$ can be derived from (17.8) *and* (17.9).

Proof. We write the binomial coefficient in terms of a contour integral,

$$(-1)^{n+1-j}\binom{2n+1}{n+1-j} = \frac{1}{2\pi i}\int_C \frac{(1-t)^{2n+1}}{t^{n+2-j}}\,dt,$$

where C is the circle of radius $1/2$ (say) about the origin, encircling the origin in positive direction. Moreover, we express j^{2s} as

$$j^{2s} = \left((x+1)\frac{d}{dx}\right)^{2s} (x+1)^j\bigg|_{x=0}.$$

Using these substitutions, the series on the left-hand side of (17.6) becomes

$$\begin{aligned}
&\sum_{n\ge0} z^n \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{2n+1}{n+1-j} j^{2s}\\
&\quad= \frac{1}{2\pi i}\int_C \sum_{n\ge0} z^n \sum_{j=0}^{n+1} \frac{(1-t)^{2n+1}}{t^{n+2-j}} \left((x+1)\frac{d}{dx}\right)^{2s} (x+1)^j\bigg|_{x=0} dt\\
&\quad= \frac{1}{2\pi i}\int_C \frac{1-t}{t^2}\sum_{n\ge0}\left(\frac{(1-t)^2z}{t}\right)^n \left((x+1)\frac{d}{dx}\right)^{2s} \sum_{j=0}^{n+1}\big(t(x+1)\big)^j\bigg|_{x=0} dt\\
&\quad= \frac{1}{2\pi i}\int_C \frac{1-t}{t^2\left(1-\frac{(1-t)^2z}{t}\right)} \left((x+1)\frac{d}{dx}\right)^{2s} \frac{1}{1-(x+1)t}\bigg|_{x=0} dt.
\end{aligned}$$

Here, in order to perform the summation over n, we had to assume that z is sufficiently small, say $|z|<1/9$, which we shall do for the moment.

In the last integral, we do the substitution $t = u/(1+u)^2$. After some simplification, this leads to

$$\begin{aligned}
\sum_{n\ge0} z^n \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{2n+1}{n+1-j} j^{2s}&\\
&= \frac{1}{2\pi i}\int_C \frac{1+u}{u\,(u^2+u-z)}\left((x+1)\frac{d}{dx}\right)^{2s} \frac{1}{1+u-(x+1)u}\bigg|_{x=0} du\\
&= \frac{1}{2\pi i}\int_C \frac{1+u}{u\,(u^2+u-z)}\sum_{m=1}^{2s}\frac{1}{m!}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{2s}\right)\\
&\qquad\qquad\cdot\left((x+1)^m\frac{d^m}{dx^m}\frac{1}{1+u-(x+1)u}\right)\bigg|_{x=0} du\\
&= \frac{1}{2\pi i}\int_C \frac{1+u}{u^2+u-z}\sum_{m=1}^{2s}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{2s}\right)u^{m-1}\,du,
\end{aligned}$$

where we have used Lemma 55 to obtain the next-to-last line.

The integral can easily be evaluated using the residue theorem. Inside the contour C, there is only one singularity, namely a simple pole at $u = -\frac{1}{2}+\frac{1}{2}\sqrt{1+4z}$. (The other pole, at $u = -\frac{1}{2}-\frac{1}{2}\sqrt{1+4z}$, lies outside the contour C.) Hence, we obtain

$$\begin{aligned}
\sum_{n\ge0} z^n \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{2n+1}{n+1-j} j^{2s}&\\
&= \frac{1+\sqrt{1+4z}}{2\sqrt{1+4z}}\left(\sum_{m=1}^{2s}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{2s}\right)u^{m-1}\right)\bigg|_{u=-\frac12+\frac12\sqrt{1+4z}}. \qquad (17.7)
\end{aligned}$$

By analytic continuation, we can now get rid of the restriction $|z| < 1/9$, and, by the transfer principle between analytic and formal power series, we may argue that Identity (17.7) holds on the level of formal power series.

Let us write the polynomial in u given by the sum over m on the right-hand side of (17.7) as a polynomial in $2u+1$, say

$$q_s^{(1)}(2u+1) := \sum_{m=1}^{2s}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{2s}\right)u^{m-1}. \qquad (17.8)$$

We claim that $q_s^{(1)}(\,.\,)$ is an odd polynomial, that is, $q_s^{(1)}(-2u-1) = -q_s^{(1)}(2u+1)$. In order to see this, we use again classical difference operator calculus. As before, we denote by E the shift operator and by $\triangle$ the forward difference operator (see the proof of Lemma 55). Furthermore, we let $\bigtriangledown$ be the backward difference operator defined by $\bigtriangledown f(y) = f(y-1) - f(y)$. The reader should observe that $\triangle = E - I$ and $\bigtriangledown = E^{-1} - I$, where I stands for the identity operator. In terms of these operators, we may rewrite

$q_s^{(1)}(2u+1)$ as

$$\begin{aligned}
q_s^{(1)}(2u+1) &= \sum_{m=1}^{2s}\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}E^k y^{2s}\Big|_{y=0}u^{m-1}\\
&= \sum_{m=1}^{2s}\triangle^m y^{2s}\Big|_{y=0}u^{m-1}\\
&= \triangle(I-u\triangle)^{-1}(I-(u\triangle)^{2s})y^{2s}\Big|_{y=0}\\
&= \triangle(I-u\triangle)^{-1}y^{2s}\Big|_{y=0},
\end{aligned}$$

where the last line is due to the standard fact that $\triangle^a y^b=0$ for all non-negative integers a and b with $a>b$. Consequently, for $q_s^{(1)}(-2u-1)$ we obtain

$$\begin{aligned}
q_s^{(1)}(-2u-1) &= \triangle(I+(u+1)\triangle)^{-1}y^{2s}\Big|_{y=0}\\
&= \triangle(E+u\triangle)^{-1}y^{2s}\Big|_{y=0}\\
&= -\bigtriangledown(I-u\bigtriangledown)^{-1}y^{2s}\Big|_{y=0}\\
&= -\bigtriangledown(I-u\bigtriangledown)^{-1}(I-(u\bigtriangledown)^{2s})y^{2s}\Big|_{y=0}\\
&= -\sum_{m=1}^{2s}u^{m-1}\bigtriangledown^m y^{2s}\Big|_{y=0}\\
&= -\sum_{m=1}^{2s}\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}E^{-k}y^{2s}\Big|_{y=0}u^{m-1}\\
&= -\sum_{m=1}^{2s}\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}(-k)^{2s}u^{m-1}=-q_s^{(1)}(-2u-1),
\end{aligned}$$

as we claimed. If we recall that (17.7) implies

$$1+p_s^{(1)}(z)=\frac{1}{\sqrt{1+4z}}q_s^{(1)}\left(\sqrt{1+4z}\right), \tag{17.9}$$

and take into account that, under the substitution $u=-\frac{1}{2}+\frac{1}{2}\sqrt{1+4z}$, we have $2u+1=\sqrt{1+4z}$, this establishes all claims of the lemma except for the special form of the polynomial $p_s^{(1)}(z)$.

First we prove that $p_s^{(1)}(0)=0$. To see this, we put $z=0$ in (17.9), and we set $u=0$ in (17.8), to get

$$1+p_s^{(1)}(0)=q_s^{(1)}(1)=1,$$

so that indeed $p_s^{(1)}(0)=0$.

Next we prove that $p_s^{(1)}(z)$ is a polynomial in z with integer coefficients. We use again the relation between $p_s^{(1)}(\,.\,)$ and $q_s^{(1)}(\,.\,)$ given in (17.9). For our purposes, it turns out that it is more convenient to work with an anti-symmetrised version of $q_s^{(1)}(2u+1)$. Namely, since we already know that $q_s^{(1)}(-2u-1) = -q_s^{(1)}(2u+1)$, we may replace $q_s^{(1)}(2u+1)$ by $\frac{1}{2}\left(q_s^{(1)}(2u+1) - q_s^{(1)}(-2u-1)\right)$. In view of (17.8), this means that

$$q_s^{(1)}(2u+1) = \frac{1}{2}\sum_{m=1}^{2s}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{2s}\right)\left(u^{m-1} - (-1-u)^{m-1}\right).$$

If this is used in (17.9), then we obtain

$$\begin{aligned}1 + p_s^{(1)}(z) &= \frac{1}{2\sqrt{1+4z}}\sum_{m=1}^{2s}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{2s}\right)\\ &\qquad\qquad\cdot\left(\left(-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1+4z}\right)^{m-1} - \left(-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1+4z}\right)^{m-1}\right)\\ &= \frac{1}{2^m\sqrt{1+4z}}\sum_{m=2}^{2s}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{2s}\right)\\ &\qquad\qquad\cdot\left(\left(-1+\sqrt{1+4z}\right)^{m-1} - \left(-1-\sqrt{1+4z}\right)^{m-1}\right).\end{aligned}$$

The reader should note that we were free to start the summation over m at $m=2$ in the last expression, since the bracketed factor vanishes for $m=1$. Use of Lemma 56 then transforms this identity into

$$1 + p_s^{(1)}(z) = \frac{(-1)^m}{2}\sum_{m=2}^{2s}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{2s}\right)\sum_{\ell\ge 0}\binom{m-\ell-2}{\ell}z^{\ell}.$$

The right-hand side is "almost" a polynomial with integer coefficients, only a denominator of 2 remains. This denominator is taken care of by the sum over k; namely we have

$$\begin{aligned}\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{2s} &\equiv \sum_{k\ge 0}(-1)^{m-2k-1}\binom{m}{2k+1} \pmod 2\\ &\equiv (-1)^{m-1}\sum_{k\ge 0}\binom{m}{2k+1} \pmod 2\\ &\equiv (-1)^{m-1}2^{m-1} \equiv 0 \pmod 2\end{aligned}$$

for $m \ge 2$. Hence, the polynomial $p_s^{(1)}(z)$ has indeed integer coefficients.

The last remaining assertion is that the coefficients of $p_s^{(1)}(z)$ are divisible by 3. Once we have shown that $q_s^{(1)}(2u+1) = 2u+1$ modulo 3, then, in view of (17.9), this last point will also follow. Consequently, in the sequel we consider $q_s^{(1)}(2u+1)$ with coefficients reduced modulo 3. Since, by assumption, s is a positive integer, we have

$$\begin{aligned} q_s^{(1)}(2u+1) &= \sum_{m=1}^{2s} u^{m-1} \sum_{\substack{k=0\\ k\not\equiv 0 \bmod 3}}^{m} (-1)^{m-k}\binom{m}{k} \quad \text{modulo } 3\\ &= -\sum_{m=1}^{2s} u^{m-1}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{3k} \quad \text{modulo } 3\\ &= \sum_{m=1}^{2s}(-u)^{m-1}\sum_{k=0}^{m}(-1)^{3k}\binom{m}{3k} \quad \text{modulo } 3. \end{aligned}$$

"Trisection" of the binomial theorem then yields

$$q_s^{(1)}(2u+1) = \frac{1}{3}\sum_{m=1}^{2s}(-u)^{m-1}\left((1-\omega)^m + (1-\omega^2)^m\right) \quad \text{modulo } 3,$$

where ω denotes a primitive third root of unity. Since, for $m \ge 3$, we have

$$\frac{1}{3}\left((1-\omega)^m + (1-\omega^2)^m\right) = \frac{1}{3}\left(\left(\tfrac{3-i\sqrt{3}}{2}\right)^m + \left(\tfrac{3+i\sqrt{3}}{2}\right)^m\right) \equiv 0 \pmod 3,$$

only the summands for $m = 1$ and $m = 2$ contribute in the above sum. Thus,

$$q_s^{(1)}(2u+1) = 1 + 2u \quad \text{modulo } 3,$$

as required. This completes the proof of the lemma. $\square$

Proposition 58. *For any positive integer s, we have*

$$\sum_{n\ge 0} z^n \sum_{j=0}^{n}(-1)^{n-j}\binom{2n}{n-j} j^{2s-1} = \frac{z}{\sqrt{1+4z}}\left(1 + p_s^{(2)}(z)\right), \tag{17.10}$$

where $p_s^{(2)}(z)$ is a polynomial in z with integer coefficients all of which are divisible by 3*, and which satisfies $p_s^{(2)}(0) = 0$. An explicit expression for $p_s^{(2)}(z)$ can be derived from* (17.11) *and* (17.12).

Proof. The proof is completely analogous to the proof of Proposition 57, so we content ourselves with just stating the crucial identities without proof.

Using the contour integration method, one obtains

$$\begin{aligned} &\sum_{n\ge 0} z^n \sum_{j=0}^{n}(-1)^{n-j}\binom{2n}{n-j} j^{2s-1}\\ &\qquad = \frac{1}{2\pi i}\int_C \frac{1+u}{u^2+u-z}\sum_{m=1}^{2s-1}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{2s-1}\right)u^m\,du. \end{aligned}$$

Application of the residue theorem then leads to

$$\sum_{n\ge0} z^n \sum_{j=0}^{n}(-1)^{n-j}\binom{2n}{n-j}j^{2s-1}\\
=\frac{z}{\sqrt{1+4z}}\left(\sum_{m=1}^{2s-1}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{2s-1}\right)u^{m-1}\right)\Bigg|_{u=-\frac12+\frac12\sqrt{1+4z}}.$$

Next one defines

$$q_s^{(2)}(2u+1):=\sum_{m=1}^{2s-1}\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{2s-1}\right)u^{m-1}, \tag{17.11}$$

so that

$$1+p_s^{(2)}(z)=q_s^{(2)}\left(\sqrt{1+4z}\right). \tag{17.12}$$

Now one proves that $q_s^{(2)}(-2u-1)=q_s^{(2)}(2u+1)$ (that is, *symmetry* of the polynomial $q_s^{(2)}(\,.\,)$), establishing that $p_s^{(2)}(z)$ is indeed a polynomial in z. By definition, it is obvious that $q_s^{(2)}(1)=1$, whence $p_s^{(2)}(0)=0$. Furthermore, using the *symmetrisation* of $q_s^{(2)}(\,.\,)$, Identity (17.5), and Relation (17.12), one shows that $p_s^{(2)}(z)$ has integer coefficients. In the final step, one proves that all these coefficients are divisible by 3, by demonstrating that

$$q_s^{(2)}(2u+1)=1\quad\text{modulo } 3.$$

□

Theorem 59. *Let $\Psi(z)$ be given by* (1.1)*, and let α be some positive integer. Then the generating function $E^{(1)}(z)=\sum_{n\ge0}A(2n,n+1)\,z^n$ for central Eulerian numbers with even first index, reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(z)$ of the form*

$$E^{(1)}(z)=a_0(z)+\sum_{i=0}^{2\cdot3^\alpha-1}a_i(z)\Psi^i(z^3)\quad\text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i=0,1,\dots,2\cdot3^\alpha-1$, are Laurent polynomials in z and $1+z$.

Proof. By (17.1), we have

$$A(2n,n+1)=\sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{2n+1}{n+1-j}j^{2n}.$$

In order to analyse this expression modulo 3^β (the reader should imagine that $\beta=3^\alpha$), we have to distinguish between several cases, depending on the congruence class of n modulo $3^{\beta-1}$. Namely, since $\varphi(3^\beta)=2\cdot3^{\beta-1}$ (with $\varphi(\,.\,)$ denoting the Euler totient function), we have

$$A(2n,n+1)\equiv\sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{2n+1}{n+1-j}j^{2s}\pmod{3^\beta}\\
\text{for}\quad n\equiv s \pmod{3^{\beta-1}}\text{ and } n,s\ge\tfrac12(\beta-1). \tag{17.13}$$

(The condition $n, s \ge \frac{1}{2}(\beta-1)$ is needed to ensure that $j^{2n} \equiv j^{2s} \equiv 0 \pmod{3^{\beta-1}}$ for all j which are divisible by 3.) From Proposition 57, we know that the generating function for the numbers on the right-hand side of the above congruence, say

$$E_s^{(1)}(z) = \sum_{n\ge 0} z^n \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{2n+1}{n+1-j} j^{2s},$$

has the form given on the right-hand side of (17.6). Consequently, it satisfies the functional equation

$$(E_s^{(1)})^2(z) - \left(1 + p_s^{(1)}(z)\right) E_s^{(1)}(z) - z\left(1 + p_s^{(1)}(z)\right)^2 = 0. \tag{17.14}$$

If we rewrite this in the form

$$(E_s^{(1)})^2(z) - E_s^{(1)}(z) - z - p_s^{(1)}(z) E_s^{(1)}(z) - z p_s^{(1)}(z)\left(2 + p_s^{(1)}(z)\right) = 0,$$

then we see that it fits into the framework of Theorem 17. Indeed, one chooses $\gamma = 1$, $c_2(z) = 1$, $c_1(z) = -1$, $c_0(z) = -z$, which implies that

$$c_1^2(z) - c_0(z)c_2(z) = 1 + z \quad \text{modulo } 3,$$

so that the constants e_1, e_2, f_1, f_2 are given by $e_1 = 0$, $e_2 = 0$, $f_1 = 0$, $f_2 = 0$, $\varepsilon = 1$, and we have

$$\mathcal{Q}(\dots) = -\tfrac{1}{3} p_s^{(1)}(z) E_s^{(1)}(z) - \tfrac{1}{3} z p_s^{(1)}(z)\left(2 + p_s^{(1)}(z)\right).$$

This is indeed a polynomial in z and $E_s^{(1)}(z)$ with integer coefficients since, by Proposition 57, all coefficients of the polynomial $p_s^{(1)}(z)$ are divisible by 3. Consequently, for each s, the generating function $E_s^{(1)}(z)$, when reduced modulo 3^β, can be written as a polynomial in the basic series $\Psi(z)$.

However, we are actually not interested in all coefficients in $E_s^{(1)}(z)$. By (17.13), the relevant coefficients are those of powers z^n, where $n \equiv s \pmod{3^{\beta-1}}$. In other words, for each fixed s, we need a particular $3^{\beta-1}$-section of $E_s^{(1)}(z)$, and these $3^{\beta-1}$-sections have then to be summed to obtain the series $E^{(1)}(z)$ modulo 3^β. To be precise, we have

$$E^{(1)}(z) = \sum_{s=\lceil \frac{1}{2}(\beta-1)\rceil}^{3^{\beta-1}+\lceil \frac{1}{2}(\beta-1)\rceil-1} \sum_{\substack{\ell\in\mathbb{Z}\\ s+3^{\beta-1}\ell\ge 0}} z^{s+3^{\beta-1}\ell} \left\langle t^{s+3^{\beta-1}\ell}\right\rangle E_s^{(1)}(t) + \mathcal{E}(z),$$

where $\langle t^m\rangle f(t)$ denotes the coefficient of t^m in the series $f(t)$, and the polynomial $\mathcal{E}(z)$ represents the possible correction which takes care of deviations from the congruence (17.13) for n's with $n < \frac{1}{2}(\beta-1)$. However, computing $3^{\beta-1}$-sections of $E_s^{(1)}(z)$ is straightforward, once we observe that

$$\Psi(z) = \Psi(z^{3^{\beta-1}}) \prod_{j=0}^{\beta-2} (1 + z^{3^j}). \tag{17.15}$$

Each $3^{\beta-1}$-section, with coefficients reduced by 3^β, is then a polynomial in $\Psi(z^{3^{\beta-1}})$, or, equivalently, if we use (17.15) in the other direction, a polynomial in $\Psi(z)$. Hence, the generating function $E^{(1)}(z)$ of central Eulerian numbers, when coefficients are reduced modulo 3^β, can be expressed as a polynomial in $\Psi(z)$, which was exactly our claim. $\square$

How the procedure explained in the above proof works for modulus 9 is described in detail in the proof of the next result.

Theorem 60. *Let* $\Psi(z)$ *be given by* (1.1). *Then, we have*

$$\sum_{n\ge0} A(2n,n{+}1)\,z^n = -1{+}3(z{+}1)\Psi(z){+}\frac{3z^4+2z^3+6z^2+2}{1+z}\Psi^3(z) \quad \text{modulo } 9. \tag{17.16}$$

Proof. We start by considering the case $n \equiv 1 \pmod 3$. Under this condition, we have

$$A(2n,n+1) \equiv \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{2n+1}{n+1-j}j^2 \pmod 9.$$

From Proposition 57, we obtain

$$E_1^{(1)}(z) = \sum_{n\ge0} z^n \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{2n+1}{n+1-j}j^2 = \frac{1}{2}\left(1+\sqrt{1+4z}\right). \tag{17.17}$$

Consequently, by applying (17.14) with $s=1$ and $p_1^{(1)}(z)=0$, we see that the series $E_1^{(1)}(z)$ satisfies the functional equation

$$(E_s^{(1)})^2(z) - E_s^{(1)}(z) - z = 0.$$

If we use the method from Section 4 in the form of Theorem 17, with the relation (4.3) replaced by (2.2), then we obtain

$$E_1^{(1)}(z) = -4+3(z+1)\Psi(z)+\left(5z^2+7z+2\right)\Psi^3(z) \quad \text{modulo } 9.$$

In this series, we are only interested in the terms involving powers z^n with $n \equiv 1 \pmod 3$. This 3-section of the right-hand side of the above congruence equals

$$6z\Psi(z^3)+(4z^4+4z)\Psi^3(z^3). \tag{17.18}$$

In a similar vein, if $n \equiv 2 \pmod 3$, then

$$A(2n,n+1) \equiv \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{2n+1}{n+1-j}j^4 \pmod 9.$$

From Proposition 57, we obtain

$$E_2^{(1)}(z) = \sum_{n\ge0} z^n \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{2n+1}{n+1-j}j^4 = \frac{1}{2}\left(1+\sqrt{1+4z}\right)(1+12z) \tag{17.19}$$

Consequently, by applying (17.14) with $s=1$ and $p_1^{(1)}(z)=12z$, we see that the series $E_2^{(1)}(z)$ satisfies the functional equation

$$(E_s^{(1)})^2(z) - (1+12z)E_s^{(1)}(z) - z(1+12z)^2 = 0.$$

The method from Section 4 then yields

$$E_2^{(1)}(z) = -3z-4+3(z+1)\Psi(z)+\left(6z^3+8z^2+4z+2\right)\Psi^3(z) \quad \text{modulo } 9.$$

The relevant part, obtained by extracting only terms involving powers z^n with $n \equiv 2 \pmod 3$ is

$$3z^2\Psi(z^3)-(z^5+z^2)\Psi^3(z^3). \tag{17.20}$$

Finally, if $n \equiv 0 \pmod 3$, then

$$A(2n, n+1) \equiv \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{2n+1}{n+1-j} j^6 \pmod 9.$$

(Although $n = 0$ violates the inequality $n \ge \frac{1}{2}(\beta - 1) = \frac{1}{2}$ in (17.13), this holds indeed for *all* $n \ge 0$ which are divisible by 3.) From Proposition 57, we obtain

$$E_3^{(1)}(z) = \sum_{n\ge 0} z^n \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{2n+1}{n+1-j} j^6 = \frac{1}{2}\left(1 + \sqrt{1+4z}\right)(1 + 60z + 360z^2) \tag{17.21}$$

Consequently, the series $E_2^{(1)}(z)$ satisfies the functional equation

$$(E_s^{(1)})^2(z) - (1 + 60z + 360z^2)E_s^{(1)}(z) - z(1 + 60z + 360z^2)^2 = 0.$$

The method from Section 4 then yields

$$E_3^{(1)}(z) = 3z - 4 + 3(z+1)\Psi(z) + \left(3z^3 + 2z^2 + z + 2\right)\Psi^3(z) \quad \text{modulo } 9.$$

The relevant part, obtained by extracting only terms involving powers z^n with $n \equiv 0 \pmod 3$ is

$$-1 + 3\Psi(z^3) + (2 + 5z^3 + 3z^6)\Psi^3(z^3). \tag{17.22}$$

Summing the expressions (17.18), (17.20), and (17.22), we arrive at (17.16). □

Applying the same procedure for the modulus 27, now involving an analysis of 9 different cases which have to be combined in the end, one is able to prove the following result.

Theorem 61. *Let $\Psi(z)$ be given by* (1.1)*. Then, we have*

$$\sum_{n\ge 0} A(2n, n+1)\, z^n = 14 + 3\left(3z^2 - 4z + 2\right)\Psi(z) + \left(21z^3 + 20z^2 + 13z + 23\right)\Psi^3(z) \\ + 3\left(6z^4 + 4z^3 + 3z^2 + 4\right)\Psi^5(z) \quad \text{modulo } 27. \tag{17.23}$$

These results generalise Theorem 5.17 in [3], which is an easy consequence.

Now we turn to the analogous theorems for the central Eulerian numbers $A(2n-1, n)$.

Theorem 62. *Let $\Psi(z)$ be given by* (1.1)*, and let α be some positive integer. Then the generating function $E^{(2)}(z) = \sum_{n\ge 0} A(2n-1, n)\, z^n$ for central Eulerian numbers with odd first index, reduced modulo 3^{3^α}, can be expressed as a polynomial in $\Psi(z)$ of the form*

$$E^{(2)}(z) = a_0(z) + \sum_{i=0}^{2\cdot 3^\alpha - 1} a_i(z)\Psi^i(z^3) \quad \text{modulo } 3^{3^\alpha},$$

where the coefficients $a_i(z)$, $i = 0, 1, \dots, 2\cdot 3^\alpha - 1$, are Laurent polynomials in z and $1 + z$.

Proof. By (17.1), we have

$$A(2n-1, n) = \sum_{j=0}^{n} (-1)^{n-j} \binom{2n}{n-j} j^{2n-1}.$$

Again, in order to analyse this expression modulo 3^β (the reader should imagine that $\beta = 3^\alpha$), we have to distinguish several cases, depending on the congruence class of n modulo $3^{\beta-1}$. Namely, we have

$$A(2n-1,n) \equiv \sum_{j=0}^{n} (-1)^{n-j} \binom{2n}{n-j} j^{2s-1} \pmod{3^\beta}$$
$$\text{for} \quad n \equiv s \pmod{3^{\beta-1}}. \text{ and } n,s \ge \tfrac{\beta}{2}. \quad (17.24)$$

(The condition $n,s \ge \frac{\beta}{2}$ is needed to ensure that $j^{2n-1} \equiv j^{2s-1} \equiv 0 \pmod{3^{\beta-1}}$ for all j which are divisible by 3.) From Proposition 58, we know that the generating function for the numbers on the right-hand side of the above congruence, say

$$E_s^{(2)}(z) = \sum_{n\ge0} z^n \sum_{j=0}^{n} (-1)^{n-j} \binom{2n}{n-j} j^{2s-1},$$

has the form given on the right-hand side of (17.10). Consequently, it satisfies the functional equation

$$(4z+1)(E_s^{(2)})^2(z) - z^2 \left(1 + p_s^{(2)}(z)\right)^2 = 0. \tag{17.25}$$

If we rewrite this in the form

$$(1+z)(E_s^{(2)})^2(z) - z^2 + 3(E_s^{(2)})^2(z) - z^2 p_s^{(2)}(z) \left(2 + p_s^{(2)}(z)\right) = 0,$$

then we see that it fits the framework of Theorem 17. Here, one chooses $\gamma = 1$, $c_2(z) = 1+z$, $c_1(z) = 0$, $c_0(z) = -z^2$, which implies that

$$c_1^2(z) - c_0(z)c_2(z) = z^2(1+z) \quad \text{modulo } 3,$$

so that the constants e_1, e_2, f_1, f_2 are given by $e_1 = 0$, $e_2 = 1$, $f_1 = 1$, $f_2 = 0$, $\varepsilon = 1$, and we have

$$\mathcal{Q}(\dots) = (E_s^{(2)})^2(z) - \tfrac{1}{3} z p_s^{(2)}(z) \left(2 + p_s^{(2)}(z)\right).$$

This is indeed a polynomial in z and $E_s^{(2)}(z)$ with integer coefficients since, by Proposition 58, all coefficients of the polynomial $p_s^{(2)}(z)$ are divisible by 3. Consequently, for each s, the generating function $E_s^{(2)}(z)$, when reduced modulo 3^β, can be written as a polynomial in the basic series $\Psi(z)$.

The rest of the argument is identical with corresponding arguments in the proof of Theorem 59. □

For the modulus 9, the procedure explained in the above proof leads to the following result.

Theorem 63. *Let $\Psi(z)$ be given by* (1.1). *Then, we have*

$$\sum_{n\ge0} A(2n-1,n)\, z^n = -3z\Psi(z) + \left(6z^4 + 6z^3 + 4z^2 + 4z\right) \Psi^3(z) \quad \text{modulo } 9. \tag{17.26}$$

Proof. As in the proof of Theorem 60, we distinguish three cases, depending on the congruence class of n modulo 3. In order to address the case where $n \equiv 1 \pmod 3$, from Proposition 58 we compute

$$\sum_{n\ge0} z^n \sum_{j=0}^{n} (-1)^{n-j} \binom{2n}{n-j} j^7 = \frac{z}{\sqrt{1+4z}} (1 + 126z + 1680z^2 + 5040z^3). \tag{17.27}$$

Consequently, by applying (17.14) with $s=1$ and $p_1^{(2)}(z)=1+126z+1680z^2+5040z^3$, we see that the series $E_2^{(2)}(z)$ satisfies the functional equation

$$(1+4z)(E_s^{(2)})^2(z)-z^2(1+126z+1680z^2+5040z^3)^2=0.$$

The method from Section 4 in the form of Theorem 17 then yields

$$E_1^{(2)}(z)=-3z\Psi(z)+\big(6z^4+6z^3+7z^2+4z\big)\,\Psi^3(z)\quad \text{modulo } 9,$$

and the relevant part of it, consisting only of terms involving powers z^n with $n\equiv 1 \pmod 3$ is

$$-3z\Psi(z^3)+\big(6z^7+4z^4+4z\big)\,\Psi^3(z^3). \tag{17.28}$$

Next, addressing the case where $n\equiv 2 \pmod 3$, we have

$$\sum_{n\ge0}z^n\sum_{j=0}^{n}(-1)^{n-j}\binom{2n}{n-j}j^3=\frac{z}{\sqrt{1+4z}}(1+6z), \tag{17.29}$$

so that this series satisfies the functional equation

$$(1+4z)(E_s^{(2)})^2(z)-z^2(1+6z)^2=0.$$

Our algorithm then yields

$$E_2^{(2)}(z)=-3z\Psi(z)+\big(6z^3+4z^2+4z\big)\,\Psi^3(z)\quad \text{modulo } 9,$$

with the relevant part being

$$-3z^2\Psi(z^3)+(22z^5+16z^2)\Psi^3(z^3). \tag{17.30}$$

Finally, in order to address the case where $n\equiv 0 \pmod 3$, we compute

$$\sum_{n\ge0}z^n\sum_{j=0}^{n}(-1)^{n-j}\binom{2n}{n-j}j^5=\frac{z}{\sqrt{1+4z}}(1+30z+120z^2), \tag{17.31}$$

so that this series satisfies the functional equation

$$(1+4z)(E_s^{(2)})^2(z)-z^2(1+30z+120z^2)^2=0.$$

Our algorithm yields

$$E_3^{(2)}(z)=-3z\Psi(z)+\big(3z^4+6z^3+z^2+4z\big)\,\Psi^3(z)\quad \text{modulo } 9,$$

with the relevant part being

$$(15z^6+21z^3)\Psi^3(z^3). \tag{17.32}$$

Summation of (17.28), (17.30), and (17.32) then leads to (17.26). $\square$

The same procedure produces the following result for the modulus 27.

Theorem 64. *Let $\Psi(z)$ be given by* (1.1). *Then, we have*

$$\sum_{n\ge0}A(2n-1,n)\,z^n=-3z\big(3z^2+5\big)\Psi(z)+z\big(24z^3+15z^2+10z+19\big)\,\Psi^3(z)\\ +3z\big(3z^4+6z^3+2z^2+7z+8\big)\,\Psi^5(z)\quad \text{modulo } 27. \tag{17.33}$$

These results generalise Theorem 5.16 in [3], which is an easy consequence.

Remark. It is obvious that we could also treat "almost" central Eulerian numbers, such as $A(2n, n-1)$ or $A(2n-1, n-1)$, modulo powers of 3 using the same approach, and this would lead to similar results.

18. Apéry numbers

In [3], Deutsch and Sagan also discuss *Apéry numbers* (and, in fact, generalised Apéry numbers), and they derive in particular explicit congruences for Apéry numbers modulo 3. The corresponding result, Theorem 5.14 in [3], has a form which seems to suggest that our method from Section 4 should be applicable here as well. It turns out however that this is not the case. As a matter of fact, we are not even able to make our method work for the modulus 3, due to the fact that we were not able to find an equation for the corresponding generating functions which would be suitable for our method.

Let us begin with the Apéry numbers of the first kind, defined by

$$A_n^{(2)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}. \tag{18.1}$$

They are associated with $\zeta(2)$, see e.g. [4].

There is a well-known recurrence relation for these numbers. Namely, Apéry computed (and the Gosper–Zeilberger algorithm (cf. [12, Ch. 6]) finds automatically) the recurrence

$$(n+2)^2 A_{n+2}^{(2)} - \left(11n^2 + 33n + 25\right) A_{n+1}^{(2)} - (n+1)^2 A_n^{(2)} = 0. \tag{18.2}$$

This translates into the linear differential equation

$$(z+3)A^{(2)}(z) + \left(3z^2 + 22z - 1\right)(A^{(2)})'(z) + \left(z^3 + 11z^2 - z\right)(A^{(2)})''(z) = 0 \tag{18.3}$$

for the generating function $A^{(2)}(z) := \sum_{n\ge0} A_n^{(2)} z^n$, with initial conditions $A^{(2)}(0) = 1$ and $(A^{(2)})'(0) = 3$. Unfortunately, this equation is *not* suitable for our method, as Equation (18.3) does not determine a unique solution modulo 3 (see the last paragraph in Section 4). Thus, in lack of a different equation, we cannot apply our method.

As it turns out, there is apparently a deeper reason why our method does not apply. By looking at enough data, we have worked out a conjectural description of the Apéry numbers of the first kind modulo 9, see the conjecture below. The patterns describing the various congruence classes do not fit with the patterns which appear in the descriptions of the coefficients of $\Psi^3(z)$ modulo 9 which one obtains by carrying out the approach sketched in Section 2 (see in particular the paragraphs after (2.6) and (2.7)), the resulting congruences being explicitly displayed in Corollary 69.

Conjecture 65. *The Apéry numbers $A_n^{(2)}$ obey the following congruences modulo 9:*

(i) $A_n^{(2)} \equiv 1 \pmod 9$ *if, and only if, the 3-adic expansion of n is an element of*

$$\{0,2\}^*;$$

(ii) $A_n^{(2)} \equiv 3 \pmod 9$ *if, and only if, the 3-adic expansion of n has exactly one occurrence of the string 01 and otherwise contains only 0's and 2's;*

(iii) $A_n^{(2)} \equiv 6 \pmod 9$ *if, and only if, the 3-adic expansion of n has exactly one occurrence of the string 21 and otherwise contains only 0's and 2's;*

(iv) *in the cases not covered by Items* (i)–(iii), $A_n^{(2)}$ *is divisible by* 9; *in particular,* $A_n^{(2)} \not\equiv 2,4,5,7,8 \pmod 9$ *for all* n.

The Apéry numbers of the second kind are defined by

$$A_n^{(3)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 .$$

They are associated with $\zeta(3)$, see e.g. [4].

These numbers satisfy also a well-known recurrence relation which Apéry computed (and the Gosper–Zeilberger algorithm finds automatically), namely

$$(n+2)^3 A_{n+2}^{(3)} - (2n+3)\left(17n^2+51n+39\right) A_{n+1}^{(3)} + (n+1)^3 A_n^{(3)} = 0.$$

This translates into the linear differential equation

$$\begin{multline}(z-5)A^{(3)}(z) + \left(7z^2-112z+1\right)(A^{(3)})'(z)\\ + 3\left(2z^3-51z^2+z\right)(A^{(3)})''(z) + \left(z^4-34z^3+z^2\right)(A^{(3)})'''(z) = 0 \quad (18.4)\end{multline}$$

for the generating function $A^{(3)}(z) := \sum_{n\ge0} A_n^{(3)} z^n$, with initial conditions $A^{(3)}(0)=1$ and $(A^{(3)})'(0)=5$. Again, this equation is *not* suitable for our method, for the same reason as before.

Also here we have worked out a conjectural description modulo 9, displaying patterns which do not fit with those appearing in the description of the coefficients of $\Psi^3(z)$ modulo 9.

Conjecture 66. *The Apéry numbers* $A_n^{(3)}$ *obey the following congruences modulo* 9*:*

(i) $A_n^{(3)} \equiv 1 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *contains* $6k$ *digits* 1*, for some* k*, and otherwise only* 0*'s and* 2*'s;*
(ii) $A_n^{(3)} \equiv 2 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *contains* $6k+5$ *digits* 1*, for some* k*, and otherwise only* 0*'s and* 2*'s;*
(iii) $A_n^{(3)} \equiv 4 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *contains* $6k+4$ *digits* 1*, for some* k*, and otherwise only* 0*'s and* 2*'s;*
(iv) $A_n^{(3)} \equiv 5 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *contains* $6k+1$ *digits* 1*, for some* k*, and otherwise only* 0*'s and* 2*'s;*
(v) $A_n^{(3)} \equiv 7 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *contains* $6k+2$ *digits* 1*, for some* k*, and otherwise only* 0*'s and* 2*'s;*
(vi) $A_n^{(3)} \equiv 8 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *contains* $6k+3$ *digits* 1*, for some* k*, and otherwise only* 0*'s and* 2*'s;*
(vii) *in the cases not covered by Items* (i)–(vi), $A_n^{(3)}$ *is divisible by* 9; *in particular,* $A_n^{(3)} \not\equiv 3,6 \pmod 9$ *for all* n.

Note

Conjectures 65 and 66 have been proven by Rowland and Yassawi [16], and independently by the authors of the present paper [8].

APPENDIX: EXPANSIONS OF POWERS OF THE CANTOR-LIKE POWER SERIES $\Psi(z)$

In this section, we indicate how to explicitly extract coefficients from powers of $\Psi(z)$ modulo 3^e, where e is a given positive integer. The reader should recall that, by (2.6) and (2.7), it suffices to explain how to extract coefficients from expressions of the form

$$\Psi(z)\frac{1}{(1+z)^K}\prod_{j=1}^{s}\left(\frac{z^{3^{k_j}}(1+z^{3^{k_j}})}{1+z^{3^{k_j+1}}}\right)^{a_j} \tag{A.1}$$

and

$$\frac{1}{(1+z)^K}\prod_{j=1}^{s}\left(\frac{z^{3^{k_j}}(1+z^{3^{k_j}})}{1+z^{3^{k_j+1}}}\right)^{a_j}. \tag{A.2}$$

We describe below how to deal with (A.1). The coefficient extraction from expressions of the form (A.2) is even simpler since these are rational functions in z.

Rather than providing a general description of how to proceed (which is possible but would be hardly comprehensible), we prefer to work through the special case where $s=2$, that is,

$$\Psi(z)\frac{1}{(1+z)^K}\left(\frac{z^{3^{k_1}}(1+z^{3^{k_1}})}{1+z^{3^{k_1+1}}}\right)^{a_1}\left(\frac{z^{3^{k_2}}(1+z^{3^{k_2}})}{1+z^{3^{k_2+1}}}\right)^{a_2}, \tag{A.3}$$

where $k_1>k_2\ge 0$ and $a_1,a_2\ge 1$. We have to distinguish several cases, depending on whether K,k_1,k_2 are "close to each other," or not.

First let $K\le k_2$ and $k_2+a_2<k_1$. In that case, the expression (A.3) can be transformed into

$$\begin{aligned}
&\prod_{i=1}^{K-1}(1-z+z^2-\dots+z^{3^i-1})\prod_{i=K}^{k_2-1}(1+z^{3^i})\\
&\times(1+z^{3^{k_2}})(z^{3^{k_2}}+z^{2\cdot3^{k_2}})^{a_2}\prod_{i=k_2+2}^{k_2+a_2}(1-z^{3^{k_2+1}}+z^{2\cdot3^{k_2+1}}-\dots+z^{3^i-3^{k_2+1}})\prod_{i=k_2+a_2+1}^{k_1-1}(1+z^{3^i})\\
&\times(1+z^{3^{k_1}})(z^{3^{k_1}}+z^{2\cdot3^{k_1}})^{a_1}\prod_{i=k_1+2}^{k_1+a_1}(1-z^{3^{k_1+1}}+z^{2\cdot3^{k_1+1}}-\dots+z^{3^i-3^{k_1+1}})\prod_{i=k_1+a_1+1}^{\infty}(1+z^{3^i}).
\end{aligned}$$

We write this as

$$P_0(z)\left(\prod_{i=K}^{k_2-1}(1+z^{3^i})\right)P_1(z)\left(\prod_{i=k_2+a_2+1}^{k_1-1}(1+z^{3^i})\right)P_2(z)\left(\prod_{i=k_1+a_1+1}^{\infty}(1+z^{3^i})\right), \tag{A.4}$$

which implicitly defines the polynomials $P_0(z)$, $P_1(z)$, and $P_2(z)$. It should be noted that $P_0(z)$ has degree at most 3^K, $P_1(z)$ has order $a_2 3^{k_2}$ and degree at most $3^{k_2+a_2}$, and $P_2(z)$ has order $a_1 3^{k_1}$ and degree at most $3^{k_1+a_1}$. We "organise" the polynomials

$P_0(z), P_1(z), P_2(z)$ in the form

$$\begin{aligned}
P_0(z) &= \sum_{\gamma\in\mathbb{Z}} \gamma \sum_{(n)_3\in Z_\gamma} z^n,\\
P_1(z) &= \sum_{\beta\in\mathbb{Z}} \beta \sum_{(n)_3\in Y_\beta 0^{k_2}} z^n,\\
P_2(z) &= \sum_{\alpha\in\mathbb{Z}} \alpha \sum_{(n)_3\in X_\alpha 0^{k_1}} z^n,
\end{aligned}$$

where $(n)_3$ denotes the 3-adic representation of the integer n, considered as a string of 0's, 1's, and 2's, Z_γ is an appropriate set of strings contained in $\{0,1,2\}^K$, Y_β is an appropriate set of strings contained in $\{0,1,2\}^{a_2+1}$, and X_α is an appropriate set of strings contained in $\{0,1,2\}^{a_1+1}$. Inspection of (A.4) makes it now obvious that the coefficients c_n in the series $\sum_{n\ge0} c_n z^n$ given by (A.3) can be described in the form

$$c_n = C \quad \text{if, and and only if,}$$
$$(n)_3 \in \bigcup_{\alpha\cdot\beta\cdot\gamma=C} \left(\bigcup_{X\in X_\alpha,\, Y\in Y_\beta,\, Z\in Z_\gamma} \{0,1\}^* X \{0,1\}^{k_1-k_2-a_2-1} Y \{0,1\}^{k_2-K} Z \right),$$

where we used again the word notation explained before Theorem 52.

Next let $K > k_2 \ge 0$ ("k_2 is close to K") and $K+a_2 \le k_1$. In this case, the expression (A.3) can be transformed into

$$\prod_{i=1}^{K-1}(1-z+z^2-\dots+z^{3^i-1})$$
$$\times (z^{3^{k_2}}+z^{2\cdot3^{k_2}})^{a_2} \prod_{i=K}^{K+a_2-1} (1-z^{3^{k_2+1}}+z^{2\cdot3^{k_2+1}}-\dots+z^{3^i-3^{k_2+1}}) \prod_{i=K+a_2}^{k_1-1}(1+z^{3^i})$$
$$\times(1+z^{3^{k_1}})(z^{3^{k_1}}+z^{2\cdot3^{k_1}})^{a_1} \prod_{i=k_1+2}^{k_1+a_1} (1-z^{3^{k_1+1}}+z^{2\cdot3^{k_1+1}}-\dots+z^{3^i-3^{k_1+1}}) \prod_{i=k_1+a_1+1}^{\infty}(1+z^{3^i}).$$

Here, we write this as

$$P_1(z)\left(\prod_{i=K+a_2}^{k_1-1}(1+z^{3^i})\right) P_2(z) \left(\prod_{i=k_1+a_1+1}^{\infty}(1+z^{3^i})\right), \tag{A.5}$$

which implicitly defines the polynomials $P_1(z)$ and $P_2(z)$. By going through the same reasoning as before, one comes to the conclusion that in the present case the coefficients c_n in the series $\sum_{n\ge0} c_n z^n$ given by (A.3) can be described in the form

$$c_n = C \quad \text{if, and and only if,} \quad (n)_3 \in \bigcup_{\alpha\cdot\beta=C}\left(\bigcup_{X\in X_\alpha,\, Y\in Y_\beta} \{0,1\}^* X\{0,1\}^{k_1-a_2-K} Y\right),$$

for appropriate sets of strings X_α and Y_β.

The other cases to be discussed are the case where $K \le k_2$ and $k_2+a_2 \ge k_1$ ("k_1 is close to k_2"; we always assume $k_2 < k_1$), and the case where $K \ge k_1 > k_2 \ge 0$ ("K, k_1, k_2 are all close to each other"). Both of them can be treated in a similar fashion.

We illustrate the above procedure by making explicit how to extract coefficients from odd powers of $\Psi(z)$ modulo 27. By Lemma 1, we have

$$\Psi^3(z) = \frac{1}{1+z}\Psi(z)\left(1 + 3\sum_{k_1\ge 0}\frac{z^{3^{k_1}}(1+z^{3^{k_1}})}{1+z^{3^{k_1+1}}} + 9\sum_{k_1>k_2\ge 0}\frac{z^{3^{k_1}+3^{k_2}}(1+z^{3^{k_1}})(1+z^{3^{k_2}})}{(1+z^{3^{k_1+1}})(1+z^{3^{k_2+1}})}\right) \quad \text{modulo } 27. \tag{A.6}$$

The following lemma provides the means for coefficient extraction from $\Psi^3(z)$ modulo any power of 3.

Lemma 67. (1) *If $k_1 = 0$, we have*

$$\frac{1}{1+z}\Psi(z)\frac{z^{3^{k_1}}(1+z^{3^{k_1}})}{1+z^{3^{k_1+1}}} = \Psi(z)\frac{z}{1+z^3} = \sum_{n\ge 0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup_{X\in\{01,02\}} \{0,1\}^* X, \\ 0, & \text{otherwise,} \end{cases}$$

while, if $k_1 \ge 1$, we have

$$\frac{1}{1+z}\Psi(z)\frac{z^{3^{k_1}}(1+z^{3^{k_1}})}{1+z^{3^{k_1+1}}} = \sum_{n\ge 0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup_{X\in\{01,10\}} \{0,1\}^* X \{0,1\}^{k_1-1}0, \\ 2, & \text{if } (n)_3 \in \{0,1\}^* 02 \{0,1\}^{k_1-1}0, \\ 0, & \text{otherwise.} \end{cases}$$

(2) *If $k_2 = 0$, we have*

$$\frac{1}{1+z}\Psi(z)\frac{z^{3^{k_2+1}+3^{k_2}}(1+z^{3^{k_2}})}{1+z^{3^{k_2+2}}} = \Psi(z)\frac{z^4}{1+z^9} = \sum_{n\ge 0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup_{X\in\{011,012,021,022\}} \{0,1\}^* X, \\ 0, & \text{otherwise,} \end{cases}$$

while, if $k_2 \ge 1$, we have

$$\frac{1}{1+z}\Psi(z)\frac{z^{3^{k_2+1}+3^{k_2}}(1+z^{3^{k_2}})}{1+z^{3^{k_2+2}}} = \sum_{n\ge 0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup_{X\in\{011,020,021,100\}} \{0,1\}^* X \{0,1\}^{k_2-1}0, \\ 2, & \text{if } (n)_3 \in \bigcup_{X\in\{012,022\}} \{0,1\}^* X \{0,1\}^{k_2-1}0, \\ 0, & \text{otherwise.} \end{cases}$$

(3) *If $k_1-1>k_2=0$, we have*

$$\frac{1}{1+z}\Psi(z)\frac{z^{3^{k_1}+3^{k_2}}(1+z^{3^{k_1}})(1+z^{3^{k_2}})}{(1+z^{3^{k_1+1}})(1+z^{3^{k_2+1}})}=\Psi(z)\frac{z^{3^{k_1}+1}(1+z^{3^{k_1}})}{(1+z^3)(1+z^{3^{k_1+1}})}=\sum_{n\ge0}c_nz^n,$$

where

$$c_n=\begin{cases}1, & \text{if } (n)_3\in\bigcup\limits_{X\in\{01,10\},\ Y\in\{01,02\}}\{0,1\}^*X\{0,1\}^{k_1-2}Y,\\ 2, & \text{if } (n)_3\in\bigcup\limits_{X\in\{01,02\}}\{0,1\}^*02\{0,1\}^{k_1-2}X,\\ 0, & \text{otherwise,}\end{cases}$$

while, if $k_1-1>k_2\ge1$, we have

$$\frac{1}{1+z}\Psi(z)\frac{z^{3^{k_1}+3^{k_2}}(1+z^{3^{k_1}})(1+z^{3^{k_2}})}{(1+z^{3^{k_1+1}})(1+z^{3^{k_2+1}})}=\sum_{n\ge0}c_nz^n,$$

where

$$c_n=\begin{cases}1, & \text{if } (n)_3\in\bigcup\limits_{X,Y\in\{01,10\}}\{0,1\}^*X\{0,1\}^{k_1-k_2-2}Y\{0,1\}^{k_2-1}0,\\ 2, & \text{if } (n)_3\in\left(\bigcup\limits_{X\in\{01,10\}}\{0,1\}^*02\{0,1\}^{k_1-k_2-2}X\{0,1\}^{k_2-1}0\right)\\ & \qquad\qquad\cup\left(\bigcup\limits_{X\in\{01,10\}}\{0,1\}^*X\{0,1\}^{k_1-k_2-2}02\{0,1\}^{k_2-1}0\right),\\ 4, & \text{if } (n)_3\in\{0,1\}^*02\{0,1\}^{k_1-k_2-2}02\{0,1\}^{k_2-1}0,\\ 0, & \text{otherwise.}\end{cases}$$

For convenience, given a (left-infinite) string $s=\dots s_2s_1s_0$ consisting of 0's, 1's, and 2's, we introduce the following notation:

$$\begin{aligned}
\#1\mathrm{str}(s)&=\#(\text{maximal strings of 1's in } s),\\
\#\mathrm{iso}_2(s)&=\#(\text{isolated 0's and 1's in } s,\text{ excluding } s_0 \text{ and } s_1),\\
\#\mathrm{iso}_3(s)&=\#(\text{isolated 0's and 1's in } s,\text{ excluding } s_0,s_1,s_2),\\
\mathrm{end}(s;e)&=\begin{cases}1 & \text{if } s \text{ ends with the string } e,\\ 0 & \text{otherwise,}\end{cases}\\
\#100_1(s)&=\#(\text{occurrences of the substring 100 in } s \text{ not involving } s_0),\\
\#011(s)&=\#(\text{occurrences of the substring 011 in } s),
\end{aligned}$$

and similarly for $\#020(s)$, $\#021(s)$, $\#102(s)$, $\#10(s)$, $\#01(s)$.

Proposition 68. *Let s denote the 3-adic expansion of the non-negative integer n. The coefficient of z^n in the series $\Psi^3(z)$, when reduced modulo 27, equals*

(1) *1, if, and only if, s is either an element of $\{0,1\}^*10$ and satisfies $\#1\mathrm{str}(s)\equiv 2 \pmod 3$ and*

$$\tfrac{2\,\#1\mathrm{str}(s)-1}{3}-\#\mathrm{iso}_2(s)+\#011(s)+\#100_1(s)\equiv0 \pmod 3,$$

or of $\{0,1\}^*00$ *and satisfies* $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)}{3} - \#\mathrm{iso}_2(s) + \#011(s) + \#100_1(s) \equiv 0 \pmod 3;$$

(2) 4*, if, and only if,* s *is either an element of* $\{0,1\}^*10$ *and satisfies* $\#1\mathrm{str}(s) \equiv 1 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-2}{3} - \#\mathrm{iso}_2(s) + \#011(s) + \#100_1(s) \equiv 0 \pmod 3,$$

or of $\{0,1\}^*00$ *and satisfies* $\#1\mathrm{str}(s) \equiv 2 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-1}{3} - \#\mathrm{iso}_2(s) + \#011(s) + \#100_1(s) \equiv 0 \pmod 3;$$

(3) 7*, if, and only if,* s *is either an element of* $\{0,1\}^*10$ *and satisfies* $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-3}{3} - \#\mathrm{iso}_2(s) + \#011(s) + \#100_1(s) \equiv 2 \pmod 3,$$

or of $\{0,1\}^*00$ *and satisfies* $\#1\mathrm{str}(s) \equiv 1 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-2}{3} - \#\mathrm{iso}_2(s) + \#011(s) + \#100_1(s) \equiv 2 \pmod 3;$$

(4) 10*, if, and only if,* s *is either an element of* $\{0,1\}^*10$ *and satisfies* $\#1\mathrm{str}(s) \equiv 2 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-1}{3} - \#\mathrm{iso}_2(s) + \#011(s) + \#100_1(s) \equiv 1 \pmod 3,$$

or of $\{0,1\}^*00$ *and satisfies* $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)}{3} - \#\mathrm{iso}_2(s) + \#011(s) + \#100_1(s) \equiv 1 \pmod 3;$$

(5) 13*, if, and only if,* s *is either an element of* $\{0,1\}^*10$ *and satisfies* $\#1\mathrm{str}(s) \equiv 1 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-2}{3} - \#\mathrm{iso}_2(s) + \#011(s) + \#100_1(s) \equiv 1 \pmod 3,$$

or of $\{0,1\}^*00$ *and satisfies* $\#1\mathrm{str}(s) \equiv 2 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-1}{3} - \#\mathrm{iso}_2(s) + \#011(s) + \#100_1(s) \equiv 1 \pmod 3;$$

(6) 16*, if, and only if,* s *is either an element of* $\{0,1\}^*10$ *and satisfies* $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-3}{3} - \#\mathrm{iso}_2(s) + \#011(s) + \#100_1(s) \equiv 0 \pmod 3,$$

or of $\{0,1\}^*00$ *and satisfies* $\#1\mathrm{str}(s) \equiv 1 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-2}{3} - \#\mathrm{iso}_2(s) + \#011(s) + \#100_1(s) \equiv 0 \pmod 3;$$

(7) 19*, if, and only if,* s *is either an element of* $\{0,1\}^*10$ *and satisfies* $\#1\mathrm{str}(s) \equiv 2 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-1}{3} - \#\mathrm{iso}_2(s) + \#011(s) + \#100_1(s) \equiv 2 \pmod 3,$$

or of $\{0,1\}^*00$ *and satisfies* $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)}{3} - \#\mathrm{iso}_2(s) + \#011(s) + \#100_1(s) \equiv 2 \pmod 3;$$

(8) 22, *if, and only if, s is either an element of* $\{0,1\}^*10$ *and satisfies* $\#1\mathrm{str}(s)\equiv 1 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-2}{3}-\#\mathrm{iso}_2(s)+\#011(s)+\#100_1(s)\equiv 2 \pmod 3,$$

or of $\{0,1\}^*00$ *and satisfies* $\#1\mathrm{str}(s)\equiv 2 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-1}{3}-\#\mathrm{iso}_2(s)+\#011(s)+\#100_1(s)\equiv 2 \pmod 3;$$

(9) 25, *if, and only if, s is either an element of* $\{0,1\}^*10$ *and satisfies* $\#1\mathrm{str}(s)\equiv 0 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-3}{3}-\#\mathrm{iso}_2(s)+\#011(s)+\#100_1(s)\equiv 1 \pmod 3,$$

or of $\{0,1\}^*00$ *and satisfies* $\#1\mathrm{str}(s)\equiv 1 \pmod 3$ *and*

$$\tfrac{2\,\#1\mathrm{str}(s)-2}{3}-\#\mathrm{iso}_2(s)+\#011(s)+\#100_1(s)\equiv 1 \pmod 3;$$

(10) 9, *if, and only if, the* 3*-adic expansion of n is an element of*

$$\left(\{0,1\}^*02\{0,1\}^*02\{0,1\}^*0\right)\cup\left(\bigcup_{X\in\{011,012,021,022\}}\{0,1\}^*X\right);$$

(11) 18, *if, and only if, the* 3*-adic expansion of n is an element of*

$$\left(\bigcup_{X\in\{01,02\}}\{0,1\}^*02\{0,1\}^*X\right)\cup\left(\bigcup_{X\in\{012,022\}}\{0,1\}^*X\{0,1\}^*0\right);$$

(12) 3, *if, and only if, the* 3*-adic expansion of n is an element of*

$$\left(\bigcup_{k\ge0,\ X\in\{01,02\}}\left(11^*00^*\right)^{3k+1}11^*X\right)\cup\left(\bigcup_{k\ge0,\ X\in\{01,02\}}\left(11^*00^*\right)^{3k+2}11^*0^*0X\right)\cup\{1,2\};$$

(13) 12, *if, and only if, the* 3*-adic expansion of n is an element of*

$$\left(\bigcup_{k\ge0,\ X\in\{01,02\}}\left(11^*00^*\right)^{3k}11^*X\right)\cup\left(\bigcup_{k\ge0,\ X\in\{01,02\}}\left(11^*00^*\right)^{3k+1}11^*0^*0X\right);$$

(14) 21, *if, and only if, the* 3*-adic expansion of n is an element of*

$$\left(\bigcup_{k\ge0,\ X\in\{01,02\}}\left(11^*00^*\right)^{3k+2}11^*X\right)\cup\left(\bigcup_{k\ge0,\ X\in\{01,02\}}\left(11^*00^*\right)^{3k}11^*0^*0X\right);$$

(15) 6, *if, and only if, s is either an element of* $\{0,1\}^*02\{0,1\}^*00$ *and satisfies*

$$\#020(s)+\#021(s)+\#102(s)-\#01(s)-\#10(s)\equiv 0 \pmod 3,$$

or an element of $\{0,1\}^*02\{0,1\}^*10$ *and satisfies*

$$\#020(s)+\#021(s)+\#102(s)-\#01(s)-\#10(s)\equiv 2 \pmod 3,$$

or an element of $\{0,1\}^*020$ *and satisfies*

$$\#020(s)+\#021(s)+\#102(s)-\#01(s)-\#10(s)-1\equiv 0 \pmod 3;$$

(16) 15, *if, and only if, s is either an element of* $\{0,1\}^*02\{0,1\}^*00$ *and satisfies*

$$\#020(s)+\#021(s)+\#102(s)-\#01(s)-\#10(s)\equiv 1 \pmod 3,$$

or an element of $\{0,1\}^*02\{0,1\}^*10$ *and satisfies*

$$\#020(s)+\#021(s)+\#102(s)-\#01(s)-\#10(s)\equiv 0 \pmod 3,$$

or an element of $\{0,1\}^*020$ *and satisfies*

$$\#020(s)+\#021(s)+\#102(s)-\#01(s)-\#10(s)-1\equiv 1 \pmod 3;$$

(17) 24, *if, and only if, s is either an element of* $\{0,1\}^*02\{0,1\}^*00$ *and satisfies*

$$\#020(s)+\#021(s)+\#102(s)-\#01(s)-\#10(s)\equiv 2 \pmod 3,$$

or an element of $\{0,1\}^*02\{0,1\}^*10$ *and satisfies*

$$\#020(s)+\#021(s)+\#102(s)-\#01(s)-\#10(s)\equiv 1 \pmod 3,$$

or an element of $\{0,1\}^*020$ *and satisfies*

$$\#020(s)+\#021(s)+\#102(s)-\#01(s)-\#10(s)-1\equiv 2 \pmod 3;$$

(18) *In all other cases, this coefficient is divisible by* 27*. In particular, it is never congruent to* 2 *modulo* 3*.*

If the above proposition is specialised to modulus 9, then one obtains the following result.

Corollary 69. *The coefficient of z^n in the series $\Psi^3(z)$, when reduced modulo* 9*, equals*

(1) 1*, if, and only if, the* 3*-adic expansion of n is an element of*

$$\{0\}\cup\bigcup_{k\ge0}\left(00^*11^*\right)^{3k+2}0\cup\bigcup_{k\ge0}\left(00^*11^*\right)^{3k}0^*00;$$

(2) 4*, if, and only if, the* 3*-adic expansion of n is an element of*

$$\bigcup_{k\ge0}\left(00^*11^*\right)^{3k+1}0\cup\bigcup_{k\ge0}\left(00^*11^*\right)^{3k+2}0^*00;$$

(3) 7*, if, and only if, the* 3*-adic expansion of n is an element of*

$$\bigcup_{k\ge0}\left(00^*11^*\right)^{3k+3}0\cup\bigcup_{k\ge0}\left(00^*11^*\right)^{3k+1}0^*00;$$

(4) 3*, if, and only if, the* 3*-adic expansion of n is an element of*

$$\{0,1\}^*01\cup\{0,1\}^*02,$$

(5) 6*, if, and only if, the* 3*-adic expansion of n is an element of*

$$\{0,1\}^*02\{0,1\}^*0.$$

(6) *In all other cases, this coefficient is divisible by* 9*. In particular, it is never congruent to* 2 *modulo* 3*.*

Now we turn our attention to coefficient extraction from $\Psi^5(z)$ modulo 27. By Lemma 1, we have

$$\Psi^5(z) = \frac{1}{(1+z)^2}\Psi(z)\left(1+6\sum_{k_1\ge0}\frac{z^{3^{k_1}}(1+z^{3^{k_1}})}{1+z^{3^{k_1+1}}}+9\sum_{k_1>k_2\ge0}\frac{z^{3^{k_1}+3^{k_2}}(1+z^{3^{k_1}})(1+z^{3^{k_2}})}{(1+z^{3^{k_1+1}})(1+z^{3^{k_2+1}})}\right.$$
$$\left.+\,9\sum_{k_1\ge0}\frac{z^{2\cdot3^{k_1}}(1+z^{3^{k_1}})^2}{(1+z^{3^{k_1+1}})^2}\right) \quad \text{modulo } 27. \tag{A.7}$$

The following lemma provides the means for coefficient extraction from $\Psi^5(z)$ modulo any power of 3.

Lemma 70. (1) *We have*

$$\frac{1}{(1+z)^2}\Psi(z) = \sum_{n\ge0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup_{X\in\{00,02\}} \{0,1\}^*X,\\ -1, & \text{if } (n)_3 \in \{0,1\}^*01,\\ 0, & \text{otherwise.}\end{cases}$$

(2) *If $k_1=0$, we have*

$$\frac{1}{(1+z)^2}\Psi(z)\frac{z^{3^{k_1}}(1+z^{3^{k_1}})}{1+z^{3^{k_1+1}}} = \Psi(z)\frac{z}{(1+z)(1+z^3)} = \sum_{n\ge0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \{0,1\}^*01,\\ 0, & \text{otherwise,}\end{cases}$$

if $k_1=1$, we have

$$\frac{1}{(1+z)^2}\Psi(z)\frac{z^{3^{k_1}}(1+z^{3^{k_1}})}{1+z^{3^{k_1+1}}} = \Psi(z)\frac{z(1-z+z^2)}{(1+z)(1+z^9)} = \sum_{n\ge0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup_{X\in\{010,012,020,022\}} \{0,1\}^*X,\\ -1, & \text{if } (n)_3 \in \bigcup_{X\in\{011,021\}} \{0,1\}^*X,\\ 0, & \text{otherwise,}\end{cases}$$

while, if $k_1\ge2$, we have

$$\frac{1}{(1+z)^2}\Psi(z)\frac{z^{3^{k_1}}(1+z^{3^{k_1}})}{1+z^{3^{k_1+1}}} = \sum_{n\ge0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{01,10\},\ Y\in\{00,02\}} \{0,1\}^* X \{0,1\}^{k_1-2} Y, \\ -1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{01,10\}} \{0,1\}^* X \{0,1\}^{k_1-2} 01, \\ 2, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{00,02\}} \{0,1\}^* 02 \{0,1\}^{k_1-2} X, \\ -2, & \text{if } (n)_3 \in \{0,1\}^* 02 \{0,1\}^{k_1-2} 01, \\ 0, & \text{otherwise.} \end{cases}$$

(3) *If $k_2 = 0$, we have*

$$\frac{1}{(1+z)^2}\Psi(z)\frac{z^{3^{k_2+1}+3^{k_2}}(1+z^{3^{k_2}})}{1+z^{3^{k_2+2}}} = \Psi(z)\frac{z^4}{(1+z)(1+z^9)} = \sum_{n\ge 0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{011,021\}} \{0,1\}^* X, \\ 0, & \text{otherwise,} \end{cases}$$

if $k_2 = 1$, we have

$$\frac{1}{(1+z)^2}\Psi(z)\frac{z^{3^{k_2+1}+3^{k_2}}(1+z^{3^{k_2}})}{1+z^{3^{k_2+2}}} = \Psi(z)\frac{z^{12}(1-z+z^2)}{(1+z)(1+z^{27})} = \sum_{n\ge 0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{0110,0112,0120,0122,0210,0212,0220,0222\}} \{0,1\}^* X, \\ -1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{0111,0121,0211,0221\}} \{0,1\}^* X, \\ 0, & \text{otherwise,} \end{cases}$$

while, if $k_2 \ge 2$, we have

$$\frac{1}{(1+z)^2}\Psi(z)\frac{z^{3^{k_2+1}+3^{k_2}}(1+z^{3^{k_2}})}{1+z^{3^{k_2+2}}} = \sum_{n\ge 0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{011,020,021,100\},\ Y\in\{00,02\}} \{0,1\}^* X \{0,1\}^{k_2-2} Y, \\ -1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{011,020,021,100\}} \{0,1\}^* X \{0,1\}^{k_2-2} 01, \\ 2, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{012,022\},\ Y\in\{00,02\}} \{0,1\}^* X \{0,1\}^{k_2-2} Y, \\ -2, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{012,022\}} \{0,1\}^* X \{0,1\}^{k_2-2} 01, \\ 0, & \text{otherwise.} \end{cases}$$

(4) *If* $k_1 - 1 > k_2 = 0$, *we have*

$$\frac{1}{(1+z)^2}\Psi(z)\frac{z^{3^{k_1}+3^{k_2}}(1+z^{3^{k_1}})(1+z^{3^{k_2}})}{(1+z^{3^{k_1+1}})(1+z^{3^{k_2+1}})} \\ = \Psi(z)\frac{z^{3^{k_1}+1}(1+z^{3^{k_1}})}{(1+z)(1+z^3)(1+z^{3^{k_1+1}})} = \sum_{n\ge 0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{01,10\}} \{0,1\}^* X \{0,1\}^{k_1-2} 01,\\ 2, & \text{if } (n)_3 \in \{0,1\}^* 02 \{0,1\}^{k_1-2} 01,\\ 0, & \text{otherwise,} \end{cases}$$

if $k_1 - 1 > k_2 = 1$, *we have*

$$\frac{1}{(1+z)^2}\Psi(z)\frac{z^{3^{k_1}+3^{k_2}}(1+z^{3^{k_1}})(1+z^{3^{k_2}})}{(1+z^{3^{k_1+1}})(1+z^{3^{k_2+1}})} \\ = \Psi(z)\frac{z^{3^{k_1}+3}(1-z+z^2)(1+z^{3^{k_1}})}{(1+z)(1+z^9)(1+z^{3^{k_1+1}})} = \sum_{n\ge 0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{01,10\},\ Y\in\{010,012,020,022\}} \{0,1\}^* X \{0,1\}^{k_1-3} Y,\\ -1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{01,10\},\ Y\in\{011,021\}} \{0,1\}^* X \{0,1\}^{k_1-3} Y,\\ 2, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{010,012,020,022\}} \{0,1\}^* 02 \{0,1\}^{k_1-3} X,\\ -2, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{011,021\}} \{0,1\}^* 02 \{0,1\}^{k_1-3} X,\\ 0, & \text{otherwise,} \end{cases}$$

while, if $k_1 - 1 > k_2 \ge 2$, *we have*

$$\frac{1}{(1+z)^2}\Psi(z)\frac{z^{3^{k_1}+3^{k_2}}(1+z^{3^{k_1}})(1+z^{3^{k_2}})}{(1+z^{3^{k_1+1}})(1+z^{3^{k_2+1}})} = \sum_{n\ge 0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \textit{if } (n)_3 \in \displaystyle\bigcup_{X,Y\in\{01,10\},\ Z\in\{00,02\}} \{0,1\}^* X \{0,1\}^{k_1-k_2-2} Y \{0,1\}^{k_2-2} Z, \\ -1, & \textit{if } (n)_3 \in \displaystyle\bigcup_{X,Y\in\{01,10\}} \{0,1\}^* X \{0,1\}^{k_1-k_2-2} Y \{0,1\}^{k_2-2} 01, \\ 2, & \textit{if } (n)_3 \in \left(\displaystyle\bigcup_{X\in\{01,10\},\ Y\in\{00,02\}} \{0,1\}^* 02 \{0,1\}^{k_1-k_2-2} X \{0,1\}^{k_2-2} Y\right) \\ & \qquad \cup \left(\displaystyle\bigcup_{X\in\{01,10\},\ Y\in\{00,02\}} \{0,1\}^* X \{0,1\}^{k_1-k_2-2} 02 \{0,1\}^{k_2-2} Y\right), \\ -2, & \textit{if } (n)_3 \in \left(\displaystyle\bigcup_{X\in\{01,10\}} \{0,1\}^* 02 \{0,1\}^{k_1-k_2-2} X \{0,1\}^{k_2-2} 01\right) \\ & \qquad \cup \left(\displaystyle\bigcup_{X\in\{01,10\}} \{0,1\}^* X \{0,1\}^{k_1-k_2-2} 02 \{0,1\}^{k_2-2} 01\right), \\ 4, & \textit{if } (n)_3 \in \displaystyle\bigcup_{X\in\{00,02\}} \{0,1\}^* 02 \{0,1\}^{k_1-k_2-2} 02 \{0,1\}^{k_2-2} X, \\ -4, & \textit{if } (n)_3 \in \{0,1\}^* 02 \{0,1\}^{k_1-k_2-2} 02 \{0,1\}^{k_2-2} 01, \\ 0, & \textit{otherwise.} \end{cases}$$

(5) *If* $k_1 = 0$, *we have*

$$\frac{1}{(1+z)^2}\Psi(z)\frac{z^{2\cdot 3^{k_1}}(1+z^{3^{k_1}})^2}{(1+z^{3^{k_1+1}})^2} = \Psi(z)\frac{z^2}{(1+z^3)^2} = \sum_{n\ge 0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \textit{if } (n)_3 \in \displaystyle\bigcup_{X\in\{002,010,022,100\}} \{0,1\}^* X, \\ -1, & \textit{if } (n)_3 \in \displaystyle\bigcup_{X\in\{012,020\}} \{0,1\}^* X, \\ 0, & \textit{otherwise,} \end{cases}$$

if $k_1 = 1$, *we have*

$$\frac{1}{(1+z)^2}\Psi(z)\frac{z^{2\cdot 3^{k_1}}(1+z^{3^{k_1}})^2}{(1+z^{3^{k_1+1}})^2} = \Psi(z)\frac{z^6(1-z+z^2)(1+z^3)}{(1+z)(1+z^9)^2} = \sum_{n\ge 0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{0020,0022,0110,0112,0121,0211,0220,0222,1010,1012\}} \{0,1\}^*X, \\ -1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{0021,0111,0120,0122,0210,0212,0221,1011\}} \{0,1\}^*X, \\ 2, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{0100,0102,0201,1000,1002\}} \{0,1\}^*X, \\ -2, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{0101,0200,0202,1001\}} \{0,1\}^*X, \\ 0, & \text{otherwise,} \end{cases}$$

while, if $k_1 \ge 2$, we have

$$\frac{1}{(1+z)^2}\Psi(z)\frac{z^{2\cdot 3^{k_1}}(1+z^{3^{k_1}})^2}{(1+z^{3^{k_1+1}})^2} = \sum_{n\ge 0} c_n z^n,$$

where

$$c_n = \begin{cases} 1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{002,102\},\ Y\in\{00,02\}} \{0,1\}^*X\{0,1\}^{k_1-2}Y, \\ -1, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{002,102\}} \{0,1\}^*X\{0,1\}^{k_1-2}01, \\ 3, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{010,011,100,101\},\ Y\in\{00,02\}} \{0,1\}^*X\{0,1\}^{k_1-2}Y \\ & \qquad \cup \bigcup\limits_{X\in\{020,021\}} \{0,1\}^*X\{0,1\}^{k_1-2}01, \\ -3, & \text{if } (n)_3 \in \bigcup\limits_{X\in\{010,011,100,101\}} \{0,1\}^*X\{0,1\}^{k_1-2}01 \\ & \qquad \cup \bigcup\limits_{X\in\{020,021\},\ Y\in\{00,02\}} \{0,1\}^*X\{0,1\}^{k_1-2}Y, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 71. *Let s denote the 3-adic expansion of the non-negative integer n. The coefficient of z^n in the series $\Psi^5(z)$, when reduced modulo 27, equals*

(1) 1, *if, and only if, s is either an element of $\bigcup_{X\in\{000,002\}}\{0,1\}^*X$ and satisfies $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ and*

$$\tfrac{4\,\#1\mathrm{str}(s)}{3} - \#\mathrm{iso}_3(s) + \#011(s) + \#100_1(s) \\ + \mathrm{end}(s;002) + \mathrm{end}(s;1000) + \mathrm{end}(s;1002) \equiv 0 \pmod 3,$$

or of $\bigcup_{X\in\{100,102\}}\{0,1\}^*X$ and satisfies $\#1\mathrm{str}(s) \equiv 2 \pmod 3$ and

$$\tfrac{4\,\#1\mathrm{str}(s)-2}{3} - \#\mathrm{iso}_3(s) + \#011(s) + \#100_1(s) \\ + \mathrm{end}(s;100) + 2\,\mathrm{end}(s;0102) \equiv 0 \pmod 3;$$

(2) 10, *if, and only if, s is either an element of $\bigcup_{X\in\{000,002\}}\{0,1\}^*X$ and satisfies $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ and*

$$\tfrac{4\,\#1\mathrm{str}(s)}{3} - \#\mathrm{iso}_3(s) + \#011(s) + \#100_1(s) \\ + \mathrm{end}(s;002) + \mathrm{end}(s;1000) + \mathrm{end}(s;1002) \equiv 1 \pmod 3,$$

or of $\bigcup_{X\in\{100,102\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s) \equiv 2 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-2}{3} - \#\mathrm{iso}_3(s) + \#011(s) + \#100_1(s) \\ + \mathrm{end}(s;100) + 2\,\mathrm{end}(s;0102) \equiv 1 \pmod 3;$$

(3) 19, *if, and only if,* s *is either an element of* $\bigcup_{X\in\{000,002\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)}{3} - \#\mathrm{iso}_3(s) + \#011(s) + \#100_1(s) \\ + \mathrm{end}(s;002) + \mathrm{end}(s;1000) + \mathrm{end}(s;1002) \equiv 2 \pmod 3,$$

or of $\bigcup_{X\in\{100,102\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s) \equiv 2 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-2}{3} - \#\mathrm{iso}_3(s) + \#011(s) + \#100_1(s) \\ + \mathrm{end}(s;100) + 2\,\mathrm{end}(s;0102) \equiv 2 \pmod 3;$$

(4) 4, *if, and only if,* s *is either an element of* $\bigcup_{X\in\{000,002\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s) \equiv 1 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-1}{3} - \#\mathrm{iso}_3(s) + \#011(s) + \#100_1(s) \\ + \mathrm{end}(s;002) + \mathrm{end}(s;1000) + \mathrm{end}(s;1002) \equiv 0 \pmod 3,$$

or of $\bigcup_{X\in\{100,102\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-3}{3} - \#\mathrm{iso}_3(s) + \#011(s) + \#100_1(s) \\ + \mathrm{end}(s;100) + 2\,\mathrm{end}(s;0102) \equiv 0 \pmod 3;$$

(5) 13, *if, and only if,* s *is either an element of* $\bigcup_{X\in\{000,002\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s) \equiv 1 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-1}{3} - \#\mathrm{iso}_3(s) + \#011(s) + \#100_1(s) \\ + \mathrm{end}(s;002) + \mathrm{end}(s;1000) + \mathrm{end}(s;1002) \equiv 1 \pmod 3,$$

or of $\bigcup_{X\in\{100,102\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-3}{3} - \#\mathrm{iso}_3(s) + \#011(s) + \#100_1(s) \\ + \mathrm{end}(s;100) + 2\,\mathrm{end}(s;0102) \equiv 1 \pmod 3;$$

(6) 22, *if, and only if,* s *is either an element of* $\bigcup_{X\in\{000,002\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s) \equiv 1 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-1}{3} - \#\mathrm{iso}_3(s) + \#011(s) + \#100_1(s) \\ + \mathrm{end}(s;002) + \mathrm{end}(s;1000) + \mathrm{end}(s;1002) \equiv 2 \pmod 3,$$

or of $\bigcup_{X\in\{100,102\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-3}{3} - \#\mathrm{iso}_3(s) + \#011(s) + \#100_1(s)$$

$$+\operatorname{end}(s;100)+2\operatorname{end}(s;0102)\equiv 2 \pmod 3;$$

(7) 7, *if, and only if,* s *is either an element of* $\bigcup_{X\in\{000,002\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s)\equiv 2 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-2}{3}-\#\mathrm{iso}_3(s)+\#011(s)+\#100_1(s)\\ +\operatorname{end}(s;002)+\operatorname{end}(s;1000)+\operatorname{end}(s;1002)\equiv 0 \pmod 3,$$

or of $\bigcup_{X\in\{100,102\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s)\equiv 1 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-4}{3}-\#\mathrm{iso}_3(s)+\#011(s)+\#100_1(s)\\ +\operatorname{end}(s;100)+2\operatorname{end}(s;0102)\equiv 0 \pmod 3;$$

(8) 16, *if, and only if,* s *is either an element of* $\bigcup_{X\in\{000,002\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s)\equiv 2 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-2}{3}-\#\mathrm{iso}_3(s)+\#011(s)+\#100_1(s)\\ +\operatorname{end}(s;002)+\operatorname{end}(s;1000)+\operatorname{end}(s;1002)\equiv 1 \pmod 3,$$

or of $\bigcup_{X\in\{100,102\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s)\equiv 1 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-4}{3}-\#\mathrm{iso}_3(s)+\#011(s)+\#100_1(s)\\ +\operatorname{end}(s;100)+2\operatorname{end}(s;0102)\equiv 1 \pmod 3;$$

(9) 25, *if, and only if,* s *is either an element of* $\bigcup_{X\in\{000,002\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s)\equiv 2 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-2}{3}-\#\mathrm{iso}_3(s)+\#011(s)+\#100_1(s)\\ +\operatorname{end}(s;002)+\operatorname{end}(s;1000)+\operatorname{end}(s;1002)\equiv 2 \pmod 3,$$

or of $\bigcup_{X\in\{100,102\}}\{0,1\}^*X$ *and satisfies* $\#1\mathrm{str}(s)\equiv 1 \pmod 3$ *and*

$$\tfrac{4\,\#1\mathrm{str}(s)-4}{3}-\#\mathrm{iso}_3(s)+\#011(s)+\#100_1(s)\\ +\operatorname{end}(s;100)+2\operatorname{end}(s;0102)\equiv 2 \pmod 3;$$

(10) 2, *if, and only if,* s *is either an element of* $\{0,1\}^*001$ *and satisfies* $\#1\mathrm{str}(s)\equiv 1 \pmod 3$ *and*

$$\tfrac{1-4\,\#1\mathrm{str}(s)}{3}+\#\mathrm{iso}_3(s)-\#011(s)-\#100_1(s)-\operatorname{end}(s;1001)\equiv 2 \pmod 3,$$

or of $\{0,1\}^*101$ *and satisfies* $\#1\mathrm{str}(s)\equiv 0 \pmod 3$ *and*

$$\tfrac{3-4\,\#1\mathrm{str}(s)}{3}+\#\mathrm{iso}_3(s)-\#011(s)-\#100_1(s)-2\operatorname{end}(s;1001)\equiv 2 \pmod 3;$$

(11) 11, *if, and only if,* s *is either an element of* $\{0,1\}^*001$ *and satisfies* $\#1\mathrm{str}(s)\equiv 1 \pmod 3$ *and*

$$\tfrac{1-4\,\#1\mathrm{str}(s)}{3}+\#\mathrm{iso}_3(s)-\#011(s)-\#100_1(s)-\operatorname{end}(s;1001)\equiv 0 \pmod 3,$$

or of $\{0,1\}^*101$ *and satisfies* $\#1\mathrm{str}(s)\equiv 0 \pmod 3$ *and*

$$\tfrac{3-4\,\#1\mathrm{str}(s)}{3}+\#\mathrm{iso}_3(s)-\#011(s)-\#100_1(s)-2\operatorname{end}(s;1001)\equiv 0 \pmod 3;$$

(12) 20, *if, and only if, s is either an element of* $\{0,1\}^*001$ *and satisfies* $\#1\mathrm{str}(s) \equiv 1 \pmod 3$ *and*
$$\tfrac{1-4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - \mathrm{end}(s;1001) \equiv 1 \pmod 3,$$
or of $\{0,1\}^*101$ *and satisfies* $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ *and*
$$\tfrac{3-4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - 2\,\mathrm{end}(s;1001) \equiv 1 \pmod 3;$$

(13) 5, *if, and only if, s is either an element of* $\{0,1\}^*001$ *and satisfies* $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ *and*
$$-\tfrac{4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - \mathrm{end}(s;1001) \equiv 2 \pmod 3,$$
or of $\{0,1\}^*101$ *and satisfies* $\#1\mathrm{str}(s) \equiv 2 \pmod 3$ *and*
$$\tfrac{2-4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - 2\,\mathrm{end}(s;1001) \equiv 2 \pmod 3;$$

(14) 14, *if, and only if, s is either an element of* $\{0,1\}^*001$ *and satisfies* $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ *and*
$$-\tfrac{4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - \mathrm{end}(s;1001) \equiv 0 \pmod 3,$$
or of $\{0,1\}^*101$ *and satisfies* $\#1\mathrm{str}(s) \equiv 2 \pmod 3$ *and*
$$\tfrac{2-4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - 2\,\mathrm{end}(s;1001) \equiv 0 \pmod 3;$$

(15) 23, *if, and only if, s is either an element of* $\{0,1\}^*001$ *and satisfies* $\#1\mathrm{str}(s) \equiv 0 \pmod 3$ *and*
$$-\tfrac{4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - \mathrm{end}(s;1001) \equiv 1 \pmod 3,$$
or of $\{0,1\}^*101$ *and satisfies* $\#1\mathrm{str}(s) \equiv 2 \pmod 3$ *and*
$$\tfrac{2-4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - 2\,\mathrm{end}(s;1001) \equiv 1 \pmod 3;$$

(16) 8, *if, and only if, s is either an element of* $\{0,1\}^*001$ *and satisfies* $\#1\mathrm{str}(s) \equiv 2 \pmod 3$ *and*
$$-\tfrac{1+4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - \mathrm{end}(s;1001) \equiv 2 \pmod 3,$$
or of $\{0,1\}^*101$ *and satisfies* $\#1\mathrm{str}(s) \equiv 1 \pmod 3$ *and*
$$\tfrac{1-4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - 2\,\mathrm{end}(s;1001) \equiv 2 \pmod 3;$$

(17) 17, *if, and only if, s is either an element of* $\{0,1\}^*001$ *and satisfies* $\#1\mathrm{str}(s) \equiv 2 \pmod 3$ *and*
$$-\tfrac{1+4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - \mathrm{end}(s;1001) \equiv 0 \pmod 3,$$
or of $\{0,1\}^*101$ *and satisfies* $\#1\mathrm{str}(s) \equiv 1 \pmod 3$ *and*
$$\tfrac{1-4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - 2\,\mathrm{end}(s;1001) \equiv 0 \pmod 3;$$

(18) 26, *if, and only if, s is either an element of* $\{0,1\}^*001$ *and satisfies* $\#1\mathrm{str}(s) \equiv 2 \pmod 3$ *and*
$$-\tfrac{1+4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - \mathrm{end}(s;1001) \equiv 1 \pmod 3,$$
or of $\{0,1\}^*101$ *and satisfies* $\#1\mathrm{str}(s) \equiv 1 \pmod 3$ *and*
$$\tfrac{1-4\,\#1\mathrm{str}(s)}{3} + \#\mathrm{iso}_3(s) - \#011(s) - \#100_1(s) - 2\,\mathrm{end}(s;1001) \equiv 1 \pmod 3;$$

(19) 9, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\Bigg(\bigcup_{X\in\{0111,0121,0211,0221\}}\{0,1\}^*X\Bigg)\cup\Bigg(\bigcup_{X\in\{012,022\}}\{0,1\}^*X\{0,1\}^*01\Bigg)\\ \cup\Bigg(\bigcup_{X\in\{011,021\}}\{0,1\}^*02\{0,1\}^*X\Bigg)\cup\Bigg(\bigcup_{X\in\{00,02\}}\{0,1\}^*02\{0,1\}^*02\{0,1\}^*X\Bigg);$$

(20) 18, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\Bigg(\bigcup_{X\in\{0110,0112,0220,0222\}}\{0,1\}^*X\Bigg)\cup\Bigg(\bigcup_{X\in\{012,022\},\ Y\in\{00,02\}}\{0,1\}^*X\{0,1\}^*Y\Bigg)\\ \cup\Bigg(\bigcup_{X\in\{010,012,020,022\}}\{0,1\}^*02\{0,1\}^*X\Bigg)\cup\Big(\{0,1\}^*02\{0,1\}^*02\{0,1\}^*01\Big);$$

(21) 6, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\Bigg(\bigcup_{k\ge0}\big(11^*00^*\big)^{3k+2}0201\Bigg)\cup\Bigg(\bigcup_{k_1+k_2\equiv1\,(\text{mod }3)}\big(11^*00^*\big)^{k_1}0200^*\big(11^*00^*\big)^{k_2}01\Bigg)\\ \cup\Bigg(\bigcup_{k_1+k_2\equiv2\,(\text{mod }3)}\big(11^*00^*\big)^{k_1}11^*0200^*\big(11^*00^*\big)^{k_2}01\Bigg)\\ \cup\Bigg(\bigcup_{k_1+k_2\equiv0\,(\text{mod }3),\,k_2>0}\big(11^*00^*\big)^{k_1}02\big(11^*00^*\big)^{k_2}01\Bigg)\\ \cup\Bigg(\bigcup_{k_1+k_2\equiv1\,(\text{mod }3),\,k_2>0}\big(11^*00^*\big)^{k_1}11^*02\big(11^*00^*\big)^{k_2}01\Bigg)\\ \cup\Bigg(\bigcup_{k_1+k_2\equiv2\,(\text{mod }3)}\big(11^*00^*\big)^{k_1}0200^*\big(11^*00^*\big)^{k_2}11^*01\Bigg)\\ \cup\Bigg(\bigcup_{k_1+k_2\equiv0\,(\text{mod }3)}\big(11^*00^*\big)^{k_1}11^*0200^*\big(11^*00^*\big)^{k_2}11^*01\Bigg)\\ \cup\Bigg(\bigcup_{k_1+k_2\equiv1\,(\text{mod }3)}\big(11^*00^*\big)^{k_1}02\big(11^*00^*\big)^{k_2}11^*01\Bigg)\\ \cup\Bigg(\bigcup_{k_1+k_2\equiv2\,(\text{mod }3)}\big(11^*00^*\big)^{k_1}11^*02\big(11^*00^*\big)^{k_2}11^*01\Bigg)\\ \cup\Big(\big(11^*00^*\big)^{3k+1}010\Big)\cup\Big(\big(11^*00^*\big)^{3k+2}012\Big)\\ \cup\Big(\big(11^*00^*\big)^{3k}020\Big)\cup\Big(\big(11^*00^*\big)^{3k+2}022\Big)\\ \cup\Big(\big(11^*00^*\big)^{3k}11^*010\Big)\cup\Big(\big(11^*00^*\big)^{3k+1}11^*012\Big)\\ \cup\Big(\big(11^*00^*\big)^{3k}11^*020\Big)\cup\Big(\big(11^*00^*\big)^{3k+2}11^*022\Big);$$

(22) 15, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\begin{aligned}
\Big(\bigcup_{k\ge0}(11^*00^*)^{3k+1}0201\Big)\cup\Big(\bigcup_{k_1+k_2\equiv0\,(\mathrm{mod}\ 3)}(11^*00^*)^{k_1}0200^*(11^*00^*)^{k_2}01\Big)\\
\cup\Big(\bigcup_{k_1+k_2\equiv1\,(\mathrm{mod}\ 3)}(11^*00^*)^{k_1}11^*0200^*(11^*00^*)^{k_2}01\Big)\\
\cup\Big(\bigcup_{k_1+k_2\equiv2\,(\mathrm{mod}\ 3),\,k_2>0}(11^*00^*)^{k_1}02(11^*00^*)^{k_2}01\Big)\\
\cup\Big(\bigcup_{k_1+k_2\equiv0\,(\mathrm{mod}\ 3),\,k_2>0}(11^*00^*)^{k_1}11^*02(11^*00^*)^{k_2}01\Big)\\
\cup\Big(\bigcup_{k_1+k_2\equiv1\,(\mathrm{mod}\ 3)}(11^*00^*)^{k_1}0200^*(11^*00^*)^{k_2}11^*01\Big)\\
\cup\Big(\bigcup_{k_1+k_2\equiv2\,(\mathrm{mod}\ 3)}(11^*00^*)^{k_1}11^*0200^*(11^*00^*)^{k_2}11^*01\Big)\\
\cup\Big(\bigcup_{k_1+k_2\equiv0\,(\mathrm{mod}\ 3)}(11^*00^*)^{k_1}02(11^*00^*)^{k_2}11^*01\Big)\\
\cup\Big(\bigcup_{k_1+k_2\equiv1\,(\mathrm{mod}\ 3)}(11^*00^*)^{k_1}11^*02(11^*00^*)^{k_2}11^*01\Big)\\
\cup\Big((11^*00^*)^{3k}010\Big)\cup\Big((11^*00^*)^{3k+1}012\Big)\\
\cup\Big((11^*00^*)^{3k+2}020\Big)\cup\Big((11^*00^*)^{3k+1}022\Big)\\
\cup\Big((11^*00^*)^{3k+2}11^*010\Big)\cup\Big((11^*00^*)^{3k}11^*012\Big)\\
\cup\Big((11^*00^*)^{3k+2}11^*020\Big)\cup\Big((11^*00^*)^{3k+1}11^*022\Big);
\end{aligned}$$

(23) 24, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\begin{aligned}
\Big(\bigcup_{k\ge0}(11^*00^*)^{3k}0201\Big)\cup\Big(\bigcup_{k_1+k_2\equiv2\,(\mathrm{mod}\ 3)}(11^*00^*)^{k_1}0200^*(11^*00^*)^{k_2}01\Big)\\
\cup\Big(\bigcup_{k_1+k_2\equiv0\,(\mathrm{mod}\ 3)}(11^*00^*)^{k_1}11^*0200^*(11^*00^*)^{k_2}01\Big)\\
\cup\Big(\bigcup_{k_1+k_2\equiv1\,(\mathrm{mod}\ 3),\,k_2>0}(11^*00^*)^{k_1}02(11^*00^*)^{k_2}01\Big)\\
\cup\Big(\bigcup_{k_1+k_2\equiv2\,(\mathrm{mod}\ 3),\,k_2>0}(11^*00^*)^{k_1}11^*02(11^*00^*)^{k_2}01\Big)
\end{aligned}$$

$$\cup\Big(\bigcup_{k_1+k_2\equiv 0\ (\mathrm{mod}\ 3)}\big(11^*00^*\big)^{k_1}0200^*\big(11^*00^*\big)^{k_2}11^*01\Big)$$
$$\cup\Big(\bigcup_{k_1+k_2\equiv 1\ (\mathrm{mod}\ 3)}\big(11^*00^*\big)^{k_1}11^*0200^*\big(11^*00^*\big)^{k_2}11^*01\Big)$$
$$\cup\Big(\bigcup_{k_1+k_2\equiv 2\ (\mathrm{mod}\ 3)}\big(11^*00^*\big)^{k_1}02\big(11^*00^*\big)^{k_2}11^*01\Big)$$
$$\cup\Big(\bigcup_{k_1+k_2\equiv 0\ (\mathrm{mod}\ 3)}\big(11^*00^*\big)^{k_1}11^*02\big(11^*00^*\big)^{k_2}11^*01\Big)$$
$$\cup\Big(\big(11^*00^*\big)^{3k+2}010\Big)\cup\Big(\big(11^*00^*\big)^{3k}012\Big)$$
$$\cup\Big(\big(11^*00^*\big)^{3k+1}020\Big)\cup\Big(\big(11^*00^*\big)^{3k}022\Big)$$
$$\cup\Big(\big(11^*00^*\big)^{3k+1}11^*010\Big)\cup\Big(\big(11^*00^*\big)^{3k+2}11^*012\Big)$$
$$\cup\Big(\big(11^*00^*\big)^{3k+1}11^*020\Big)\cup\Big(\big(11^*00^*\big)^{3k}11^*022\Big);$$

(24) 3, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\Big(\bigcup_{k\ge 0,\ X\in\{00,02\}}\big(11^*00^*\big)^{3k+1}02X\Big)$$
$$\cup\Big(\bigcup_{k_1+k_2\equiv 0\ (\mathrm{mod}\ 3),\ X\in\{00,02\}}\big(11^*00^*\big)^{k_1}0200^*\big(11^*00^*\big)^{k_2}X\Big)$$
$$\cup\Big(\bigcup_{k_1+k_2\equiv 1\ (\mathrm{mod}\ 3),\ X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*0200^*\big(11^*00^*\big)^{k_2}X\Big)$$
$$\cup\Big(\bigcup_{k_1+k_2\equiv 2\ (\mathrm{mod}\ 3),\ k_2>0,\ X\in\{00,02\}}\big(11^*00^*\big)^{k_1}02\big(11^*00^*\big)^{k_2}X\Big)$$
$$\cup\Big(\bigcup_{k_1+k_2\equiv 0\ (\mathrm{mod}\ 3),\ k_2>0,\ X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*02\big(11^*00^*\big)^{k_2}X\Big)$$
$$\cup\Big(\bigcup_{k_1+k_2\equiv 1\ (\mathrm{mod}\ 3),\ X\in\{00,02\}}\big(11^*00^*\big)^{k_1}0200^*\big(11^*00^*\big)^{k_2}11^*X\Big)$$
$$\cup\Big(\bigcup_{k_1+k_2\equiv 2\ (\mathrm{mod}\ 3),\ X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*0200^*\big(11^*00^*\big)^{k_2}11^*X\Big)$$
$$\cup\Big(\bigcup_{k_1+k_2\equiv 0\ (\mathrm{mod}\ 3),\ X\in\{00,02\}}\big(11^*00^*\big)^{k_1}02\big(11^*00^*\big)^{k_2}11^*X\Big)$$
$$\cup\Big(\bigcup_{k_1+k_2\equiv 1\ (\mathrm{mod}\ 3),\ X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*02\big(11^*00^*\big)^{k_2}11^*X\Big)$$

$$\cup\Big(\big(11^*00^*\big)^{3k+1}011\Big)\cup\Big(\big(11^*00^*\big)^{3k+2}021\Big)$$
$$\cup\Big(\big(11^*00^*\big)^{3k}11^*011\Big)\cup\Big(\big(11^*00^*\big)^{3k+2}11^*021\Big);$$

(25) 12, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\Bigg(\bigcup_{k\ge0,\,X\in\{00,02\}}\big(11^*00^*\big)^{3k+2}02X\Bigg)$$
$$\cup\Bigg(\bigcup_{k_1+k_2\equiv1\,(\text{mod }3),\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}0200^*\big(11^*00^*\big)^{k_2}X\Bigg)$$
$$\cup\Bigg(\bigcup_{k_1+k_2\equiv2\,(\text{mod }3),\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*0200^*\big(11^*00^*\big)^{k_2}X\Bigg)$$
$$\cup\Bigg(\bigcup_{k_1+k_2\equiv0\,(\text{mod }3),\,k_2>0,\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}02\big(11^*00^*\big)^{k_2}X\Bigg)$$
$$\cup\Bigg(\bigcup_{k_1+k_2\equiv1\,(\text{mod }3),\,k_2>0,\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*02\big(11^*00^*\big)^{k_2}X\Bigg)$$
$$\cup\Bigg(\bigcup_{k_1+k_2\equiv2\,(\text{mod }3),\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}0200^*\big(11^*00^*\big)^{k_2}11^*X\Bigg)$$
$$\cup\Bigg(\bigcup_{k_1+k_2\equiv0\,(\text{mod }3),\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*0200^*\big(11^*00^*\big)^{k_2}11^*X\Bigg)$$
$$\cup\Bigg(\bigcup_{k_1+k_2\equiv1\,(\text{mod }3),\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}02\big(11^*00^*\big)^{k_2}11^*X\Bigg)$$
$$\cup\Bigg(\bigcup_{k_1+k_2\equiv2\,(\text{mod }3),\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*02\big(11^*00^*\big)^{k_2}11^*X\Bigg)$$
$$\cup\Big(\big(11^*00^*\big)^{3k+2}011\Big)\cup\Big(\big(11^*00^*\big)^{3k}021\Big)$$
$$\cup\Big(\big(11^*00^*\big)^{3k+1}11^*011\Big)\cup\Big(\big(11^*00^*\big)^{3k}11^*021\Big);$$

(26) 21, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\Bigg(\bigcup_{k\ge0,\,X\in\{00,02\}}\big(11^*00^*\big)^{3k}02X\Bigg)$$
$$\cup\Bigg(\bigcup_{k_1+k_2\equiv2\,(\text{mod }3),\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}0200^*\big(11^*00^*\big)^{k_2}X\Bigg)$$
$$\cup\Bigg(\bigcup_{k_1+k_2\equiv0\,(\text{mod }3),\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*0200^*\big(11^*00^*\big)^{k_2}X\Bigg)$$

$$\cup\left(\bigcup_{k_1+k_2\equiv 1\ (\mathrm{mod}\ 3),\, k_2>0,\, X\in\{00,02\}}\big(11^*00^*\big)^{k_1}02\big(11^*00^*\big)^{k_2}X\right)$$
$$\cup\left(\bigcup_{k_1+k_2\equiv 2\ (\mathrm{mod}\ 3),\, k_2>0,\, X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*02\big(11^*00^*\big)^{k_2}X\right)$$
$$\cup\left(\bigcup_{k_1+k_2\equiv 0\ (\mathrm{mod}\ 3),\, X\in\{00,02\}}\big(11^*00^*\big)^{k_1}0200^*\big(11^*00^*\big)^{k_2}11^*X\right)$$
$$\cup\left(\bigcup_{k_1+k_2\equiv 1\ (\mathrm{mod}\ 3),\, X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*0200^*\big(11^*00^*\big)^{k_2}11^*X\right)$$
$$\cup\left(\bigcup_{k_1+k_2\equiv 2\ (\mathrm{mod}\ 3),\, X\in\{00,02\}}\big(11^*00^*\big)^{k_1}02\big(11^*00^*\big)^{k_2}11^*X\right)$$
$$\cup\left(\bigcup_{k_1+k_2\equiv 0\ (\mathrm{mod}\ 3),\, X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*02\big(11^*00^*\big)^{k_2}11^*X\right)$$
$$\cup\Big(\big(11^*00^*\big)^{3k}011\Big)\cup\Big(\big(11^*00^*\big)^{3k+1}021\Big)$$
$$\cup\Big(\big(11^*00^*\big)^{3k+2}11^*011\Big)\cup\Big(\big(11^*00^*\big)^{3k+1}11^*021\Big).$$

(27) *In all other cases, this coefficient is divisible by* 27.

If the above proposition is specialised to modulus 9, then one obtains the following result.

Corollary 72. *Let s denote the 3-adic expansion of the non-negative integer n. The coefficient of z^n in the series $\Psi^5(z)$, when reduced modulo 9, equals*

(1) 1, *if, and only if, s is an element of*

$$\{0\}\cup\bigcup_{k\ge0}\big(00^*11^*\big)^{3k}000\cup\bigcup_{k\ge0}\big(00^*11^*\big)^{3k}002$$
$$\cup\bigcup_{k\ge0}\big(00^*11^*\big)^{3k+2}00\cup\bigcup_{k\ge0}\big(00^*11^*\big)^{3k+2}02;$$

(2) 4, *if, and only if, s is an element of*

$$\bigcup_{k\ge0}\big(00^*11^*\big)^{3k+1}000\cup\bigcup_{k\ge0}\big(00^*11^*\big)^{3k+1}002\cup\bigcup_{k\ge0}\big(00^*11^*\big)^{3k+3}00\cup\bigcup_{k\ge0}\big(00^*11^*\big)^{3k+3}02;$$

(3) 7, *if, and only if, s is an element of*

$$\bigcup_{k\ge0}\big(00^*11^*\big)^{3k+2}000\cup\bigcup_{k\ge0}\big(00^*11^*\big)^{3k+2}002\cup\bigcup_{k\ge0}\big(00^*11^*\big)^{3k+1}00\cup\bigcup_{k\ge0}\big(00^*11^*\big)^{3k+1}02;$$

(4) 2, *if, and only if, s is an element of*

$$\bigcup_{k\ge0}\big(00^*11^*\big)^{3k+1}001\cup\bigcup_{k\ge0}\big(00^*11^*\big)^{3k+3}01;$$

(5) 5, *if, and only if, s is an element of*

$$\bigcup_{k\ge 0}\big(00^*11^*\big)^{3k}001\cup\bigcup_{k\ge 0}\big(00^*11^*\big)^{3k+2}01;$$

(6) 8, *if, and only if, s is an element of*

$$\bigcup_{k\ge 0}\big(00^*11^*\big)^{3k+2}001\cup\bigcup_{k\ge 0}\big(00^*11^*\big)^{3k+1}01;$$

(7) 6, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\begin{aligned}\Bigg(\bigcup_{k\ge 0}\big(11^*00^*\big)^{k}0201\Bigg)&\cup\Bigg(\bigcup_{k_1,k_2\ge 0}\big(11^*00^*\big)^{k_1}0200^*\big(11^*00^*\big)^{k_2}01\Bigg)\\
&\cup\Bigg(\bigcup_{k_1,k_2\ge 0}\big(11^*00^*\big)^{k_1}11^*0200^*\big(11^*00^*\big)^{k_2}01\Bigg)\\
&\cup\Bigg(\bigcup_{k_1\ge 0,\,k_2>0}\big(11^*00^*\big)^{k_1}02\big(11^*00^*\big)^{k_2}01\Bigg)\\
&\cup\Bigg(\bigcup_{k_1\ge 0,\,k_2>0}\big(11^*00^*\big)^{k_1}11^*02\big(11^*00^*\big)^{k_2}01\Bigg)\\
&\cup\Bigg(\bigcup_{k_1,k_2\ge 0}\big(11^*00^*\big)^{k_1}0200^*\big(11^*00^*\big)^{k_2}11^*01\Bigg)\\
&\cup\Bigg(\bigcup_{k_1,k_2\ge 0}\big(11^*00^*\big)^{k_1}11^*0200^*\big(11^*00^*\big)^{k_2}11^*01\Bigg)\\
&\cup\Bigg(\bigcup_{k_1,k_2\ge 0}\big(11^*00^*\big)^{k_1}02\big(11^*00^*\big)^{k_2}11^*01\Bigg)\\
&\cup\Bigg(\bigcup_{k_1,k_2\ge 0}\big(11^*00^*\big)^{k_1}11^*02\big(11^*00^*\big)^{k_2}11^*01\Bigg)\\
\cup\Big(\big(11^*00^*\big)^{k}010\Big)\cup\Big(\big(11^*00^*\big)^{k}012\Big)&\cup\Big(\big(11^*00^*\big)^{k}020\Big)\cup\Big(\big(11^*00^*\big)^{k}022\Big)\\
&\cup\Big(\big(11^*00^*\big)^{k}11^*010\Big)\cup\Big(\big(11^*00^*\big)^{k}11^*012\Big)\\
&\qquad\cup\Big(\big(11^*00^*\big)^{k}11^*020\Big)\cup\Big(\big(11^*00^*\big)^{k}11^*022\Big);\end{aligned}$$

(8) 3, *if, and only if, the* 3*-adic expansion of* n *is an element of*

$$\begin{aligned}&\Bigg(\bigcup_{k\ge 0,\,X\in\{00,02\}}\big(11^*00^*\big)^{k}02X\Bigg)\\
&\qquad\cup\Bigg(\bigcup_{k_1,k_2\ge 0,\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}0200^*\big(11^*00^*\big)^{k_2}X\Bigg)\\
&\qquad\cup\Bigg(\bigcup_{k_1,k_2\ge 0,\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*0200^*\big(11^*00^*\big)^{k_2}X\Bigg)\end{aligned}$$

$$\cup\Bigg(\bigcup_{k_1,k_2\ge0,\,k_2>0,\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}02\big(11^*00^*\big)^{k_2}X\Bigg)$$

$$\cup\Bigg(\bigcup_{k_1,k_2\ge0,\,k_2>0,\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*02\big(11^*00^*\big)^{k_2}X\Bigg)$$

$$\cup\Bigg(\bigcup_{k_1,k_2\ge0,\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}0200^*\big(11^*00^*\big)^{k_2}11^*X\Bigg)$$

$$\cup\Bigg(\bigcup_{k_1,k_2\ge0,\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*0200^*\big(11^*00^*\big)^{k_2}11^*X\Bigg)$$

$$\cup\Bigg(\bigcup_{k_1,k_2\ge0,\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}02\big(11^*00^*\big)^{k_2}11^*X\Bigg)$$

$$\cup\Bigg(\bigcup_{k_1,k_2\ge0,\,X\in\{00,02\}}\big(11^*00^*\big)^{k_1}11^*02\big(11^*00^*\big)^{k_2}11^*X\Bigg)$$

$$\cup\Big(\big(11^*00^*\big)^{k}011\Big)\cup\Big(\big(11^*00^*\big)^{k}021\Big)\cup\Big(\big(11^*00^*\big)^{k}11^*011\Big)\cup\Big(\big(11^*00^*\big)^{k}11^*021\Big);$$

(9) *In all other cases, this coefficient is divisible by* 9.

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